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Cheney–Sharma Type Operators on a Triangle with Straight Sides

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Abstract: We consider two types of Cheney–Sharma operators for functions defined on a triangle with all straight sides. We construct their product and Boolean sum, we study their interpolation properties and the orders of accuracy and we give different expressions of the corresponding remainders, highlighting the symmetry between the methods. We also give some illustrative numerical examples.

Keywords: Cheney–Sharma operator; triangle; product and Boolean sum operators; modulus of continuity; degree of exactness; Peano's theorem; error evaluation

MSC: 41A35; 41A36; 41A25; 41A80

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1. Introduction

In order to match all the boundary information on a domain, there were considered interpolation operators on triangles with straight sides (see, e.g., [1–7]) and on triangles with curved sides (see, e.g., [8–21]).

Here, we construct two kind of Cheney–Sharma type operators (see, e.g., [22–25]) defined on a triangle with all straight sides and study the interpolation properties, the orders of accuracy, their products and boolean sums and the remainders of the corresponding approximation formulas, using the modulus of continuity and Peano's theorem. There is a symmetrical connection between the two methods introduced here. Using the interpolation properties of such operators, blending function interpolants can be constructed that exactly match the function on some sides of the given region. Applications of these blending functions are in computer-aided geometric design, in the finite element method for differential equations problems and for the construction of surfaces that satisfy some given conditions (see, e.g., [1,14,17,20,21,26–34]).

We have considered the standard triangle T_h (see Figure 1), with vertices $V_1 = (0, h)$, $V_2 = (h, 0)$ and $V_3 = (0, 0)$ and sides $\Gamma_1, \Gamma_2, \Gamma_3$.

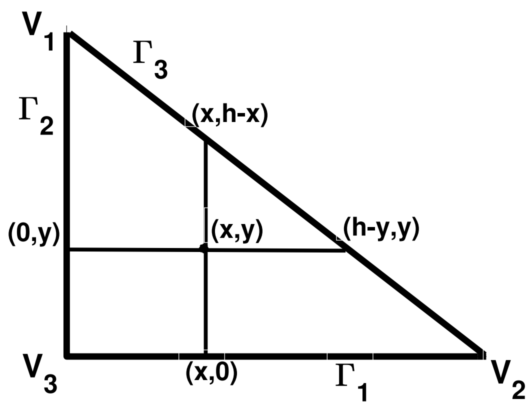

Figure 1. Triangle T_h.

2. Cheney–Sharma Operator of the Second Kind

Let $m \in \mathbb{N}$ and β be a nonnegative parameter. The Cheney–Sharma operator of the second kind $Q_m : C[0,1] \to C[0,1]$, introduced in [23], is given by

$$(Q_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{i}{m}), \tag{1}$$

with

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}(1-x)[1-x+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}.$$

The following results are useful in the sequel.

Remark 1. *(1) Notice that for $\beta = 0$, the operator Q_m becomes the Bernstein operator.*

(2) In [25], it has been proved that the Cheney–Sharma operator Q_m interpolates a given function at the endpoints of the interval.

(3) In [23,25], it has been proved that the Cheney–Sharma operator Q_m reproduces the constant and the linear functions, so its degree of exactness is 1 (denoted $\mathrm{dex}(Q_m) = 1$).

(4) In [23], the following result is given:

$$\begin{aligned}(Q_m e_2)(x) =& x(1+m\beta)^{1-m}[S(2, m-2, x+2\beta, 1-x) \\ & - (m-2)\beta S(2, m-3, x+2\beta, 1-x+\beta)],\end{aligned} \tag{2}$$

where $e_i(x) = x^i$, $i \in \mathbb{N}$, and

$$S(j, m, x, y) = \sum_{k=0}^{m} \binom{m}{k} (x+k\beta)^{k+j-1}[y+(m-k)\beta]^{m-k}, \tag{3}$$

$j = 0, \ldots, m$, $m \in \mathbb{N}$, $x, y \in [0,1]$, $\beta > 0$.

