symmetry

MDPI

Article

An Innovative Hybrid Multi-Criteria Decision-Making Approach under Picture Fuzzy Information

Ghous Ali [1,*], Muhammad Zain Ul Abidin [1], Qin Xin [2] and Ferdous M. O. Tawfiq [3]

1 Department of Mathematics, Division of Science and Technology, University of Education, Lahore 54770, Pakistan
2 Faculty of Science and Technology, University of the Faroe Islands, Vestarabryggja 15, FO 100 Torshavn, Faroe Islands, Denmark
3 Department of Mathematics, College of Science, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia
* Correspondence: ghous.ali@ue.edu.pk

Abstract: These days, multi-criteria decision-making (MCDM) approaches play a vital role in making decisions considering multiple criteria. Among these approaches, the picture fuzzy soft set model is emerging as a powerful mathematical tool for handling various kinds of uncertainties in complex real-life MCDM situations because it is a combination of two efficient mathematical tools, namely, picture fuzzy sets and soft sets. However, the picture fuzzy soft set model is deficient; that is, it fails to tackle information symmetrically in a bipolar soft environment. To overcome this difficulty, in this paper, a model named picture fuzzy bipolar soft sets (P_RFBSSs, for short) is proposed, which is a natural hybridization of two models, namely, picture fuzzy sets and bipolar soft sets. An example discussing the selection of students for a scholarship is added to illustrate the initiated model. Some novel properties of P_RFBSSs such as sub-set, super-set, equality, complement, relative null and absolute P_RFBSSs, extended intersection and union, and restricted intersection and union are investigated. Moreover, two fundamental operations of P_RFBSSs, namely, the AND and OR operations, are studied. Thereafter, some new results (De Morgan's law, commutativity, associativity, and distributivity) related to these proposed notions are investigated and explained through corresponding numerical examples. An algorithm is developed to deal with uncertain information in the P_RFBSS environment. To show the efficacy and applicability of the initiated technique, a descriptive numerical example regarding the selection of the best graphic designer is explored under P_RFBSSs. In the end, concerning both qualitative and quantitative perspectives, a detailed comparative analysis of the initiated model with certain existing models is provided.

Keywords: picture fuzzy soft set; bipolarity; score function; algorithm; multi-criteria decision-making

check for updates

Citation: Ali, G.; Abidin, M.Z.U.; Xin, Q.; Tawfiq, F.M.O. An Innovative Hybrid Multi-Criteria Decision-Making Approach under Picture Fuzzy Information. *Symmetry* **2022**, *14*, 2434. https://doi.org/10.3390/sym14112434

Academic Editors: Jan Awrejcewicz, Palle E.T. Jorgensen and Sergei D. Odintsov

Received: 24 October 2022
Accepted: 12 November 2022
Published: 17 November 2022

1. Introduction

The development and improvement of technology have changed a lot of things in the past few decades. These changes include new trends and adoptions adding to the problems and complications emerging with these new possibilities. Conditions often emerge in which one must make decisions considering many different criteria and dependencies. This is where the multi-criteria decision-making (MCDM) techniques come in handy. The MCDM tools and methods allow the consideration of multiple criteria in one place, thus making the decision-making process smoother and easier.

For dealing with complicated decision problems and undeniable uncertainties in information, decision making is considered an important branch of science that provides tools and techniques to tackle such confusions and complications effectively. Early theories, e.g., the classical set theory, could only handle a limited set of problems and failed to deal with uncertain and incomplete information. This issue was of prime significance for decision makers keeping in view the wide area of problems facing uncertainty and needing a solution. Following the 16th century works on the development of modern probability theory, Zadeh [1] solved this problem in 1965 by introducing his revolutionary fuzzy set theory. Zadeh's fuzzy sets (FSs) considered an extended range of memberships for the consideration of the truthfulness of an event, i.e., he mapped the set of objects to the close interval of $[0, 1]$ rather than the discrete set $\{0, 1\}$ as in classical sets. This interval-valued range allowed for the illustration of partial truthfulness in between the true and false bounds (1 and 0, respectively). This theory found many applications and great interest in the decision sciences. Some recent works include the application of FSs in risk assessment and excavation management by Lin et al. [2]; a discussion on the applications and contributions of FS theory in human reliability by Gholamizadeh et al. [3]; evidential fuzzy multi-criteria decision making (MCDM) based on belief entropy by Xiao [4]; and more.

