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Ulam–Hyers Stability via Fixed Point Results for Special Contractions in *b*-Metric Spaces

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Abstract: In this paper, we present some fixed point results for Subrahmanyan contraction in the setting of a b-metric space. We consider the case of multivalued operators. We also deduce the Ulam–Hyers stability property of the fixed point inclusion. The notion of b-metric generalizes the one of a metric, as in the third condition, the right-hand side is multiplied by a real number greater than 1. We remark that the second axiom, i.e., the one which shows the symmetry of the b-metric, remains unchanged. The findings presented in this paper extend some recent results which were proved in the context of a metric space. Some open questions are presented at the end of the paper.

Keywords: fixed points; Subrahmanyan contractions; multivalued operators; b-metric spaces; Ulam–Hyers stability

MSC: 46T99; 47H10; 54H25

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1. Introduction

In this paper, we present some fixed point results for Subrahmanyan contraction in the setting of a b-metric space. We consider the case of multivalued operators. These results expand on some recent theorems proved in classical metric spaces.

Let us recall the fact that a b-metric is a generalization of the notion of metric and thus, the results obtained in the context of b-metric space are more general than those proved in the case of metric spaces. The symmetry of the b-metric should also be mentioned.

An interesting paper regarding the roots of the notion of a b-metric, as well as a brief survey on this concepts and related results, is the very recent paper of Berinde and Pacurar [1]. The authors show that a very impressive research work has been devoted in the last two decades to obtaining fixed point theorems in b-metric spaces (also called quasimetric spaces). Other results and examples regarding this notion can be found in [2–6].

The well-known Banach contraction principle is generalized in two main directions: on the one hand, the work space is changed, on the other, one can consider single-valued or multivalued operators satisfying different contraction conditions.

We consider both directions and work in a b-metric space. For the operator, we choose to work with a multivalued Subrahmanyan contraction. This notion was introduced for the single-valued case in [7], and for the multivalued operators in [8]. Fixed point results are presented in the context of a metric space. One can easily notice that this notion generalizes the notion of graph contraction.

Section 2 of the paper provides a review of some definitions, examples and results that will be needed in the next part.

In Section 3, we obtain some fixed point theorems for multivalued Subrahmanyan contractions. The theoretical results are then used to derive a local Ulam–Hyers stability result for the fixed point inclusion.

Another important problem is that of giving explicit conditions for the existence of a strict fixed point of a multivalued operator. This issue is addressed in Section 4. An open problem is also stated.

The last section summarizes the main findings of the present study.

The main contribution of this paper is presenting fixed points and strict fixed points results for Subrahmanyan contractions, as well as deriving Ulam–Hyers stability, in the context of b-metric spaces.

2. Preliminaries

First, let us recall some notions and results that will be needed later.

Definition 1 (Bakhtin [9], Czerwik [5])**.** *Let M be a nonempty set and consider $s \geq 1$, a given real number. A functional $d : M \times M \to \mathbb{R}_+$ is called a b-metric (which is named also in some papers quasi-metric) with constant $s \geq 1$ if the first two Fréchet axioms of the metric are satisfied. The third one (the triangle inequality axiom), is different, in the sense that it has the following form:*

$$(\star) \;\; d(x,z) \leq s[d(x,y) + d(y,z)], \text{ for all } x,y,z \in M.$$

The pair (M,d) with these properties is called a b-metric space with constant $s \geq 1$.

Some classical examples of b-metric space are the following, given by Berinde in [2].

Example 1. *The space $L^p[0,1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional $d(x,y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}$, is a b-metric space. Notice that $s = 2^{1/p}$.*

Example 2. *For $0 < p < 1$, the set $l^p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} | \sum\limits_{n=1}^{\infty} |x_n|^p < \infty\}$ together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$, $d(x,y) := (\sum\limits_{n=1}^{\infty} |x_n - y_n|^p)^{1/p}$, is a b-metric space with coefficient $s = 2^{1/p} > 1$. Notice that the above result holds for the general case $l^p(X)$ with $0 < p < 1$, where X is a Banach space.*

Other interesting examples of b-metric spaces can be found in [2–6]. It is known that some topological properties in the setting of b-metric spaces are the same as in metric spaces.

