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Distances and Similarity Measures of Q-Rung Orthopair Fuzzy Sets Based on the Hausdorff Metric with the Construction of Orthopair Fuzzy TODIM

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Abstract: In recent years, q-rung orthopair fuzzy sets (q-ROFSs), a novel and rigorous generalization of the fuzzy set (FS) coined by Yager in 2017, have been used to manage inexplicit and indefinite information in daily life with a high precision and greater accuracy than intuitionistic fuzzy sets (IFSs) and Pythagorean fuzzy sets (PFSs). The characterization of a measure of similarity between q-ROFSs is important, as they have applications in different areas, including pattern recognition, clustering, image segmentation and decision making. Therefore, this article is dedicated to the construction of a measure of similarity between q-ROFSs based on the Hausdorff metric. This is a very useful tool for establishing the similarity between two objects. Furthermore, some axiomatic definitions of the distances and similarity measures of q-ROFSs are also presented. In this article, we first present a novel method to calculate the distance between q-ROFSs based on the Hausdorff metric. We then utilize our proposed distance measure to construct the degree of similarity between q-ROFSs. We provide some properties for the proposed similarity measures. We offer several numerical examples related to pattern recognition and characterization linguistic variables to demonstrate the usefulness of the proposed similarity measures. We construct an algorithm for orthopair fuzzy TODIM (interactive and multi-criteria decision making, in Portuguese) based on our proposed methods. Finally, we use the constructed orthopair fuzzy TODIM method to address problems related to daily life settings involving multi-criteria decision making (MCDM). The numerical results show that the proposed similarity measures are suitable, applicable and well-suited to the contexts of pattern recognition, queries with fuzzy linguistic variables and MCDM.

Keywords: fuzzy sets; intuitionistic fuzzy sets; Pythagorean fuzzy sets; q-rung orthopair fuzzy sets; similarity measure; linguistic variable; multicriteria decision making; orthopair fuzzy TODIM

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1. Introduction

Fuzzy sets (FSs), pioneered by Zadeh [1], are an extension of the conventional or hard sets that are binary in their construction, with an element either belonging or not belonging to the set. FSs are a collection of objects related to the expression of uncertainty of the characterization of objects with membership grades or a degree of belonging in the unit interval $[0,1]$, in which FSs have been applied in different areas [2–4]. On the other hand, fuzzy sets are characterized by their grade of memberships. In many daily life settings, we may compare two objects based on their fuzzy similarity. This describes the similarity between two sets. Consequently, the question of formalizing the process of such a comparison has great significance. In general, similarity measures of fuzzy sets are based on their distances. There are several models used to measure fuzzy similarities between two objects. Zwick et al. [5] conducted a comparative analysis of the measures of similarity between fuzzy concepts. Pappis and Karacapilidis [6] provided further similarity measures based on the union and intersection operations, maximum difference, difference and sum of membership grades. Wang [7] considered measures of similarity between fuzzy sets and between elements. Candan et al. [8] performed an application in a multimedia database query and applied the concept of similarity to conduct query processing with different fuzzy semantics.

The characterization of fuzzy sets (FSs) is based on a membership function with a value in the unit interval $[0, 1]$. The non-membership degree of a fuzzy set is considered as one minus the membership value. Nevertheless, if our intuition expresses the degree of membership of a given element in a fuzzy set, then very often it does not express a corresponding degree of non-membership as one minus the membership value. This embellishes a well-known psychological fact that linguistic negation does not always comply with logical negation (see Grzegorzewski [9]). This notion led to a fascinating story, whose point of departure was such a concept of intuitionistic FSs (IFSs), developed by Atanassov [10], which were found to be very useful for modeling uncertain and vague concepts with a better performance than FSs. The knowledge and semantic representation of this new invention is more meaningful, useful and applicable [11]. In this particular area, IFSs were investigated and applied to various areas by researchers [12–15]. Often, people feel hesitant and dubious in many real-life situations, which makes it difficult to reach final results. To handle such situations, Torra [16] proposed hesitant fuzzy sets (HFSs) which allow for possible values in the unit interval $[0, 1]$ for each alternative in order to tackle these hesitant issues more accurately and precisely. Rodríguez et al. [17] further considered the state of the art and future directions for HFSs. Some research regarding the distance, similarity and entropy of HFSs has been published in the literature, such as the studies of Xu and Xia [18], Hussain and Yang [19,20] and Zhang et al. [21].

In 2013, Yager and Abbasov [22] and Yager [23] further developed the concept of Pythagorean fuzzy sets (PFSs), which are more flexible and applicable than IFSs. Pythagorean fuzzy sets are characterized by a membership degree and a non-membership degree, with the condition that the sum of their squares is equal to one. For instance, IFS is denoted as an ordered pair (μ, ν) satisfying the condition of $0 \leq \mu + \nu \leq 1$. However, PFS changes the condition $0 \leq \mu + \nu \leq 1$ to $0 \leq \mu^2 + \nu^2 \leq 1$, and thus the PFS membership values are the ordered pair (μ, ν) that fulfill the required condition with different aggregation operators and applications in pattern recognition, image processing and multi-criteria decision making (MCDM). For example, if $\mu = 0.81$ and $\nu = 0.5$, then IFSs cannot be used in this case, because $\mu + \nu > 1$, but we can use PFSs, because $\mu^2 + \nu^2 \leq 1$. The space of PFSs is wider than the space of IFSs. Therefore, they can model uncertain and vague information related to daily life situations more effectively. Many researchers have contributed to this field, such as Zhang [24], who provided a novel approach based on the similarity measure of Pythagorean fuzzy MCDM. Biswas and Sarkar [25] examined Pythagorean fuzzy MCDM through similarity measures based on point operators, and Hussain et al. [26] considered Aczel-Alsina operators for PFSs with applications in multi-attribute decision making.

Furthermore, Yager [27] generalized PFSs to q-rung orthopair fuzzy sets (q-ROFSs), which are more flexible and efficient in handling the uncertainty, vagueness and fuzziness with a high precision and accuracy. The q-ROFSs have pair (μ, ν) of membership and non-membership grades satisfying the condition $0 \leq \mu^q + \nu^q \leq 1$. Thus, FSs, IFSs and PFSs are the subclasses of q-ROPFSs. Further developments in this area were provided by many researchers. Yager et al. [28] considered further aspects of the q-ROFSs. Peng and Liu [29] proposed information measures for q-ROFSs. Yang et al. [30] created three-way decisions based on q-ROFSs. Joshi et al. [31] considered interval-valued q-ROFSs, along with their properties, and Joshi and Gegov [32] provided confidence levels for q-rung orthopair fuzzy aggregation operators and their applications to MCDM problems. It is known that distance and similarity measures are often used to determine the degrees of difference and similarity between two objects, respectively, and they have been widely used in many fields, such as pattern recognition, decision making, clustering, etc. We know that q-ROFSs are characterized by their membership degree μ and non-membership degree ν, such that the sum of the qth power of then membership degree and the qth power of the non-membership degree are less than or equal to 1, i.e., $\mu^q + \nu^q \leq 1$. In this sense, the distance and similarity measures of q-ROFSs are more capable than those of IFSs and PFSs in handling vague, incomplete and uncertain information. This is because q-ORFSs are more flexible than IFSs and PFSs with q powers. Due to the wider range space of q-ROFSs, the distance and similarity between q-ROFSs can be very useful and helpful in a variety of fields involving uncertain situations related to day-to-day life. Different types of distance and similarity measures for q-ROFSs have been widely used in various fields. For example, Pinar and Boran [33] provided distance measures for q-ROFSs and then applied them for the purpose of a supplier selection. Wang et al. [34] and Liu et al. [35] proposed distance and similarity measures of q-ROFSs based on cosine functions, along with their applications. Farhadinia et al. [36] provided a group of similarity measures for q-ROFSs and their applications to MCDM. However, we have not observed the use of the Hausdorff metric to the define distance and similarity measures for q-ROFSs thus far in the literature. In light of these aspects, we were motivated to propose these novel distance and similarity measures for q-ROFSs based on the Hausdorff metric.

In this paper, we first construct some measures of similarity between q-ROFSs based on the Hausdorff metric. We then present a novel method that can be used to calculate the distance between q-ROFSs based on the Hausdorff metric. We also present some axiomatic definitions of the distances and similarity measures of q-ROFSs. Based on the proposed distance and similarities, we construct an orthopair fuzzy TODIM. We apply the constructed orthopair fuzzy TODIM to pattern recognition, linguistic variables and multicriteria decision making. The rest of the paper is organized as follows. In Section 2, we briefly examine some basic definitions of the IFSs, PFSs and q-ROFSs, respectively, and then provide a brief description of the concept of the Hausdorff metric. In Section 3, we propose a novel method for calculating the distance between q-ROFSs based on the Hausdorff metric. Then, we describe several new similarity measures for q-ROFSs based on the Hausdorff metric. Section 4 is dedicated to the presentation of certain applications of the proposed similarity measures of q-ROFSs in pattern recognition, queries with fuzzy linguistic variables and MCDM for the purpose of validation. In Section 5, we construct an algorithm for TODIM based on our proposed similarity measures generated by the weighted Hausdorff metric. This leads us to select the best alternative among the many alternatives so as to solve problems related to daily life situations. In Section 6, we offer our conclusions.

2. Preliminaries

In this section, we briefly review the definitions of IFSs, PFSs, q-ROFSs and the Hausdorff metric.

2.1. Q-Rung Orthopair Fuzzy Sets

First, we concisely review the definitions of intuitionistic fuzzy sets (IFSs), Pythagorean fuzzy sets (PFSs) and q-rung orthopair fuzzy sets (q-ROFSs) to show the transition from IFSs to PFSs and then from PFSs to q-ROFSs.

Definition 1 ([10]). *An IFS $\widetilde{A}$ in X is defined and represented by $\widetilde{A} = \{\langle x,\ \mu_{\widetilde{A}}(x),\ \nu_{\widetilde{A}}(x)\rangle : x \in X\}$ with $0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$, where the functions $\mu_{\widetilde{A}}(x) :\ X \rightarrow [0,1]$ and $\nu_{\widetilde{A}}(x) :\ X \rightarrow [0,1]$ represent the degree of membership and non-membership of x in X as belonging to the IFS $\widetilde{A}$, respectively. In general, $\pi_{\widetilde{A}}(x) :\ X \rightarrow [0,1]$ is used to indicate the intuitionistic fuzzy index of x with $\pi_{\widetilde{A}}(x) = 1 - (\mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x))$.*

To address fuzzy and ambiguous situations more widely, PFSs, a generalized form of IFSs, are defined as follows.

