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On the Convergence of Two-Step Kurchatov-Type Methods under Generalized Continuity Conditions for Solving Nonlinear Equations

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Abstract: The study of the microworld, quantum physics including the fundamental standard models are closely related to the basis of symmetry principles. These phenomena are reduced to solving nonlinear equations in suitable abstract spaces. Such equations are solved mostly iteratively. That is why two-step iterative methods of the Kurchatov type for solving nonlinear operator equations are investigated using approximation by the Fréchet derivative of an operator of a nonlinear equation by divided differences. Local and semi-local convergence of the methods is studied under conditions that the first-order divided differences satisfy the generalized Lipschitz conditions. The conditions and speed of convergence of these methods are determined. Moreover, the domain of uniqueness is found for the solution. The results of numerical experiments validate the theoretical results. The new idea can be used on other iterative methods utilizing inverses of divided differences of order one.

Keywords: convergence; Banach spaces; Fréchet derivative; divided difference; Kurchatov's method

MSC: 65G99; 47H99; 49M15; 65H10

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1. Introduction

Let X and Y stand for Banach spaces and Ω be a convex and nonempty subset of X. A plethora of applications from diverse disciplines can be solved if reduced to a nonlinear equation of the form

$$F(x) = 0. \tag{1}$$

This reduction takes place using Mathematical Modeling [1,2]. Then, a solution denoted by $x^* \in \Omega$ is to be found that answers the application. The solution may be a number or a vector or a matrix or a function. This task is very challenging in general. Obviously, the solution x^* is desired in closed form. However, in practice, this is achievable only in rare cases. That is why researchers mostly develop iterative methods convergent to x^* under some conditions on the initial data.

A popular method is the Newton's method [2–5] defined, respectively, for a starting point $x_0 \in \Omega$ and all $n = 0, 1, 2, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n). \tag{2}$$

Here, F' is the notation for the Fréchet derivative of the operator F. The convergence rate of Newton's method is quadratic. However, this method requires the calculation of the derivative of the operator F [1–3]. It is not always easy or impossible to do, in particular, in the case when the operator is not given analytically, but only the algorithm for its calculation on the computer is known. Then, Newton's method (2) and its modifications [4–8] using derivatives are not suitable for solving (1). In this case, we can use difference methods [1,3,9–11]. The simplest of them is the Secant method [2,3,6,7]

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n) \tag{3}$$

for all $n = 0, 1, 2, \ldots$, x_{-1}, x_0 are starting points. The Secant method was extended for the solution of (1) in Banach spaces by J.W. Schmidt [9]. This method under different conditions was studied in many papers [2,7]. The convergence order of the method (3) is equal to $\dfrac{1+\sqrt{5}}{2} = 1,618\ldots$. The method with a higher quadratic convergence is described for all $n = 0, 1, 2, \ldots$ by the formula

$$x_{n+1} = x_n - [x_{n-1}, 2x_n - x_{n-1}; F]^{-1} F(x_n). \tag{4}$$

This method is famous as the method of the linear interpolation or the Kurchatov's method. It does not interfere with Newton's method in the convergence order, and it does not require analytically given derivatives as the Secant method does. The method (4) was proposed for the first time by V.A. Kurchatov in [12] for the one-dimensional case. In the Banach space, the method (4) was first presented in the works of S.M. Shakhno [13,14]. In addition, this method was studied by many authors I.K. Argyros, H. Ren, J.A. Ezquerro, and M.A. Hernández [15–17]. The Kurchatov method uses only first-order divided differences in its iterative formula. However, often the studying of its convergence additionally requires conditions for the second-order divided differences. This ensures theoretically obtaining the second order of convergence. Kurchatov's two-step methods were studied by I.K. Argyros, S. George, H. Kumar, P.K. Parida, and S.M. Shakhno [18,19].

In this article, we propose the following modification of the method (4).

Let $x_{-1}, x_0 \in \Omega$. Define the two-step Kurchatov-type methods for all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} A_n &= [x_{n-1}, 2x_n - x_{n-1}; F], \\ y_n &= x_n - A_n^{-1} F(x_n), \\ x_{n+1} &= y_n - A_n^{-1} F(y_n) \end{aligned} \tag{5}$$

and

$$\begin{aligned} y_n &= x_n - A_n^{-1} F(x_n), \\ B_n &= [x_n, 2y_n - x_n; F], \\ x_{n+1} &= y_n - B_n^{-1} F(y_n). \end{aligned} \tag{6}$$

It is known that multi-step methods converge faster than the corresponding one-step methods. Therefore, there is a growing interest in the development and theoretical studying of the convergence of such algorithms. It is worth noting that the method (5) uses the same inverse operator in both steps. This helps to reduce the total number of calculations compared to the corresponding one-step method, especially for large scale problems.