One can define the notions of a compact subset and closed subset of a b-metric space in the same manner as that in which the context of a metric space is defined.

It is also known that in a b-metric space (M,d), a convergent sequence has a unique limit and each convergent sequence is Cauchy.

However, there are some important distance-type differences: the b-metric on M may not be continuous, open balls in b-metric spaces need not be open sets, the closed ball is not necessary a closed set, to recall a few.

It is also important to emphasize the symmetry of the b-metric.

Throughout the paper, $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}^*$ is the set of non-zero natural numbers, $\mathbb{R}$ is the set of all real numbers and $\mathbb{R}_+$ is the set of all real non-negative numbers.

For the convenience of the reader, we briefly recall the definition of the following notions, which are well known in nonlinear analysis.

Let (M,d) be a b-metric space and $P(M)$ be the family of all nonempty subsets of M. We denote by $P_{cl}(M)$ the family of all nonempty closed subsets of M, by $P_b(M)$ the family of all nonempty bounded subsets of M and by $P_{cp}(M)$ the family of all nonempty compact subsets of M.

We denote by $B(y_0;r) := \{y \in M | d(y_0,y) < r\}$, for $y_0 \in X$ and $r > 0$ the open ball and by $\tilde{B}(y_0;r) := \{y \in M | d(y_0,y) \leq r\}$, for $y_0 \in M$ and $r > 0$ the closed ball, centered in y_0 with radius r.

In the context of a b-metric space, the functionals used in the multivalued analysis theory are defined in the same way as in the context of a usual metric space. We will use the following notations:
(1) The gap functional, i.e., the distance between a point $a \in M$ and a set $B \subset M$:

$$D(a, B) := \inf\{d(a, b) \mid b \in B\};$$

(2) The excess functional of A over B generated by d:

$$e(A, B) := \sup\{D(a, B) \mid a \in A\};$$

(3) The Hausdorff–Pompeiu functional generated by d:

$$H(A, B) = \max\{e(A, B), e(B, A)\}.$$

For the definitions and important properties of these functionals, see [10–12].

The following Lemma presents some properties of the functionals which will be used later in the proof of our main result.

Lemma 1. *If (M, d) is a b-metric space with $s \geq 1$, then we have:*
(a) $D(x, A) \leq sd(x, y) + sD(y, A)$, for all $x, y \in M$ and $A \in P(M)$;
(b) If $A \in P_{cl}(M)$ and $x \in M$ are such that $D(x, A) = 0$, then $x \in A$.
(c) If $A, B \in P(M)$ and $q > 1$, then, for each $a \in A$ there is $b \in B$ such that $d(a, b) \leq qH(A, B)$.

Recall that if (M, d) is a b-metric space, then a set $Y \in P(M)$ is said to be proximinal if for every $x \in M$ there exists $y \in Y$ such that $d(x, y) = D(x, Y)$.

Additionally, let us recall that if M is a nonempty set and $S : M \to P(M)$ is a multivalued operator, then we denote by $Fix(S) := \{x \in M : x \in S(x)\}$ the fixed point set for S, by $SFix(S) := \{x \in M : \{x\} = S(x)\}$ the strict fixed point set for S, by $Graph(S) := \{(x, y) \in M \times M | y \in S(x)\}$ the graph of S.

Moreover, for arbitrary $(x_0, x_1) \in Graph(S)$, the sequence $(x_n)_{n \in \mathbb{N}}$ with $x_{n+1} \in S(x_n)$ (for $n \in \mathbb{N}^*$) is called the sequence of successive approximations for S staring from (x_0, x_1).

We present now the notion of multivalued Subrahmanyan contraction in the setting of b-metric space. This was introduced in [8] for the case of metric space. For the single-valued case, the reader is referred to [7].