Considering the partitions $\Delta_m^x = \left\{ i\frac{h-y}{m} \,\middle|\, i = 0, \ldots, m \right\}$ and $\Delta_n^y = \left\{ j\frac{h-x}{n} \,\middle|\, j = 0, \ldots, n \right\}$ of the intervals $[0, h-y]$ and $[0, h-x]$, the real-valued function F defined on T_h (Figure 1), for $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, we introduce the following extensions to the triangle T_h of the Cheney–Sharma operator given in (1):

$$\begin{aligned}(Q_m^x F)(x,y) &= \sum_{i=0}^{m} q_{m,i}(x,y) F\left(i\tfrac{h-y}{m}, y\right), \\ (Q_n^y F)(x,y) &= \sum_{j=0}^{n} q_{n,j}(x,y) F\left(x, j\tfrac{h-x}{n}\right),\end{aligned} \tag{4}$$

with

$$\begin{aligned}q_{m,i}(x,y) &= \binom{m}{i} \frac{\frac{x}{h-y}(\frac{x}{h-y}+i\beta)^{i-1}(1-\frac{x}{h-y})[1-\frac{x}{h-y}+(m-i)\beta]^{m-i-1}}{(1+m\beta)^{m-1}}, \\ q_{n,j}(x,y) &= \binom{n}{j} \frac{\frac{y}{h-x}(\frac{y}{h-x}+jb)^{j-1}(1-\frac{y}{h-x})[1-\frac{y}{h-x}+(n-j)b]^{n-j-1}}{(1+nb)^{n-1}}.\end{aligned}$$

Remark 2. *As the Cheney–Sharma operator of the second kind interpolates a given function at the endpoints of the interval, we may use the operators Q_m^x and Q_n^y as interpolation operators.*

Theorem 1. *If F is a real-valued function defined on T_h , then*

(i) $Q_m^x F = F$ on $\Gamma_1 \cup \Gamma_3$,
(ii) $Q_n^y F = F$ on $\Gamma_2 \cup \Gamma_3$.

Proof. (i) We may write

$$\begin{aligned}(Q_m^x F)(x,y) = &\frac{1}{(1+m\beta)^{m-1}}\Big\{(1-\tfrac{x}{h-y})[1-\tfrac{x}{h-y}+m\beta]^{m-1}F(0,y) \\ &+\tfrac{x}{h-y}(1-\tfrac{x}{h-y})\sum_{i=1}^{m-1}\tbinom{m}{i}(\tfrac{x}{h-y}+i\beta)^{i-1} \\ &\cdot[1-\tfrac{x}{h-y}+(m-i)\beta]^{m-i-1}F\Big(i\tfrac{h-y}{m},y\Big) \\ &+\tfrac{x}{h-y}(\tfrac{x}{h-y}+m\beta)^{m-1}F(h-y,y)\Big\}.\end{aligned} \tag{5}$$

Considering (5), it follows that

$$\begin{aligned}(Q_m^x F)(0,y) &= F(0,y),\\ (Q_m^x F)(h-y,y) &= F(h-y,y).\end{aligned}$$

(ii) Similarly, writing

$$\begin{aligned}(Q_n^y F)(x,y) = &\frac{1}{(1+nb)^{n-1}}\Big\{(1-\tfrac{y}{h-x})[1-\tfrac{y}{h-x}+nb]^{n-1}F(x,0) \\ &+\tfrac{y}{h-x}(1-\tfrac{y}{h-x})\sum_{j=1}^{n-1}\tbinom{n}{j}(\tfrac{y}{h-x}+jb)^{j-1} \\ &\cdot[1-\tfrac{y}{h-x}+(n-j)b]^{n-j-1}F\Big(x,j\tfrac{h-x}{n}\Big) \\ &+\tfrac{y}{h-x}(\tfrac{y}{h-x}+nb)^{n-1}F(x,h-x)\Big\},\end{aligned}$$

we find that

$$\begin{aligned}(Q_n^y F)(x,0) &= F(x,0),\\ (Q_n^y F)(x,h-x) &= F(x,h-x).\end{aligned}$$

□

Theorem 2. *The operators Q_m^x and Q_n^y have the following orders of accuracy:*

