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Typical Structure of Oriented Graphs and Digraphs with Forbidden Blow-Up Transitive Triangles

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Abstract: Transitive tournament (including transitive triangle) and its blow-up have some symmetric properties. In this work, we establish an analogue result of the Erdös-Stone theorem of weighted digraphs with a forbidden blow-up of the transitive tournament. We give a stability result of oriented graphs and digraphs with forbidden blow-up transitive triangles and show that almost all oriented graphs and digraphs with forbidden blow-up transitive triangles are almost bipartite, which reconfirms and strengthens the conjecture of Cherlin.

Keywords: forbidden digraph; Erdös-Stone theorem; transitive triangle; blow-up

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Citation: Liang, M.; Liu, J. Typical Structure of Oriented Graphs and Digraphs with Forbidden Blow-Up Transitive Triangles. *Symmetry* **2022**, *14*, 2551. https://doi.org/10.3390/sym14122551

Academic Editor: Manuel Lafond

Received: 27 October 2022
Accepted: 26 November 2022
Published: 2 December 2022

1. Introduction

Given a fixed graph H, a graph is called H-free if it does not contain a subgraph isomorphic to H. For brevity's sake, other mentioned notations in the section are provided later in Section 2. In the study history of extremal graph theory, there are two types of important problems: (1) What are the maximum edges among H-free graphs on n vertices? (2) What is the typical structure of H-free graphs on n vertices? It is natural to consider graphs with some symmetry property, for example, the complete graphs K_r, cycles C_k on r vertices, respectively. The significant progress of the first problem was made by Turán in 1941 who determined the maximum edges among K_{r+1}-free graphs on n vertices and the corresponding extremal graphs. In 1946 Erdös and Stone [1] extended Turán's theorem by replacing the forbidden subgraph K_{r+1} by its blow-up and asymptotically determined the maximum edges. The second problem started in 1976 when Erdös, Kleitman and Rothschild [2] showed that almost all K_3-free graphs are bipartite and asymptotically determined the logarithm of the number of K_{r+1}-free graphs on n vertices, for every integer $r \geq 2$. This was strengthened by Kolaitis, Prömel and Rothschild [3], who showed that almost all K_{r+1}-free graphs are r-partite, for every integer $r \geq 2$. These work inspired a vast body of works concerning the maximum edges, the number and structure of H-free graphs among H-free graphs respectively (see, e.g., [3–11]). More recently, some related results have been proved for hypergraphs (see, e.g., [12,13]).

All the works mentioned above dealt with undirected graphs. It is natural to generalize those results to digraphs or oriented graphs. The study of the first problem of digraphs and oriented graphs started by Brown and Harary [14] in 1970. They considered and determined the n-vertex digraphs with maximum edges and not containing the transitive tournament T_{r+1} in [14]. In 2017, Kühn, Osthus, Townsend and Zhao [15] extended this result to weighted digraphs. However, the similar result of the Erdös and Stone [1] theorem for weighted digraphs is still open. In this work, we will establish an analogue Erdös and Stone result of weighted digraphs.

For the study of the second problem of digraphs and oriented graphs, Cherlin [16] gave a classification of countable homogeneous oriented graphs. He remarked that 'the striking work of [3] does not appear to go over to the directed case' and conjectured that almost all T_3-free oriented graphs are tripartite in 1998. Kühn, Osthus, Townsend and

Zhao [15] verified this conjecture and showed that almost all T_{r+1}-free oriented graphs and almost all T_{r+1}-free digraphs are r-partite.

It is natural to ask a similar question: what are the typical structures of digraphs and oriented graphs not containing the blow-up of T_{r+1}? In this work, we shall reconfirm and generalize Cherlin's conjecture [15]. We show that almost all T_3^t-free oriented graphs and almost all T_3^t-free digraphs are almost bipartite for any positive integer t, where T_3^t is the blow-up of the transitive triangle T_3.

The rest of the paper is organized as followed. Some notations and useful tools and our results will be laid out in Section 2. The Regularity Lemma of digraphs and the proof of our first result will be given in Section 3. A stability result of digraphs and the proof of our second result will be given in Section 4. The concluding remarks will be given in Section 5.

2. Notations, Tools and Results

We shall give some notions before we start to state some relevant results. A *digraph* is a pair (V, E) where V is a set of vertices and E is a set of ordered pairs of distinct vertices in V (note that this means we do not allow loops or multiple arcs in the same direction in a digraph). An *oriented graph* is a digraph with at most one arc between two vertices, so may be considered as an orientation of a simple undirected graph.

Given a class of graphs (digraphs or oriented graphs, respectively) $\mathcal{A}$, we let $\mathcal{A}_n$ denote the set of all graphs (digraphs or oriented graphs, respectively) in $\mathcal{A}$ that have precisely n vertices, and we say that *almost* all graphs (digraphs or oriented graphs, respectively) in $\mathcal{A}$ have property $\mathcal{B}$ or, the typical structure of $\mathcal{A}$ is $\mathcal{B}$ if

$$\lim_{n\to\infty} \frac{|\{G \in \mathcal{A}_n : G \text{ has property } \mathcal{B}\}|}{|\mathcal{A}_n|} = 1.$$