Definition 2 ([8]). *Let (M, d) be a b-metric space and $S : M \to P(M)$ be a multivalued operator. We say that S is a multivalued Subrahmanyan contraction if there exists $\psi : X \to [0, 1[$ such that:*
(i) $H(S(x), S(y)) \leq \psi(x)d(x, y)$, for all $(x, y) \in Graph(S)$;
(ii) $\psi(y) \leq \psi(x)$, for every $(x, y) \in Graph(S)$.

Remark 1. *For a constant $k \in [0, 1[$, let $\psi(x) := k$ for each $x \in M$. Then, we obtain the notion of a multivalued graph contraction with constant k.*

In 1969, S.B. Nadler Jr. proved the first metric fixed point principle for multivalued operators in complete metric spaces [13,14]. Later, in 1998, S. Czerwik generalized the results in the context of a b-metric space. The result is stated as follows (see Theorem 5 in [15]).

Theorem 1. *Let (M, d) be a complete b-metric space. If $S : M \to P_{cl}(M)$ is a multivalued α-contraction where α is a real constant and $0 \leq \alpha < s^{-1}$, then $Fix(S) \neq \emptyset$.*

A useful remark regarding the above theorem can be found in [16].

3. Fixed Point Results and Ulam–Hyers Stability

Let us present an important result which allows us to prove the existence of a fixed point in b-metric spaces without any additional conditions on the constant s. We can also obtain an a priori estimate from this Lemma.

Lemma 2 ([17]). *Every sequence $(x_n)_{n\in\mathbb{N}}$ of elements from a b-metric space (M,d) with constant s, for which there exists $\gamma \in [0,1)$ such that $d(x_{n+1},x_n) \le \gamma d(x_n,x_{n-1})$, $n \in \mathbb{N}$ is a Cauchy sequence. Moreover, the following estimate holds*

$$d(x_{n+1},x_{n+p}) \le \frac{\gamma^n S}{1-\gamma} d(x_0,x_1), \text{ for all } n,p \in \mathbb{N},$$

where $S := \sum_{i=1}^{\infty} \gamma^{2i\log_\gamma s + 2^{i-1}}$.

The following theorem is the first main result of this paper.

Theorem 2. *Let (M,d) be a complete b-metric space and let $S : M \to P(M)$ be a multivalued Subrahmanyan contraction with closed graph. Then, we have the following:*

(a) $Fix(S) \neq \emptyset$;

(b) For every $(x_0,x_1) \in Graph(S)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for S starting with (x_0,x_1), which converges to a fixed point $x^*(x_0,x_1)$ of S and the following a priori estimate holds:

$$d(x_{n+1},x^*(x_0,x_1)) \le \frac{(q\psi(x_0))^n sS}{1-q\psi(x_0)} d(x_0,x_1), n,k \in \mathbb{N}.$$

where $q \in (1, \frac{1}{\psi(x_0)})$ *and* $S := \sum_{i=1}^{\infty} (q\psi(x_0))^{2i\log_{q\psi(x_0)} s + 2^{i-1}}$;

(c) For all $(x_0,x_1) \in Graph(S)$, the following retraction displacement type condition holds:

$$d(x_0,x^*(x_0,x_1)) \le (s + \frac{s^2 S\psi(x_0)}{1-s\psi(x_0)}) d(x_0,x_1),$$

where $S := \sum_{i=1}^{\infty} (q\psi(x_0))^{2i\log_{q\psi(x_0)} s + 2^{i-1}}$.

Proof. We prove parts (a) and (b) together.

Let $(x_0,x_1) \in Graph(S)$ be arbitrary and let $1 < q < \frac{1}{\psi(x_0)}$.

The first step of the proof is to construct a sequence of successive approximations for S starting with $(x_0,x_1) \in Graph(S)$.