We provide the local as well as the semi-local convergence analysis for these methods under generalized conditions. Moreover, these conditions include only operators that appear in methods. The local convergence is given in Section 2. The semi-local convergence is presented in Section 3, followed by the examples and the concluding remarks in Sections 4 and 5, respectively.

2. Local Convergence

It is convenient for the study of the local convergence for the methods to introduce some parameters and real functions. Set $M = [0, \infty)$.

Suppose:

(1) There exists a function $\varphi_0 : M \times M \to \mathbb{R}$ which is continuous and nondecreasing in both variables such that the equation

$$\varphi_0(t,3t)-1=0$$

has the smallest solution $\varrho \in M-\{0\}$.

Set $M_0=[0,\rho)$.

(2) There exists a function $\varphi : M_0 \times M_0 \to \mathbb{R}$, which is continuous and nondecreasing in both variables such that the equation

$$h_1(t)-1=0$$

has a smallest solution $r_1 \in M_0-\{0\}$, where the function $h_1 : M_0 \to \mathbb{R}$ is given by

$$h_1(t)=\frac{\varphi(2t,3t)}{1-\varphi_0(t,3t)}.$$

Define the function $h_2 : M_0 \to \mathbb{R}$ by

$$h_2(t)=\frac{\varphi((1+h_1(t))t,3t)}{1-\varphi_0(t,3t)}h_1(t).$$

(3) The equation

$$h_2(t)-1=0$$

has a smallest solution $r_2 \in M_0-\{0\}$.

Define the parameter

$$r=\min\{t_j\}, \quad j=1,2. \tag{7}$$

This parameter will be shown to be a radius of convergence in Theorem 1 for the method (5).

Set $M_1=[0,1)$. Then, follows by definition (7) that for all $t \in M_1$

$$0 \le \varphi_0(t,3t) < 1 \tag{8}$$

and

$$0 \le h_j(t) < 1. \tag{9}$$

Let $U(v,d), U[v,d]$ stand for the open and closed ball in X, respectively, of center $v \in X$ and radius $d>0$. By $£(X,Y)$ we denote the space of bounded linear operators from X into Y.

The convergence analysis uses the conditions (C) for both methods.

Suppose:

(C_1) The equation $F(x)=0$ has a simple solution $x^* \in \Omega$ such that $F'(x^*)^{-1} \in £(Y,X)$.

(C_2) $\|F'(x^*)^{-1}([u_1,u_2;F]-F'(x^*))\| \le \varphi_0(\|u_1-x^*\|,\|u_2-x^*\|)$ for all $u_1,u_2 \in \Omega$.

Set $\Omega_0=U(x^*,\varrho)\cap\Omega$.

(C_3) $\|F'(x^*)^{-1}([u_3,u_4;F]-[u_5,x^*;F])\| \le \varphi(\|u_3-u_5\|,\|u_4-x^*\|)$ for all $u_3,u_4,u_5 \in \Omega_0$.

(C_4) $U[x^*,3r] \subset \Omega$.

Next, the local convergence is established for the method (5).

Theorem 1. *Suppose that the conditions (C) hold. Moreover, if the starting points $x_{-1},x_0 \in U(x^*,r)-\{x^*\}$, then the sequence $\{x_n\}$ generated by Formula (5) exists in $U(x^*,r)$, stays in $U(x^*,r)$ for all $n=0,1,2,\dots$ and is convergent to x^*. Moreover, the following assertions hold*

$$\|y_n-x^*\| \le h_1(r)\|x_n-x^*\| \le \|x_n-x^*\| < r \tag{10}$$

and

$$\|x_{n+1}-x^*\| \le h_2(r)\|x_n-x^*\| \le \|x_n-x^*\| < r, \tag{11}$$

where the radius r is given by Formula (7) and the functions h_j are as previously defined.

Proof. By hypothesis $x_{-1}, x_0 \in U(x^*, r) - \{x^*\}$. Then, by applying conditions $(C_1), (C_2)$, the definition of the radius r and (8), we have

$$\begin{aligned} \|F'(x^*)^{-1}(A_0 - F'(x^*))\| &\leq \varphi_0(\|x_{-1} - x^*\|, 2\|x_0 - x^*\| + \|x_{-1} - x^*\|) \\ &= q_0 \leq \varphi_0(r, 3r) < 1. \end{aligned} \tag{12}$$

It follows by (12) and the Banach lemma on the invertible operator [4] that $A_0^{-1} \in \pounds(Y, X)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - \varphi_0(\|x_{-1} - x^*\|, 2\|x_0 - x^*\| + \|x_{-1} - x^*\|)}. \tag{13}$$