Let $G = (V, E)$ be a digraph, we write uv for the arc directed from vertex u to vertex v. Denote by $N_G^+(v) := \{u \in V : vu \in E\}$ and $N_G^-(v) := \{u \in V : uv \in E\}$ the out-neighborhood and the in-neighborhood of v, respectively. Denote by $d_G^+(v) := |N_G^+(v)|$ and $d_G^-(v) := |N_G^-(v)|$ the out-degree and the in-degree $d_G^-(v)$ of v, respectively. Denote by $N_G(v) := N_G^- \bigcup N_G^+$ and $N_G^\pm(v) := N_G^- \bigcap N_G^+$ the neighborhood and the intersection neighborhood of v. Denote by $\Delta(G), \Delta^+(G)$ and $\Delta^-(G)$ the maximum of $|N_G(v)|, |N_G^+(v)|$ and $|N_G^-(v)|$ among all $v \in G$, respectively. Denote by $\Delta^0(G)$ the maximum of $d^+(v)$ and $d^-(v)$ among all $v \in V$. For $A \subset V(G)$, denote by $G[A]$ and $G - A$ the sub-digraph of G induced by A and the digraph obtained from G by deleting all vertices in A and all arcs incident to A, respectively. Given two disjoint subsets A and B of vertices of G, an $A \to B$ arc is an arc ab where $a \in A$ and $b \in B$. We denote by $E(A, B)$ for the set of all these arcs and $e_G(A, B) := |E(A, B)|$. Denote by $(A, B)_G$ the bipartite oriented subgraph of G whose vertex class are A and B and whose arc set is $E(A, B)$. The density of $(A, B)_G$ is defined to be

$$d_G(A, B) := \frac{e_G(A, B)}{|A||B|}.$$

Given $\epsilon > 0$, we call $(A, B)_G$ an *ϵ-regular* pair if $|d(X, Y) - d(A, B)| < \epsilon$ holds for all subsets $X \subseteq A$ and $Y \subseteq B$ with $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$. Note that (B, A) may not necessarily be an ϵ-regular pair since the order matters.

Given a digraph $G = (V, E)$, denote by $f_1(G)$ the number of pairs $u, v \in V$ such that exactly one of uv and vu is an arc of G, and $f_2(G)$ the number of pairs $u, v \in V$ such that both uv and vu are arcs of G(we call uv a double edge for convenience). For a vertex v, denote by $f_1(v)$ the number $u \in V$ such that exactly one of uv and vu is an arc of G, and $f_2(v)$ the number double edges incident with v. For a real number $a \geq 1$, the *weighted size* of G is defined by $e_a(G) := a \cdot f_2(G) + f_1(G)$. For convenience, we write $e(H) := e_a(H)$ for oriented graph H. For a vertex v, its weight is defined by $e_a(v) := a \cdot f_2(v) + f_1(v)$. This definition allows for a unified approach to extremal problems on undirected graphs, oriented graphs and digraphs. Because for an undirected graph, $e(G) := e_1(G)$ is the number of its edges when we equate an undirected edge in an undirected graph with a double edge in a digraph. Furthermore, a digraph G contains $4^{f_2(G)}2^{f_1(G)} = 2^{e_2(G)}$ labelled sub-digraphs and $3^{f_2(G)}2^{f_1(G)} = 2^{e_{\log 3}(G)}$ labelled oriented subgraphs if we set $a = 2$ and $a = \log 3$, respectively.

The *Turán graph* $Tu_r(n)$ is an undirected graph of n vertices which is formed by partitioning the set of n vertices into r-parts of nearly equal size, and connecting two vertices by an edge whenever they belong to different parts. Denote by $t_r(n)$ the edge size (number of edges) of $Tu_r(n)$. Denote by $DT_r(n)$ the digraph obtained from $Tu_r(n)$ by replacing each undirected edge of $Tu_r(n)$ with a double edge. Denote by DK_r the digraph obtained from the complete graph K_r on r vertices by replacing each edge of K_r by a double edge.

Given a digraph H, the *weighted Turán number* $ex_a(n, H)$ is the maximum $e_a(G)$ among all H-free digraphs G on n vertices. It is easy to see that $DT_r(n)$ is T_{r+1}-free, so $ex_a(n, T_{r+1}) \geq e_a(DT_r(n)) = a \cdot t_r(n)$.

A *transitive oriented graph* is an oriented graph such that whenever it contains arcs uv and vw then it contains uw too. A *transitive tournament* T_n on n vertices is the orientation of K_n such that it is transitive. Denote by T^t_{r+1} the blow-up of T_{r+1} for some positive integer t by replacing every vertex v_i of T_{r+1} with an independent set of t vertices and connecting every pair of vertices whenever they belong to different independent sets with the homogeneous direction in accordance with that of T_{r+1}. For a positive integer k we write $[k] := \{1, \ldots, k\}$. For convenience, we drop the subscripts of all notions if they are unambiguous.

We need the following result of the forbidden digraphs container of Kühn et al. [15], which allows us to reduce an asymptotic counting problem to an extremal problem. Given an oriented graph H with $e(H) \geq 2$, we let

$$m(H) = \max_{H' \subset H, e(H') > 1} \frac{e(H') - 1}{v(H') - 2}.$$

Theorem 1 (Theorem 3.3 [15])**.** *Let H be an oriented graph with $h := v(H)$ and $e(H) \geq 2$, and let $a \in \mathbb{R}$ with $a \geq 1$. For every $\epsilon > 0$, there exists $c > 0$ such that for all sufficiently large N, there exists a collection $\mathcal{C}$ of digraphs on the vertex set $[n]$ with the following properties.*

(a) For every H-free digraph I on $[N]$ there exists $G \in \mathcal{C}$ such that $I \subset G$.
(b) Every digraph $G \in \mathcal{C}$ contains at most ϵN^h copies of H, and $e_a(G) \leq ex_a(N, H) + \epsilon N^2$.
(c) $\log |\mathcal{C}| \leq cN^{2-1/m(H)} \log N$.