Definition 2 ([22]). *A PFS $\widetilde{P}$ in X is defined by $\widetilde{P} = \{\langle x, \mu_{\widetilde{P}}(x), \nu_{\widetilde{P}}(x)\rangle : x \in X\}$ with $0 \leq \mu^2_{\widetilde{P}}(x) + \nu^2_{\widetilde{P}}(x) \leq 1$, where the functions $\mu_{\widetilde{P}}(x) : X \rightarrow [0,1]$ and $\nu_{\widetilde{P}}(x) : X \rightarrow [0,1]$ denote the degree of membership and non-membership of x in X as belonging to the PFS $\widetilde{P}$, re-*

spectively. For each $x \in X$, $\pi_{\widetilde{P}}(x) = \sqrt{1 - \left\{\mu^2_{\widetilde{P}}(x) + \nu^2_{\widetilde{P}}(x)\right\}}$ *is called the Pythagorean fuzzy index of the element* x, *representing the degree of hesitancy.*

To generalize PFSs, Yager [27] proposed a definition of q-ROFSs, as follows:

Definition 3 ([27]). *A q-ROFS* $\widetilde{M}$ *in* X *is defined by* $\widetilde{M} = \left\{\left\langle x,\ \mu_{\widetilde{M}}(x),\ \nu_{\widetilde{M}}(x)\right\rangle : x \in X\right\}$ *with* $0 \le \mu^q_{\widetilde{M}}(x) + \nu^q_{\widetilde{M}}(x) \le 1, 1 \le q < \infty$, *where the functions* $\mu_{\widetilde{A}}(x) : X \to [0,1]$ *and* $\pi_{\widetilde{A}}(x) : X \to [0,1]$ *denote the degree of membership and non-membership of* x *in* X *as belonging to the q-ROFS* $\widetilde{M}$, *respectively. For each* $x \in X$, $\pi_{\widetilde{P}}(x) = \sqrt[q]{1 - \left(\mu^q_{\widetilde{P}}(x) + \nu^q_{\widetilde{P}}(x)\right)}$ *is used and called the orthopairian fuzzy index of* x, *representing the degree of hesitancy.*

It is obvious from the characterization of q-ROFSs that they are more flexible and efficient than IFSs and PFSs in handling the fuzziness and uncertainty of daily life issues. We should mention that IFSs and PFSs are not capable of handling some fuzziness, but q-ROFSs can handle these situations effectively. Therefore, to solve daily life problems with a greater accuracy and high precision, q-ROFSs are much better than IFSs and PFSs, because q-ROFSs are more generalized than IFSs and PFSs.

Definition 4. *Let* $\widetilde{M} = \left\{\left\langle x, \mu_{\widetilde{M}}(x), \nu_{\widetilde{M}}(x)\right\rangle : x \in X\right\}$ *be a q-ROFS in* X. *For any positive real number* n, *the q-ROFS* $\widetilde{M}^n$ *is defined as:*

$$\widetilde{M}^n = \left\{\left\langle x, \left(\mu^q_{\widetilde{M}}(x)\right)^n, \sqrt[q]{1 - \left(1 - \nu^q_{\widetilde{M}}(x)\right)^n}\right\rangle : x \in X\right\}, n > 0 \text{ and } 2 < q < \infty. \quad (1)$$

It is obviously that, for a real positive number n, $0 \le \left[\mu^q_{\widetilde{M}}(x)\right]^n + \left[\sqrt[q]{1 - \left(1 - \nu^q_{\widetilde{M}}(x)\right)^n}\right]$ ≤ 1. From Equation (1), the concentration and dilation of a q-ROFS $\widetilde{M}$ can be defined, respectively, as follows:

$$CON\left(\widetilde{M}\right) = \left\{\left\langle x, \mu_{CON(\widetilde{M})}(x), \nu_{CON(\widetilde{M})}(x)\right\rangle : x \in X\right\} \quad (2)$$

where $\mu_{CON(\widetilde{M})}(x) = \left[\mu^q_{\widetilde{M}}(x)\right]^2$, $\nu_{CON(\widetilde{M})}(x) = \sqrt[q]{1 - \left[1 - \nu^q_{\widetilde{M}}(x)\right]^2}$; and

$$DIL\left(\widetilde{M}\right) = \left\{\left\langle x, \mu_{DIL(\widetilde{M})}(x), \nu_{DIL(\widetilde{M})}(x)\right\rangle : x \in X\right\} \quad (3)$$

where $\mu_{DIL(\widetilde{M})}(x) = \left[\mu^q_{\widetilde{M}}(x)\right]^{\frac{1}{2}}$, $\nu_{DIL(\widetilde{M})}(x) = \sqrt[q]{1 - \left[1 - \nu^q_{\widetilde{M}}(x)\right]^{\frac{1}{2}}}$.

Definition 5 (Peng and Liu [29]). *If* $\widetilde{M}$ *and* $\widetilde{N}$ *are two q-ROFSs of* X, *then*

(i) $\widetilde{M}^c = \left\{\left\langle x, \nu_{\widetilde{M}}(x),\ \mu_{\widetilde{M}}(x)\right\rangle : x \in X\right\}$;
(ii) $\widetilde{M} \subseteq \widetilde{N}$ *if and only if* $\forall\ x \in X$, $\mu_{\widetilde{M}}(x) \le \mu_{\widetilde{N}}(x)$ *and* $\nu_{\widetilde{M}}(x) \ge \nu_{\widetilde{N}}(x)$;
(iii) $\widetilde{M} = \widetilde{N}$ *if and only if* $\forall\ x \in X$, $\mu_{\widetilde{M}}(x) = \mu_{\widetilde{N}}(x)$ *and* $\nu_{\widetilde{M}}(x) = \nu_{\widetilde{N}}(x)$;
(iv) $\widetilde{M} \cup \widetilde{N} = \left\{max\left(\mu_{\widetilde{M}}(x),\ \mu_{\widetilde{N}}(x)\right), min\left(\nu_{\widetilde{M}}(x),\ \nu_{\widetilde{N}}(x)\right)\right\}, \forall\ x \in X$;
(v) $\widetilde{M} \cap \widetilde{N} = \left\{min\left(\mu_{\widetilde{M}}(x),\ \mu_{\widetilde{N}}(x)\right), max\left(\nu_{\widetilde{M}}(x),\ \nu_{\widetilde{N}}(x)\right)\right\}, \forall\ x \in X$.

2.2. Hausdorff Metric

The Hausdorff metric is a well-known measure of the distance between two nonempty closed and bounded (compact) subsets S and T in a Banach space K. It is defined as the maximum value of the two direct forward and backward distances of Hausdorff [37–39]. Let $d(x, y)$ be a metric between a point x in the set S and a point y in the set T. The forward distance is defined as $h(S,T) = \underset{x\in S}{max}\left\{\underset{y\in T}{min}\,(\|x - y\|)\right\}$, and the backward distance is defined as $h(T,S) = \underset{y\in T}{max}\left\{\underset{x\in S}{min}\,(\|x - y\|)\right\}$, respectively. Hausdorff metric is then defined as follows:

$$H(S,T) = max\{h(S,T), h(T,S)\}. \tag{4}$$

We should mention that the Hausdorff metric is asymmetric. That is, in general, $h(S,T) \neq h(T,S)$. For example, if $K = \Re$, $S = [\alpha_1, \alpha_2]$ and $T = [\beta_1, \beta_2]$ are the two intervals, and then according to Equation (4) we have:

$$H(S,T) = max\{|\alpha_1 - \beta_1|, |\alpha_2 - \beta_2|\}. \tag{5}$$

Equation (5) is a popular measure of the distance between two intervals. We know that a similarity measure can be defined by the distance measure. In general, distance and similarity measures play an important role in verifying the resemblance between two sets or objects. They have earned considerable popularity due to their wide-ranging applications in many areas. Although there are distance and similarity measures of PFSs and HFSs based on the Hausdorff metric in the literature [20,40], there are no distance and similarity measures for q-ROFSs based on the Hausdorff metric. Therefore, based on the above-defined Hausdorff metric, we propose some new measures of the distance and similarity between q-ROFSs in the next section.

3. Measures of the Distance and Similarity between q-ROFSs Based on the Hausdorff Metric

In this section, we define a distance between q-ROFSs according to the Hausdorff metric and then provide measures of similarity between q-ROFSs.

3.1. A Distance for q-ROFSs Based on the Hausdorff Metric

We define a generalized interval type of q-ROFSs, which we then utilize to define a new distance between q-ROFSs according to the Hausdorff metric. It is natural that this metric can be applied to intervals that can be directly utilized for q-ROFSs. Suppose that $\widetilde{M}$ and $\widetilde{N}$ are any two q-ROFSs in $X = \{x_1, x_2, \ldots, x_n\}$ with $\widetilde{M} = \{\langle x,\ \mu_{\widetilde{M}}(x),\ \nu_{\widetilde{M}}(x)\rangle : x \in X\}$ and $\widetilde{N} = \{\langle x,\ \mu_{\widetilde{N}}(x),\ \nu_{\widetilde{N}}(x)\rangle : x \in X\}$. Let $\widetilde{I}_{\widetilde{M}}(x_i)$ and $\widetilde{I}_{\widetilde{N}}(x_i)$ be subintervals on $[0,1]$, denoted by $\widetilde{I}_{\widetilde{M}}(x_i) = \left[\mu^q_{\widetilde{M}}(x_i), 1 - \nu^q_{\widetilde{M}}(x_i)\right]$ and $\widetilde{I}_{\widetilde{N}}(x_i) = \left[\mu^q_{\widetilde{N}}(x_i), 1 - \nu^q_{\widetilde{N}}(x_i)\right]$, where $q \geq 1,\ i = 1, 2, \ldots, n$. We define the distance between $\widetilde{M}$ and $\widetilde{N}$ by using the distance between $\widetilde{I}_{\widetilde{M}}(x_i)$ and $\widetilde{I}_{\widetilde{N}}(x_i)$ based on the Hausdorff metric with $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ I_{\widetilde{N}}(x_i)\right) = max\left\{\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right|, \left|1 - \nu^q_{\widetilde{M}}(x_i) - \left(1 - \nu^q_{\widetilde{N}}(x_i)\right)\right|\right\}$. Thus, we give a novel Hausdorff metric $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)$ between the q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ as follows:

$$\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) = \frac{1}{n}\sum_{i=1}^{n} max\left\{\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right|, \left|1 - \nu^q_{\widetilde{M}}(x_i) - \left(1 - \nu^q_{\widetilde{N}}(x_i)\right)\right|\right\} \tag{6}$$

Next, we provide the main theorem with some properties for the proposed Hausdorff metric $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)$.

