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Long-Wave Anti-Plane Motion in a Pre-Stressed Compressible Elastic Laminate with One Fixed and One Free Face

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Abstract: In this paper, long-wave anti-plane shear motion in a multilayered laminate composed of pre-stressed compressible elastic layers is investigated. The layers of the laminate are perfectly bonded, while a fixed-free boundary condition is prescribed on the outer faces of the laminate. The solution of the model is determined analytically via the propagator matrix and numerically through the asymptotic approach. Moreover, the numerical results featuring harmonic curves are presented graphically, together with an asymptotic long-wave analysis of the vibration modes. As a special case of materials, linear isotropic with one shear modulus is considered. A polynomial long-wave low-frequency approximation of the related dispersion relation is also studied. It governs dispersion curves including the lowest harmonic. It is revealed that a low-frequency mode exists in both the two- and three-layered laminates, which are taken as prototypical structures. Lastly, comparisons between the exact and approximate asymptotic results are presented, and excellent agreement is observed.

Keywords: anti-plane motion; pre-stressed; compressible; laminates; fixed-free; long-wave; cut-off frequency; asymptotic

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1. Introduction

Addressing the problem of wave propagation in multilayered elastic media is a crucial subject of much concern. Thus, we accordingly attribute the early theoretical works on the propagation of waves with regard to the material behaviour and dispersion characteristics of such structures to the works of Lamb in 1917, Tolstoy and Usdin in 1957, and Mindlin in 1960; for more on these studies, interested reader(s) are refereed to [1–6] and the references therewith, where the propagation of waves was associated with higher modes for a plane section of an isotropic plate. In addition, many of the findings of the previous researches are tackled via the application of numerical methods, including, for instance, the works by Rogerson and Sandiford in 1997 [7], 2000 [8], and 2002 [9], respectively, with regard to the computational analysis of the frequency equations arising in multilayered composites; moreover, cases of the symmetric and anti-symmetric vibrations were considered in favor of incompressible symmetric three-layered panels.

Asymptotic analysis, based on the ratio of the materials parameters has been discussed by a considerable number of research for compressible, incompressible and nearly incompressible materials, we may cite here the early wark by Willson in 1973 [10] who examined wave propagation in an incompressible pre-stressed plate and Ogden and Roxburgh [11], who investigated in detail plane incremental waves and vibrations in a pre-stressed incompressible elastic plate.

In the counterpart, studies focused on the compressible materials examined for infinitesimal surface waves by Chadwick and Jarvis 1979 [12]. More contributions included in some papers, see for example the work by Ogden [13] and the recent paper by Helmi and Rogreson [14] investigated long wave motion in a compressible pre-stressed materials. Some aspects of wave motion problems, in slightly compressible materials can be found in the work done by Chattopadhyay and Rogerson [15], and the references [16–19].

Nevertheless, the asymptotic analysis approach has equally been utilized in the past and has further been proven to be an expensive tool for the analytical scrutiny of different frequency equations or, alternatively, the dispersion relations. For more on the use and application of the asymptotic analysis approach, we make mention of the famous book of Kaplunov et al. [20] on the dynamics of thin-walled elastic media via the asymptotic method, the work of Rogerson et al. [21] on the vibration of waves on thin elastic plates through the asymptotic method, the good paper by Andrianov et al. [22] on the vibration of waves in periodic composite media via higher-order asymptotic homogenization, and the relevant study by Daya and Potier-Ferry [23] on the propagation of waves in repetitive structures via the application of the double-scaled asymptotic approach, to mention a few. Equally, we mention the recent developments with regard to the application of the asymptotic approach to both the plane and anti-plane dynamic problems, which are associated with the multilayered and composite structures, to include [24–30] and the references therein. Furthermore, unlike just the mere analytical or numerical approach, the asymptotic procedure is able to provide the complete characterization of the dispersion relation by deeply unraveling some salient features of the propagating wave. For instance, we recall various studies by Rogerson et al. [31], Nolde and Rogerson [32], Pichugin and Rogerson [33], and Knowles [34], to mention a few, that lead to the complete understanding of the dispersion relation of a propagating wave in diverse heterogeneous elastic media. In fact, an asymptotic solution generally involves donations from each mode of the frequency equation; read the works of Rogerson [35] and Yang [27] for more on the approximation of the fundamental mode within the low-frequency region. Additionally, further studies on vibration stability, fiber-reinforced composites, long-wave motions in multilayered media, and material nonhomogeneity, which are momentous in their own right, can be accessed via the reported results in [36–42] and the references enclosed therein.

In particular, as the anti-plane shear motion would be the base for the formulation of the governing model of multilayered laminate, let us recall that the equation of the anti-plane motion according to Horgan in 1995 [43] is "an interesting two-dimensional mathematical model arising in solid mechanics involving a single second-order linear or quasi-linear partial differential equation. This model has the virtue of relative mathematical simplicity without a loss of essential physical relevance. Anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo". Thus, recent considerations of elastodynamical models through anti-plane dynamic motion comprise the study of the dispersion of elastic waves in inhomogeneous three- and five-layered panels of various structural configurations [44–48]. Certainly, the four types of material contrasts, involving low- and high-contrast material, which are typical for the classical sandwich, were analyzed. We also made mention of the recent mechanically loaded multilayered plates that were modeled via anti-plane shear motion in [49]. In fact, these mechanical loads happened to be due to Winkler elastic foundations—the famous simple class of elastic foundations, which found its application in a variety of engineering applications. In addition, multilayered structures with high material contrast layers are extensively utilized in contemporary engineering; take a look at, for instance, photovoltaic panels, laminated glass, vibration filters, and smart periodic structures, to mention a few, see [50] and the references therein.