Note that this result is essentially a consequence of a recent and very powerful result of Balogh, Morris and Samotij [7] and Saxton and Thomason [17], which introduces the notion of hypergraph containers to give an upper bound on the number of independent sets in hypergraphs, and a digraph analogue [15] of the well-known supersaturation result of Erdös and Simonovits [1].

As mentioned in Section 1, Brown and Harary [14] first determined the extremal digraph with the maximum edge among T_{r+1}-free digraphs with n vertices. Kühn, Osthus, Townsend and Zhao [15] extended this result to weighted digraphs.

Lemma 1 ([15])**.** *Let $a \in (\frac{3}{2}, 2]$ be a real number and let $r, n \in \mathbb{N}$. Then $ex_a(n, T_{r+1}) = a \cdot t_r(n)$, and $DT_r(n)$ is the unique extremal T_{r+1}-free digraph on n vertices.*

We generalize the above result in the following.

Theorem 2. *For all positive integers r, t, every real numbers $a \in (\frac{3}{2}, 2]$ and $\gamma > 0$, there exists an integer n_0 such that every digraph G with $n \geq n_0$ vertices and*

$$e_a(G) \geq a \cdot t_r(n) + \gamma n^2$$

contains T_{r+1}^t as a sub-digraph.

Example 1. *For a given digraph H, it is contained in its blow-up H^t for each positive integer t. Thus a H-free digraph (oriented graph) G is also a H^t-free digraph (oriented graph), but a H^t-free digraph (oriented graph) is not necessarily a H-free digraph (oriented graph). For example, T_{r+1} is a sub-digraph of T_{r+1}^t and a T_{r+1}-free digraph (oriented graph) G is also a T_{r+1}^t-free digraph (oriented graph), but a T_{r+1}^t-free digraph (oriented graph) is not necessary a T_{r+1}-free digraph (oriented graph). However, the above results show that the extremal weighted sizes are asymptotically equal. One can also see that there are many other sub-digraphs contained in T_{r+1}^t, for example, the stars with the orientation all going away from the centre or going toward the centre, respectively. For those digraph-free digraphs, the corresponding extremal problems are still open. Our result may shed some light on them.*

In 1998 Cherlin [16] gave the following conjecture.

Conjecture 1 (Cherlin [16])**.** *Almost all T_3-free oriented graphs are tripartite.*

Kühn, Osthus, Townsend and Zhao [15] verified this conjecture and showed that almost all T_{r+1}-free oriented graphs and almost all T_{r+1}-free digraphs are r-partite. We will strengthen Cherlin's conjecture and show that almost all T_3^t-free oriented graphs and almost all T_3^t-free digraphs are almost bipartite for any positive integer t. Let $f(n, T_{r+1}^t)$ and $f^*(n, T_{r+1}^t)$ denote the number of labelled T_{r+1}^t-free oriented graphs and digraphs on n vertices, respectively.

Theorem 3. *For every positive integer $t \in \mathbb{N}$ and any $\alpha > 0$ there exists $\epsilon > 0$ such that the following holds for all sufficiently large n.*

(i) All but at most $f(n, T_3^t)2^{-\epsilon n^2}$ T_3^t-free oriented graphs on n vertices can be made bipartite by changing at most αn^2 edges.
(ii) All but at most $f^*(n, T_3^t)2^{-\epsilon n^2}$ T_3^t-free digraphs on n vertices can be made bipartite by changing at most αn^2 edges.

3. The Regularity Lemma and the Proof of Theorem 2

In this section, we shall give the proof of Theorem 2. First, we need the regularity lemma of digraphs of Alon and Shapira [18]. See [19] for a survey on the Regularity Lemma.

Given partitions $V_0, V_1, \ldots, V_k$ and $U_0, U_1, \ldots, U_\ell$ of the vertex set of a digraph, we say that $V_0, V_1, \ldots, V_k$ *refines* $U_0, U_1, \ldots, U_\ell$ if for all V_i with $i \geq 1$ there is some U_j with $j \geq 0$ such that $V_i \subseteq U_j$. Note that V_0 need not be contained in any U_j.

Lemma 2 (Degree form of the Regularity Lemma of Digraphs [18])**.** *For every $\epsilon \in (0,1)$ and all integers M', M'' there are integers M and n_0 such that if*

- *G is a digraph on $n \geq n_0$ vertices,*
- *$U_0, \ldots, U_{M''}$ is a partition of the vertex set of G,*
- *$d \in [0,1]$ is any real number,*

then there is a partition of the vertex set of G into $V_0, \ldots, V_k$ and a spanning sub-digraph G' of G such that the following holds:

(1) $M' \leq k \leq M$,
(2) $|V_0| \leq \epsilon \cdot n$,
(3) $|V_1| = \ldots = |V_k| = \ell$,
(4) $V_0, \ldots, V_k$ *refines the partition* $U_0, \ldots, U_{M''}$,
(5) $d_{G'}^+(x) > d_G^+(x) - (d+\epsilon)n$ *for all vertices x of G,*
(6) $d_{G'}^-(x) > d_G^-(x) - (d+\epsilon)n$ *for all vertices x of G,*
(7) $G'[V_i]$ *is empty for all $i = 1, \ldots, k$,*

(8) *the bipartite oriented graph* $(V_i, V_j)_{G'}$ *is* ϵ*-regular and has density either* 0 *or density at least* d *for all* $1 \leq i, j \leq k$ *and* $i \neq j$.