Moreover, the iterates y_0 and x_1 are well defined by the two substeps of the method (5). In view of that, we can write in that

$$y_0 - x^* = x_0 - x^* - A_0^{-1}(A_0 - [x_0, x^*; F])(x_0 - x^*). \tag{14}$$

Using (7) and (9) (for $j = 1$), (C_3), (C_4), (13) and (14) we obtain

$$\|y_0 - x^*\| \leq \frac{p_0}{1 - q_0}\|x_0 - x^*\| \leq h_1(r)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \tag{15}$$

where we also used

$$\begin{aligned} \|F'(x^*)^{-1}(A_0 - [x_0, x^*; F])\| &\leq \varphi(\|x_{-1} - x_0\|, \|2x_0 - x_{-1} - x^*\|) \\ &\leq \varphi(\|x_{-1} - x_*\| + \|x_0 - x^*\|, 2\|x_0 - x^*\| + \|x_{-1} - x^*\|) \\ &= p_0 \leq \varphi(2r, 3r), \end{aligned} \tag{16}$$

and

$$\|2x_0 - x_{-1} - x^*\| \leq 2\|x_0 - x^*\| + \|x_{-1} - x^*\| \leq 3r.$$

Similarly, by the second substep of the method (5), we can write

$$x_1 - x^* = y_0 - x^* - A_0^{-1}F(y_0) = A_0^{-1}(A_0 - [y_0, x^*; F])(y_0 - x^*),$$

so

$$\begin{aligned} \|x_1 - x^*\| &\leq \frac{\varphi(\|x_{-1} - y_0\|, \|2\|x_0 - x_{-1} - x^*\|)}{1 - q_0}\|y_0 - x^*\| \\ &\leq h_2(r)\|x_0 - x^*\| \leq \|x_0 - x^*\|. \end{aligned} \tag{17}$$

Hence, the estimates (10) and (11) hold for $n = 0$. By simply replacing the role of x_{-1}, x_0, y_0, x_1 by $x_m, x_{m+1}, y_{m+1}, x_{m+2}$, $m = -1, 0, 1, \dots$ in the preceding calculations the induction for the estimates (10) and (11) is terminated. It follows that

$$\|x_{m+1} - x^*\| \leq c\|x_m - x^*\| < r, \tag{18}$$

where $c = h_2(r) \in [0, 1)$. Thus, we conclude $\lim_{m \to \infty} x_m = x^*$. □

Next, a unique result is presented for the solution of the equation $F(x) = 0$.

Proposition 1. *Suppose:*
(a) There exists a solution $u^* \in U(x^*, \varrho_1)$ of the equation $F(x) = 0$ for some $\varrho_1 > 0$.
(b) The conditions (C_1) and (C_2) hold.
(c) There exists $\varrho_2 \geq \varrho_1$ such that

$$\varphi_0(\varrho_2, 0) < 1. \tag{19}$$

Set $\Omega_1 = U[x^*, \varrho_2] \cap \Omega$.
Then, the equation $F(x) = 0$ is uniquely solvable by the element x^* in the region Ω_1.

Proof. Let $T = [u^*, x^*; F]$. If then follows by (a)–(c) in turn that

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \varphi_0(\|u^* - x^*\|, 0) \leq \varphi_0(\varrho_2, 0) < 1.$$

Hence, the operator T is invertible. Then, by the identity

$$u^* - x^* = T^{-1}(F(u^*) - F(x^*)) = T^{-1}(0 - 0) = 0,$$

we conclude that $u^* = x^*$. $\square$

Concerning the local convergence analysis of the method (6), clearly the function h_1 is the same, whereas the function h_3 corresponding to h_2 is given by

$$h_3(t) = \frac{\varphi((1 + h_1(t))t, (2h_1(t) + 1)t)h_1(t)}{1 - \varphi_0(t, (2h_1(t) + 1)t)}. \tag{20}$$

This is due to the similar computation

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|B_n^{-1}(B_n - [y_n, x^*; F])\|\|y_n - x^*\| \\ &\leq \|B_n^{-1}F'(x^*)\|\|F'(x^*)^{-1}(B_n - [y_n, x^*; F])\|\|y_n - x^*\| \\ &\leq \frac{\varphi(\|y_n - x_n\|, 2\|y_n - x^*\| + \|x_n - x^*\|)}{1 - \varphi_0(\|x_n - x^*\|, 2\|y_n - x^*\| + \|x_n - x^*\|)}\|y_n - x^*\| \\ &\leq h_3(r)\|x_n - x^*\|, \end{aligned} \tag{21}$$

where

$$\bar{r} = \min\{r_1, r_3\} \tag{22}$$

and r_3 is the smallest solution of the equation $h_3(t) - 1 = 0$ in the interval $[0, \bar{\varrho})$, where $\bar{\varrho} = \min\{\varrho, \varrho_3\}$ and ϱ_3 is the smallest positive solution of the equation

$$\varphi_0(t, (2h_1(t) + 1)t) - 1 = 0$$

(if it exists). Hence, we arrived at the corresponding semi-local convergence result for the method (6).