Hence, for $x_1 \in S(x_0)$, there exists $x_2 \in S(x_1)$ such that

$$d(x_1,x_2) \le qH(S(x_0),S(x_1)) \le q\psi(x_0)d(x_0,x_1).$$

Next, for $x_2 \in S(x_1)$, there exists $x_3 \in S(x_2)$ such that

$$d(x_2,x_3) \le qH(S(x_1),S(x_2)) \le q\psi(x_1)d(x_1,x_2) \le (q\psi(x_0))^2 d(x_0,x_1).$$

Therefore, we obtain a sequence which has the following property:

$$d(x_n,x_{n+1}) \le (q\psi(x_0))^n d(x_0,x_1).$$

Applying Lemma 2 with $\gamma = q\psi(x_0)$, we deduce that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

From the same Lemma, we obtain the inequality:

$$d(x_{n+1}, x_{n+p}) \leq \frac{(q\psi(x_0))^n S}{1 - q\psi(x_0)} d(x_0, x_1), \text{ for all } n, p \in \mathbb{N}, \quad (1)$$

where $S := \sum_{i=1}^{\infty} (q\psi(x_0))^{2i \log_{q\psi(x_0)} s + 2^{i-1}}$.

Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, using the completeness of the b-metric, it follows that the sequence converges to $x^*(x_0, x_1) \in X$. From the hypothesis that S has a closed graph, we have that $x^*(x_0, x_1)$ is a fixed point of S.

Moreover, from (1), we obtain

$$d(x_{n+1}, x^*) \leq s(d(x_{n+1}, x_{n+k}) + d(x_{n+k}, x^*) \leq$$

$$\frac{(q\psi(x_0))^n sS}{1 - q\psi(x_0)} d(x_0, x_1) + sd(x_{n+k}, x^*), n, k \in \mathbb{N},$$

which provides the a priori estimate.

(c) By letting $k \to \infty$, we obtain

$$d(x_{n+1}, x^*) \leq \frac{(q\psi(x_0))^n sS}{1 - q\psi(x_0)} d(x_0, x_1), n \in \mathbb{N}.$$

Taking $n = 0$ in the relation above, it follows that

$$d(x_1, x^*(x_0, x_1)) \leq \frac{q\psi(x_0) sS}{1 - q\psi(x_0)} d(x_0, x_1), n \in \mathbb{N}.$$

For the retraction displacement type condition, using the above relation, we obtain:

$$d(x_0, x^*(x_0, x_1)) \leq s(d(x_0, x_1)) + d(x_1, x^*(x_0, x_1) \leq s(d(x_0, x_1)) + \frac{q\psi(x_0) sS}{1 - q\psi(x_0)} d(x_0, x_1))$$

Letting $q \to 1$, we obtain the desired conclusion. □

As a consequence of this first main result, we can obtain the local Ulam–Hyers stability property in b-metric spaces. For more details regarding the Ulam–Hyers stability, see [18,19].

Definition 3. *Let (M, d) be a b-metric space with constant $s \geq 1$ and let $S : M \to P(M)$ be an operator. Then, the fixed point inclusion*

$$x \in S(x), x \in X, \quad (2)$$

is said to be local Ulam–Hyers stable if there exists $c > 0$ such that for any $\varepsilon > 0$ and any ε-solution z of the fixed point inclusion (2), i.e.,

$$D(z, S(z)) \leq \varepsilon,$$

there exists $x^* \in Fix(S)$ with $d(z, x^*) \leq c\varepsilon$.

Using the above notion, we can state the Ulam–Hyers stability result as follows.

Theorem 3. *Let (X, d) be a complete b-metric space and let $S : M \to P(M)$ be a multivalued operator, which satisfies the hypothesis in Theorem 2. If, in addition, the operator has proximinal values, we obtain that the fixed point inclusion (2) is local Ulam–Hyers stable.*

Proof. Let $\varepsilon > 0$ and consider p, an ε-solution of (2). From the hypothesis, we know that the operator S has proximinal values.

By this property, there is $w \in S(p)$ such that $d(p,w) = D(p,S(p)) \leq \epsilon$.

Then, from Theorem 2, it follows that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for S starting with$(p,w) \in Graph(S)$ which converges to a fixed point $x^* = x^*(p,w)$ of S and

$$d(z,x^*) \leq C(z)d(p,w) \leq C(p)\varepsilon.$$

□

The next theorem is an extension of Theorem 2, in the sense that the second assumption is given in the terms of the functional e_d.