We call $V_1, \ldots, V_k$ *clusters* and V_0 the *exceptional set*. The last condition of the lemma says that all pairs of clusters are ϵ-regular in both directions (but possibly with different densities). We call the spanning digraph $G' \subseteq G$ in the lemma the *pure digraph* with parameters ϵ, d, ℓ. Given clusters $V_1, \ldots, V_k$ and a digraph G', the reduced digraph R with parameters ϵ, d, ℓ is the digraph whose vertices are $V_1, \ldots, V_k$ and whose arcs are all the $V_i \to V_j$ arcs in G' that is ϵ-regular and has a density of at least d.

Note that a simple consequence of the ϵ-regular pair (A, B): for any subset $Y \subseteq B$ that is not too small, most vertices of A have about the expected number of out-neighbours in Y; and similarly, for any subset $X \subseteq A$ that is not too small, most vertices of B have about the expected number of in-neighbours in X.

Lemma 3. *Let* (A, B) *be an* ϵ*-regular pair, of density* d *say, and* $X \subseteq A$ *has size* $|X| \geq \epsilon|A|$ *and* $Y \subseteq B$ *has size* $|Y| \geq \epsilon|B|$. *Then all but at most* $\epsilon|A|$ *of vertices in* A *have (each) at least* $(d - \epsilon)|Y|$ *out-neighbors in* Y *and all but at most* $\epsilon|B|$ *of vertices in* B *have (each) at least* $(d - \epsilon)|X|$ *in-neighbors in* X.

Proof. Let A' be the set of vertices with out-neighbors in Y less than $(d - \epsilon)|Y|$. Then $e(A', Y) < |A'|(d - \epsilon)|Y|$, so

$$d(A', Y) = \frac{e(A', Y)}{|A'||Y|} < d - \epsilon = d(A, B) - \epsilon.$$

Since (A, B) is ϵ-regular, this implies that $|A'| < \epsilon|A|$.

Similarly, let B' be the set of vertices with in-neighbours in X less than $(d - \epsilon)|X|$. Then $e(X, B') < |X|(d - \epsilon)|B'|$, so

$$d(X, B') = \frac{e(X, B')}{|X||B'|} < d - \epsilon = d(X, B) - \epsilon.$$

Since (A, B) is ϵ-regular, this implies that $|B'| < \epsilon|B|$. $\square$

The following lemma says that the blow-up R^s of the reduced digraph R can be found in G, provided that ϵ is small enough and the V_i is large enough.

Lemma 4. *For all* $d \in (0, 1)$ *and* $\Delta \geq 1$, *there exists an* $\epsilon_0 > 0$ *such that if* G *is any digraph,* s *is an integer and* R *is a reduced digraph of* G', *where* G' *is the pure digraph of* G *with parameters* $\epsilon \leq \epsilon_0$, $\ell \geq s/\epsilon_0$ *and* d. *For any digraph* H *with* $\Delta(G') \leq \Delta$, *then*

$$H \subseteq R^s \Rightarrow H \subseteq G' \subseteq G.$$

Proof. The proof is similar with that of Lemma 7.3.2 in [20]. Given d and Δ, choose $\epsilon_0 < d$ small enough such that

$$\frac{\Delta + 1}{(d - \epsilon_0)^{\Delta}} \epsilon_0 \leq 1; \tag{1}$$

such a choice is possible, since $\frac{\Delta+1}{(d-\epsilon)^{\Delta}} \epsilon \to 0$ as $\epsilon \to 0$. Now let G, H, s, R be given as stated. Let $\{V_0, V_1, \ldots, V_k\}$ be the ϵ-regular partition of G' that give rise to R. Thus, $\epsilon < \epsilon_0, V(R) = \{V_1, \ldots, V_k\}$ and $|V_1| = \ldots = |V_k| = \ell$. Let us assume that H is actually a sub-digraph of R^s, with vertices $u_1, \ldots, u_h$. Each vertex u_i lies in one of the s-sets V_j^s of R^s. This defines a map $\sigma : i \mapsto j$. We aim to define an embedding $u_i \mapsto v_i \in V_{\sigma(i)}$ of H in G'; thus, $v_1, \ldots, v_h$ will be distinct, and $v_i v_j$ will be an arc of G' whenever $u_i u_j$ is an arc of H.

We will choose the vertices $v_1, \ldots, v_h$ inductively. Throughout the induction, we shall have a "target set" $Y_i \subseteq V_{\sigma(i)}$ assigned to each i; this contains the vertices that are still candidates for the choice of v_i. Initially, Y_i is the entire set $V_{\sigma(i)}$. As the embedding proceeds, Y_i will get smaller and smaller (until it collapses to $\{v_i\}$): whenever we choose a vertex v_j with $j < i$ and if

Case (i): u_i are both out-neighbor and in-neighbor of u_j in H, we delete all those vertices from Y_i that are not adjacent to v_j with double edges.
Case (ii): u_i is just the out-neighbor of u_j in H, we delete all those vertices from Y_i that are not the out-neighbor of v_j.
Case (iii): u_i is just the in-neighbour of u_j in H, we delete all those vertices from Y_i that are not the in-neighbour of v_j.