(i) $\Big(Q_m^x e_{kj}\Big)(x,y) = x^k y^j,\ \ k=0,1;\ j\in\mathbb{N};$

(ii) $\Big(Q_n^y e_{kj}\Big)(x,y) = x^k y^j,\ \ k\in\mathbb{N};\ j=0,1,$ where $e_{kj}(x,y)=x^k y^j,\ \ k,j\in\mathbb{N}.$

Proof. (i) We have

$$(Q_m^x e_{kj})(x,y) = y^j\sum_{i=0}^{m} q_{m,i}(x,y)(i\tfrac{h-y}{m})^k,$$

and by Remark 1, the result follows.

Similarly, (ii) follows. □

We consider the approximation formula

$$F = Q_m^x F + R_m^x F,$$

where $R_m^x F$ denotes the approximation error.

Theorem 3. *If $F(\cdot,y)\in C[0,h-y]$, the following holds:*

$$\big|(R_m^x F)(x,y)\big| \le (1+\tfrac{1}{\delta}\sqrt{A_m - x^2})\,\omega(F(\cdot,y);\delta), \quad \forall\delta>0, \tag{6}$$

where $\omega(F(\cdot,y);\delta)$ is the modulus of continuity and $A_m = x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]$, with S given in (3).

Proof. By Theorem 2, it follows that $\text{dex}(Q_m^x) = 1$; thus, we may apply the following property of linear operators (see, for example, [35]):

$$\left|(Q_m^x F)(x,y) - F(x,y)\right| \le [1 + \delta^{-1}\sqrt{(Q_m^x e_{20})(x,y) - x^2}]\omega(F(\cdot,y);\delta), \ \ \forall \delta > 0;$$

thus, taking into account (2), we obtain (6). □

Theorem 4. *If* $F(\cdot,y) \in C^2[0,h-y]$*, then*

$$\begin{aligned}(R_m^x F)(x,y) =& \tfrac{1}{2}F^{(2,0)}(\xi,y)\{x^2 - x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x) \\ &- (m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\},\end{aligned} \tag{7}$$

for $\xi \in [0,h-y]$ *and* $\beta > 0$.

Proof. Taking into account the fact that $\text{dex}(Q_m^x) = 1$, by Theorem 2 and applying Peano's theorem (see, e.g., [36]), it follows that

$$(R_m^x F)(x,y) = \int_0^{h-y} K_{20}(x,y;s)F^{(2,0)}(s,y)ds,$$

where

$$K_{20}(x,y;s) = (x-s)_+ - \sum_{i=0}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right)_+.$$

For a given $\nu \in \{1,\ldots,m\}$, one denotes by $K_{20}^{\nu}(x,y;\cdot)$ the restriction of the kernel $K_{20}(x,y;\cdot)$ to the interval $\left[(\nu-1)\frac{h-y}{m}, \nu\frac{h-y}{m}\right]$, i.e.,

$$K_{20}^{\nu}(x,y;\nu) = (x-s)_+ - \sum_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right),$$

whence,

$$K_{20}^{\nu}(x,y;s) = \begin{cases} x - s - \sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right), & s < x \\ -\sum\limits_{i=\nu}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right), & s \ge x. \end{cases}$$

It follows that $K_{20}^{\nu}(x,y;s) \le 0$, for $s \ge x$.

For $s < x$, we have

$$K_{20}^{\nu}(x,y;s) = x - s - \textstyle\sum_{i=0}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right) + \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right).$$

Applying Theorem 2, we get

$$\sum_{i=0}^{m} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right) = (Q_m^x e_{10})(x,y) - s(Q_m^x e_{00})(x,y) = x - s;$$

it then follows that

$$K_{20}^{\nu}(x,y;s) = \sum_{i=0}^{\nu-1} q_{m,i}(x,y)\left(i\frac{h-y}{m} - s\right) \le 0.$$

Thus, $K_{20}^{\nu}(x,y;\cdot) \le 0$, for any $\nu \in \{1,\ldots,m\}$, i.e., $K_{20}(x,y;s) \le 0$, for $s \in [0,h-y]$.