Theorem 2. *Suppose that the conditions (C) hold with $\bar{r}$ replacing r. Then, the conclusions of Theorem 1 hold for the method (6) with the function h_2.*

Clearly, the uniqueness of the solution results of Proposition 1 holds for the method (6).

3. Semi-Local Convergence

The analysis is based on majorizing sequences. Let $c \geq 0$ and $\eta \geq 0$ be given parameters. Suppose that there exists a function $\psi_0 : M \to \mathbb{R}$ which is continuous and nondecreasing such that the equation

$$\psi_0(t - c, t - c) - 1 = 0$$

has a smallest solution $\varrho_4 > c$. Set $M_2 = [0, \varrho_4)$. Moreover, suppose that there exist a function $\psi : M_2 \to \mathbb{R}$ which is continuous and nondecreasing.

Define the sequence $\{t_n\}$ for $t_{-1} = 0, t_0 = c, s_0 = c + \eta$ and all $n = 0, 1, 2, \ldots$ by

$$\begin{aligned} t_{n+1} &= s_n + \frac{\psi(s_n - t_{n-1}, t_n - t_{n-1})(s_n - t_n)}{1 - \psi_0(t_{n-1} - c, 2t_n - t_{n-1} - c)}, \\ s_{n+1} &= t_{n+1} + \frac{b_{n+1}}{1 - \psi_0(t_n - c, 2t_{n+1} - t_n - c)}, \end{aligned} \tag{23}$$

where $b_{n+1} = (1 + \psi_0(t_{n+1}, s_n))(t_{n+1} - s_n) + \alpha_n$.

Next we present a convergence result for the sequence $\{t_n\}$.

Lemma 1. *Suppose that for all $n = 0, 1, 2, \ldots$*

$$0 \leq \psi_0(t_n - c, 2t_n - t_{n-1} - c) < 1, \quad c \leq t_n < \lambda \ for \ some \ \lambda > 0. \tag{24}$$

Then, the sequence $\{t_n\}$ given by Formula (23) is nondecreasing and convergent to its unique least upper bound $t_* \in [0, 1]$.

Proof. The sequence $\{t_n\}$ is nondecreasing and bounded from above by λ and as such it is convergent to t_*. □

The condition (H) shall be used in the semi-local convergence analysis first of the method (5).

Suppose:

(H_1) There exist points $x_{-1}, x_0 \in \Omega$, parameters $c \geq 0, \eta \geq 0$ such that $F'(x_0)^{-1}, A_0^{-1} \in \pounds(Y, X), \|x_0 - x_{-1}\| \leq c$ and $\|A_0^{-1}F(x_0)\| \leq \eta$.

(H_2) $\|F'(x_0)^{-1}([x, y; F] - F'(x_0))\| \leq \psi_0(\|x - x_0\|, \|y - x_0\|)$ for all $x, y \in \Omega$.

Set $\Omega_2 = U(x_0, \varrho_4) \cap \Omega$.

(H_3) $\|F'(x_0)^{-1}([x, y; F] - [z, u; F])\| \leq \psi(\|x - z\|, \|y - u\|)$ for all $x, y, z, u \in \Omega_2$.

(H_4) Conditions (24) holds

and

(H_5) $U[x_0, 3t_*] \in \Omega$.

Next, the semi-local convergence of the method (5) is presented based on the conditions (H) and the preceding terminology.

Theorem 3. *Suppose that the conditions (H) hold. Then, the sequence $\{x_n\}$ generated by the method (5) is well defined in $U(x_0, t_*)$, remains in $U(x_0, t_*)$ for all $n = -1, 0, 1, 2, \ldots$ and is convergent to a solution $x^* \in U[x_0, t_*]$ of the equation $F(x) = 0$. Moreover, the following error estimates hold*

$$\|x^* - x_n\| \leq t_* - t_n. \tag{25}$$

Proof. It follows as in the proof of Theorem 1 but there are some small differences. Iterates y_0 and x_0 are well defined by the condition (H_1) and the first substep of the method (5) for $n = 0$. We also have

$$\|y_0 - x_0\| = \|A_0^{-1}F(x_0)\| \leq \eta = s_0 - t_0 < t_*, \tag{26}$$

so the iterate $y_0 \in U(x_0, t_*)$. Then, as in Theorem 1 but using the $''\psi'', x_0$ instead of $''\psi'', x^*$, we obtain the estimates

$$\begin{aligned}\|F'(x_0)^{-1}(A_n - F'(x_0))\| &\leq \psi_0(\|x_{n-1} - x_0\|, \|2x_n - x_{n-1} - x_0\|) \\ &\leq \psi_0(t_{n-1} - t_0, t_n - t_{n-1} + t_n - t_0) < 1, \quad n = 1, 2, \ldots\end{aligned}$$

so

$$\|A_n^{-1}F'(x_0)\| \leq \frac{1}{\psi_0(t_{n-1} - c, t_n - t_{n-1} + t_n - c)}, \tag{27}$$