Theorem 4. *Let (M,d) be a complete b-metric space and $S : M \to P(M)$ be an operator with closed graph. Assume that there exists $\phi : X \to [0,1[$ such that:*

(i) $e(S(x),S(y)) \leq \phi(x)d(x,y)$, for every $(x,y) \in Graph(S)$;

(ii) $\phi(y) \leq \phi(x)$, for all $(x,y) \in Graph(S)$.

Then, we have the following:

(a) $Fix(S) \neq \emptyset$;

(b) For every $(x_0,x_1) \in Graph(S)$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations for S, starting with (x_0,x_1), which converges to a fixed point of S, $x^*(x_0,x_1)$. Moreover, the following a priori estimate holds

$$d(x_{n+1},x^*(x_0,x_1)) \leq \frac{(q\psi(x_0))^n sS}{1-q\psi(x_0)}d(x_0,x_1), \text{ for every } n \in \mathbb{N},$$

where $q \in (1,\frac{1}{\psi(x_0)})$ and $S := \sum_{i=1}^{\infty}(q\psi(x_0))^{2i\log_{q\psi(x_0)} s+2^{i-1}}$.

Remark 2. *An important open question is obtaining other stability properties (well-posedness in the sense of Reich and Zaslavski, see [20–23]), Ostrowski stability property [22,23]) for the fixed point inclusion $x \in S(x)$.*

4. Strict Fixed Point Results

First, let us remark that a strict fixed point (end-point) for S is a special fixed point for the operator $S : M \to P(M)$. In this framework, we can state a strict fixed point result for multivalued Subrahmanyan contractions in b-metric spaces, which generalizes one of the main theorems in [24].

Theorem 5. *Let (M,d) be a complete b-metric space and $S : M \to P(M)$ be a multivalued operator which satisfies the hypothesis in Theorem 2. Further, suppose that:*

(1) $S(S(x)) \subset S(x)$, for each $x \in M$;

(2) If $Y \in P_{cl}(M)$ with $S(Y) = Y$, then Y is a singleton.

Then, $Fix(S) = SFix(S) \neq \emptyset$.

Proof. Applying Theorem 2, we obtain the existence of a fixed point. Let us denote by p the fixed point of S. By the first hypothesis, we obtain that $S(p) \subset S(S(p)) \subset S(p)$. Thus, $S(S(p)) = S(p)$ and so $S(p)$ is a fixed set for S. By the second hypothesis, we obtain that $S(p)$ is a singleton. Hence, $S(p) = \{p\}$.

Hence, we see that $Fix(S) \subset SFix(S)$. Thus, $Fix(S) = SFix(S) \neq \emptyset$. □

The next theorem is a generalization of Theorem 1 from [24] for the case of a b-metric space.

For the proof of the theorem, the following well-known Cantor's Lemma is very useful. This is given here in the setting of a b-metric space.

Lemma 3. *Let (M,d) be a b-metric space. Suppose that (M,d) is complete. Then, for every descending sequence $\{S_n\}_{n\geq 1}$ of nonempty closed subsets of M such that*

$$\mathrm{diam}(\mathrm{S_n}) \to 0 \textit{ as } \mathrm{n} \to \infty, \tag{3}$$

the intersection $\bigcap_{n=1}^{\infty} S_n$ contains exactly one point.

Let (M,d) be a b-metric space, $M : X \to P_{cl}(M)$ be a multivalued operator and $A \subset M$. Define $S(A) := \bigcup_{x\in M} S(x)$.

The strict fixed point result for the case of a multivalued operator is the following.

Theorem 6. *Let (M,d) be a complete b-metric space and $S : M \to P_{cl}(M)$ be a multivalued operator satisfying the following hypotheses:*

(i) $S(S(x)) \subset S(x)$, for each $x \in M$;

(ii) For any $x \in M$ and $\varepsilon > 0$, there exists $y \in S(x)$ such that $diam(S(y)) < \varepsilon$.