The FS theory considered only the memberships declaring how much an object satisfies some particular aspect. However, decision makers had to face issues considering problems requiring knowledge of the degree of dissatisfaction degrees as well. To overcome this issue, Atanassov [5] introduced the intuitionistic fuzzy sets (IFSs) (analogous to the type-I fuzzy sets) as a natural extension of FSs. An IFS offers two memberships, i.e., a membership η^+ declaring the degree of satisfaction and a non-membership η^- declaring the degree of dissatisfaction of an object concerning some particular parameter, along with the restriction that bounds the sum of these two degrees by unity. Later extensions of the IFS model include Pythagorean fuzzy sets (analogous to the type-II fuzzy sets) [6] that softened the restrictions of IFSs. These Pythagorean fuzzy sets allowed for decision making in situations where the sum of squares of the two degrees (membership and non-membership) are bounded by unity, i.e., $0 \le (\eta^+)^2 + (\eta^-)^2 \le 1$. Many researchers adopted this idea of non-membership degrees. In 2019, Xiao [7] provided a distance measure for IFSs and applied it to pattern classification problems. Recently, Wang et al. [8] discussed interval-valued intuitionistic fuzzy Jenson–Shannon divergence and its application in multi-attribute decision making. Yu et al. [9] discussed the evolution of IFSs with time through a deep exploration of the literature. Wang et al. [10] presented some uncertainty measures for Pythagorean fuzzy sets and discussed their MCDM applications. The discussed theories could not effectively deal with multi-parameterized data sets. However, while making decisions, one often encounters problems considering the alternatives regarding several criteria or attributes. To deal with this issue, Molodtsov [11] launched the theory of soft sets, which acted as parameterized families of sets. After the initiation of the soft set model, many researchers investigated different operations for soft sets and explored several applications related to soft sets [12]. Early hybridizations of models based on soft sets include the hybridization of soft sets and fuzzy sets known as fuzzy soft sets (FSSs) [13], which was an approach introduced by Maji et al. [14]. The further improvements and extensions of the soft sets contributed significantly to the the development of MCDM (see [15–17]).

Apart from these scenarios, there arise many problems and uncertain situations in daily life requiring the involvement of neutral behavior of alternatives in the data sets. For example, in an election, voters may be divided into three groups, those who vote for, those who vote against, and those who refuse to vote. To solve such problems, Cuong [18–20] introduced the concept of picture fuzzy sets (P_RFSs). These P_RFSs can deal with membership, non-membership, and neutral degrees of given alternatives. After this powerful invention of P_RFS theory, several researchers were attracted to the notion of P_RFS and proposed various new operations and hybrid soft set models under the picture fuzzy environment. For instance, Ganie et al. [21] developed certain correlation coefficients of P_RFSs to verify how much P_RFSs correlate with each other. The prominent applications of P_RFSs include decision making, pattern recognition, clustering analysis, medical diagnosis, etc. Further extending P_RFS theory, Tchier et al. [22] introduced the notion of picture fuzzy soft expert sets (P_RFSESs) and discussed their effective applications towards group decision making. The literature of P_RFS theory is very rich in solving complex MCDM problems. For instance, Saraji and Streimikiene [23] evaluated the adoption of circular supply chains in manufacturing sectors under the picture fuzzy approach. Rong et al. [24]

presented a novel MCDM P_RFS-based method for estimating emergency management schemes. Simic et al. [25] proposed a novel picture fuzzy extension of the CODAS method for the multiple criteria of a vehicle shredding facility location (see also Wang et al. [26]). Some more works related to P_RFSs include the fuzzy logic operators for P_RFSs [27], picture fuzzy soft sets (P_RFSSs) [28], correlation coefficients for P_RFSs [29,30], similarity measures for P_RFSs and their applications towards MCDM [31,32], distance measures for P_RFSs [33], bipolar P_RFSs [34], 2-tuple linguistic complex q-rung picture fuzzy sets [35], and picture fuzzy Aczel–Alsina average aggregation operators [36], etc. Some recently explored applications of P_RFSs include online-review-based sentiment analysis for the evaluation of new electric vehicles under P_RFSs by He and Wang [37]; the evaluation of citizens' satisfaction level regarding municipality services using the P_RF VIKOR method by Yildirim and Yildirim [38]; and the evaluation of pedagogic systems via P_RFS-based group decision making by Van Pham et al. [39]. Figure 1 elaborates on the differences between IFSs and P_RFSs pictorially, and Table 1 discusses the recent MCDM contributions under picture fuzzy information.