and

$$\begin{aligned}\|F'(x_0)^{-1}F(y_n)\| &\leq \|F'(x_0)^{-1}([y_n, x_n; F] - A_n)(y_n - x_n)\| \\ &\leq \psi(\|y_n - x_{n-1}\|, \|x_n - 2x_n + x_{n-1}\|)\|y_n - x_n\| \\ &\leq \psi(s_n - t_{n-1}, t_n - t_{n-1}\|)(s_n - t_n)\end{aligned}$$

leading to

$$\begin{aligned}
\|x_{n+1}-y_n\| &\leq \|A_n^{-1}F'(x_0)\|\|F'(x_0)^{-1}([y_n,x_n;F]-A_n)(y_n-x_n)\| \\
&\leq \frac{\psi(s_n-t_{n-1},t_n-t_{n-1})(s_n-t_n)}{1-\psi_0(t_{n-1}-c,2t_n-t_{n-1}-c)} \\
&= \frac{\alpha_n}{1-\psi_0(t_{n-1}-c,2t_n-t_{n-1}-c)} \\
&= t_{n+1}-s_n,
\end{aligned} \tag{28}$$

and

$$\|x_{n+1}-x_0\| \leq \|x_{n+1}-y_n\|+\|y_n-x_0\| \leq t_{n+1}-s_n+s_n-t_0=t_{n+1}-c<t_*,$$

where we also used

$$\|y_n-x_{n-1}\| \leq \|y_n-x_n\|+\|x_n-x_{n-1}\| \leq s_n-t_n+t_n-t_{n-1}=s_n-t_{n-1},$$

$$\|2x_n-x_{n-1}-x_0\| \leq 2\|x_n-x_0\|+\|x_{n-1}-x_0\| \leq 3t_*.$$

Moreover, we can write

$$F(x_{n+1})=F(x_{n+1})-F(y_n)+F(y_n)=[x_{n+1},y_n;F](x_{n+1}-y_n)+F(y_n),$$

so

$$\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| &\leq \|F'(x_0)^{-1}(([x_{n+1},y_n;F]-F'(x_0))+F'(x_0))\|\|x_{n+1}-y_n\|+\alpha_n \\
&\leq (1+\psi_0(\|x_{n+1}-x_0\|,\|y_n-x_0\|))\|x_{n+1}-y_n\|+\alpha_n \\
&\leq (1+\psi_0(t_{n+1}-t_0,s_n-t_0))(t_{n+1}-s_n)+\alpha_n=b_{n+1}.
\end{aligned} \tag{29}$$

Thus, we obtain by (23) and the second substep of the method (5) that

$$\begin{aligned}
\|y_{n+1}-x_{n+1}\|\| &\leq \|A_{n+1}^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\
&\leq \frac{b_{n+1}}{1-\psi_0(t_n-c,2t_{n+1}-t_n-c)} \\
&= s_{n+1}-t_{n+1}
\end{aligned} \tag{30}$$

and

$$\|y_{n+1}-x_0\| \leq \|y_{n+1}-x_{n+1}\|+\|x_{n+1}-x_0\| \leq s_{n+1}-t_{n+1}+t_{n+1}-c=s_{n+1}-c<t_*.$$

It follows from (28) and (30) that the sequence $\{t_n\}$ is complete (since (28) is also complete as convergent) in a Banach space X and as such is convergence to some point $x^* \in U[x_0,t_*]$. Furthermore, by letting $n \to \infty$ in (29) and using the continuity of F we conclude that $F(x^*)=0$. Then, from the estimate

$$\begin{aligned}
\|x_{n+i}-x_n\| &\leq \|x_{n+i}-x_{n+i-1}\|+\ldots+\|x_{n+1}-x_n\| \\
&\leq t_{n+i}-t_{n+i-1}+\ldots+t_{n+1}-t_n=t_{n+i}-t_n
\end{aligned}$$

and letting $i \to \infty$ we show (25). □

Proposition 2. *Suppose:*
(1) There exists a solution $v_* \in U(x_0,\varrho_6)$ of the equation $F(x)=0$ for some $\varrho_6>0$.
(2) Conditions on $F'(x_0)^{-1}$ and (H_2) hold on $U(x_0,\varrho_6)$.
(3) There exist $\varrho_7>\varrho_6$ such that

$$\psi_0(\varrho_7,\varrho_6)<1. \tag{31}$$

Set $\Omega_4=U[x_0,\rho_7]\cap\Omega$.

Then, the equation $F(x) = 0$ is uniquely solvable by x^* in the region Ω_4.