However, within this article we investigate the anti-plane dynamic shear of pre-stressed compressible multilayered structures, considering the two- and three-layered laminates as prototypical structures. A fixed-free type of boundary condition will be examined with material contrast and non-contrast properties for each layer. Moreover, as the structural performance of multilayered laminate depends not only on the properties of shear modulus and the geometrical dimensions of the components, the present study will adopt both the analytical and approximation approaches for a solution, to deeply examine the resulting dispersion relations. A comparative investigation between the two approaches is also set to be carried out. For this reason, we arrange the present study as follows: the formulated problem is given in Section 2, while Section 3 determines the solution and the analytical dispersion relations. Section 4 makes use of the asymptotic analysis approach to derive the corresponding approximate results. Additionally, the cut-off frequency and the polynomial dispersion relation, corresponding to the long-wave low-frequency limit, are equally derived in Section 4. Lastly, Section 5 provides some concluding remarks.

2. Statement of the Problem

The first concern in this paper is the modelling and analysis of a 2-layered laminate formed by the ground layer of thickness h_1 and perfectly bonded to the second layer of thickness h_2. Then, we model a 3-layered laminate of thickness $h_1 + h_2 + h_3$. Both layers of 2-layered laminate are assumed to be composed of a pre-stressed compressible material. The structures are finite in x_2 direction and of infinite in the two remaining lateral extent in both the x_1 and x_3 directions; see Figure 1 for the 3-layered prototypical structure.

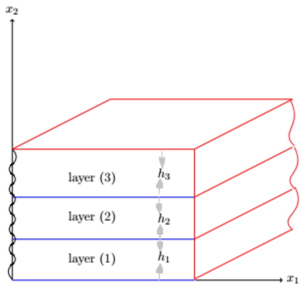

Figure 1. A 3-layered structure.

Moreover, for the sake of simplicity, we consider a state of anti-plane strain for this layered problem, thus, the only non-zero displacement component is orthogonal to the plane $x_1\,x_2$ and taking the form: $(u_1, u_2, u_3) = (0, 0, u_3)$. Hence, the equations of motion for the 2-layered and 3-layered pre-stressed compressible structures can be written in the following form [20,21,25]

$$C^{(n)}_{1313}u_{3,11} + C^{(n)}_{2323}u_{3,22} = \rho_n \ddot{u}_3, \tag{1}$$

where $n = 1, 2$ for the 2-layered laminate and $n = 1, 2, 3$ for the 3-layered laminate.

In addition, perfectly bonded interfaces are considered between the layers of the governing structures; thus ensuring continuity of displacements and tractions across the interfaces. The boundary conditions of zero traction at one surface and zero displacement of the other one will be applied. Thus, in what follows, the exact dispersion relations in both structures will be determined analytically via the application of the propagator matrix technique, and subsequently, asymptotically.

3. Exact Dispersion Relation

To determine the resulting dispersion relations, the solution of the above equation is sought as a travelling wave in the following form

$$u^{(n)}_3(x_1, x_2, t) = U e^{kq_n x_2} e^{ik(x_1 - vt)}, \tag{2}$$

where k is the wave number, U is an arbitrary constant, t is time, $C^{(n)}_{2323}$, $C^{(n)}_{1313}$ are material parameters, ρ_n are the volume mass densities of layers, v is the phase wave speed, in which the subscript (n) denotes to the layer number, for $n = 1, 2$ or $n = 1, 2, 3$, and q_n is to be determined. After substituting the above solution into (1), we obtain a linearised equation, which possesses a non-trivial solution provided in the following form

$$C^{(n)}_{2323}q_n^2 - C^{(n)}_{1313} + \rho_n v^2 = 0, \tag{3}$$

from which we deduce that the solutions for q_n are given by

$$q_n^2 = \frac{C_{1313}^{(n)} - \rho_n\, v^2}{C_{2323}^{(n)}}. \tag{4}$$

Then, the displacement can be written after separating $e^{ik(x_1 - vt)}$ factor as linear combinations, associated with the two solutions indicates in (4) as

$$u_3^{(n)} = U_n e^{kq_n x_2} + V_n e^{-kq_n x_2}. \tag{5}$$

In addition, the non-zero increment traction may be defined in component form by

$$\hat{\tau}^{(n)} = \frac{\tau_3^{(n)}}{k} = C_{2323}^{(n)}\, q_n \left(U_n e^{kq_n x_2} - V_n e^{-kq_n x_2} \right). \tag{6}$$

Accordingly, the appropriate matrix form for both the two layers (for the 2-layered laminate) may be introduced as

$$\begin{pmatrix} u_3^{(n)} \\ \hat{\tau}^{(n)} \end{pmatrix} = \begin{pmatrix} e^{kq_n h_n} & e^{-kq_n h_n} \\ q_n C_{2323}^{(n)} e^{kq_n h_n} & -q_n C_{2323}^{(n)} e^{-kq_n h_n} \end{pmatrix} \begin{pmatrix} U_n \\ V_n \end{pmatrix}. \tag{7}$$