By the Mean Value Theorem, one obtains

$$(R_m^x F)(x,y) = F^{(2,0)}(\xi,y)\int_0^{h-y} K_{20}(x,y;s)ds, \ \ \text{for } 0 \le \xi \le h-y,$$

with

$$\int_0^{h-y} K_{20}(x,y;s)ds = \tfrac{1}{2}[x^2 - (Q_m^x e_{20})(x,y)],$$

and using (2) we get (7). □

Remark 3. *Analogous results with the ones in Theorems 3 and 4 can be obtained for the remainder $R_n^y F$ of the formula $F = Q_n^y F + R_n^y F$.*

2.1. Product Operators

Let $P_{mn}^1 = Q_m^x Q_n^y$, respectively, $P_{nm}^2 = Q_n^y Q_m^x$ be the products of the operators Q_m^x and Q_n^y, given by

$$\left(P_{mn}^1 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y) q_{n,j}\left(i\tfrac{h-y}{m}, y\right) F\left(i\tfrac{h-y}{m}, j\tfrac{(m-i)h+iy}{mn}\right),$$

respectively,

$$\left(P_{nm}^2 F\right)(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}\left(x, j\tfrac{h-x}{n}\right) q_{n,j}(x,y) F\left(i\tfrac{(n-j)h+jx}{mn}, j\tfrac{h-x}{n}\right).$$

Theorem 5. *If F is a real-valued function defined on T_h, then*

(i) $(P_{mn}^1 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{mn}^1 F)(\Gamma_3) = F(\Gamma_3),$
(ii) $(P_{nm}^2 F)(V_i) = F(V_i), \quad i = 1,2,3;$
$(P_{nm}^2 F)(\Gamma_3) = F(\Gamma_3),$

Proof. By a straightforward computation, we obtain the following properties:

$$\begin{aligned}
&(P_{mn}^1 F)(x,0) = (Q_m^x F)(x,0),\\
&(P_{mn}^1 F)(0,y) = (Q_n^y F)(0,y),\\
&(P_{mn}^1 F)(x,h-x) = F(x,h-x), \quad x,y \in [0,h]
\end{aligned}$$

and

$$\begin{aligned}
&(P_{nm}^2 F)(x,0) = (Q_m^x F)(x,0),\\
&(P_{nm}^2 F)(0,y) = (Q_n^y F)(0,y),\\
&(P_{nm}^2 F)(h-y,y) = F(h-y,y), \quad x,y \in [0,h],
\end{aligned}$$

and, taking into account Theorem 1, these imply (i) and (ii). □

We consider the following approximation formula:

$$F = P_{mn}^1 F + R_{mn}^{P^1} F,$$

where $R_{mn}^{P^1}$ is the corresponding remainder operator.

Theorem 6. *If $F \in C(T_h)$ then*

$$\left|(R_{mn}^{P^1} F)(x,y)\right| \le (A_m + B_n - x^2 - y^2 + 1)\omega\left(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}\right), \ \forall (x,y) \in T_h, \tag{8}$$

where

$$\begin{aligned}A_m =&x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)\\&-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\\B_n =&y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]\end{aligned} \tag{9}$$

and $\omega(F;\delta_1,\delta_2)$, with $\delta_1>0,\delta_2>0$, is the bivariate modulus of continuity.

Proof. Using a basic property of the modulus of continuity, we have

$$\begin{aligned}\left|(R^{P1}_{mn}F)(x,y)\right| \le \Bigg[&\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)\left|x-\tfrac{i}{m}(h-y)\right|\\&+\frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)\left|y-\tfrac{j}{n}\tfrac{(m-i)h+iy}{m}\right|\\&+\sum_{i=0}^{m}\sum_{j=0}^{n} q_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)\Bigg]\omega(F;\delta_1,\delta_2),\ \ \forall\delta_1,\delta_2>0.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)\left|x-\tfrac{i}{m}(h-y)\right| \le \sqrt{(Q^x_m e_{20})(x,y)-x^2},\\&\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)\left|y-\tfrac{j}{n}\tfrac{(m-i)h+iy}{m}\right| \le \sqrt{(Q^y_n e_{02})(x,y)-y^2},\\&\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y)q_{n,j}\left(\tfrac{i}{m}(h-y),y\right)=1,\end{aligned}$$