Proof. Let $w_* \in \Omega_4$ with $F(w_*) = 0$. Define the linear operator G by $G = [w_*, w_*; F]$. By applying (2) and (31), we obtain

$$\|F'(x_0)^{-1}(G - F'(x_0))\| \leq \psi_0(\|w_* - x_0\|, \|v_* - x_0\|) \leq \psi_0(\varrho_7, \varrho_6) < 1.$$

So, the linear operator G is invertible. Therefore, from the identity

$$w_* - v_* = G^{-1}(F(w_*) - F(v_*)) = G^{-1}(0 - 0) = 0,$$

we conclude that $w_* = v_*$. $\square$

The majorizing sequence $\{t_n\}$ for the method (6) is defined similarly by

$$t_{n+1} = s_n + \frac{\alpha_n}{1 - \psi_0(t_n - c, 2s_n - t_n - c)}$$
and
$$s_{n+1} = t_{n+1} + \frac{b_{n+1}}{1 - \psi_0(t_n - c, 2t_{n+1} - t_n - c)}. \tag{32}$$

Lemma 2. *Suppose that for all $n = -1, 0, 1, 2, \ldots$*

$$0 \leq \psi_0(t_n - c, 2s_n - t_n - c) < 1, \quad \psi_0(t_n - c, 2t_{n+1} - t_n - c) < 1, \\ t_n \leq \mu \text{ for some } \mu > 0. \tag{33}$$

Then, the sequence $\{t_n\}$ given by Formula (32) is nondecreasing and convergent to its unique least upper bound $\bar{t}_* \in [0, \mu]$.

Theorem 4. *Suppose that the conditions (H) hold with (32), $\bar{t}_*$ replacing (24) and t_*, respectively. Then, the conclusions of Theorem 3 hold for the method (6).*

The uniqueness of the solution x^* is given in Proposition 2.

Remark 1. *(1) Proposition 2 is shown without using all the conditions of the Theorem 3; however, if all conditions are used, we can set $\varrho_6 = t_*$. In this case $x^* = t_*$.*
(2) If $\Omega = X$, then we have $2x_n - x_{n-1} \in \Omega$ for all $x_n, x_{n-1} \in \Omega$. Consequently, the conditions (C_4) or (H_5) can be replaced by $(C_4)'$ $U[x^*, r] \subset \Omega$ for the method (5) or $U[x_0, \bar{t}_*] \subset \Omega$ for the method (6) and similarly $(H_5)'$ $U[x_0, t_*] \subset \Omega$ for the method (5) or $U[x^*, \bar{r}] \subset \Omega$ for the method (6).
(3) The parameter ϱ_4 given in closed form can be replaced t_* or $\bar{t}_*$ in the condition (H_5) or $(H_5)'$.

4. Numerical Examples

In this section, we provide examples to verify the theoretical result.

Example 1. *Let $X = Y = \mathbb{R}$ and $\Omega \subseteq R$. Define the function F on Ω by*

$$F(x) = x^3 - q,\ q \geq 0,\ x^* = \sqrt[3]{q}.$$

Then,

$$\varphi_0(t_1, t_2) = A_0 t_1 + B_0 t_2,\ A_0 = \max_{x \in \Omega} \frac{|x + 2x^*|}{3(x^*)^2},\ B_0 = \max_{x \in \Omega} \frac{|2x + x^*|}{3(x^*)^2},$$

$$\varphi(t_1, t_2) = A_1 t_1 + B_1 t_2,\ A_1 = \max_{x \in \Omega_0} \frac{|x|}{(x^*)^2},\ B_1 = \max_{x \in \Omega_0} \frac{|2x + x^*|}{3(x^*)^2},$$

$$\psi_0(t_1, t_2) = A_0 t_1 + B_0 t_2,\ A_0 = \max_{x \in \Omega} \frac{|x + 2x_0|}{3x_0^2},\ B_0 = \max_{x \in \Omega} \frac{|2x + x_0|}{3x_0^2},$$

$$\psi(t_1, t_2) = A_1 t_1 + B_1 t_2,\ A_1 = B_1 = \max_{x \in \Omega_2} \frac{|x|}{x_0^2}.$$

Local case Let $\Omega = (0, 1.5)$ and $q = 0.9$. Then, $x^* \approx 0.9655$, $\Omega_0 \approx (0.7830, 1.1479)$, $r = \bar{r} \approx 0.0874$, $U[x^*, 3r] \approx [0.7033, 1.2277] \subset \Omega$.

Semi-local case. Let $\Omega = (0, 1.5)$, $q = 0.9$, $x_0 = 1$, $x_{-1} = 1.05$. Then, $\Omega_2 = (0.55, 1.45)$. Majorizing sequences for method (5) and (6) are

$$\{t_n\} = \{0,\ 0.0500,\ 0.0898,\ 0.1084,\ 0.1152,\ 0.1164,\ \dots, 0.1165\},$$

$$\{\bar{t}_n\} = \{0,\ 0.0500,\ 0.0904,\ 0.1101,\ 0.1177,\ 0.1192,\ \dots, 0.1193\},$$

respectively. So, $U[x^*, 3t_*] \approx [0.6506, 1.3494] \subset \Omega$ and $U[x^*, 3\bar{t}_*] \approx [0.6422, 1.3578] \subset \Omega$.