Table 1. Summary some recent contrubution to picture fuzzy MCDM.

Reference	Decision Model	Contribution
Simic et al. [25]	P_RFS-based CODAS method	MCDM CODAS method extended for picture fuzzy information and applied to a vehicle shredding facility location.
Sindhu et al. [34]	Bipolar P_RFS-based operators	Aggregation operator, TOPSIS, and VIKOR MCDM approaches based on bipolar picture fuzzy sets applied to an MCDM investment problem.
Senapati [36]	P_RFS-based Aczel–Alsina operators	MADM method based on picture fuzzy Aczel–Alsina aggregation operators applied to an investment problem.
Yildirim and Yildirim [38]	P_RFS-based VIKOR method	Picture fuzzy VIKOR method applied to evaluate the satisfaction level of citizens with municipality services.
Akram et al. [40]	q-Rung P_RFS-based Einstein operators	Einstein operators based aggregation operators applied to MADM problems for the selection of business sites and suppliers under q-rung picture fuzzy sets.
Akram [41]	q-Rung P_RFS-based VIKOR and TOPSIS methods	q-Rung picture fuzzy VIKOR and TOPSIS methods applied to the selection of housing society and industrial robots.
Haktanir and Kahraman [42]	P_RFS-based CRITIC and REGIME MCDM methods	Picture fuzzy CRITIC and REGIME MCDM methods applied to selection of wearable health technology.
Shit et al. [43]	Trapezoidal P_RFS-based harmonic operators	MADM technique based on trapezoidal picture fuzzy harmonic aggregation operators applied to site selection for a telecom tower.
Karamti et al. [44]	P_RFS-based divergence measure	Picture fuzzy divergence measure-based similarity MCDM applications to dengue sickness and pattern identification.
Rehman and Mahmood [45]	Picture fuzzy N-soft set	Picture fuzzy N-soft set-based MCDM applications for the selection of coronavirus vaccine and next-generation firewall.
Li et al. [46]	q-Rung picture linguistic set	MAGDM technique based on q-rung picture Heronian mean operators applied to choosing an enterprise resource planning system.
Mahmood et al. [47]	Complex picture fuzzy N-soft set	Complex picture fuzzy N-soft sets-based MADM algorithm applied to the performance assessment of e-waste recycling program and winner prediction for FIFA world cup 2022.

For other important results related to P_RFSs, the readers are referred to [48–50].

In addition, many real-life problems involve positive parameters (favoring parameters) and symmetrically negative parameters (opposing parameters). Because of this, Shabir and Naz [51] developed the idea of bipolar soft sets (B_RSSs). This structure considers two sets of parameters; one set contains the favorable parameters, while the other set contains the parameters opposite to those in the favorable set. In this way, B_RSSs effectively depict the bipolarity of parameters in complicated decision-making problems. In addition, Naz and Shabir [52] combined FSs with B_RSSs to formulate a hybridized model called fuzzy B_RSSs. Currently, the theory of B_RSSs is playing a vital role in different domains to solve several MCDM problems in the form of various hybridized uncertainty theories (see [53–57]).

Considering the research gaps in the literature (i.e., limitations of models such as P_RFSSs failing to consider the bipolarity of decision attributes and inefficacy of B_RSSs in considering uncertainties in positive, negative and neutral degrees), this work introduces a model capable of combining the above-discussed qualities in one method.