Theorem 1. *Suppose that the set $X = \{x_1, \ldots, x_n\}$ is a finite universe of discourses. The metric $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)$ between the two q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ has the following results of $(P1) - (P5)$:*

$(P1)$ $(Nonnegativity)$ $0 \leq \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) \leq 1;$

$(P2)$ $(Separability)$ $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) = 0\ if\ and\ only\ if\ \widetilde{M} = \widetilde{N};$

(P3) (*Symmetric*) $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)=\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{M}\right)$;

(P4) (*Containment*) *If* $\tilde{M}\subseteq N\subseteq\tilde{O}$ *then* $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$ *and* $\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{O}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$;

(P5) (*Triangle Inequality*) *For any* $\tilde{M},\tilde{N}$ *and* $\tilde{O}$, *then* $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)+\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{O}\right)$.

Proof. We provide the proof of the properties $(P1)-(P5)$ of Theorem 1 as follows:

$(P1)$: Since $\tilde{M}$ and $\tilde{N}$ are two q-ROFSs on $X=\{x_1,\ldots,x_n\}$, $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)$, given by Equation (6), is obviously positive, i.e., $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)\geq 0$. On the other hand, $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)$ is defined by its normalization with $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)\leq 1$. Thus, the result $(P1)$ is proved.

$(P2)$: If $\tilde{M}=\tilde{N}$, then for every $x_i\in X$, we have $\mu^q_{\tilde{M}}(x_i)=\mu^q_{\tilde{N}}(x_i)$ and $\nu^q_{\tilde{M}}(x_i)=\nu^q_{\tilde{N}}(x_i)$, and so $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)=0$. Conversely, if $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)=0$, then for every $x_i\in X$, we have $max\left\{\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|,\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|\right\}=0$, and so $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|=0$ and $\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|=0$. Thus, we obtain $\tilde{M}=\tilde{N}$, and the result $(P2)$ is proved.

$(P3)$: It is obvious that $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)=\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{M}\right)$ holds because, for each $x_i\in X$, $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|=\left|\mu^q_{\tilde{N}}(x_i)-\mu^q_{\tilde{M}}(x_i)\right|$ and $\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|=\left|\nu^q_{\tilde{N}}(x_i)-\nu^q_{\tilde{M}}(x_i)\right|$ are held. Thus, the result $(P3)$ is proved.

$(P4)$: If $\tilde{M}\subseteq N\subseteq\tilde{O}$, then we have $\mu^q_{\tilde{M}}(x_i)\leq\mu^q_{\tilde{N}}(x_i)\leq\mu^q_{\tilde{O}}(x_i)$ and $\nu^q_{\tilde{M}}(x_i)\geq\nu^q_{\tilde{N}}(x_i)\geq\nu^q_{\tilde{O}}(x_i)$, for each $x_i\in X$. Thus, we can obtain $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{N}}(x_i)\right)=max\left\{\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|,\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|\right\}$, $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)=max\left\{\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|,\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\right\}$, and $\tilde{H}\left(\tilde{I}_{\tilde{N}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)=max\left\{\left|\mu^q_{\tilde{N}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|,\left|\nu^q_{\tilde{N}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\right\}$. We consider the following two cases:

(i) If $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\geq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$, then $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)=\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|$. However, we have $\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|$ and $\left|\nu^q_{\tilde{N}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|$. On the other hand, we have $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|$ and $\left|\mu^q_{\tilde{N}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|$. By combining the above inequalities, we can obtain $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{N}}(x_i)\right)\leq\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)$ and $\tilde{H}\left(\tilde{I}_{\tilde{N}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)\leq\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)$. Hence, we have $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$ and $\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{O}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$. We next consider the second case.

(ii) If $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$, then $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)=\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$. However, we have $\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{N}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$ and $\left|\mu^q_{\tilde{N}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\leq\left|\mu^q_{\tilde{M}}(x_i)-\mu^q_{\tilde{O}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$. On the other hand, we have $\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{N}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$ and $\left|\nu^q_{\tilde{N}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|\leq\left|\nu^q_{\tilde{M}}(x_i)-\nu^q_{\tilde{O}}(x_i)\right|$. Based on the previous inequalities, we have $\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{N}}(x_i)\right)\leq\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)$ and $\tilde{H}\left(\tilde{I}_{\tilde{N}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)\leq\tilde{H}\left(\tilde{I}_{\tilde{M}}(x_i),\ \tilde{I}_{\tilde{O}}(x_i)\right)$. Hence, we have $\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{N}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$ and $\tilde{d}_{\tilde{H}}\left(\tilde{N},\tilde{O}\right)\leq\tilde{d}_{\tilde{H}}\left(\tilde{M},\tilde{O}\right)$. Therefore, the cases (i) and (ii) complete the verification of the result $(P4)$.

$(P5)$: For any three q-ROFSs $\tilde{M},\tilde{N},\tilde{O}$ on X with membership functions $\mu^q_{\tilde{M}}(x_i),\mu^q_{\tilde{N}}(x_i),\mu^q_{\tilde{O}}(x_i)$ and non-memberships functions $\nu^q_{\tilde{M}}(x_i),\nu^q_{\tilde{N}}(x_i),\nu^q_{\tilde{O}}(x_i)$, respectively, we also consider the following two cases:

(i) If $\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right| \geq \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|$, then we have $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) = \left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right|$, $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) = \left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i) + \mu^q_{\widetilde{N}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right|$, and $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) \leq \left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right| + \left|\mu^q_{\widetilde{N}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right| = max\left\{\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right|, \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{N}}(x_i)\right|\right\} + max\left\{\left|\mu^q_{\widetilde{N}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right|, \left|\nu^q_{\widetilde{N}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|\right\}$ $= \widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i), \widetilde{I}_{\widetilde{N}}(x_i)\right) + \widetilde{H}\left(\widetilde{I}_{\widetilde{N}}(x_i), \widetilde{I}_{\widetilde{O}}(x_i)\right) = \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) \leq \widetilde{d}_{\widetilde{H}}\left(\widetilde{N}, \widetilde{O}\right)$. Similarly, we have the following second case:
(ii) If $\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right| \leq \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|$, then $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) = \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|$, $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) = \left|\nu^q_{\widetilde{M}}(x_i) - \nu^2_{\widetilde{N}}(x_i) + \nu^q_{\widetilde{N}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|$, and $\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) \leq \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{N}}(x_i)\right| + \left|\nu^q_{\widetilde{N}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right| = max\left\{\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right|, \left|\nu^q_{\widetilde{M}}(x_i) - \nu^q_{\widetilde{N}}(x_i)\right|\right\} + max\left\{\left|\mu^q_{\widetilde{N}}(x_i) - \mu^q_{\widetilde{O}}(x_i)\right|, \left|\nu^q_{\widetilde{N}}(x_i) - \nu^q_{\widetilde{O}}(x_i)\right|\right\} = \widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i),\ \widetilde{I}_{\widetilde{N}}(x_i)\right) + \widetilde{H}\left(\widetilde{I}_{\widetilde{N}}(x_i),\ \widetilde{I}_{\widetilde{O}}(x_i)\right) = \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) + \widetilde{d}_{\widetilde{H}}\left(\widetilde{N}, \widetilde{O}\right)$. Thus, we have $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{O}\right) \leq \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) + \widetilde{d}_{\widetilde{H}}\left(\widetilde{N}, \widetilde{O}\right)$. Based on the two cases (i) and (ii), we prove the triangle inequality result $(P5)$.

Next, we utilize Equation (6) to establish the weighted Hausdorff metric and then create different similarities measures. Let a weight vector w of each element $x_i \in X$ be $w_i(i = 1, 2, 3, \ldots, n)$, such that $\sum_{i=1}^{n} w_i = 1$, where $0 \leq w_i \leq 1$. We define the orthopairian weight Hausdorff metric as follows:

$$\widetilde{d}_{w\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) = \sum_{i=1}^{n} w_i max\left\{\left|\mu^q_{\widetilde{M}}(x_i) - \mu^q_{\widetilde{N}}(x_i)\right|, \left|1 - \nu^q_{\widetilde{M}}(x_i) - \left(1 - \nu^q_{\widetilde{N}}(x_i)\right)\right|\right\} \tag{7}$$

In general, Equation (7) becomes Equation (6) if we replace $w_i = 1/n$, for $i = 1, 2, \ldots, n$, i.e., Equation (6) is the special case of Equation (7).

3.2. Similarity Measures for q-ROFSs Based on the Hausdorff Distance

It is well-known that the relationship between the distance and similarity can be established through a dual concept. Thus, the distance between q-ROFSs can be used to define the similarity between q-ROFSs according to the Hausdorff metric. Let g be a monotone decreasing function. Because $0 \leq \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right) \leq 1$, we have $g(1) \leq g\left(\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)\right) \leq g(0)$. This implies $0 \leq \frac{g\left(\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)\right) - g(1)}{g(0) - g(1)} \leq 1$. Therefore, the measure of similarity between q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ can be defined as follows.