Then, $SFix(S) \neq \emptyset$.

Suppose that, additionally, the following assumption is satisfied:

(iii) $Y \in P_{cl}(S)$ with $S(Y) = Y$, implies that Y has exactly one fixed point.

Then, $Fix(S) = SFix(S) \neq \emptyset$.

Proof. For proving the first assumption, fix $x_0 \in M$. By condition (ii), there exists $x_1 \in S(x_0)$ such that $diam(S(x_1)) < 1$. From assumption (i), we have $S(x_1) \subseteq S(S(x_0)) \subseteq S(x_0)$.

Using an inductive argument, we obtain a sequence (x_n) such that for any $n \in \mathbb{N}$, $diam(S(x_n)) < \frac{1}{n}$ and $S(x_n) \subseteq S(x_{n-1})$.

By Cantor's Lemma 3, there exists $x^* \in M$ such that $\{x^*\} = \bigcap_{n\in\mathbb{N}} S(x_n)$. Since $x^* \in S(x_n)$, condition (i) implies that $S(x^*) \subseteq S(x_n)$, i.e., $S(x^*) \subseteq \bigcap_{n\in\mathbb{N}} S(x_n) = \{x^*\}$.

Hence, $SFix(S) \neq \emptyset$.

For the second conclusion, let us show that $Fix(S) \subset SFix(S)$. Let $p \in Fix(S)$. Then, by hypothesis (i), we obtain $S(p) \subset S(S(p)) \subset S(p)$. Hence, $S(S(p)) = S(p)$. By the third assumption, it follows that $S(p)$ has exactly one fixed point. Since $p \in S(p)$, we immediately obtain that $S(p) = \{p\}$. Hence, $p \in SFix(S)$ and $Fix(S) = SFix(S) \neq \emptyset$. □

We can state a similar result where the operator has the "approximate endpoint property", see [25].

Definition 4. *Let $S : M \to P(M)$ be a multivalued operator. We say that S has the approximate endpoint property if $\inf_{x\in M} H(x, S(x)) = 0$.*

Theorem 7. *Let (M,d) be a complete b-metric space with $s \geq 1$. Let $S : M \to P_{b,cl}(M)$ be a multivalued operator such that*

(i) $H(S(x), S(y)) \leq \psi(d(x,y))$, for all $x, y \in X$, where $\psi : [0,\infty) \to [0,\infty)$ is upper semicontinous, $\psi(t) < t, \forall t > 0$ and $\liminf_{t\to\infty}(t - s\psi(t)) > 0$

(ii) $\inf_{x\in X} H(x, S(x)) = 0$.

Then, $Fix(S) = SFix(S) = \{x^*\}$.

Proof. Let us consider the following set

$$B_n = \{x \in M : H(\{x\}, S(x)) = \sup_{y\in S(x)} d(x,y) \leq \frac{1}{n}\},$$

for every $n \in \mathbb{N}$.

We observe that for each $n \in \mathbb{N}$, $B_{n+1} \subseteq B_n$. As the mapping $x \to H(\{x\}, S(x))$ is continuous, we note that B_n is closed.

Next, we have to show that for every $n \in \mathbb{N}$, B_n is bounded.

To this end, let us suppose that diam $B_{n_0} = \infty$, for some $n_0 \in \mathbb{N}$. Then, there exist $x_n, y_n \in B_{n_0}$ such that $d(x_n, y_n) \geq n_0$. By assumption (i) and the triangle inequality, we obtain:

$$d(x_n, y_n) = H(\{x_n\}, \{y_n\}) \leq s[H(\{x_n\}, S(x_n))H(S(x_n), S(y_n)) + H(\{y_n\}, S(y_n))] \leq s\frac{2}{n} + s\psi(d(x_n, y_n)).$$

Hence,

$$d(x_n, y_n) - s\psi(d(x_n, y_n)) \leq \frac{2s}{n}, \text{ for each } x_n, y_n \in B_{n_0}.$$

Thus,

$$0 \leq d(x_n, y_n) - s\psi(d(x_n, y_n)) \leq \frac{2s}{n}.$$

Since $\lim_{n\to\infty}(d(x_n, y_n) - s\psi(d(x_n, y_n))) = 0$ and $d(x_n, y_n) \to \infty$, we reach a contradiction.