Then, the solution can be rewritten in the following form

$$\mathbf{Y}(x_2^{l_n}) = \mathbf{Q}^{(n)}\ \mathbf{U}, \tag{8}$$

where $\mathbf{U} = (U_n, V_n)^T$, $\mathbf{Y}(x_2^{l_n}) = (u_3^{(n)}, \hat{\tau}^{(n)})^T$ and $\mathbf{Q}^{(n)}$ is the 2×2 matrix introduced in (7). Next, the arbitrary constant vector $\mathbf{U}$ may be eliminated from the solution shown in (7) to yield

$$\begin{pmatrix} u_3^{(n)} \\ \hat{\tau}^{(n)} \end{pmatrix} = \begin{pmatrix} \cosh(kq_n h_n) & \dfrac{1}{C_{2323}^{(n)} q_n} \sinh(kq_n h_n) \\ C_{2323}^{(n)} q_n \sinh(kq_n h_n) & \cosh(kq_n h_n) \end{pmatrix} \begin{pmatrix} \tilde{u}_3^{(n)} \\ \tilde{\tau}^{(n)} \end{pmatrix}. \tag{9}$$

Further, this can be rewritten as

$$\mathbf{Y}(x_2) = \mathbf{P}^{(n)}\ \tilde{\mathbf{Y}}(x_2), \tag{10}$$

in which, the displacements and tractions for the lower layer are denoted by imposing an over tilde, and $\mathbf{P}^{(n)}$ is the so-called propagator matrix for each layer, given by

$$\mathbf{P}^{(n)} = \begin{pmatrix} \cosh(kq_n h_n) & \dfrac{1}{C_{2323}^{(n)} q_n} \sinh(kq_n h_n) \\ C_{2323}^{(n)} q_n \sinh(kq_n h_n) & \cosh(kq_n h_n) \end{pmatrix}. \tag{11}$$

Then, the components of the above matrix for the whole 2-layered structure will be provided by multiplying $\mathbf{P} = \mathbf{P}^{(2)}\ \mathbf{P}^{(1)}$.

Similarly, we can generate the propagator matrix for the 3-layered structure, which has been built by adding one more layer with the same material parameters and volume mass density of the first layer but of different thickness h_3, that is, the third layer occupied $h_2 \leq x_2 \leq h_3$. Thus, the upper surface is at $x_2 = (h_1 + h_2 + h_3)$, and the lower surface at $x_2 = 0$. Therefore, the components of the propagator matrix can be introduced for the whole 3-layered structure by multiplying $\mathbf{P} = \mathbf{P}^{(3)}\, \mathbf{P}^{(2)}\ \mathbf{P}^{(1)}$, that is

$$\mathbf{P} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}. \tag{12}$$

In general, if we add more layers to the structure under consideration, we will be able to reformulate the solution by the propagator matrix as $\mathbf{Y}^{(upper)} = \mathbf{P}\,\mathbf{Y}^{(lower)}$, with

$$\mathbf{P} = \mathbf{P}^{(n)}\,\mathbf{P}^{(n-1)}...\mathbf{P}^{(1)}.$$

Therefore, to derive the dispersion relation for the fixed-free boundary condition, we satisfy the boundary conditions of zero traction on one surface and the zero displacement on the other surface, in conjunction with the condition of continuity across the interface, see [51].

Under the above assumptions, either $P_{11} = 0$ or $P_{22} = 0$ yields the dispersion relation. Thus, by considering $kq_n h_n = \alpha_n$, $\Gamma_{23}^{(n)} = C_{2323}^{(n)}$, it can be easily shown that the dispersion relation for the 2-layered laminate takes the following exact form

$$\frac{\alpha_2\,\Gamma_{23}}{\alpha_1}\tanh(\alpha_1\,h_{12})\tanh(\alpha_2) + 1 = 0. \tag{13}$$

Following similar steps, the associated dispersion relation for the 3-layered laminate can equally be expressed as follows

$$1 + \tanh(\alpha_1\,h_{32})\left(\tanh(\alpha_1\,h_{12}) + \frac{\alpha_1}{\Gamma_{23}\,\alpha_2}\tanh(\alpha_2)\right) + \frac{\Gamma_{23}\,\alpha_2}{\alpha_1}\tanh(\alpha_1\,h_{12})\tanh(\alpha_1) = 0, \tag{14}$$

with,

$$\alpha_1 = \sqrt{\frac{\gamma}{\Gamma}K^2 - \frac{\Gamma_{23}}{\rho}\Omega^2}, \qquad \alpha_2 = \sqrt{\gamma K^2 - \Omega^2}, \tag{15}$$

where the dimensionless quantities including the frequency Ω and wave number K are defined as follows

$$\Omega = \frac{\omega\,h_2}{\Gamma_{23}^{(2)}/\rho_2}, \qquad K = kh_2, \tag{16}$$

with the following introduced dimensionless basic parameters

$$h_{12} = \frac{h_1}{h_2}, h_{32} = \frac{h_3}{h_2}, \rho = \frac{\rho_2}{\rho_1}, \Gamma_{23} = \frac{\Gamma_{23}^{(2)}}{\Gamma_{23}^{(1)}}, \Gamma_{13} = \frac{\Gamma_{13}^{(2)}}{\Gamma_{13}^{(1)}}, \gamma = \frac{\Gamma_{13}^{(2)}}{\Gamma_{23}^{(2)}}, \Gamma = \frac{\Gamma_{13}}{\Gamma_{23}}.$$