Definition 6. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be the universe of discourse and $\widetilde{M} = \left\{\left(x,\ \mu_{\widetilde{M}}(x),\ \nu_{\widetilde{M}}(x)\right) \middle| x \in X\right\}$ with $0 \leq \mu^q_{\widetilde{M}}(x) + \nu^q_{\widetilde{M}}(x) \leq 1$, and $\widetilde{N} = \left\{\left(x,\ \mu_{\widetilde{N}}(x),\ \nu_{\widetilde{N}}(x)\right) \middle| x \in X\right\}$ with $\mu^q_{\widetilde{N}}(x) + \nu^q_{\widetilde{N}}(x) = 1$, $1 \leq q < \infty$, be two q-ROFSs on X . Let g be a monotone decreasing function. Then, a new similarity measure $\widetilde{S}\left(\widetilde{M}\ ,\ \widetilde{N}\right)$ between the two q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ is defined as:*

$$\widetilde{S}\left(\widetilde{M}\ ,\ \widetilde{N}\right) = \frac{g\left(\widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right)\right) - g(1)}{g(0) - g(1)}. \tag{8}$$

According to Equation (8), different similarity measures can be provided by selecting an appropriate function g. For example, we may choose the simplest linear function g with $g(x) = 1 - x$. Thus, the measure of similarity between the two q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ using Equation (8) can be denoted as:

$$\widetilde{S}_l\left(\widetilde{M}, \widetilde{N}\right) = 1 - \widetilde{d}_{\widetilde{H}}\left(\widetilde{M}, \widetilde{N}\right). \tag{9}$$

On the other hand, we consider the rational function $g(x) = 1/(1+x)$. Then, the defined measure of similarity between the two q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ can be represented as follows:

$$\widetilde{S}_r\left(\widetilde{M},\widetilde{N}\right) = \frac{1-\widetilde{d}_{\widetilde{H}}\left(\widetilde{M},\widetilde{N}\right)}{1+\widetilde{d}_{\widetilde{H}}\left(\widetilde{M},\widetilde{N}\right)}. \tag{10}$$

$$\widetilde{S}_e\left(\widetilde{M},\widetilde{N}\right) = \frac{e^{-\widetilde{d}_{\widetilde{H}}(\widetilde{M},\widetilde{N})}-e^{-1}}{1-e^{-1}}. \tag{11}$$

Furthermore, if we consider the continuous universe of discourses with $X = [a,b]$, then we can obtain the following results. Let $\widetilde{M} = \left\{\left(x,\ \mu_{\widetilde{M}}(x),\ \nu_{\widetilde{M}}(x)\right)\middle|x\in X\right\}$ with $\mu^q_{\widetilde{N}}(x) + \nu^q_{\widetilde{N}}(x) = 1$, and $\widetilde{N} = \left\{\left(x,\ \mu_{\widetilde{N}}(x),\ \nu_{\widetilde{N}}(x)\right)\middle|x\in X\right\}$ with $\mu^q_{\widetilde{N}}(x) + \nu^q_{\widetilde{N}}(x) = 1$, $1 \le q < \infty$, be any two q-ROFSs on X. Then, we define a new distance measure $\widetilde{d}_{\widetilde{R}\widetilde{H}\widetilde{c}}$ between the two q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ as follows:

$$\widetilde{d}_{q\widetilde{R}\widetilde{H}\widetilde{C}}\left(\widetilde{M},\widetilde{N}\right) = \frac{1}{b-a}\int_a^b \widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i)\quad,\ \widetilde{I}_{\widetilde{N}}(x_i)\right)dx, \tag{12}$$

In pattern recognition, the elements in the set of the universe can differ in their importance. Therefore, it is necessary for us to consider the weight vector w of each element $x \in X$. For example, we may assume that the weight of each $x \in X$ in $X = [a,b]$ of the continuous case is $w(x)$, where $0 \le w(x) \le 1$ and $\int_a^b w(x) = 1$. Then, the weighted Hausdorff metric between the q-ROFSs $\widetilde{M}$ and $\widetilde{N}$ can be defined as:

$$\widetilde{d}_{\widetilde{R}\widetilde{H}w\widetilde{C}}\left(\widetilde{M},\widetilde{N}\right) = \int_a^b w(x)\widetilde{H}\left(\widetilde{I}_{\widetilde{M}}(x_i)\quad,\ \widetilde{I}_{\widetilde{N}}(x_i)\right)dx, \tag{13}$$

We should mention that Equation (13) becomes Equation (12) if we substitute $w(x) = 1/(b-a)$ for $x \in [a,b]$. Therefore, Equation (12) is a special case of Equation (13). Obviously, we can obtain measures of similarity between $\widetilde{M}$ and $\widetilde{N}$ by replacing $\widetilde{d}_{w\widetilde{H}}\left(\widetilde{M},\widetilde{N}\right)$, $\widetilde{d}_{q\widetilde{R}\widetilde{H}\widetilde{C}}\left(\widetilde{M},\widetilde{N}\right)$ and $\widetilde{d}_{\widetilde{R}\widetilde{H}w\widetilde{C}}\left(\widetilde{M},\widetilde{N}\right)$ with $\widetilde{d}_{\widetilde{H}}\left(\widetilde{M},\widetilde{N}\right)$ in Equation (8).

4. Examples and Comparisons

In this section, we provide some examples to demonstrate the practicality and applicability of the proposed methods. We first present examples in the context of pattern recognition. Then, we apply it to queries with fuzzy linguistic variables.

4.1. Pattern Recognition

Here, we use the proposed similarity measures of Equations (9)–(11) to display some examples related to pattern recognition.

Example 1. *In this example, we consider three patterns represented by q-ROFSs in the universe of discourse* $X = \{x_1,x_2,x_3\}$ *. The three q-ROFSs are denoted as follows:* $\widetilde{M}_1 = \{\langle x_1,0.5,0.5\rangle, \langle x_2,0.7,0.7\rangle,\langle x_3,0.8,0.8\rangle\}$, $\widetilde{M}_2 = \{\langle x_1,0.8,0.8\rangle,\langle x_2,0.8,0.8\rangle,\langle x_3,0.8,0.8\rangle\}$ *and* $\widetilde{M}_3 = \{\langle x_1,0.7,0.7\rangle,\langle x_2,0.7,0.7\rangle,\langle x_3,0.7,0.7\rangle\}$*. Let a given sample be* $\widetilde{N} = \{\langle x_1,0.5,0.5\rangle,\langle x_2,0.7,0.7\rangle,\langle x_3,0.8,0.8\rangle\}$ *. By utilizing Equations (9)–(11), we obtain* $\widetilde{S}_l\left(\widetilde{M}_1,\widetilde{N}\right) = 1$, $\widetilde{S}_l\left(\widetilde{M}_2,\widetilde{N}\right) = 0.8278$, $\widetilde{S}_l\left(\widetilde{M}_3,\widetilde{N}\right) = 0.8843$, $\widetilde{S}_r\left(\widetilde{M}_1,\widetilde{N}\right) = 1$, $\widetilde{S}_r\left(\widetilde{M}_2,\widetilde{N}\right) = 0.7062$, $\widetilde{S}_r\left(\widetilde{M}_3,\widetilde{N}\right) = 0.7926$, $\widetilde{S}_e\left(\widetilde{M}_1,\widetilde{N}\right) = 1$, $\widetilde{S}_e\left(\widetilde{M}_2,\widetilde{N}\right) = 0.7497$, *and* $\widetilde{S}_e\left(\widetilde{M}_3,\widetilde{N}\right) = 0.8272$. *Thus, we have* $\widetilde{M}_1 = \widetilde{N}$*. Intuitively, we can see that the sample* $\widetilde{N}$ *is close to the pattern* $\widetilde{M}_3$ *as* *compared to the pattern $\widetilde{M}_2$. Thus, the proposed similarity measures exhibit the correct classification according to the principle of the maximum degree of similarity between q-ROFSs.*

Example 2. *Assume that three patterns are represented by q-ROFSs in the universe of discourse $X = \{x_1, x_2, x_3\}$. The two q-ROFSs are denoted as follows: $\widetilde{M}_1 = \{\langle x_1, 0.7, 0.7\rangle, \langle x_2, 0.7, 0.7\rangle, \langle x_3, 0.7, 0.7\rangle\}$; and $\widetilde{M}_2 = \{\langle x_1, 0.9, 0.9\rangle, \langle x_2, 0.9, 0.9\rangle, \langle x_3, 0.9, 0.9\rangle\}$. Let a given sample be $\widetilde{N} = \{\langle x_1, 0.8, 0.8\rangle, \langle x_2, 0.8, 0.8\rangle, \langle x_3, 0.8, 0.8\rangle\}$. By utilizing Equations (9)–(11), we obtain $\widetilde{S}_l\left(\widetilde{M}_1, \widetilde{N}\right) = 0.8305$, $\widetilde{S}_l\left(\widetilde{M}_2, \widetilde{N}\right) = 0.7314$, $\widetilde{S}_r\left(\widetilde{M}_1, \widetilde{N}\right) = 0.7101$, $\widetilde{S}_r\left(\widetilde{M}_2, \widetilde{N}\right) = 0.5765$, $\widetilde{S}_e\left(\widetilde{M}_1, \widetilde{N}\right) = 0.7533$, and $\widetilde{S}_e\left(\widetilde{M}_2, \widetilde{N}\right) = 0.6274$. According to the above analysis, it can be seen that $\widetilde{N}$ is similar to $\widetilde{M}_1$. Thus, the proposed similarity measures show the correct classification based on the maximum degree principle of similarity between q-ROFSs.*

Example 3. *Suppose that three patterns are denoted by q-ROFSs in the universe of discourse $X = \{x_1, x_2, x_3\}$. The three q-ROFSs are denoted as follows: $\widetilde{M}_1 = \{\langle x_1, 0.1, 0.5\rangle, \langle x_2, 0.7, 0.9\rangle, \langle x_3, 0.5, 0.8\rangle\}$; $\widetilde{M}_2 = \{\langle x_1, 0.5, 0.9\rangle, \langle x_2, 0.0, 0.8\rangle, \langle x_3, 0.3, 0.5\rangle\}$; and $\widetilde{M}_3 = \{\langle x_1, 0.7, 0.3\rangle, \langle x_2, 0.5, 0.9\rangle, \langle x_3, 0.3, 0.2\rangle\}$. Let us assume a given sample $\widetilde{N} = \{\langle x_1, 0.5, 0.5\rangle, \langle x_2, 0.7, 0.7\rangle, \langle x_3, 0.9, 0.9\rangle\}$. By utilizing Equations (9)–(11), we obtain $\widetilde{S}_l\left(\widetilde{M}_1, \widetilde{N}\right) = 0.7086$, $\widetilde{S}_l\left(\widetilde{M}_2, \widetilde{N}\right) = 0.6564$, $\widetilde{S}_l\left(\widetilde{M}_3, \widetilde{N}\right) = 0.7909$, $\widetilde{S}_r\left(\widetilde{M}_1, \widetilde{N}\right) = 0.5487$, $\widetilde{S}_r\left(\widetilde{M}_2, \widetilde{N}\right) = 0.4885$, $\widetilde{S}_r\left(\widetilde{M}_3, \widetilde{N}\right) = 0.6541$, $\widetilde{S}_e\left(\widetilde{M}_1, \widetilde{N}\right) = 0.6001$, $\widetilde{S}_e\left(\widetilde{M}_2, \widetilde{N}\right) = 0.5400$, and $\widetilde{S}_e\left(\widetilde{M}_3, \widetilde{N}\right) = 0.7015$. According to above results, it is seen that $\widetilde{N}$ is similar to $\widetilde{M}_3$ based on the maximum degree principle of similarity between q-ROFSs.*

The abovementioned Examples 1–3 of the different types display the practicality and reliability of our proposed similarity measures. Next, we provide an example pertaining to characterization of the similarity between linguistic variables using our proposed similarity measures in Equations (9)–(11).