applying (2), we get

$$\begin{aligned}&\left|(R^{P1}_{mn}F)(x,y)\right| \le \left\{\tfrac{1}{\delta_1}[x(1+m\beta)^{1-m}]^{\frac12}\right.\\&\cdot\left\{[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]-x^2\right\}^{\frac12}\\&+\tfrac{1}{\delta_2}[y(1+nb)^{1-n}]^{\frac12}\\&\cdot\left\{[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]-y^2\right\}^{\frac12}+1\}\omega(F;\delta_1,\delta_2).\end{aligned}$$

Denoting

$$\begin{aligned}A_m &= x(1+m\beta)^{1-m}[S(2,m-2,x+2\beta,1-x)-(m-2)\beta S(2,m-3,x+2\beta,1-x+\beta)]\\B_n &= y(1+nb)^{1-n}[S(2,n-2,y+2b,1-y)-(n-2)bS(2,n-3,y+2b,1-y+\beta)]\end{aligned}$$

and taking $\delta_1=\frac{1}{\sqrt{A_m-x^2}}$ and $\delta_2=\frac{1}{\sqrt{B_n-y^2}}$, we get (8). □

2.2. Boolean Sum Operators

The Boolean sums of the operators Q^x_m and Q^y_n are given by

$$\begin{aligned}S^1_{mn} &:= Q^x_m \oplus Q^y_n = Q^x_m + Q^y_n - Q^x_m Q^y_n,\\S^2_{nm} &:= Q^y_n \oplus Q^x_m = Q^y_n + Q^x_m - Q^y_n Q^x_m.\end{aligned}$$

Theorem 7. *If F is a real-valued function defined on T_h, then*

$$\begin{aligned} S^1_{mn}F|_{\partial T_h} &= F|_{\partial T_h}, \\ S^2_{mn}F|_{\partial T_h} &= F|_{\partial T_h}. \end{aligned}$$

Proof. By

$$\begin{aligned} \left(Q^x_m Q^y_n F\right)(x,0) =& (Q^x_m F)(x,0), \\ \left(Q^y_n Q^x_m F\right)(0,y) =& \left(Q^y_n F\right)(0,y), \\ (Q^x_m F)(x,h-x) =& \left(Q^y_n F\right)(x,h-x) \\ &= (P^1_{mn}F)(x,h-x) = (P^2_{nm}F)(x,h-x) = F(x,h-x), \end{aligned}$$

and, taking into account Theorem 1, the conclusion follows. □

We consider the following approximation formula:

$$F = S^1_{mn}F + R^{S^1}_{mn}F,$$

where $R^{S^1}_{mn}$ is the corresponding remainder operator.

Theorem 8. *If $F \in C(T_h)$, then*

$$\begin{aligned} &\left|(R^{S^1}_{mn}F)(x,y)\right| \leq \\ &\leq (1 + A_m - x^2)\omega(F(\cdot,y); \tfrac{1}{\sqrt{A_m - x^2}}) + (1 + B_n - y^2)\omega(F(x,\cdot); \tfrac{1}{\sqrt{B_n - y^2}}) \\ &+ (A_m + B_n - x^2 - y^2 + 1)\omega(F; \tfrac{1}{\sqrt{A_m - x^2}}, \tfrac{1}{\sqrt{B_n - y^2}}), \end{aligned} \tag{10}$$

with A_m and B_n given in (9).