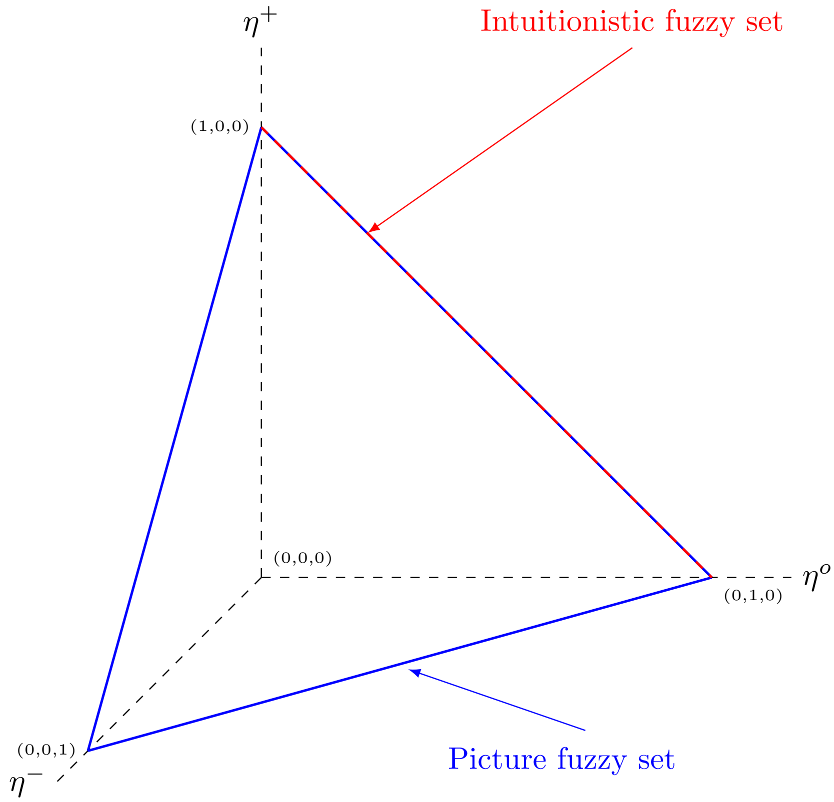

Figure 1. Comparison among the IFS [5] and PFS [20] models.

The following are the motivations of our proposed study:

1. Existing decision-making structures, including P_RFSSs, are unable to deal with the bipolarity of decision attributes efficiently.
2. Models such as B_RSSs and fuzzy B_RSSs which are capable of depicting the bipolarity of attributes fail to handle uncertainties effectively since they fail to consider positive, negative, and neutral degrees of opinion.
3. Problems such as selection of a fashion designer specific to the company's needs are complicated MCDM problems requiring a strong decision-making algorithm.

The contributions of this paper are provided below:

1. A model called picture fuzzy bipolar soft set (or P_RFBSS), a natural hybrid extension of P_RFS and B_RSS, is proposed.
2. Some novel properties and two fundamental operations of P_RFBSSs are presented and illustrated via corresponding numerical examples.
3. Important results including the commutative, associative, and distributive properties are presented. Furthermore, De Morgan's Laws for the proposed properties and operations are shown.
4. A P_RFBSS-based algorithm using score functions for picture fuzzy numbers is presented to deal with MCDM problems considering the decision attributes in symmetry.

5. An MCDM application of P_RFBSSs, i.e., the selection of a fashion designer for a studio, is presented and solved using the newly proposed algorithm based on P_RFBSSs.
6. Finally, a detailed comparative analysis concerning both qualitative and quantitative perspectives of the proposed model with certain existing models is provided.

Organization of Paper

The remaining formulation of the paper is given as follows: In Section 2, firstly, some basic notions, including B_RSSs and P_RFSs, and some of their fundamental results are recalled. In Section 3, the main notion of our study called picture fuzzy bipolar soft sets (or P_RFBSSs) is proposed. Some novel properties and fundamental operations of P_RFBSSs are presented and explained with numerical examples. In Section 4, an MCDM application related to the initiated model is explored, supported by a novel algorithm. In Section 5, a detailed comparative analysis of the proposed model with certain existing models is studied. Finally, in Section 6, the concluding remarks are discussed along with future directions.

2. Preliminaries

This section recalls some fundamental notions, including B_RSSs and P_RFSs, along with their score and accuracy functions, which will support the developments in coming sections. The following is the definition of B_RSSs.

Definition 1 ([51]). *For a given universal set $\mathscr{P}$ and a universe of attributes $\mathscr{R}$, a triple $(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ is called a bipolar soft set, or B_RSS, over $\mathscr{P}$, where $\mathscr{Q} \subseteq \mathscr{R}$ if the functions $\mathscr{G}$ and $\mathscr{H}$ are provided by $\mathscr{G} : \mathscr{Q} \to \mathscr{F}(\mathscr{P})$ and $\mathscr{H} : \neg\mathscr{Q} \to \mathscr{F}(\mathscr{P})$, such that*

$$\mathscr{G}(\mathfrak{p}) \cap \mathscr{H}(\neg\mathfrak{p}) = \emptyset, \quad \forall\, \mathfrak{p} \in \mathscr{Q}, \neg\mathfrak{p} \in \neg\mathscr{Q}.$$

Here $\mathscr{F}(\mathscr{P})$ represents the power set of the universal set $\mathscr{P}$, and $\neg\mathscr{Q}$ serves as the collection of opposite attributes as compared to the attributes in $\mathscr{Q}$.