4.2. Queries with Fuzzy Linguistic Variables

Tahani [41] first created a framework for fuzzy query processing using fuzzy sets, and then Kacprzyk and Ziolkowski [42] provided database queries with fuzzy linguistic quantifiers. In due course, a fuzzy database with its principles and applications was provided in the book by Petry [43]. For query processing in multimedia databases, Candan et al. [8] used similarity measures as an important method, with a similarity-based ranking technique. Similar to Hussain and Yang [40], we consider measures of the similarity between linguistic hedges. In the following Example 4, the similarities between linguistic variables are characterized utilizing our proposed measures of similarity between q-ROFSs.

Example 4. *Let $\widetilde{M} = \left\{\langle x, \mu_{\widetilde{M}}(x), \nu_{\widetilde{M}}(x)\rangle : x \in X\right\}$ be a q-rung orthopair fuzzy set on X . For a given positive real number n , we have the q-ROFS $\widetilde{M}^n$ from Definition 4, with*

$$\widetilde{M}^n = \left\{\left\langle x, \left(\mu^q_{\widetilde{M}}(x)\right)^n, \sqrt[q]{1 - \left(1 - \nu^q_{\widetilde{M}}(x)\right)^n}\right\rangle : x \in X\right\}, 2 < q < \infty.$$

Moreover, the two linguistic operators, the dilation and concentration of the q-ROFS $\widetilde{M}$, are $DIL\left(\widetilde{M}\right) = \widetilde{M}^{\frac{1}{2}}$ and $CON\left(\widetilde{M}\right) = \widetilde{M}^2$, respectively. These linguistic operations may have different presentations, where $DIL\left(\widetilde{M}\right)$ may be considered as "more or less $\left(\widetilde{M}\right)$", but $CON\left(\widetilde{M}\right)$ may be regarded as "very $\left(\widetilde{M}\right)$". Now, we take the q-ROFS $\widetilde{M}$ in $X = \{5, 10, 15, 20, 25\}$ as:

$$\widetilde{M} = \{\langle 5, 0.9, 0.5\rangle, \langle 10, 0.8, 0.6\rangle, \langle 15, 0.7, 0.5\rangle, \langle 20, 0.6, 0.9\rangle, \langle 25, 1.0, 0.0\rangle\}$$

where the q-ROFS $\widetilde{M}$ represents "LARGE" in X. The more generalized q-ROFSs of $\widetilde{M}$ can be regarded as linguistic hedges, such as $\widetilde{M}$ as "LARGE" in X. In the same manner, the linguistic operations $DIL\left(\widetilde{M}\right)$ and $CON\left(\widetilde{M}\right)$ can be utilized as linguistic hedges, such as "More or less LARGE", "Very LARGE", and "Very very LARGE". Therefore, we can express linguistic hedges in terms of q-ROPFSs, as follows: $\widetilde{M}^{\frac{1}{2}}$ is regarded as "More or less LARGE", $\widetilde{M}^2$ is regarded as "Very LARGE" and $\widetilde{M}^4$ is regarded as "Very very LARGE".

For the sake of brevity, the following acronyms are used: L for $LARGE$, $M.L.L$ for $More\ or\ less\ LARGE$, $V.L.$ for $Very\ LARGE$ and $V.V.L.$ for $Very\ very\ LARGE$. The proposed measures of similarity between the q-ROFSs from Equations (9)–(11) are utilized to compute the degrees of similarities between the abovementioned q-ROFSs. The computational results are displayed in Table 1. Thus, we obtain the following requirements according to the degrees of similarity between the q-ROFSs from Table 1, with $\widetilde{S}(L., M.L.L.) > \widetilde{S}(L., V.L.) > \widetilde{S}(L., V.V.L.)$, $\widetilde{S}(M.L.L., L.) > \widetilde{S}(M.L.L., V.L.) > \widetilde{S}(M.L.L., V.L.L.)$, $\widetilde{S}(V.L., V.V.L.) > \widetilde{S}(V.L., L.) > \widetilde{S}(V.L., M.L.L.)$, and $\widetilde{S}(V.V.L., V.L) > \widetilde{S}(V.V.L., L.) > \widetilde{S}(V.V.L., M.L.L.)$. According to the above order of similarities from Table 1, based on the proposed similarity measures of Equations (9)–(11), we find that there is a good ordering between $L.$, $V.L.$, $M.L.L.$ and $V.V.L.$ That is, the proposed similarity measures in Equations (9)–(11) are well-suited, valid and efficient in the context of compound linguistic variables.

Table 1. Calculation of similarity measures using Equations (9)–(11).

	L.	*M.L.L.*	*V.L*	*V.V.L.*
L.	$1.000_{(9)}$ $1.000_{(10)}$ $1.000_{(11)}$	$0.8358_{(9)}$ $0.7179_{(10)}$ $0.7640_{(11)}$	$0.8259_{(9)}$ $0.7034_{(10)}$ $0.7472_{(11)s}$	$0.7456_{(9)}$ $0.5944_{(10)}$ $0.6446_{(11)}$
M.L.L.	$0.8358_{(9)}$ $0.7179_{(10)}$ $0.7640_{(11)}$	$1.000_{(9)}$ $1.000_{(10)}$ $1.000_{(11)}$	$0.6618_{(9)}$ $0.4945_{(10)}$ $0.5461_{(11)}$	$0.5815_{(9)}$ $0.4099_{(10)}$ $0.4591_{(11)}$
V.L	$0.8259_{(9)}$ $0.7034_{(10)}$ $0.7472_{(11)s}$	$0.6618_{(9)}$ $0.4945_{(10)}$ $0.5461_{(11)}$	$1.000_{(9)}$ $1.000_{(10)}$ $1.000_{(11)}$	$0.9151_{(9)}$ $0.8435_{(10)}$ $0.8712_{(11)}$
V.V.L.	$0.7456_{(9)}$ $0.5944_{(10)}$ $0.6446_{(11)}$	$0.5815_{(9)}$ $0.4099_{(10)}$ $0.4591_{(11)}$	$0.9151_{(9)}$ $0.8435_{(10)}$ $0.8712_{(11)}$	$1.000_{(9)}$ $1.000_{(10)}$ $1.000_{(11)}$

$_{(i)}$ Indicates the similarity degree obtained by Equation (i).

4.3. Comparison Analysis

Recently, Farhadinia et al. [36] proposed the following similarity measure:

$$S_{\mathrm{F}}\left(\widetilde{M}, \widetilde{N}\right) = \frac{S_{PL1}\left(\widetilde{M}, \widetilde{N}\right) + 1 - d_w\left(\widetilde{M}, \widetilde{N}\right)}{2}$$

where $S_{PL1}\left(\widetilde{M}, \widetilde{N}\right) = 1 - \frac{1}{2n}\sum_{i=1}^{n}\left\{\left|\mu_{\widetilde{M}}^q - \mu_{\widetilde{N}}^q\right| + \left|\nu_{\widetilde{M}}^q - \nu_{\widetilde{N}}^q\right| + \left|\pi_{\widetilde{M}}^q - \pi_{\widetilde{N}}^q\right|\right\}$ and

$$d_w\left(\widetilde{M}, \widetilde{N}\right) = 1 - \frac{1}{n}\sum_{i=1}^{n}\cos\left\{\frac{\pi}{2}\left\{\max\left(\left|\mu_{\widetilde{M}}^q - \mu_{\widetilde{N}}^q\right|, \left|\nu_{\widetilde{M}}^q - \nu_{\widetilde{N}}^q\right|\right)\right\}\right\}$$

In this subsection, we conduct a numerical analysis of our proposed similarity measures, comparing them with those of Farhadinia et al. [36] to show the superiority of our proposed methods.

Example 5. *Let us consider a single universe $X = \{x_1\}$ of discourse. Let the three q-ROFSs in X be $\widetilde{M} = \{\langle x_1, 0.3, 0.4\rangle\}$, $\widetilde{N} = \{\langle x_1, 0.4, 0.3\rangle\}$ and $\widetilde{O} = \{\langle x_1, 0.0, 0.45\rangle\}$. Intuitively, we see* *that $\widetilde{M}$ is closer to $\widetilde{N}$ than to $\widetilde{O}$. Thus, we expect that the measure of the similarity between $\widetilde{M}$ and $\widetilde{N}$ should be larger than the measure of similarity between $\widetilde{M}$ and $\widetilde{O}$. According to the similarity measure S_{F} of Farhadinia* et al. [36], *we obtain* $S_{\mathrm{F}}\left(\widetilde{M}, \widetilde{N}\right) = 0.9480 < S_{\mathrm{F}}\left(\widetilde{M}, \widetilde{O}\right) = 0.9858$. *On contrary, the similarity results obtained by S_{F} cannot correctly classify these three q-ROFSs, $\widetilde{M}$, $\widetilde{N}$ and $\widetilde{O}$. On the other hand, the results of our proposed similarity measures in Equations (9)–(11) are obtained as follows:* $S_l\left(\widetilde{M}, \widetilde{N}\right) = 0.963 > S_l\left(\widetilde{M}, \widetilde{O}\right) = 0.936$; $S_r\left(\widetilde{M}, \widetilde{N}\right) = 0.9286 > S_r\left(\widetilde{M}, \widetilde{O}\right) = 0.8797$ *and* $S_e\left(\widetilde{M}, \widetilde{N}\right) = 0.9425 > S_e\left(\widetilde{M}, \widetilde{O}\right) = 0.9019$. *It can be seen that the similarity measures of Equations (9)–(11) give the following result: $\widetilde{M}$ is closer to $\widetilde{N}$ as compared to $\widetilde{O}$. Thus, our proposed similarity measures correctly classify the three q-ROFSs $\widetilde{M}$, $\widetilde{N}$ and $\widetilde{O}$ according to the principle of the maximum degree of similarity between q-ROFSs.*

5. Application to Multi-Criteria Decision Making Related to Daily Life

In daily life, decision making is a process of hand-picking the best option or options from an available finite number of viable options. For the purpose of better decision making, the best alternative is always chosen among the many alternatives. Decision making is a kind of day-to-day activity in our daily lives. It plays an important role in many fields, including business intelligence, biological sciences, computer science, engineering, financial management, medical sciences, and social and political sciences. Multi-criteria decision making (MCDM) is the process of selecting an appropriate and reasonable option from a finite number of available options under the influences of several criteria. To acquire the most preferred option, decision makers apply their preference information to the available options. Inexactness is a reality of daily life that requires very careful attention, because it may have effects on management and decision making. In real-life matters, available information is often vague, fuzzy and incomplete, and it may be difficult to form a complete and exactly accurate decision. The q-ROFSs are found to be a very rigorous tool for solve these kinds of decision-making problems involving such incomplete and vague information with a high accuracy and exactness.