4. Asymptotic Approach of Pre-Stressed Materials

In this section, we determine the governing approximate dispersion relations asymptotically with regard to both the 2- and 3-layered laminates. More so, in what follows, we examine the effect of material contrast within the low-frequency vibration modes, in relation to the obtained exact and approximate dispersion relations in both scenarios. Hence, the material contrasts of interest for the typical sandwich laminates include [47–49].

$$(i)\ \Gamma_{23} \ll 1,\ h_{12} \sim 1,\ \rho \sim \Gamma_{23}^2.$$

$$(ii)\ \Gamma_{23} \ll 1,\ h_{12} \sim 1,\ \rho \sim \Gamma_{23},$$

4.1. 2-Layered Laminate

Let us begin here with the case of a 2-layered laminate. Therefore, Guided by the numerical analysis of the dispersion relation (13) in the long wave region, we set $\alpha_n = i\hat{\alpha}_n$, in which $\hat{\alpha}_n$ is real and positive, see [9]. This leads us to rewrite the dispersion relation obtained in the case of a 2-layered laminate in (13) as

$$\frac{\alpha_1}{\Gamma_{23}\,\alpha_2}\tan(\alpha_1\,h_{12})\tan(\alpha_2) - 1 = 0, \tag{17}$$

with,

$$\alpha_1 = \Omega\left(1 - \frac{\gamma K^2}{2\Gamma\,\Omega^2} + ..\right)\sqrt{\frac{\Gamma_{23}}{\rho}}, \qquad \alpha_2 = \Omega\left(1 - \frac{\gamma K^2}{2\Omega^2} + ...\right), \tag{18}$$

Then, with the help of (16), we may adopt a near cut-off asymptotic expansion in the form

$$\Omega^2 = \Omega_0 + \Omega_1 K^2 + O(K^4), \tag{19}$$

such that upon substituting the latter equation into (17), we obtain at the leading order (first) the following

$$\sqrt{\Gamma_{23}\,\rho} - T(\Omega_0^*)\,T(\Omega_0) = 0, \tag{20}$$

where Ω_0 are the cut-off frequency values, that will be solved numerically.

Then, the second order Ω_1 can be calculated from

$$\Omega_1 = \frac{T(\Omega_0^*)\,T(\Omega_0)\big((\Omega_0\,T(\Omega_0) + T(\Omega_0^*)) + \Omega_0(1 + h_{12}\Gamma_{23}\rho) + \sqrt{\Omega_0} + (1 + h_{12}\,\Gamma_{23}^2)\;T(\Omega_0)\big)}{T(\Omega_0^*)\,T(\Omega_0)\big(h_{12}\,\Gamma_{23}^2 T(\Omega_0^*) + \sqrt{\rho}T(\Omega_0)\big) + T(\Omega_0)\,h_{12}\,\Gamma_{23}^2) + \sqrt{\rho}\,T(\Omega_0^*)}, \tag{21}$$

where

$$T(\Omega_0^*) = \tan\left(\sqrt{\frac{\Gamma}{\rho}\,\Omega_0}\,h_{12}\right), \quad T(\Omega_0) = \tan(\sqrt{\Omega_0}).$$

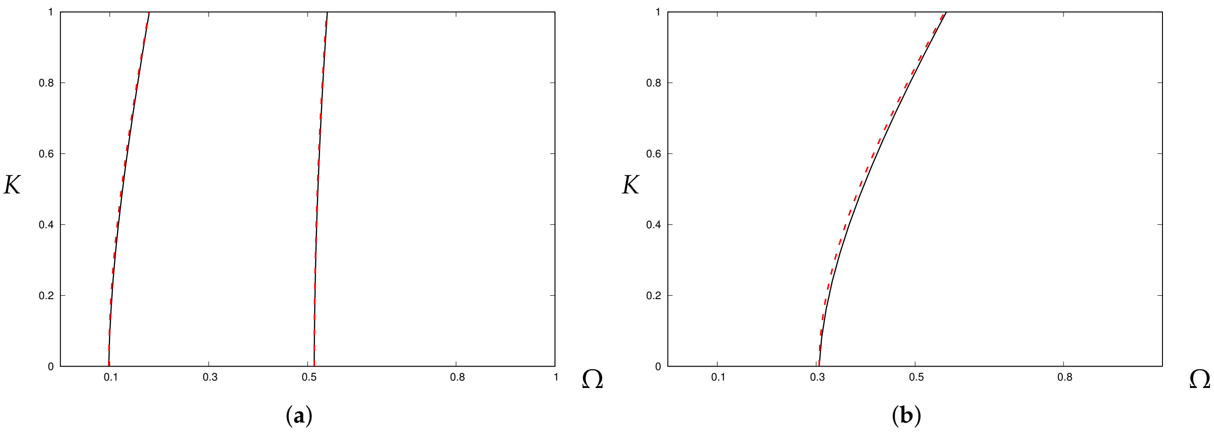

Figure 2. Harmonic curves for a 2-layered pre-stressed laminate through the dispersion relation (13) (black solid line) and asymptotic solution (19) (red dashed line) when (**a**) $\Gamma_{23} = 0.5$, $\Gamma_{13} = 0.45$, $\rho = 0.01$, and $h_{12} = 0.9$, and (**b**) $\Gamma_{23} = 0.5$, $\Gamma_{13} = 0.45$, $\rho = 0.1$, and $h_{12} = 0.9$.