Example 2. *Consider the system of m equations*

$$\sum_{j=1}^{m} x_j + e^{x_i} - 1 = 0, \quad i = 1, \dots, m.$$

Here $X = Y = \mathbb{R}^n$, $\Omega \subseteq R$ *and* $x^* = (0, \dots, 0)^T$.

Then

$$\varphi_0(t_1, t_2) = A_0 t_1 + B_0 t_2,\ A_0 = B_0 = \max_{x \in \Omega} \frac{e^x - 1}{2\gamma},$$

$$\varphi(t_1, t_2) = A_1 t_1 + B_1 t_2,\ A_1 = B_1 = \max_{x \in \Omega_0} \frac{e^x}{2\gamma},\ \gamma = |(F'(x^*))^{-1}_{1,1}|,$$

$$\psi_0(t_1, t_2) = A_0 t_1 + B_0 t_2,\ A_0 = B_0 = \max_{x \in \Omega} \frac{e^x}{2\gamma},$$

$$\psi(t_1, t_2) = A_1 t_1 + B_1 t_2,\ A_1 = B_1 = \max_{x \in \Omega_2} \frac{e^x}{2\gamma},\ \gamma = |(F'(x_0))^{-1}_{1,1}|.$$

Local case. Let $m = 5$, $\Omega = U(x^*, 1)$. Then, $\Omega_0 \approx (x^*, 0.3492)$, $r = \bar{r} \approx 0.1719$, $U[x^*, 3r] \approx [-0.5157, 0.5157] \subset \Omega$.

Semi-local case. Let $m = 5$, $\Omega = U(x^*, 1)$, $x_0 = (0.02, \dots, 0.02)$ and $x_{-1} = x_0 + 0.0001$. Then, $\Omega_2 \approx (x_0, 0.4502)$. Majorizing sequences for method (5) and (6) are

$$\{t_n\} = \{0,\ 0.0001,\ 0.1047,\ 0.1266,\ 0.1372,\ \dots, 0.1387\},$$

$$\{\bar{t}_n\} = \{0,\ 0.0001,\ 0.1064,\ 0.1323,\ 0.1460,\ \dots, 0.1487\},$$

respectively. So, $U[x^*, 3t_*] \approx [-0.3962, 0.4362] \subset \Omega$ and $U[x^*, 3\bar{t}_*] \approx [-0.4260, 0.4660] \subset \Omega$.

Let us apply methods (4)–(6) for solving considered nonlinear problems under different initial approximations x_0. All these methods require addition approximation x_{-1}. It is computed by the rule $x_{-1} = x_0 + 10^{-4}$. The stopping conditions for the iterative process are $\|x_{n+1} - x_n\| \leq 10^{-8}$.

Tables 1 and 2 show number of iterations that are needed for solving one equation and system of equations for $m = 10$.

Figures 1 and 2 demonstrate that norms $F(x_n)$ and $x_n - x_{n-1}$ for the two-step Kurchatov's methods (5) and (6) decrease faster than for Kurchatov's method (4).