In this section, we apply the proposed similarity measures in Equations (9)–(11) to manage daily life issues involving complex MCDM processes. In 1992, Gomes and Lima [44] first provided a practical application of the TODIM method (the name TODIM is a Portuguese acronym for interactive and multicriteria decision making), and then Trotta et al. [45] and Gomes and Rangel [46] developed further applications of TODIM in healthcare and the rental evaluation of residential properties. We now provide an orthopairian fuzzy TODIM (OF-TODIM) approach that extends TODIM based on our proposed similarity measures Equations (9)–(11) so that it can be used for TODIM in the case of q-ROFSs. We then apply the proposed OF-TODIM to the MCDM process for daily life problems. Let $A = \left\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_i\right\}$ represent the set of alternatives under the set of criteria $C = \left\{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_j\right\}$. According to the concept of the TODIM method, the gain and loss of each alternative $\widetilde{A}_i$, with respect to the criteria $\widetilde{C}_j$, is assessed. Then, we calculate the dominance degree of each alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_j$. The overall value of each alternative $\widetilde{A}_i$ is obtained and ranks the alternatives $\widetilde{A}_i$ in descending order. The steps used to develop an algorithm for OF-TODIM so as to solve problems related to daily life settings involving the MCDM process utilizing our proposed similarity measures in Equations (9)–(11) are as follows:

Step 1: Construction of the orthopairian fuzzy decision matrix

Let $\widetilde{A} = \left\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_i\right\}$ represent the set of alternatives and let the set of criteria be represented by $\widetilde{C} = \left\{\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_j\right\}$. Assume that the orthopairian fuzzy decision matrix (OFDM) is represented by $\widetilde{R} = (r_{ij})_{m \times n}$, given by the decision makers (DMs) in the MCDM problems. Let r_{ij} be a q-rung orthopair fuzzy number (q-ROFN) and $r_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$, $i = \{1, 2, \ldots, m\}$, $j = \{1, 2, \ldots, n\}$ be a criteria value provided by the decision maker to each alternative. The value μ_{ij} indicates the degree to which the alternative $\widetilde{A}_i$ satisfies the criteria $\widetilde{C}_j$, with ν_{ij} representing the degree to which the alternative $\widetilde{A}_i$ does not satisfy the criteria $\widetilde{C}_j$, such that $0 \leq \mu^q_{\widetilde{M}}(x) + \nu^q_{\widetilde{M}}(x) \leq 1$. The quantities $\mu^q_{\widetilde{M}}(x) \in [0,1]$ and $\nu^q_{\widetilde{M}}(x) \in [0,1]$, respectively, are mapped. The decision matrix is constructed as follows:

$$\widetilde{R} = (r_{ij})_{m\times n} = \begin{array}{c} \\ \widetilde{A}_1 \\ \widetilde{A}_2 \\ . \\ . \\ . \\ \widetilde{A}_i \end{array} \begin{array}{c} \begin{array}{cccccc} \widetilde{C}_1 & \widetilde{C}_2 & . & . & . & \widetilde{C}_j \end{array} \\ \begin{pmatrix} r_{11} & r_{12} & . & . & . & r_{1j} \\ r_{21} & r_{22} & . & . & . & r_{2j} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ r_{i1} & r_{i2} & . & . & . & r_{ij} \end{pmatrix} \end{array}$$

Step 2: Construction of the normalized orthopairian fuzzy decision matrix

Transform the decision matrix $\widetilde{R} = (r_{ij})_{m\times n}$ into a normalized orthopair fuzzy decision matrix (NOFDM):

$$\widetilde{L} = (l_{ij})_{m\times n} = \begin{cases} r_{ij} & for\ beneficial\ type\ attribute \\ (r_{ij})^c & for\ cost\ type\ attribute \end{cases}$$

In this step, we transform the OFDM $\widetilde{R} = (r_{ij})_{m\times n}$ into NOFDM. If the criteria are benefits, then we write the original matrix, but if the criterion is cost, then we take the complement of criteria $(r_{ij})^c$.

Step 3: Calculation of the relative weight of each criterion

We calculate the relative weight w_{jr} of each criterion $\widetilde{C}_j$ using $w_{jr} = w_j/w_r$, where w_j is the individual weight $w_j(1,2,\ldots,n)$ of each criterion $\ddot{C}_j$, satisfying the condition $\sum_{j=1}^{n} w_j = 1$. In the OF-TODIM method, we select the heaviest weight w_r as a reference weight and divide the reference weight by all the weights w_j:

$$w_r = max\left(w_j : j = \{1,2,\ldots,n\}\right)$$

Step 4: Dominance degree of each alternative $\widetilde{A}_i$

The dominance degree of each alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_j$ is calculated as follows:

$$\phi_j\left(\widetilde{A}_i, \widetilde{A}_t\right) = \begin{cases} \sqrt{\dfrac{w_{rj}\ \widetilde{S}(I_{ij}, I_{tj})}{\sum\limits_{j=1}^{n} w_{jr}}} & if\ \ I_{ij} > I_{tj} \\ 0 & if\ \ I_{ij} = I_{tj} \\ -\frac{1}{\theta}\sqrt{\dfrac{\sum\limits_{j=1}^{n} w_{rj}\widetilde{S}(I_{ij}, I_{tj})}{w_{jr}}} & if\ \ I_{ij} < I_{tj} \end{cases}$$

where I_{ij} and I_{tj} are the performance of alternatives $\widetilde{A}_i$ and $\widetilde{A}_t$, respectively, in relation to j, and $\phi_j\left(\widetilde{A}_i, \widetilde{A}_t\right)$ represents the dominance degree of each alternative $\widetilde{A}_i$ individually over alternative $\widetilde{A}_t$ with respect to the criterion $\ddot{C}_j$, equating alternative $\widetilde{A}_i$ with alternative $\widetilde{A}_t$. θ represents the attenuation factor of the loss. If $I_{ij} > I_{tj}\ or\ \ I_{ij} - I_{tj} > 0$, then we conclude that it is the dominance degree of gain. $\widetilde{S}(I_{ij}, I_{tj})$ represents the proposed similarity measures given in Equations (9)–(11), and w_{jr} is the relative weight of each criterion $\widetilde{C}_j$. If $I_{ij} < I_{tj}\ or\ I_{ij} - I_{tj} < 0$, then it represents the dominance degree of the loss.

If $I_{ij} > I_{tj}$ means gain, then we use $\sqrt{\frac{w_{jr}\ \widetilde{S}(I_{ij}, I_{tj})}{\sum_{i=1}^{n} w_{jr}}}$. If $I_{ij} < I_{tj}$ means loss, then we use $-\frac{1}{\theta}\sqrt{\sum_{i=1}^{n} \frac{w_{jr}\widetilde{S}(I_{ij}, I_{tj})}{w_{jr}}}$, where $I_{ij} = I_{tj}$ is considered as nonsensical.

Step 5: Calculate the overall dominance degree of $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$

In this step, we calculate the overall dominance degree of each $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ using $\delta\left(\widetilde{A}_i, \widetilde{A}_t\right) = \sum_{j=1}^{n} \phi_j\left(\widetilde{A}_i, \widetilde{A}_t\right)$, where $\delta\left(\widetilde{A}_i, \widetilde{A}\right)$ denotes the measurement of the dominance of each alternative $\widetilde{A}_i$ over alternative $\widetilde{A}_t$.

Step 6: Overall value of each alternative $\widetilde{A}_i$

We calculate the overall value of each alternative $\widetilde{A}_i$ using:

$$\psi\left(\widetilde{A}_i\right) = \frac{\sum_{i=1}^{n} \delta\left(\widetilde{A}_i, \widetilde{A}_t\right) - \min_i\left(\sum_{i=1}^{m} \delta\left(\widetilde{A}_i, \widetilde{A}_t\right)\right)}{\max_i\left\{\sum_{i=1}^{m} \delta(\widetilde{A}_i, \widetilde{A}_t)\right\} - \min_i\left(\sum_{i=1}^{m} \delta\left(\widetilde{A}_i, \widetilde{A}_t\right)\right)}$$

Clearly, $0 \leq \psi_i \leq 1$, and we need to select the greater value of ψ_i, which will be considered as the best alternative A_i.

Step 7: Ranking of the alternatives

In this step, we determine the ranking of the alternatives according to the overall values. The alternatives are arranged in descending order. The alternative in which $\psi(A_i)$ takes the greatest value is considered as the best alternative.

Example 6. *In the present era, the internet has become an integral part of human life. It plays an important role in connecting people regardless of physical barriers and helps people to enhance their lives. It also provides a facility to human beings that can be used to access useful data, various things, information, knowledge and learning and is convenient for individuals and socio-economic development. In the present COVID-19 pandemic, the use of the internet has become more demanding, and online education systems are becoming increasingly popular. Despite its importance, many areas of developing countries still lack this very important facility for one reason or another. Therefore, the planning commission of a developing country invites national and international internet-providing companies (IPCs) to provide fast and uninterrupted internet facilities to their people. Assume that the commission has to select the IPC that is supposed to provide the best internet facility according to wishes of the people. There are five available IPCs as alternatives, (A1) IPC 1, (A2) IPC 2, (A3) IPC 3, (A4) IPC 4 and (A5) IPC 5, under the following five identified criteria used to assess these five IPCs. The orthopairian fuzzy decision matrix of the alternative over the criteria is displayed in the Table 2 below. The calculation of the weight criteria using* $w_j = \frac{max\left(\mu_A^3, \mu_B^3\right) + min\left(\nu_A^3, \nu_B^3\right)}{2}$ *is shown in Table 3, and the calculation of the relative weights is given in Table 4.*

Table 2. Q-rung orthopairian fuzzy decision matrix.