Harmonic curves computed from Equation (13) are shown in Figure 2a,b for the two cases of material contrasts. This provides the lowest cut-off frequencies as $\Omega_0 \approx 0.1$, 0.51 and $\Omega_0 \approx 0.3$. Moreover, the figures also portrayed comparative curves between the exact dispersion relation earlier determined and that of the asymptotic result found in (19). In fact, from these figures a good agreement between the two approaches has been realized.

4.2. 3-Layered Laminate

Equally, we introduce the asymptotic approach to (14). In doing so, we first re-write the equation in the following form

$$\tan(\alpha_1\,h_{32})\left(\tan(\alpha_1\,h_{12}) + \frac{\Gamma_{23}\,\alpha_2}{\alpha_1}\tan(\alpha_2)\right) + \frac{\alpha_1}{\Gamma_{23}\,\alpha_2}\tan(\alpha_1\,h_{12})\tan(\alpha_2) - 1 = 0, \tag{22}$$

then, in asymptotic form, we get

$$T(\Omega_0^*)\,T(\Omega_0^{**}) + T(\Omega_0)\left(\sqrt{\rho\,\Gamma_{23}}\,T(\Omega_0^{**}) + T(\Omega_0^*)\right) - 1 = 0, \tag{23}$$

where,
$$T(\Omega_0^{**}) = \tan\left(\sqrt{\frac{\Gamma\,\Omega_0}{\rho}}\;h_{32}\right).$$

Then the (19) may be written as
$$\tilde{\Omega}^2 = \tilde{\Omega}_0 + \tilde{\Omega}_1\,K^2 + O(K^4), \tag{24}$$

where $\tilde{\Omega}_0$ is the cut-off frequencies for 3-layered laminate, which can be obtained numerically from (23) and $\tilde{\Omega}_1 = \dfrac{F_1(\Omega_0)}{F_2(\Omega_0)}$, where $F_1(\Omega_0)$ and $F_2(\Omega_0)$ are found to be as follow

$$\begin{aligned}F_1(\Omega_0) &= T(\Omega_0)\Big(\Omega_0\sqrt{\rho\,\Gamma_{23}}\Big(h_{32}\,T^2(\Omega_0^{**}) + h_{12}\,T^2(\Omega_0^{*})\Big) + \Gamma_{23}\,\Gamma\sqrt{\Omega_0}(T(\Omega_0^{**}) + T(\Omega_0^{*}))\\ &+\sqrt{\rho\,\Gamma_{23}}\Omega_0(h_{12}+h_{32}) - \rho\sqrt{\Omega_0}(T(\Omega_0^{**}) + T(\Omega_0^{*}))\Big) + T^2(\Omega_0)(\Omega_0\,\Gamma_{23}(T(\Omega_0^{**}) + T(\Omega_0^{*})))\\ &+T(\Omega_0^{*})(h_{12}+\Gamma_{23}\,\Gamma)\,\Omega_0 + T(\Omega_0^{**})\,(h_{32}+\Gamma_{23}\,\Gamma)\,\Omega_0,\end{aligned}$$

$$\begin{aligned}F_2(\Omega_0) &= T(\Omega_0)\Big(\Omega_0\sqrt{\rho\,\Gamma_{23}}\Big(h_{32}\,T^2(\Omega_0^{**}) + h_{12}\,T^2(\Omega_0^{*})\Big) + \Gamma_{23}\,\Gamma\sqrt{\Omega_0}(T(\Omega_0^{**}) + T(\Omega_0^{*}))\\ &+\sqrt{\rho\,\Gamma_{23}}\Omega_0(h_{12}+h_{32}) + T(\Omega_0^{*}))\Big)\Omega_0\Gamma_{23} + T^2(\Omega_0)\rho((T(\Omega_0^{**}) + T(\Omega_0^{*})))\\ &+T(\Omega_0^{*})(h_{12}+\rho) + T(\Omega_0^{**})\,(h_{32}+\rho).\end{aligned}$$

4.3. Special Cases

In this regard, we consider a special case of linear isotropic material in both layers of the structures (of course, the structures are composed of different materials in each layer). Therefore, upon assuming the 2- and 3-layered laminate under consideration to be isotropic materials, several of the existing results are set to be obtained and analyzed.

4.3.1. 2-Layered Isotropic Laminate

Therefore, the dispersion relation found (17) with isotropic material parameters can be expressed as follows
$$\frac{\alpha_1}{\mu\,\alpha_2}\tan(\alpha_1\,h_{12})\tan(\alpha_2) - 1 = 0, \tag{25}$$

in which,
$$\alpha_1 = \sqrt{\frac{\mu}{\rho}\Omega^2 - K^2}, \quad \alpha_2 = \sqrt{\Omega^2 - K^2}, \quad \mu = \frac{\mu_2}{\mu_1}.$$

Thus, the frequency equation in this case can only be computed for two material parameters ρ and μ, in which $\Gamma_{23} = \Gamma_{13} = \mu$. Hence, for such a structure, the dispersion relation (13) can be introduced as
$$\sqrt{\rho\,\mu} = \tan(\Omega)\tan\left(h_{12}\sqrt{\frac{\mu}{\rho}}\Omega\right). \tag{26}$$