To make this approach work, we have to ensure that the target set Y_i does not get too small. When we come to embed a vertex u_j, we consider all the indices $i > j$ such that u_i is adjacent to u_j in H; there are at most Δ such i. For each of these i, we wish to select v_j so that

$$Y_i^j = N^*(v_j) \bigcap Y_i^{j-1} \tag{2}$$

is large, where

$$N^*(v_j) = \begin{cases} N^{\pm}(v_j) & \text{if } u_i \text{ are both out-neighbor and in-neighbor of } u_j; \\ N^{+}(v_j) & \text{if } u_i \text{ is out-neighbor of } u_j; \\ N^{-}(v_j) & \text{if } u_i \text{ is in-neighbor of } u_j. \end{cases}$$

Now this can be done by Lemma 3: unless Y_i^{j-1} is tiny (of size less than $\epsilon\ell$), all but at most $\epsilon\ell$ choices of v_j will be such that (2) implies

$$|Y_i^j| \geq (d-\epsilon)|Y_i^{j-1}| \tag{3}$$

Doing this simultaneously for all of at most Δ values of i considered, we find that all but at most $\Delta\epsilon\ell$ choices of v_j from $V_{\sigma(j)}$, and in particular from $Y_j^{j-1} \subseteq V_{\sigma(j)}$, satisfy (3) for all i.

It remains to show that $|Y^{j-1}| - \Delta\epsilon\ell \geq s$ to ensure that a suitable choice for v_j exists: since $\sigma(j') = \sigma(j)$ for at most $s-1$ of the vertices $u_{j'}$ with $j' < j$, a choice between s suitable candidates for v_j will suffice to keep v_j distinct from $v_1, \ldots, v_{j-1}$. But all this follows from our choice of ϵ_0. Indeed, the initial target sets Y_i^0 have size ℓ, and each Y_i has vertices deleted from it only when some v_j with $j < i$ and u_j and u_i are adjacent in H, which happens at most Δ times. Thus,

$$|Y_i^j| - \Delta\epsilon\ell \geq (d-\epsilon)^{\Delta} - \Delta\epsilon\ell \geq (d-\epsilon_0)^{\Delta} - \Delta\epsilon_0\ell \geq \epsilon_0\ell \geq s$$

whenever $j < i$, so in particular $|Y_i^j| - \Delta \geq \epsilon_0\ell \geq \epsilon\ell$ and $|Y_j^{j-1}| - \Delta \geq \epsilon\ell \geq s$. □

We can now prove Theorem 2 using Lemma 1, Lemma 4 and the Regularity Lemma of digraphs.

Proof of Theorem 2. Let $d := \gamma, \Delta = \Delta(K_{r+1}^s)$, then Lemma 4 returns an $\epsilon_0 > 0$. Assume

$$\epsilon_0 < \gamma/2 < 1 \tag{4}$$

Let $M', M'' > 1/\gamma$, choose $\epsilon > 0$ small enough that $\epsilon \le \epsilon_0$ and $\delta := (a-1)d - \epsilon - a\epsilon^2/2 - a\epsilon > 0$. The Regularity Lemma of digraphs returns an integer M. Assume

$$n \ge \frac{Ms}{\epsilon_0(1-\epsilon)},$$

Since $\frac{Ms}{\epsilon_0(1-\epsilon)} \ge M', M''$. Let $\{U_0, U_1, \cdots, U_{M''}\}$ be a partition of vertex set of G. The Regularity Lemma of digraphs provided us with an $\epsilon-$regular partition $\{V_0, V_1, \ldots, V_k\}$ of G', the pure digraph of G, with parameters ϵ, d, ℓ and $M' \le k \le M$. That is $|V_1| = \ldots = |V_k| = \ell$ and $|V_0| < \epsilon n$. Then

$$n \ge k\ell \tag{5}$$

$$\ell = \frac{n - |V_0|}{k} \ge \frac{n - \epsilon n}{M} = n\frac{1-\epsilon}{M} \ge \frac{s}{\epsilon_0}$$

by the choice of n. Let R be the regularity digraph of G' with parameters ϵ, ℓ, d corresponding to the above partition. Since $\epsilon \le \epsilon_0, \ell \ge s/\epsilon_0$. R satisfies the premise of Lemma 4 and $\Delta(K^s_{r+1}) = \Delta$. Thus, to conclude by Lemma 4 that $T^s_{r+1} \subseteq G'$, all that remains to be checked is that $T_{r+1} \subseteq R$.

Our plan was to show $T_{r+1} \subseteq R$ by Lemma 1. We thus have to check that the weight of R is large enough.

By (5) and (6) of the regularity Lemma of digraphs, we have

$$\| G \|_a \le \| G' \|_a + (d+\epsilon)n^2 \tag{6}$$

At most $\binom{|V_0|}{2}$ double edges lie inside V_0, and at most $|V_0|k\ell \le \epsilon nk\ell$ double edges join $|V_0|$ to other partition sets. The $\epsilon-$regular pairs in G' of 0 density contribute nothing to the weight of G'. Since each edge of R corresponds to at most ℓ^2 edges of G', we thus have in total

$$\| G' \|_a \le \frac{1}{2}a\epsilon^2 n^2 + a\epsilon nk\ell + \| R \|_a \ell^2.$$

Combined with (6), we get

$$\begin{aligned} \| R \|_a &\ge k^2 \cdot \frac{a(\frac{r-1}{r} + \gamma)n^2 - (d+\epsilon)n^2 - \frac{1}{2}a\epsilon^2 n^2 - a\epsilon nk\ell}{k^2\ell^2} \\ &\ge a\frac{r-1}{r}k^2 + \delta k^2 \\ &= a \cdot t_r(k) + \delta k^2 \\ &> a \cdot t_r(k), \end{aligned}$$

for all sufficiently large n. Therefore $T_{r+1} \subseteq R$ by Lemma 1, as desired. □

Similar to the Erdös-Stone theorem of undirected graphs, the Erdös-Stone theorem of digraphs is interesting not only in its own right, but also has an interesting corollary. For an oriented graph H, its chromatic number is defined as the chromatic number of its underlying graph. An oriented graph H with chromatic number $\chi(H)$ is called homogeneous if there is a colouring of its vertices by $[\chi(H)]$ such that either $E(V_i, V_j) = \emptyset$ or $E(V_j, V_i) = \emptyset$ for every $1 \le i \ne j \le \chi(H)$, where V_i is the vertex set with colour i.