Proof. The identity

$$F - S^1_{mn}F = (F - Q^x_m F) + (F - Q^y_n F) - (F - P^1_{mn}F)$$

implies that

$$\left|(R^{S^1}_{mn}F)(x,y)\right| \leq \left|(R^x_m F)(x,y)\right| + \left|(R^y_n F)(x,y)\right| + \left|(R^{P^1}_{mn}F)(x,y)\right|,$$

and, applying Theorems 3 and 6, we get (10). □

3. Cheney–Sharma Operator of the First Kind

Let $m \in N$ and β be a nonnegative parameter. In [23], based on the following Jensen's identity,

$$(x + y + m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} x(x + k\beta)^{k-1}[y + (m-k)\beta]^{m-k}, \ \ (\forall)\ (x,y) \in \mathbb{R}^2,$$

the Cheney–Sharma operators of the first kind $G_m : C[0,1] \to C[0,1]$ were introduced, given by

$$(G_m f)(x) = \sum_{i=0}^{m} q_{m,i}(x) f(\tfrac{k}{m}),$$

with

$$q_{m,i}(x) = \binom{m}{i} \frac{x(x+i\beta)^{i-1}[1-x+(m-i)\beta]^{m-i}}{(1+m\beta)^m}.$$

For F, a real-valued function defined on T_h, $m, n \in \mathbb{N}$, $\beta, b \in \mathbb{R}_+$, and the uniform partitions Δ_m^x and Δ_n^y of the intervals $[0, h-y]$ and $[0, h-x]$, we consider here the new extensions of the Cheney–Sharma operator of the first kind,

$$\begin{aligned}
(G_m^x F)(x,y) &= \sum_{i=0}^{m} r_{m,i}(x,y) F\left(i\tfrac{h-y}{m}, y\right), \\
(G_n^y F)(x,y) &= \sum_{j=0}^{n} r_{n,j}(x,y) F\left(x, j\tfrac{h-x}{n}\right),
\end{aligned} \tag{11}$$

with

$$\begin{aligned}
r_{m,i}(x,y) &= \binom{m}{i} \frac{\frac{x}{h-y}\left(\frac{x}{h-y}+i\beta\right)^{i-1}\left[1-\frac{x}{h-y}+(m-i)\beta\right]^{m-i}}{(1+m\beta)^m}, \\
r_{n,j}(x,y) &= \binom{n}{j} \frac{\frac{y}{h-x}\left(\frac{y}{h-x}+jb\right)^{j-1}\left[1-\frac{y}{h-x}+(n-j)b\right]^{n-j}}{(1+nb)^n}.
\end{aligned}$$

We denote by $P_{mn}^G = G_m^x G_n^y$ the product and by $S_{mn}^G := G_m^x \oplus G_n^y = G_m^x + G_n^y - G_m^x G_n^y$, respectively, the Boolean sum of the operators G_m^x and G_n^y.

Remark 4. *The new extensions of the Cheney–Sharma operator of the first kind, G_m^x and G_n^y, and their product and Boolean sum, P_{mn}^G and S_{mn}^G, introduced here, have similar properties as the ones of the Cheney–Sharma operator of the second kind from the previous section.*

4. Numerical Examples

In this section, we consider two test functions for which we plot the graphs of the approximants using the methods presented here, and also we study the maximum approximation errors for the corresponding approximants.

Example 1. *Consider the following test functions, generally used in the literature (see, e.g., [37]):*

$$\begin{aligned}
&\textit{Gentle: } F_1(x,y) = \tfrac{1}{3}\exp\left[-\tfrac{81}{16}\left((x-0.5)^2+(y-0.5)^2\right)\right], \\
&\textit{Saddle: } F_2(x,y) = \frac{1.25+\cos 5.4y}{6+6(3x-1)^2}.
\end{aligned} \tag{12}$$

Considering $h = 1, m = 5$, $n = 6$, $\beta = 1$, in Table 1, one can see the maximum errors for approximating F_i by $Q_m^x F_i$, $Q_n^y F_i$, $P_{mn}^1 F_i$, $S_{mn}^1 F_i$, $G_m^x F_i$, $G_n^y F_i$, $P_{mn}^G F_i$, $S_{mn}^G F_i$, $i = 1, 2$; in Figures 2 and 3, we have plotted the graphs of F_i, $Q_m^x F_i$, $Q_n^y F_i$, $P_{mn}^1 F_i$, $S_{mn}^1 F_i$, $G_m^x F_i$, $S_{mn}^G F_i$, $i = 1, 2$ on T_h.

Table 1. Maximum approximation errors.