Further, within the low-frequency regime, both arguments in the above equation must satisfy the following relations, that is,
$$\Omega \ll 1, \qquad \text{and} \qquad h_{12}\sqrt{\frac{\mu}{\rho}}\Omega \ll 1,$$

over the low-frequency range
$$\Omega\left(1 + h_{12}\sqrt{\frac{\mu}{\rho}}\right) \ll 1.$$

Furthermore, from (26), we can write

$$\sqrt{\rho\,\mu} = \Omega\left(\Omega\, h_{12}\sqrt{\frac{\mu}{\rho}}\right), \tag{27}$$

upon which $\Omega \approx \sqrt{\dfrac{\rho}{h_{12}}} \ll 1$ and $h_{12}\sqrt{\dfrac{\mu}{\rho}}\Omega \ll 1$, which further gives the predicted single cut-off frequency as follows

$$\rho \ll h_{12} \ll \mu^{-1}. \tag{28}$$

Hence, the formulated problem under consideration does not support the fundamental mode with a zero cut-off frequency.

Besides, on the long-wave motion, that is, when $K(1+h) \ll 1$, and in conjunction with the low-frequency range expressed in (28), the shortened polynomial dispersion relation from (26) is further got to be

$$\mu + h_{12}\,K^2 - \frac{\mu\,h_{12}}{\rho}\Omega^2 + h_{12}\left(\frac{h_{12}^2+1}{3}\right)K^4 - \frac{h_{12}}{3}\left(1+\frac{\mu}{\rho}(2h_{12}^2+1)\right)\Omega^2\,K^2 + ... = 0. \tag{29}$$

Equation (29) will be analysed for two cases of contrast. In case (a), $\mu \ll 1$, $\rho \sim \mu^2$, $h_{12} \sim 1$, then $\rho^* = \dfrac{\rho}{\mu^2} \sim 1$ and $h_{12}^* = \dfrac{h_{12}}{\mu}$. As a result, we obtain the shortened polynomial cut-off frequency equation as follows

$$\mu + \mu\left(\frac{1}{2} + h_{12}^*\right)K^2 - \frac{h_{12}^*}{3\rho^*}\,K^2\,\Omega^2 - \left(\frac{\mu}{2} + \frac{h_{12}^*}{\rho^*}\right)\Omega^2 + ... = 0, \tag{30}$$

In order to get the shortened polynomial in this case, we first provide Ω in the following form

$$\Omega_*^2 = \Omega_0^2 + \mu\Omega_2^2 + .. \tag{31}$$

Then, after substituting the later into (30), we finally arrived at

$$\Omega_0^2 = \frac{\rho^*}{h_{12}^*},\quad \Omega_2^2 = \frac{\rho^*}{h_{12}^*}\left(\frac{1}{3} + h_{12}^*\right)K^2 - \frac{1}{3}\left(\frac{\rho^*}{h_{12}^*}\right)^2. \tag{32}$$

Next, for the parameters in case (ii), we set $\rho^* = \frac{\rho}{\mu} \sim 1$, and further express the shortened dispersion formula as follows

$$\frac{\mu}{h_{12}} + K^2 - \frac{1}{\rho^*}\Omega^2 = 0. \tag{33}$$

Then, we introduce $\Omega^2 = \mu^\eta\,\Omega_*^2$ and $K^2 = \mu^\eta\,K_*^2$, in which, $\Omega_* \sim K_* \sim 1$, $0 < \eta \leq 1$. In the long-wave low-frequency regime, equation written in (33) may be re-expressed as

$$\Omega_*^2 = \rho^*\left(K_*^2 + \frac{\mu^{1-\eta}}{h_{12}}\right). \tag{34}$$

Amazingly, the above special case (results) for the 2-layered linear isotropic laminate exactly corresponds to the findings of the symmetric three-layered plate that was recently studied by Prikazchikova et al. (2018) [48]. More so, it is very relevant to state here that the case of antisymmetric vibrational modes was analyzed in [41], owing to the fact that only antisymmetric dispersion relation was found to satisfy the global low-frequency region. Further, we examine two cases of material contrasts, comprising case (i) when $(\mu = 0.2, h_{12} = 0.9, \rho = 0.01)$, and case (ii) when $(\mu = 0.2, h_{12} = 0.9,\ \rho = 0.1)$. Additionally, the dispersion curves have been presented for the first case of contrasting setup in Figure 3a and for the second case in Figure 3b, respectively. Furthermore, it is observed from these figures that the lowest cut-off frequency associated with cases (i) and (ii) are approximately realized when at $\Omega \approx 0.102$ and $\Omega \approx 0.31$, correspondingly.