Given an acyclic homogeneously oriented graph H and an integer n, consider the number $h_n := ex_a(n, H)/(a\binom{n}{2})$: the maximum weighted density that an $n-$vertex digraph can have without containing a copy of H.

Theorem 2 implies that the limit of h_n as $n \to \infty$ is determined by a very simple function of a natural invariant of H–its chromatic number!

Corollary 1. *For every acyclic homogeneously oriented graph H with at least one edge,*

$$\lim_{n\to\infty} \frac{ex_a(n,H)}{a\binom{n}{2}} = \frac{\chi(H)-2}{\chi(H)-1}.$$

Proof of Corollary 1. Let $r := \chi(H)$. Since H can not be coloured with $r-1$ colours, we have $H \nsubseteq DT_{r-1}(n)$ for all $n \in \mathbb{N}$, and hence

$$a \cdot t_{r-1}(n) \leq ex_a(n,H).$$

On the other hand, we have $H \subseteq T_r^t$ for all sufficiently large t. Thus

$$ex_a(n,H) \leq ex_a(n,T_r^t)$$

for sufficiently large t. Fix such t, Theorem 2 implies that eventually (i.e., for large enough n)

$$ex_a(n,T_r^t) < at_{r-1}(n) + \epsilon n^2.$$

Hence for large enough n,

$$\begin{aligned}
\frac{t_{r-1}(n)}{\binom{n}{2}} &\leq \frac{ex_a(n,H)}{a\binom{n}{2}} \\
&\leq \frac{ex_a(n,T_r^t)}{a\binom{n}{2}} \\
&< \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{\epsilon n^2}{a\binom{n}{2}} \\
&< \frac{t_{r-1}(n)}{\binom{n}{2}} + \frac{2\epsilon}{a(1-1/n)} \\
&\leq \frac{t_{r-1}(n)}{\binom{n}{2}} + 4\epsilon
\end{aligned}$$

Since $\frac{t_{r-1}(n)}{\binom{n}{2}}$ converges to $\frac{r-2}{r-1}$ as $n \to \infty$, we get $\frac{ex_a(n,H)}{a\binom{n}{2}} = \frac{r-2}{r-1}$, for every $\epsilon > 0$. $\square$

4. Stability of Digraphs and Proof of Theorem 3

In this section, we will establish the stability of T_{r+1}^t-free digraphs and give a proof of Theorem 3. First, we need the following lemma.

Lemma 5. *Let G be a digraph obtained by adding a new vertex u and connecting it to each vertex of DK_{r-1} with an arc, then G contains T_r.*

Proof. We divide the vertices of DK_{r-1} into two classes, i.e., $N_G^-(u)$ and $N_G^+(u)$. Then we can arbitrarily order the vertices of $N_G^-(u)$ and $N_G^+(u)$ respectively. Assume $|N_G^-(u)| = p$, $|N_G^+(u)| = r-p-1$, and $N_G^+(u) = \{v_1,\ldots,v_p\}, N_G^-(u) = \{u_1,\ldots,u_{r-p-1}\}$. Then we order the vertices of G as $\{v_1,\ldots,v_p,u,u_1,\ldots,u_{r-p-1}\}$. A T_r can be chosen by choosing the vertices and all their out-edges which connect to all the vertices behind it in this order. Obviously, the result is true if $p=0$ or $r-p-1=0$. $\square$

We now give the result of stability of T_{r+1}^t-free digraphs.

Theorem 4. *(Stability) Let $t \in \mathbb{N}$ and $a \in \mathbb{R}$ with $t \geq 1, a \in (\frac{3}{2},2]$. Then for any T_3^t-free digraph G with*

$$e_a(G) = a\left(\frac{1}{2}+o(1)\right)\frac{n^2}{2}$$

satisfies $G = DT_2(n) \pm o(n^2)$.

Proof. We claim that there are $\Omega(n^2)$ double edges in G. For otherwise, we assume that there are $o(n^2)$ double edges in G. Delete all double edges from G and denote the resulting digraph by G'. Then the digraph G' is T_3^t-free and contains no double edges. And $e_a(G') = a\left(\frac{1}{2} + o(1)\right)\frac{n^2}{2} - a \cdot o(n^2) = a\left(\frac{1}{2} + o(1)\right)\frac{n^2}{2}$. Then G' contains a DT_2^t by Theorem 2 which contradicts the fact that G' contains no double edges.

Second we can assume that all but $o(n)$ vertices of G have weight at least $\frac{an}{2}(1 + o(1))$. For otherwise let $v_1, \ldots, v_k, k = \lfloor \epsilon \cdot n \rfloor$ (ϵ is a small positive number independent of n) be the vertices of G each of which has weight less than $\frac{an}{2}(1 - c)$, where $0 < c(\epsilon) < c < 1$. But then we have

$$
\begin{aligned}
e_a(G[v_{k+1}, \ldots, v_n]) &\geq a(\frac{1}{2} + o(1))\frac{n^2}{2} - \frac{akn}{2}(1 - c) \\
&= a\left(\frac{1}{4}(n^2 - 2kn + k^2) - \frac{k^2}{4} + \frac{ckn}{2} + \frac{n^2}{2}o(1)\right) \\
&> \frac{a}{4}(n - k)^2\big(1 + \delta(\epsilon, c)\big),
\end{aligned}
$$

where $\delta(\epsilon, c) > 0$. By Theorem 2, we conclude that $G[v_{k+1}, \ldots, v_n]$ contains a T_3^t and so is G, which contradicts our assumption.