	$\widetilde{C}_1$	$\widetilde{C}_2$	$\widetilde{C}_3$	$\widetilde{C}_4$	$\widetilde{C}_5$
$\widetilde{A}_1$	(0.9, 0.5)	(0.8, 0.6)	(0.7, 0.5)	(0.6, 0.6)	(0.8, 0.8)
$\widetilde{A}_2$	(0.7, 0.7)	(0.9, 0.3)	(0.6, 0.3)	(0.7, 0.9)	(0.5, 0.7)
$\widetilde{A}_3$	(0.8, 0.5)	(0.7, 0.5)	(0.5, 0.6)	(0.5, 0.7)	(0.7, 0.7)
$\widetilde{A}_4$	(0.6, 0.6)	(0.9, 0.5)	(0.8, 0.6)	(0.7, 0.5)	(0.8, 0.6)
$\widetilde{A}_5$	(0.8, 0.6)	(0.7, 0.7)	(0.9, 0.5)	(0.8, 0.5)	(0.7, 0.7)

Table 3. Weight of each criterion.

	w_1	w_2	w_3	w_4	w_5
max/min	0.229	0.203	0.203	0.171	0.195

Table 4. Relative weights.

	w_{1r}	w_{2r}	w_{3r}	w_{4r}	w_{5r}
wjr	1.000	0.886	0.886	0.747	0.852

The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_1$ using Equation (9) is denoted by $\phi_1(A_i, A_t)$. Table 5 reflects the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect to criterion $\widetilde{C}_1$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_2$ using Equation (9) is denoted by $\phi_2(A_i, A_t)$. Table 6 reflects the evaluation of the dominance degree of the alternatives $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criteria $\widetilde{C}_2$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_3$ using Equation (10) is denoted by $\phi_3(A_i, A_t)$. Table 7 reflects the evaluation of the dominance degree of the alternatives $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_3$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_4$ using Equation (9) is denoted by $\phi_4(A_i, A_t)$. Table 8 reflects the evaluation of the dominance degree of the alternatives $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_4$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_5$ using Equation (9) is denoted by $\phi_5(A_i, A_t)$. Table 9 reflects the evaluation of the dominance degree of the alternatives $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_5$.

Table 5. The matrix for criterion $\widetilde{C}_1$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.375	0.423	0.334	0.423
$\widetilde{A}_2$	−0.375	0.000	−0.423	0.000	−0.436
$\widetilde{A}_3$	−0.423	0.423	0.000	0.401	0.456
$\widetilde{A}_4$	−0.334	0.000	−0.401	0.000	−0.401
$\widetilde{A}_5$	−0.423	0.436	−0.456	0.401	0.000

Table 6. The matrix for criterion $\widetilde{C}_2$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	−0.450	0.401	−0.450	0.401
$\widetilde{A}_2$	0.389	0.000	0.353	0.374	0.353
$\widetilde{A}_3$	−0.463	−0.398	0.000	−0.398	0.398
$\widetilde{A}_4$	0.398	−0.477	0.352	0.000	0.353
$\widetilde{A}_5$	−0.463	−0.398	−0.449	−0.398	0.000

Table 7. The matrix for criterion $\widetilde{C}_3$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.421	0.398	−0.463	−0.398
$\widetilde{A}_2$	−0.475	0.000	0.405	−0.426	−0.355
$\widetilde{A}_3$	−0.449	−0.458	0.000	−0.398	−0.320
$\widetilde{A}_4$	0.410	0.378	0.352	0.000	−0.450
$\widetilde{A}_5$	0.353	0.314	0.283	0.398	0.000

Table 8. The matrix for criterion $\widetilde{C}_4$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.288	0.386	−0.517	−0.464
$\widetilde{A}_2$	−0.368	0.000	−0.434	−0.348	−0.348
$\widetilde{A}_3$	−0.517	0.324	0.000	−0.434	−0.433
$\widetilde{A}_4$	0.386	0.360	0.324	0.000	−0.433
$\widetilde{A}_5$	0.347	0.260	0.323	0.297	0.000

Table 9. The matrix for criterion $\widetilde{C}_5$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.346	0.000	−0.435	0.000
$\widetilde{A}_2$	−0.406	0.000	−0.406	−0.406	−0.406
$\widetilde{A}_3$	0.000	0.346	0.000	−0.472	0.000
$\widetilde{A}_4$	0.370	0.346	0.402	0.000	−0.402
$\widetilde{A}_5$	0.000	0.346	0.000	−0.472	0.000

The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_1$ using Equation (10) is denoted by $\phi_1(A_i, A_t)$. Table 10 shows the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_1$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_2$ using Equation (10) is denoted by $\phi_2(A_i, A_t)$. Table 11 represents the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_2$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_3$ using Equation (10) is denoted by $\phi_3(A_i, A_t)$. Table 12 reflects the evaluation of the dominance degree of the alternatives $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_3$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_4$ using Equation (10) is denoted by $\phi_4(A_i, A_t)$. Table 13 exhibits the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_4$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_5$ using Equation (10) is denoted by $\phi_5(A_i, A_t)$. Table 14 reflects the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_5$.

Table 10. The matrix for criterion $\widetilde{C}_1$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.318	0.383	0.271	0.383
$\widetilde{A}_2$	−0.318	0.000	−0.383	0.000	−0.403
$\widetilde{A}_3$	−0.383	0.383	0.000	0.352	0.436
$\widetilde{A}_4$	−0.271	0.000	−0.352	0.000	−0.352
$\widetilde{A}_5$	−0.383	0.403	−0.436	0.352	0.000

Table 11. The matrix for criterion $\widetilde{C}_2$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	−0.407	0.379	−0.407	0.379
$\widetilde{A}_2$	0.361	0.000	0.299	0.400	0.299
$\widetilde{A}_3$	−0.428	−0.338	0.000	−0.338	0.361
$\widetilde{A}_4$	0.361	−0.452	0.299	0.000	0.299
$\widetilde{A}_5$	−0.428	−0.338	−0.408	−0.338	0.000

Table 12. The matrix for criterion $\widetilde{C}_3$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.396	0.361	−0.428	−0.338
$\widetilde{A}_2$	−0.441	0.000	0.372	−0.374	−0.288
$\widetilde{A}_3$	−0.408	−0.419	0.000	−0.338	−0.252
$\widetilde{A}_4$	0.379	0.332	0.299	0.000	−0.407
$\widetilde{A}_5$	0.299	0.255	0.224	0.361	0.000

Table 13. The matrix for criterion $\widetilde{C}_4$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.235	0.364	−0.487	−0.408
$\widetilde{A}_2$	−0.314	0.000	−0.368	−0.275	−0.275
$\widetilde{A}_3$	−0.487	0.275	0.000	−0.368	−0.368
$\widetilde{A}_4$	0.364	0.205	0.275	0.000	−0.467
$\widetilde{A}_5$	0.305	0.205	0.275	0.348	0.000

Table 14. The matrix for criterion $\widetilde{C}_5$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.293	0.000	−0.382	0.000
$\widetilde{A}_2$	−0.344	0.000	−0.345	−0.344	−0.345
$\widetilde{A}_3$	0.000	0.293	0.000	−0.437	0.000
$\widetilde{A}_4$	0.325	0.293	0.372	0.000	0.372
$\widetilde{A}_5$	0.000	0.293	0.000	−0.437	0.000

The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_1$ using Equation (11) is denoted by $\phi_1(A_i, A_t)$. Table 15 reflects the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_1$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_2$ using Equation (11) is denoted by $\phi_2(A_i, A_t)$. Table 16 denotes the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_2$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_3$ using Equation (11) is denoted by $\phi_3(A_i, A_t)$. Table 17 represents the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_3$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_4$ using Equation (11) is denoted by $\phi_4(A_i, A_t)$. Table 18 reflects the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criteria $\widetilde{C}_4$. The calculation of the dominance degree of $\widetilde{A}_i$ over $\widetilde{A}_t$ with respect to the criterion $\widetilde{C}_5$ using Equation (11) is denoted by $\phi_5(A_i, A_t)$. Table 19 exhibits the evaluation of the dominance degree of the alternative $\widetilde{A}_i$ over each alternative $\widetilde{A}_t$ with respect criterion $\widetilde{C}_5$.

Table 15. The matrix for criterion $\widetilde{C}_1$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.336	0.398	0.289	0.398
$\widetilde{A}_2$	−0.336	0.000	−0.397	0.000	−0.415
$\widetilde{A}_3$	−0.398	0.397	0.000	0.369	0.444
$\widetilde{A}_4$	−0.289	0.000	−0.369	0.000	−0.369
$\widetilde{A}_5$	−0.398	0.415	−0.444	0.369	0.000

Table 16. The matrix for criterion $\widetilde{C}_2$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	−0.422	0.391	−0.422	0.391
$\widetilde{A}_2$	0.374	0.000	0.316	0.410	0.316
$\widetilde{A}_3$	−0.441	−0.357	0.000	−0.357	0.374
$\widetilde{A}_4$	0.374	−0.416	0.316	0.000	0.316
$\widetilde{A}_5$	−0.441	−0.357	−0.422	−0.357	0.000

Table 17. The matrix for criterion $\widetilde{C}_3$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.405	0.375	−0.441	−0.357
$\widetilde{A}_2$	−0.458	0.000	0.384	−0.392	−0.307
$\widetilde{A}_3$	−0.422	−0.434	0.000	−0.356	−0.270
$\widetilde{A}_4$	0.391	0.347	0.316	0.000	−0.422
$\widetilde{A}_5$	0.316	0.272	0.239	0.374	0.000

Table 18. The matrix for criterion $\widetilde{C}_4$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.247	0.372	−0.498	−0.427
$\widetilde{A}_2$	−0.331	0.000	−0.389	−0.294	−0.294
$\widetilde{A}_3$	−0.498	0.290	0.000	−0.389	−0.388
$\widetilde{A}_4$	0.372	0.219	0.290	0.000	−0.480
$\widetilde{A}_5$	0.319	0.219	0.290	0.359	0.000

Table 19. The matrix for criterion $\widetilde{C}_5$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.309	0.000	−0.399	0.000
$\widetilde{A}_2$	−0.363	0.000	−0.364	−0.363	−0.363
$\widetilde{A}_3$	0.000	0.309	0.000	−0.450	0.000
$\widetilde{A}_4$	0.341	0.309	0.383	0.000	0.383
$\widetilde{A}_5$	0.000	0.309	0.000	−0.450	0.000

The overall dominance degrees of A_i over A_t using $\delta(A_i, A_t) = \sum_{j=1}^{n} \phi_j(A_i, A_t)$ for Equations (9)–(11) are shown in Tables 20–22, respectively.