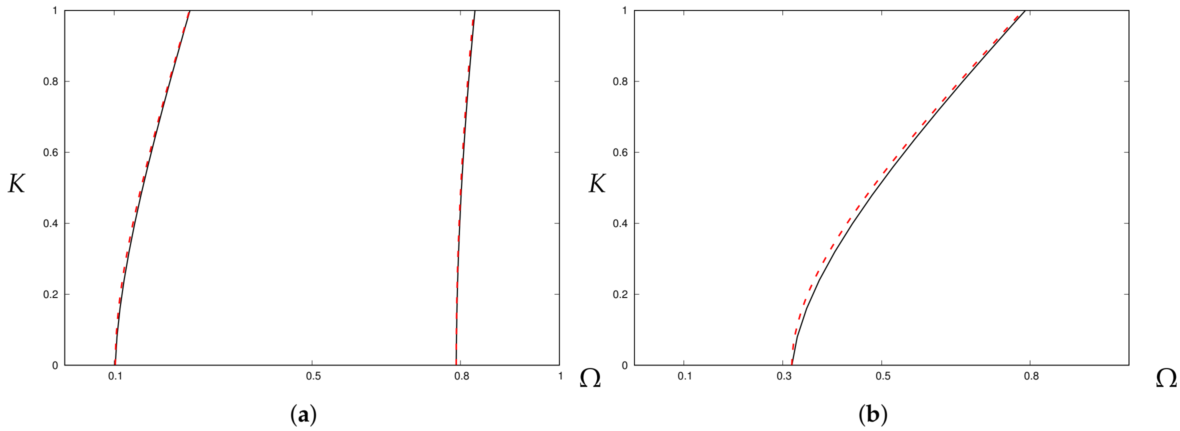

Figure 3. Harmonic curves for a 2-layered linear isotropic laminate through the dispersion relation (26) (black solid line) and shortened polynomial (33) (red dashed line) when (**a**) $\mu = 0.2$, $\rho = 0.01$, and $h_{12} = 0.9$, and (**b**) $\mu = 0.2$, $\rho = 0.1$, and $h_{12} = 0.9$.

4.3.2. 3-Layered Isotropic Laminate

In the same manner, we deduce from the obtained dispersion relation for the pre-stressed 3-layered structure in (14) the corresponding/reduced dispersion relation for a 3-layered linear isotropic laminate as follows

$$\tan(\alpha_1 h_{32})\,\tan(\alpha_1 h_{12}) + \frac{\alpha_1}{\mu\,\alpha_2}\tan(\alpha_1 h_{32})\tan(\alpha_2) + \frac{\mu\,\alpha_2}{\alpha_1}\tan(\alpha_1 h_{12})\tan(\alpha_2) = 1. \tag{35}$$

Accordingly, by setting $K = 0$ in (35), we have for the low frequency range

$$\Omega \approx \left(\frac{\rho}{\mu}\right)^{\frac{1}{2}} \frac{1}{\sqrt{h_{12}h_{32} + \rho\,(h_{12} + h_{32})}} \ll 1. \tag{36}$$

Consequently, without further delay, the harmonic curves computed from the exact and approximate frequency equations in this scenario are equally portrayed in Figure 4a,b, respectively, for the two cases of material contrasts. Indeed, a very good agreement has been graphically realized between the two approaches.

Next, expanding the trigonometric functions in (35) in Taylor series about $\Omega = K = 0$, and assuming (3), an approximate polynomial dispersion relation may be obtained in the form

$$\begin{aligned} &1 + \gamma_1 K^2 + \gamma_2\,\Omega^2 + \gamma_3\Omega^2 K^2 + \gamma_4 K^4 + \gamma_5 K^4\Omega^2 + \gamma_6 K^2\Omega^4 + \gamma_7\Omega^4 \\ &+ \gamma_8\,K^6 + \gamma_9\,\Omega^6 + .. = 0, \end{aligned} \tag{37}$$

where the coefficients γ_i for $i = 1, 2, \ldots, 9$, are expressed as follows

$$
\begin{aligned}
\gamma_1 &= \mu(h_{12}+h_{32}) + h_{12}\,h_{32},\\
\gamma_2 &= -\mu(h_{12}+h_{32}) - \frac{\mu}{\rho}h_{12}\,h_{32},\\
\gamma_3 &= \frac{\mu}{3}\,(h_{12}+h_{32}) + \frac{\mu}{6}\left(\frac{\mu}{\rho}+1\right)\left(h_{12}^3+h_{32}^3\right)\\
\gamma_4 &= -\frac{\mu}{6}\left(h_{12}+h_{12}^3+h_{32}+h_{32}^3\right),\\
\gamma_5 &= -\frac{\mu}{36}\left(\frac{\mu}{\rho}+2\right)\left(h_{12}^3+h_{32}^3\right),\\
\gamma_6 &= -\frac{\mu}{36}\left(\frac{2\mu}{\rho}+1\right)\left(h_{12}^3+h_{32}^3\right),\\
\gamma_7 &= -\frac{\mu^2}{6\,\rho^2}\,(h_{12}\,h_{32}^3+h_{32}\,h_{12}^3),\\
\gamma_8 &= \frac{\mu}{36}\left(h_{12}^3+h_{32}^3\right),\\
\gamma_9 &= -\frac{1}{36}\,\frac{\mu^2}{\rho}\left(h_{12}^3+h_{32}^3\right).
\end{aligned}
\tag{38}
$$

Comparatively, the findings of the present special case are similitude to that of the recent study reported by Kaplunov et al. 2020 [44], Alkinidri et al. 2020 [45], and Nuruddeen et al. 2021 [46] with regard to the vibration of waves on an asymmetric elastic three-layered sandwich plate. However, we state the exception of the geometric arrangement of the layers and the traction-free faces in [44–46], which differ from the present fixed-free structure.