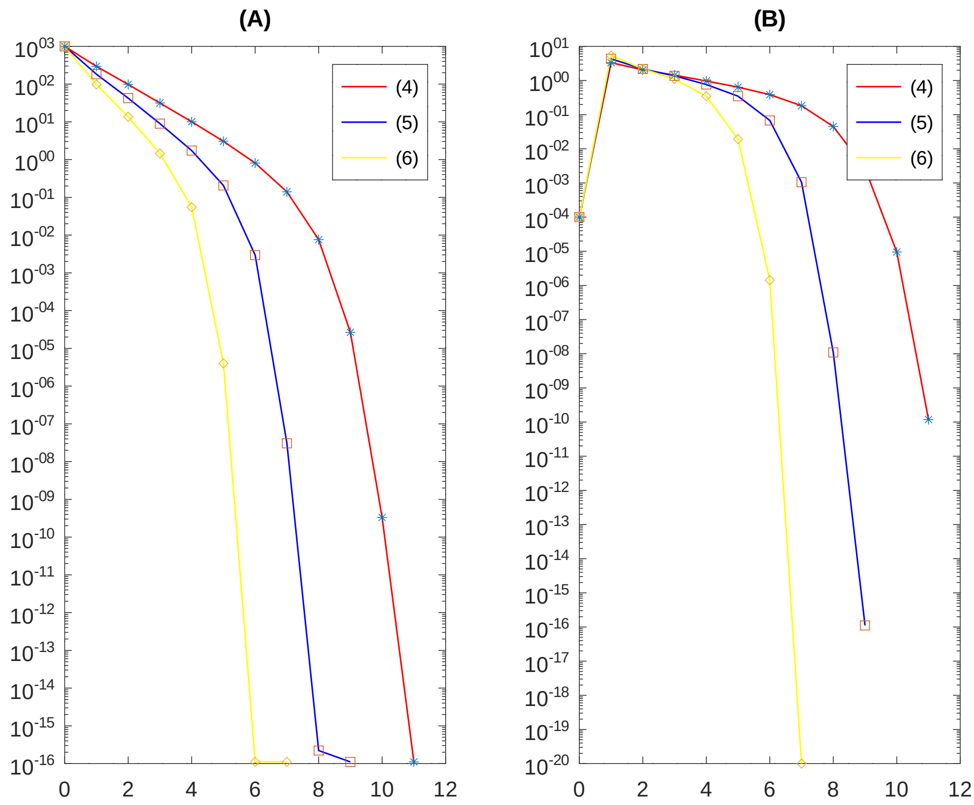

Figure 1. Example 1: norm of residual—(**A**) and norm of correction—(**B**) at each iteration.

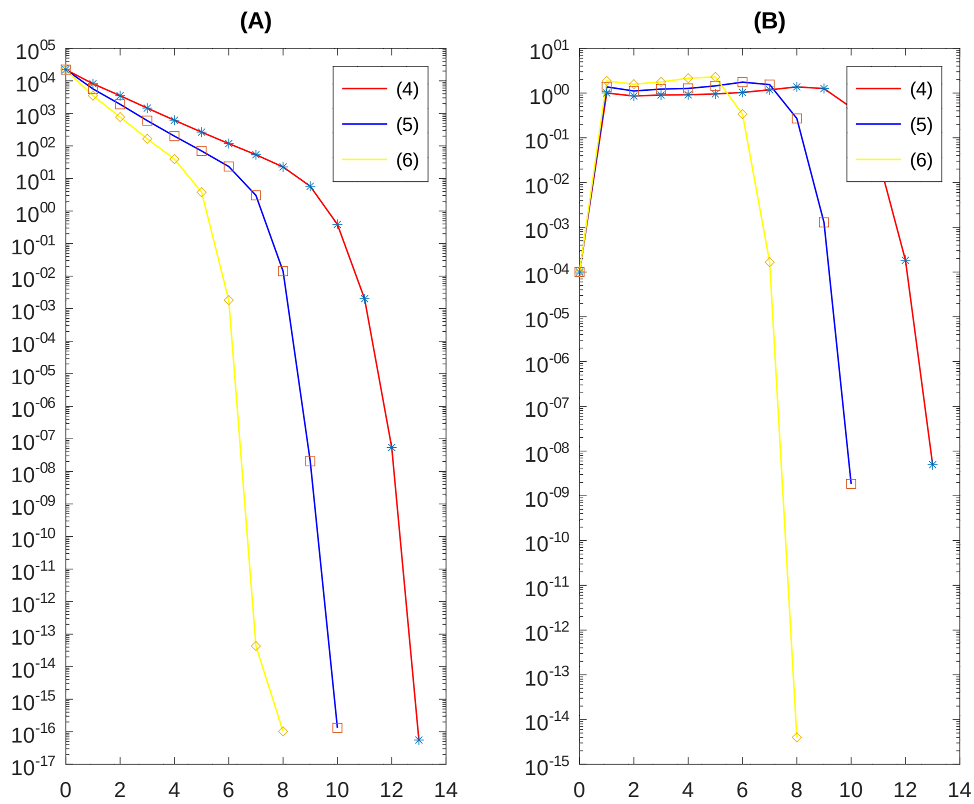

Figure 2. Example 2: norm of residual—(**A**) and norm of correction—(**B**) at each iteration.

Table 1. Results for Example 1.

x_0	**Method (4)**	**Method (5)**	**Method (6)**
1	4	3	3
10	11	9	7
100	17	13	10

Table 2. Results for Example 2.

x_0	**Method (4)**	**Method (5)**	**Method (6)**
$(1,\ldots,1)^T$	5	4	3
$(10,\ldots,10)^T$	13	10	8
$(20,\ldots,20)^T$	25	19	14

5. Conclusions

The objective in this work is to develop a process for studying the convergence of iterative methods containing inverses of linear operators under weak conditions. These conditions involve only operators appearing in the methods. In particular, a local and a semi-local convergence analysis of the two-step Kurchatov-type methods is provided under the generalized Lipschitz conditions for only divided differences of order one. Regions of convergence and uniqueness of the solution are established. The results of the numerical experiment are given. The developed technique does not rely on the studied methods. That is why it can also be used on other methods that contain inverses of divided differences or inverses of linear operators in general.

The future work involves the application of this process on other single step, multi-step iterative methods with inverses [14,15,17,20,21]. We will also study the analogs of the studied methods when the Fréchet is replaced by the Gateaux derivative.

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