Table 20. Overall dominance degree.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.980	1.617	−1.531	−0.029
$\widetilde{A}_2$	−1.244	0.000	−0.505	−0.806	−1.192
$\widetilde{A}_3$	−1.852	0.273	0.000	−1.301	0.101
$\widetilde{A}_4$	1.230	0.507	1.029	0.000	−0.600
$\widetilde{A}_5$	−0.186	0.958	−0.299	0.226	0.000

Table 21. Overall dominance degree.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.835	1.487	−1.433	0.016
$\widetilde{A}_2$	−1.056	0.000	−0.425	−0.593	−1.012
$\widetilde{A}_3$	−1.706	0.194	0.000	−1.129	0.177
$\widetilde{A}_4$	1.158	0.378	0.893	0.000	−0.555
$\widetilde{A}_5$	−0.207	0.818	−0.345	0.286	0.000

Table 22. Overall dominance degree.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.875	1.535	−1.471	0.005
$\widetilde{A}_2$	−1.114	0.000	−0.450	−0.639	−1.064
$\widetilde{A}_3$	−1.759	0.205	0.000	−1.183	0.160
$\widetilde{A}_4$	1.189	0.414	0.936	0.000	−0.572
$\widetilde{A}_5$	−0.204	0.858	−0.337	0.295	0.000

The overall values of A_i over each alternative A_t using

$$\psi\left(\widetilde{A}_i\right) = \frac{\sum_{i=1}^{m} \delta(A_i, A_t) - \min_i \left(\sum_{i=1}^{m} \delta(A_i, A_t)\right)}{\max_i \left\{\sum_{i=1}^{m} \delta(A_i, A_t)\right\} - \min_i \left(\sum_{i=1}^{m} \delta(A_i, A_t)\right)}$$

for the proposed similarity measures of Equations (9)–(11) are shown in Table 23. Finally, we rank the alternatives using Table 23 in descending order according to the values of ψ_i. The final ranking is shown in Table 24. From Table 24, we find that there is no conflict in the ranking of the alternatives using the proposed similarity measures of Equations (9)–(11). They all rank in the order of $A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$, which is unanimously confirmed by the proposed similarity measures. Thus, our analysis shows that the best possible alternative is A_4.

Table 23. Overall values of A_i over each alternative A_t.

$\widetilde{S}_l$	Ψ_i	$\widetilde{S}_r$	Ψ_i	$\widetilde{S}_e$	Ψ_i
$\widetilde{A}_1$	0.8091	$\widetilde{A}_1$	0.7100	$\widetilde{A}_1$	0.8045
$\widetilde{A}_2$	0.000	$\widetilde{A}_2$	0.000	$\widetilde{A}_2$	0.000
$\widetilde{A}_3$	0.1637	$\widetilde{A}_3$	0.1106	$\widetilde{A}_3$	0.1318
$\widetilde{A}_4$	1.0000	$\widetilde{A}_4$	1.0000	$\widetilde{A}_4$	1.0000
$\widetilde{A}_5$	0.7519	$\widetilde{A}_5$	0.6472	$\widetilde{A}_5$	0.7411

Table 24. Ranking of the alternatives.

Similarities	Ranking
$\widetilde{S}_l$	$\widetilde{A}_4 > \widetilde{A}_1 > \widetilde{A}_5 > \widetilde{A}_3 > \widetilde{A}_2$
$\widetilde{S}_r$	$\widetilde{A}_4 > \widetilde{A}_1 > \widetilde{A}_5 > \widetilde{A}_3 > \widetilde{A}_2$
$\widetilde{S}_e$	$\widetilde{A}_4 > \widetilde{A}_1 > \widetilde{A}_5 > \widetilde{A}_3 > \widetilde{A}_2$

In Example 5 in Section 4.3, we compared our proposed similarity measures with the measure of similarity $S_{\mathrm{F}}\left(\widetilde{M}, \widetilde{N}\right)$ proposed by Farhadinia et al. [36]. Next, we provided a further comparison of our proposed (orthopairian fuzzy) TODIM with the TODIM based on the similarity S_{F}. We used the application example, Example 6, for our comparison. It is obvious that Steps 1, 2 and 3 based on the similarity S_F provided by Farhadinia et al. [36] are the same as those of our proposed (Orthopairian fuzzy) TODIM. Here, we use S_F in Steps 4, 5, 6 and 7 as follows:

In Step 4, the dominance degree of A_i over A_t with respect to the criteria $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$, $\widetilde{C}_4$ and $\widetilde{C}_5$ using $S_F(A_i, A_t)$ are shown in Tables 25–29, respectively.

Table 25. The matrix for criterion $\widetilde{C}_1$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.194	0.232	0.151	0.196
$\widetilde{A}_2$	−0.423	0.000	−0.464	0.000	−0.436
$\widetilde{A}_3$	−0.464	0.199	0.000	0.177	0.456
$\widetilde{A}_4$	−0.383	0.000	−0.446	0.000	−0.446
$\widetilde{A}_5$	−0.464	0.196	−0.456	0.177	0.000

Table 26. The matrix for criterion $\widetilde{C}_2$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	−0.464	0.199	−0.464	0.200
$\widetilde{A}_2$	0.200	0.000	0.353	0.192	0.158
$\widetilde{A}_3$	−0.462	−0.398	0.000	−0.423	0.196
$\widetilde{A}_4$	0.212	−0.488	0.182	0.000	0.165
$\widetilde{A}_5$	−0.474	−0.398	−0.464	−0.390	0.000

Table 27. The matrix for criterion $\widetilde{C}_3$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.165	0.199	−0.462	−0.423
$\widetilde{A}_2$	−0.384	0.000	0.197	−0.419	−0.368
$\widetilde{A}_3$	−0.464	−0.458	0.000	−0.464	−0.333
$\widetilde{A}_4$	0.211	0.181	0.199	0.000	−0.452
$\widetilde{A}_5$	0.194	0.158	0.299	0.191	0.000

Table 28. The matrix for criterion $\widetilde{C}_4$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.165	0.205	−0.469	−0.447
$\widetilde{A}_2$	−0.384	0.000	−0.390	−0.349	−0.349
$\widetilde{A}_3$	−0.481	0.168	0.000	−0.434	−0.390
$\widetilde{A}_4$	0.215	0.151	0.187	0.000	−0.466
$\widetilde{A}_5$	0.204	0.151	0.168	0.187	0.000

Table 29. The matrix for criterion $\widetilde{C}_5$.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.169	0.000	−0.447	0.000
$\widetilde{A}_2$	−0.398	0.000	−0.413	−0.424	−0.464
$\widetilde{A}_3$	0.000	0.198	0.000	−0.474	0.000
$\widetilde{A}_4$	0.204	0.180	0.204	0.000	0.181
$\widetilde{A}_5$	0.000	0.196	0.000	−0.457	0.000

In Step 5, the overall dominance degrees of A_i over A_t using the similarity S_F are shown in Table 30.

Table 30. Overall dominance degrees.

	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
$\widetilde{A}_1$	0.000	0.229	0.835	−1.691	−0.474
$\widetilde{A}_2$	−1.389	0.000	−0.717	−1.000	−1.459
$\widetilde{A}_3$	−1.871	−0.291	0.000	−1.625	0.071
$\widetilde{A}_4$	0.459	0.024	0.326	0.000	−1.018
$\widetilde{A}_5$	−0.544	−0.303	−0.462	−0.292	0.000

In Step 6, the overall values of A_i over A_t using $\psi(A_i)$ for the similarity measure S_{F} are shown in Table 31.

Table 31. Overall values of A_i over each alternative A_t.

$\widetilde{S}_F$	$\widetilde{A}_1$	$\widetilde{A}_2$	$\widetilde{A}_3$	$\widetilde{A}_4$	$\widetilde{A}_5$
Ψ_i	0.7950	0.0000	0.1949	1.0000	0.6804

Finally, we rank the alternatives using Table 31 in descending order according to the values of ψ_i. The final ranking is shown in Table 32. From Table 32, we find that they ranking in the order of $A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$.

Table 32. Ranking of the alternatives.

Similarity	Ranking
$\widetilde{S}_l$	$\widetilde{A}_4 > \widetilde{A}_1 > \widetilde{A}_5 > \widetilde{A}_3 > \widetilde{A}_2$

By comparing our proposed methods in Equations (9)–(11) with the method proposed by Farhadinia et al. [36], we can see that they all provide the same results for this application example. That is, they demonstrate that there is no conflict in the ranking of the alternatives and the selection of the best alternative A_4, with $A_4 \succ A_1 \succ A_5 \succ A_3 \succ A_2$. We should mention that if we compare the formula form of our proposed Equations (9)–(11) with that of Farhadinia et al. [36], we can see that our proposed methods are simpler and more intuitive as compared to that of Farhadinia et al. [36]. On the other hand, the comparative analysis of Example 5 actually indicates that our proposed methods perform better than that of Farhadinia et al. [36].

6. Conclusions

Many extensions and generalizations of fuzzy sets have been suggested in the literature that can be used to model uncertain, vague and fuzzy information with a high accuracy and precision. In this paper, we considered the generalization of fuzzy sets, called q-rung orthopair fuzzy sets (q-ROFSs), to construct distance and similarity measures according to the Hausdorff metric. We considered the Hausdorff metric to calculate the distance between q-ROFSs. We then used the proposed distances between the q-ROFSs to develop new similarity measures for calculating the degrees of similarity between q-ROFSs. Our consideration of the measures of the distance and similarity between q-ROFSs was based on finite universes of discourses, which are used not only in computing environments but also in more general cases for large universal sets. The proposed methods are natural and easy to apply in a variety of applications and well-suited to the q-ROFSs environment. We presented several examples to show the reasonability and applicability of our proposed methods. We applied the proposed measures to pattern recognition and queries with fuzzy linguistic variables. Based on the proposed methods, we also constructed a new orthopairian fuzzy TODIM method to address problems related to daily life settings involving multi-criteria decision making (MCDM). The results show that the proposed methods are reasonable, applicable and well-suited to pattern recognition, linguistic variables and MCDM problems.

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