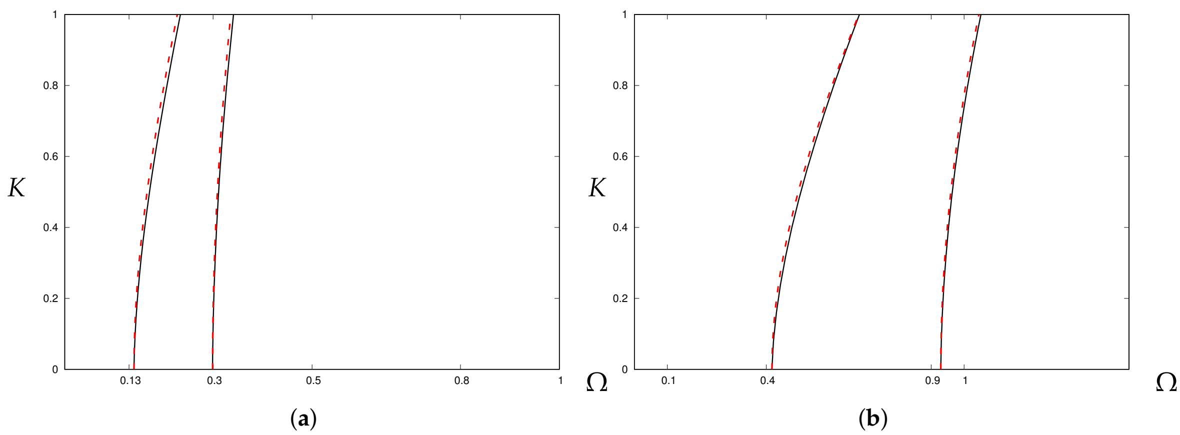

Figure 4. Harmonic curves for a 3-layered pre-stressed laminate through the dispersion relation (14) (black solid line) and asymptotic solution (24) (red dashed line) when (**a**) $\Gamma_{23} = 0.5$, $\Gamma_{13} = 0.45$, $\rho = 0.01$, and $h_{12} = 0.9$, $h_{32} = 0.3$ and (**b**) $\Gamma_{23} = 0.5$, $\Gamma_{13} = 0.45$, $\rho = 0.1$, and $h_{12} = 0.9$, $h_{32} = 0.3$.

In the same passion, Figure 5 presented comparative harmonic curves, comparing the exact and approximate results via (35) with respect to linear isotropic material parameter, for fixed-free faces laminate. Also, from the figures, the two lowest harmonics curves with the two cut-off frequency values at 0.16, and 0.41, respectively, are showed a good agreement over the long-wave region.

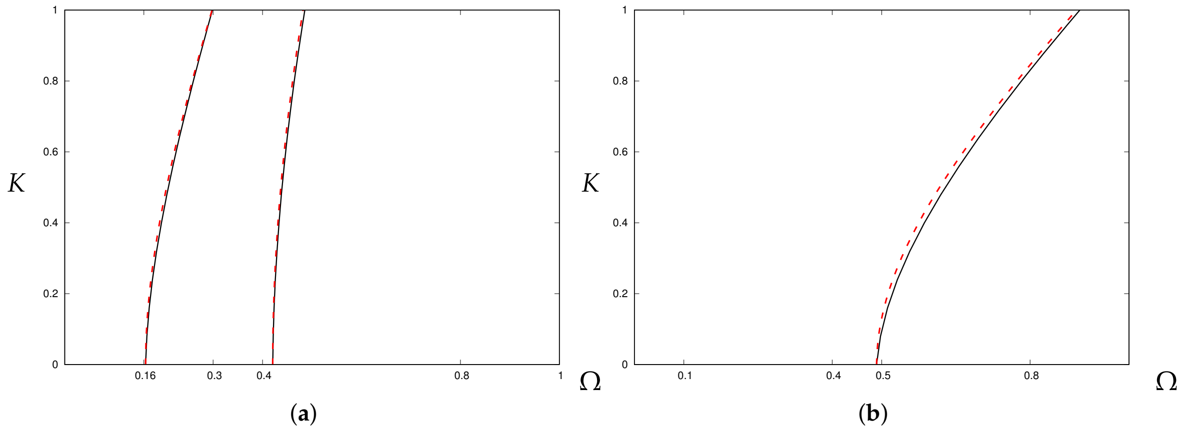

Figure 5. Harmonic curves for a 3-layered linear isotropic laminate through the dispersion relation (35) (black solid line) and asymptotic expansion (37) (red dashed line) when (**a**) $\mu = 0.2$, $\rho = 0.01$, and $h_{12} = 0.9, h_{32} = 0.3$, and (**b**) $\mu = 0.2$, $\rho = 0.1$, and $h_{12} = 0.9, h_{32} = 0.3$.

5. Some Concluding Remarks

The present study employed both the analytical and approximation approaches to examine the dispersion of multilayered elastic laminates with fixed-free outer faces. More precisely, the dispersion of small amplitude waves in multilayered structures, which are made up of pre-stressed compressible elastic materials has been investigated, with the 2- and 3-layered laminates as prototypical structures. Analytically, the propagator matrix technique has been utilized to get hold of the relevant exact frequency equations, with the method offering a simpler derivation procedure; while on the other hand, an asymptotic analysis approach has been deployed as the efficient approximation method of concern to divulge the analogous approximate frequency equations. Moreover, two cases of material contrasts have been incorporated in the structures to further facilitate the long-wave low-frequency vibration. Besides, in an attempt to validate the results of the present study, some well-known results of the classical linear isotropic scenario have been deduced from the governing pre-stressed compressible material. Long-wave approximations of the exact and analogous asymptotic dispersion relations have been established in each case—and graphically shown to be in a good agreement. Moreover, an n-layered generalized anisotropic structure will be the target of the future undertaking, in fact, the both the geometric and material properties are heterogeneous in nature.

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