The next step is to show that $\lim_{n\to\infty} \text{diam } B_n = 0$.

Suppose that

$$\lim_{n\to\infty} \text{diam } B_n = t_0 > 0.$$

We can remark that the sequence $(\text{diam } B_n)_{n\in\mathbb{N}}$ is non-increasing, as well as bounded from below. Hence, it has a limit.

Let

$$\rho = \inf_{n\in\mathbb{N}}\{\liminf_{k\to\infty}(t_{n,k} - s\psi(t_{n,k})) : (x_{n,k}, y_{n,k}) \in B_n$$

and

$$t_{n,k} = d(x_{n,k}, y_{n,k}) \to \text{diam } B_n, \text{ as } n \to \infty\}.$$

We want to show that $\rho > 0$. By contradiction, let us suppose that $\rho = 0$. Using its definition, it follows that there is a sequence t_n such that $t_n \to t_0$ and $\lim_{n\to\infty}(t_n - \psi(t_n)) = 0$. Then, $\lim_{n\to\infty} \psi(t_n) = t_0$. However, since ψ is upper semicontinuous and $t_0 > 0$, it follows that

$$t_0 = \lim_{n\to\infty} \psi(t_n) \leq \psi(t_0) < t_0.$$

Hence, we obtain a contradiction, which implies that $\rho > 0$.

Now, for every $n \in \mathbb{N}$, let $(x_k, y_k) \in B_n$ be a sequence such that $d(x_k, y_k) \to \text{diam } B_n$, as $k \to \infty$.

We obtain that:

$$\rho < \liminf_{k\to\infty}(d(x_k, y_k)) - s\psi(d(x_k, y_k)) \leq \frac{2s}{n},$$

for every $n \in \mathbb{N}$.

This implies that $\rho = 0$, which is a contradiction. Hence, $\lim\limits_{n\to\infty} \text{diam } B_n = 0$.

By Cantor's Intersection Lemma 3, it follows that $\cap_{n\in\mathbb{N}} B_n = \{x_0\}$.

Then,

$$H(\{x_0\}, S(x_0)) = \sup_{y\in S(x_0)} d(x_0, y) = 0$$

and thus,

$$S(x_0) = \{x_0\}.$$

The uniqueness of the strict fixed point follows immediately. □

Corollary 1. *Let (M, d) be a complete b-metric space with $s \geq 1$ and $S : M \to P_{b,cl}(M)$ be a multivalued α contraction with $\alpha < \frac{1}{s}$, which satisfies the approximate endpoint property. Then, S has a unique strict fixed point.*

Proof. We apply the theorem above for $\psi(t) = \alpha t$. □

Remark 3. *In the above theorems, the main tool for proving the existence of the strict fixed point is Cantor's intersection theorem in b-metric spaces.*

Open Problem. Obtaining similar results for multivalued Subrahmanyan contractions under the "approximate endpoint property" is still an open question.

5. Conclusions

The present paper discusses fixed point theory in *b*-metric spaces. The first part is dedicated to some preliminary notions and results, which are useful for the readers. The first main result is a fixed point theorem for a multivalued Subrahmanyan contraction. We give a proof of this result using a very recent useful lemma by Miculescu and Mihail. Not just the existence of the fixed point can be obtained using this result, but also an a priori estimate and a retraction-displacement condition. We can apply the main result to derive local Ulam–Hyers stability for fixed point inclusion. The second part of the paper is dedicated to the notion of strict fixed point multivalued Subrahmanyan contraction. A partial answer is given regarding this notion and an open question is also stated. The problem of given explicit conditions for the existence of a strict fixed point is important because many iteration methods for multivalued operators are working under this assumption.

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