Now assume that $v_1, \ldots, v_p, p = \big(1 + o(1)\big)n$ be the vertices of G with weight not less than $\frac{an}{2}\big(1 + o(1)\big)$. Then the weight of each vertex of $G[v_1, \cdots, v_p]$ in $(G[v_1, \cdots, v_p])$ is at least $ap\big(\frac{1}{2} + o(1)\big) = an\big(\frac{1}{2} + o(1)\big)$. And $e_a(G[v_1, \cdots, v_p]) = \frac{ap^2}{2}\big(\frac{1}{2} + o(1)\big) = \frac{an^2}{2}\big(\frac{1}{2} + o(1)\big)$. Thus to prove our theorem it will suffice to show that $G[v_1, \cdots, v_p] = DT_2(p) \pm o(p^2)$.

Thus it is clear that without loss of generality we can assume that the weight of every vertex in G is at least $an\big(\frac{1}{2} + o(1)\big)$. Note that we now no longer have to use the assumption of $e_a(G) = \frac{an^2}{2}\big(\frac{1}{2} + o(1)\big)$. Since our assumption that $e_a(v_i) \geq an\big(\frac{1}{2} + o(1)\big), i = 1, \ldots, n$ and G is T_3^t-free already implies that $e_a(G) = \frac{an^2}{2}\big(\frac{1}{2} + o(1)\big)$.

We shall show that if G is T_3^t-free digraph with $e_a(G) = \frac{an^2}{2}\big(\frac{1}{2} + o(1)\big)$ for some fixed t, then $G = DT_2(n) \pm o(n^2)$.

We call a double edge *bad* if it is contained in only $o(n)$ of T_3 in G. Assume first that G has at least ϵn^2 bad double edges. By Theorem 2, G contains DT_2^t with two classes of vertices, say $u_1, \ldots, u_t$ and $v_1, \ldots, v_t$, so that all the double edges of DT_2^t are bad. Now since the weight of each vertex in $\{u_1, \ldots, u_t, v_1, \ldots, v_t\}$ is at least $an(\frac{1}{2} + o(1))$ and each double edge $(u_i, v_j)(1 \leq i, j \leq t)$ is contained in only $o(n)$ of T_3, a simple argument shows that the remaining $n - 2t$ vertices of G can be divided into two classes (neglecting $o(n)$ vertices), say $z_1, \ldots, z_{u_1}; w_1, \ldots, w_{u_2}, u_1 = (1 + o(1))n/2, u_2 = (1 + o(1))n/2$, so that all the u_i are connect to all the z's with double edges and all the y_j are connect to all the w's.

If $e_a(G[z_1, \ldots, z_{u_1}]) = o(n^2)$ and $e_a(G[w_1, \ldots, w_{u_2}]) = o(n^2)$, then a simple computation shows that $DT_2(u_1, u_2)$ with the vertex set $\{z_1, \ldots, z_{u_1}; w_1, \ldots, w_{u_2}\}$ differs from G by $o(n^2)$ edges, which prove our theorem (the remaining $n - u_1 - u_2 = o(n)$ vertices can be clearly ignored).

If, say $e_a(G[z_1, \ldots, z_{u_1}])$ is not $o(n^2)$, then by Theorem 2 it contains a DT_2^t with two classes of vertices, say $z_1, \ldots, z_t$ and $z_{t+1}, \ldots, z_{2t}$. But then the digraph

$$
G[u_1, \ldots, u_t; z_1, \ldots, z_t; z_{t+1}, \ldots, z_{2t}]
$$

clearly contains a T_3^t, which contradicts with our assumption.

Henceforth we can assume that there are $o(n^2)$ bad double edges. Bearing in mind that G contains $\Omega(n^2)$ double edges. Let $e_1, \ldots, e_s, s > \alpha n^2$ be the double edges each of which is contained in at least βn of T_3, where $\alpha, \beta > 0$. We now deduce from this assumption that G contains a T_3^t. Let $v_1^{(i)}, \ldots, v_{r_i}^{(i)}$ be the vertices which form a T_3 with $e_i, r_i \geq \beta n, s \geq i \geq 1$.

Since there are 2^r orientations of a star S_{r+1} of $r+1$ vertices. Therefore there are at least $\beta' n := \beta n/2^r$ vertices of $\{v_j^{(i)}, r_i \geq j \geq 1\}$ formed with e_i with homogeneous T_3, say $\{v_j^{(i)}, r_i' \geq j \geq 1\}, r_i' \geq \beta' n$ connect to both end vertices of e_i in the same way. Similarly there are at least $\alpha' n^2 := \alpha n^2/2^r$ double edges of $\{e_i, s \geq i \geq 1\}$ each formed with at least $\beta' n$ vertices with homogeneous T_3. And all those T_3 formed with those at least $\alpha' n^2$ double edges e_i' are homogeneous.