Max Error	F_1	F_2
$Q_m^x F_i$	0.0862	0.1922
$Q_n^y F_i$	0.1264	0.1529
$P_{mn}^1 F_i$	0.1680	0.1926
$S_{mn}^1 F_i$	0.0152	0.0235
$G_m^x F_i$	0.1523	0.1695
$G_n^y F_i$	0.1560	0.2364
$P_{mn}^G F_i$	0.2444	0.1697
$S_{mn}^G F_i$	0.0676	0.0750

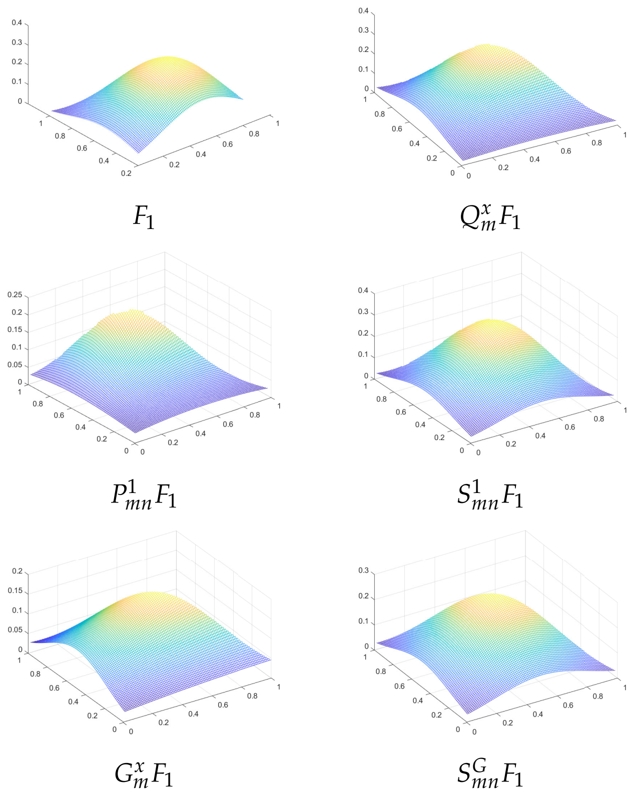

Figure 2. Graphs of F_1 and its interpolants on T_1.

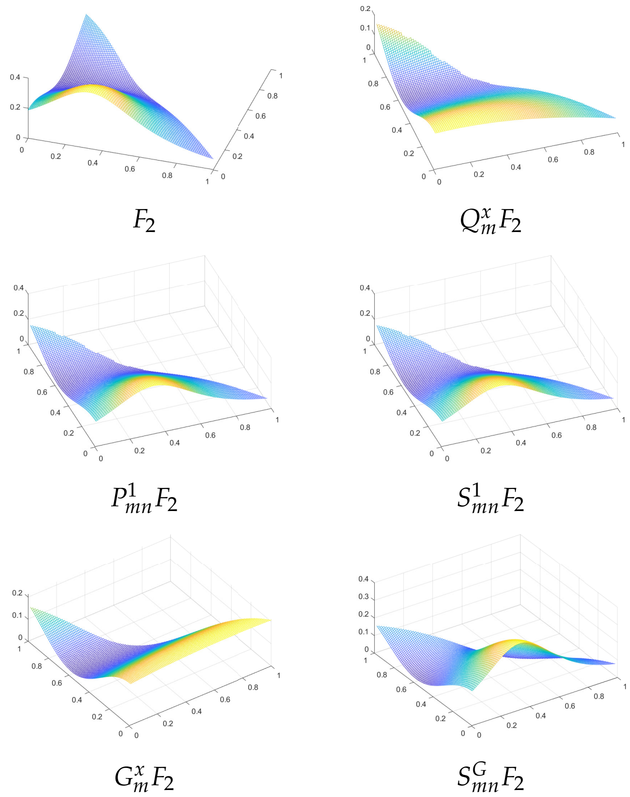

Figure 3. Graphs of F_2 and its interpolants on T_1.

5. Conclusions

According to Table 1 and Figures 2 and 3, we note the good approximation properties of the two types of Cheney–Sharma operators considered here, especially of the Boolean sum operators, which interpolate on the entire frontier of the domain.

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