Definition 2 ([20]). *For a given universal set $\mathscr{P}$ and a universe of attributes $\mathscr{R}$, a picture fuzzy set (or P_RFS) $\mathscr{G}$ over the universal set $\mathscr{P}$ is provided as below:*

$$\mathscr{G} = \{\langle \mathfrak{p}, \eta^{+}(\mathfrak{p}), \eta^{o}(\mathfrak{p}), \eta^{-}(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P}\},$$

which satisfies the following condition:

$$0 \le \eta^{+}(\mathfrak{p}) + \eta^{o}(\mathfrak{p}) + \eta^{-}(\mathfrak{p}) \le 1, \tag{1}$$

where $\eta^{+}(\mathfrak{p})$, $\eta^{o}(\mathfrak{p})$ and $\eta^{-}(\mathfrak{p})$ are the degrees of positive, neutral and negative memberships, respectively. For simplicity, we call the triple $\mathscr{F} = (\eta^{+}_{\mathscr{F}}, \eta^{o}_{\mathscr{F}}, \eta^{-}_{\mathscr{F}})$ a picture fuzzy number (PFN).

From now on, we represent the collection of all P_RFSs over $\mathscr{P}$ as $\mathscr{F}^{\mathscr{P}}$.

Definition 3 ([48]). *For a PFN $\mathscr{F} = (\eta^{+}_{\mathscr{F}}, \eta^{o}_{\mathscr{F}}, \eta^{-}_{\mathscr{F}})$, we define its score function $s(\mathscr{F})$ as below:*

$$s(\mathscr{F}) = \frac{1}{2}(1 + (2\eta^{+}_{\mathscr{F}}) - (\frac{\eta^{o}_{\mathscr{F}}}{2}) - (\eta^{-}_{\mathscr{F}})). \tag{2}$$

Similarly, for a PFN $\mathscr{F} = (\eta^{+}_{\mathscr{F}}, \eta^{o}_{\mathscr{F}}, \eta^{-}_{\mathscr{F}})$, we define its accuracy function of $\mathcal{A}(\mathscr{F})$ as follows:

$$\mathcal{A}(\mathscr{F}) = (\eta^{+}_{\mathscr{F}}) + (\eta^{o}_{\mathscr{F}}) + (\eta^{-}_{\mathscr{F}}). \tag{3}$$

Definition 4 ([48]). *Let $s(\mathscr{F}_1)$, $s(\mathscr{F}_2)$ and $\mathcal{A}(\mathscr{F}_1)$, $\mathcal{A}(\mathscr{F}_2)$ be the score and accuracy functions of two PFNs $\mathscr{F}_1 = (\eta^{+}_{\mathscr{F}_1}, \eta^{o}_{\mathscr{F}_1}, \eta^{-}_{\mathscr{F}_1})$ and $\mathscr{F}_2 = (\eta^{+}_{\mathscr{F}_2}, \eta^{o}_{\mathscr{F}_2}, \eta^{-}_{\mathscr{F}_2})$, respectively. Then, $\mathscr{F}_1 < \mathscr{F}_2$ if and only if:*

(i) $s(\mathscr{F}_1) < s(\mathscr{F}_2)$; or
(ii) $s(\mathscr{F}_1) = s(\mathscr{F}_2)$ and $\mathcal{A}(\mathscr{F}_1) < \mathcal{A}(\mathscr{F}_2)$.

3. Picture Fuzzy Bipolar Soft Sets

In this section, we first present a new hybrid MCDM model, namely, picture fuzzy bipolar soft sets (or P_RFBSS). Then, we discuss some basic properties and operations along with examples. We start with the definition of P_RFBSS as below:

Definition 5. *For a given universal set $\mathscr{P}$ and a universe of attributes $\mathscr{R}$, a picture fuzzy bipolar soft set or P_RFBSS over $\mathscr{P}$, denoted by $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ where $\mathscr{Q} \subseteq \mathscr{R}$ and $\neg\mathscr{Q} \subseteq \neg\mathscr{R}$, is a mixture of two mappings $\mathscr{G} : \mathscr{Q} \to \mathscr{F}^{\mathscr{P}}$ and $\mathscr{H} : \neg\mathscr{Q} \to \mathscr{F}^{\mathscr{P}}$, which are respectively defined as:*

$$\mathscr{G}(\mathfrak{q}) = \{\langle \mathfrak{p}, \eta^{+}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}), \eta^{\circ}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}), \eta^{-}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P}\},$$
$$\mathscr{H}(\neg\mathfrak{q}) = \{\langle \mathfrak{p}, \zeta^{+}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}), \zeta^{\circ}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}), \zeta^{-}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P}\},$$