Form all possible t-tuple from those homogeneous vertices $v_{r_i}^{(i)}$. We get at least

$$\sum_{i=1}^{\alpha' n^2} \binom{r_i'}{t} \geq \sum_{i=1}^{\alpha' n^2} \binom{\beta' n}{t} \geq \alpha' n^2 \frac{(\beta' n)^t}{3^t t!} > \alpha' n^2 (\frac{\beta'}{3})^t \binom{n}{t}$$

t-tuples. Since the total number of t-tuples formed from n elements is $\binom{n}{t}$, there is a t-tuple say $z_1, \ldots, z_t$ which corresponds to at least $\alpha' n^2 (\frac{\beta'}{3})^t$ double edges e_i. By Theorem 2 these double edges determine a DT_2^t with vertices $x_1, \ldots, x_t; y_1, \ldots, y_t$. Thus finally $G[x_1, \ldots, x_t; y_1, \ldots, y_t; z_1, \ldots, z_t]$ contains a T_3^t as stated. But by assumption, G is T_3^t-free. This contradiction completes the proof. □

To keep all symbols consistent, we reshape Theorem 4 as follows:

Theorem of Stability. Let $t \in \mathbb{N}$ and $a \in \mathbb{R}$ with $t \geq 1, a \in (\frac{3}{2}, 2]$. Then for any $\beta > 0$ there exists $\gamma > 0$ such that the following holds for all sufficiently large n. If a T_3^t-free digraph G on n vertices satisfies

$$e_a(G) = a\left(\frac{1}{2} - \gamma\right)\frac{n^2}{2},$$

then $G = DT_2(n) \pm \beta n^2$.

We also need the Digraph Removal Lemma of Alon and Shapira [18].

Lemma 6. *(Removal Lemma). For any fixed digraph H on h vertices, and any $\gamma > 0$ there exists $\epsilon' > 0$ such that the following holds for all sufficiently large n. If a digraph G on n vertices contains at most $\epsilon' n^h$ copies of H, then G can be made H-free by deleting at most γn^2 edges.*

Now we are ready to show that almost all T_3^t-free oriented graphs and almost all T_3^t-free digraphs are almost bipartite.

Proof of Theorem 3. We only prove (i) here; the proof of (ii) is almost identical. Let $a := \log 3$. Choose $n_0 \in \mathbb{N}$ and $\epsilon, \gamma, \beta > 0$ such that $1/n_0 \ll \epsilon \ll \gamma \ll \beta \ll \alpha, 1/r$. Let $\epsilon' := 2\epsilon$ and $n \geq n_0$. By Theorem 1 (with T_3^t, n and ϵ taking the roles of H, N and ϵ respectively) there is a collection $\mathcal{C}$ of digraphs on vertex set $[n]$ satisfying properties $(a) - (c)$. In particular, every T_3^t-free oriented graph on vertex set $[n]$ is contained in some digraph $G \in \mathcal{C}$. Let $\mathcal{C}_1$ be the family of all those $G \in \mathcal{C}$ for which $e_{\log 3}(G) \geq ex_{\log 3}(n, T_3^t) - \epsilon' n^2$. Then the number of T_3^t-free oriented graphs not contained in some $G \in \mathcal{C}_1$ is at most

$$|\mathcal{C}| 2^{ex_{\log 3}(n, T_3^t) - \epsilon' n^2} \leq 2^{-\epsilon n^2} f(n, T_3^t),$$

because $|\mathcal{C}| \leq 2^{n^{2-\epsilon'}}$ and $f(n, T_3^t) \geq 2^{ex_{\log 3}(n, T_3^t)}$. Thus it suffices to show that every digraph $G \in \mathcal{C}_1$ satisfies $G = DT_2(n) \pm \alpha n^2$. By (b), each $G \in \mathcal{C}_1$ contains at most $\epsilon' n^{3t}$ copies of T_3^t. Thus by Lemma 6 we obtain a T_3^t-free digraph G' after deleting at most γn^2 edges from G. Then $e_{\log 3}(G') \geq ex_{\log 3}(n, T_3^t) - (\epsilon' + \gamma) n^2$. We next apply the Theorem of Stability to G' and derive that $G' = DT_2(n) \pm \beta n^2$. As a result, the original digraph G satisfies $G = DT_2(n) \pm (\beta + \gamma) n^2$, hence $G = DT_2(n) \pm \alpha n^2$ as required. □

5. Concluding Remarks

In the work, we first give an analogue result of the Erdös-Stone theorem for weighted T^t_{r+1}-free digraphs. We then give a stability result of T^t_3-free oriented graphs and T^t_3-free digraphs. These results reconfirmed and strengthen Cherlin's conjecture. However, we can not get the exact typical structures of T^t_{r+1}-free oriented graphs and digraphs. From our study experience and clues from other research, such as Kühn, Osthus, Townsend and Zhao [15], we believe that the exact structures are the same as those of T_{r+1}-free oriented graphs and digraphs. Therefore, we give the following conjecture at the end of this work:

Conjecture 2. *Let $r, t \in \mathbb{N}$ with $r \geq 2, t \geq 1$. Then the following hold.*
(i) Almost all T^t_{r+1}-free oriented graph are r-partite.
(ii) Almost all T^t_{r+1}-free digraph are r-partite.

Author Contributions: Conceptualization, M.L. and J.L.; methodology, J.L.; formal analysis, J.L.; writing—original draft preparation, J.L.; writing—review and editing, M.L.; funding acquisition, J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research partially funded by the Natural Science Foundation of Guangdong Province (No. 2018A030313267).

Data Availability Statement: Not applicable.

Acknowledgments: We would like to thank the anonymous reviewers for their careful work and constructive suggestions that have helped improve this paper substantially.

Conflicts of Interest: The authors declare no conflict of interest.

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