for all $\mathfrak{q} \in \mathscr{Q}$ and $\neg\mathfrak{q} \in \neg\mathscr{Q}$ with the following conditions:

$$0 \le \eta^{+}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) + \eta^{\circ}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) + \eta^{-}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) \le 1, \tag{4}$$
$$0 \le \zeta^{+}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) + \zeta^{\circ}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) + \zeta^{-}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) \le 1, \tag{5}$$

where

$$0 \le \eta^{+}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) + \zeta^{+}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) \le 1, \tag{6}$$
$$0 \le \eta^{\circ}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) + \zeta^{\circ}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) \le 1, \tag{7}$$
$$0 \le \eta^{-}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}) + \zeta^{-}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}) \le 1, \tag{8}$$

and the degrees of positive memberships are $\eta^{+}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p})$ and $\zeta^{+}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p})$, the degrees of neutral memberships are $\eta^{o}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p})$ and $\zeta^{o}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p})$ and those of negative memberships are $\eta^{-}_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p})$ and $\zeta^{-}_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p})$, respectively.

Notice that the set $\neg\mathscr{Q}$ contains opposite parameters as compared to the parameters in $\mathscr{Q}$.

The following example explains the Definition 5.

Example 1. *Suppose a university ABC has some merit scholarships and to choose the most appropriate student from the five shortlisted candidates, a committee of senior professors is designed to perform this task. Let $\mathscr{R} = \{\mathfrak{q}_1 = \text{intelligent}, \mathfrak{q}_2 = \text{hard working}, \mathfrak{q}_3 = \text{regular}, \mathfrak{q}_4 = \text{cooperative}, \mathfrak{q}_5 = \text{well} - \text{mannered}\}$ be the collection of favorable attributes and let the respective not-set be $\neg\mathscr{R} = \{\neg\mathfrak{q}_1 = \text{dull}, \neg\mathfrak{q}_2 = \text{lazy}, \neg\mathfrak{q}_3 = \text{irregular}, \neg\mathfrak{q}_4 = \text{non} - \text{cooperative}, \neg\mathfrak{q}_5 = \text{bad} - \text{mannered}\}$. After a detailed discussion between committee members, they decide to take the favorable set of decision attributes $\mathscr{Q} = \{\mathfrak{q}_1, \mathfrak{q}_3, \mathfrak{q}_5\} \subset \mathscr{R}$ and its corresponding 'not-set' in the evaluation process. Consider the set $\mathscr{P}=\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_5\}$ represents five students bearing different qualities. The evaluation reports for these students are provided by the committee in the form of a P_RFBSS $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ as follows, describing the qualities and weaknesses of students:*

$$\begin{aligned}
\mathscr{G}(\mathfrak{q}_1) = \big\{&(\mathfrak{p}_1, 0.13, 0.03, 0.43), (\mathfrak{p}_2, 0.23, 0.33, 0.19), (\mathfrak{p}_3, 0.53, 0.13, 0.23),\\
&(\mathfrak{p}_4, 0.63, 0.06, 0.13), (\mathfrak{p}_5, 0.73, 0.13, 0.12)\big\},\\
\mathscr{G}(\mathfrak{q}_3) = \big\{&(\mathfrak{p}_1, 0.44, 0.13, 0.02), (\mathfrak{p}_2, 0.45, 0.13, 0.32), (\mathfrak{p}_3, 0.13, 0.12, 0.11),\\
&(\mathfrak{p}_4, 0.46, 0.26, 0.13), (\mathfrak{p}_5, 0.11, 0.16, 0.39)\big\},\\
\mathscr{G}(\mathfrak{q}_5) = \big\{&(\mathfrak{p}_1, 0.68, 0.21, 0.01), (\mathfrak{p}_2, 0.11, 0.56, 0.23), (\mathfrak{p}_3, 0.44, 0.23, 0.22),\\
&(\mathfrak{p}_4, 0.56, 0.12, 0.26), (\mathfrak{p}_5, 0.78, 0.01, 0.12)\big\},
\end{aligned}$$

$$\begin{aligned}\mathscr{H}(\neg\mathfrak{q}_1) = \{&(\mathfrak{p}_1, 0.26, 0.36, 0.23), (\mathfrak{p}_2, 0.02, 0.43, 0.12), (\mathfrak{p}_3, 0.33, 0.23, 0.43),\\ &(\mathfrak{p}_4, 0.13, 0.02, 0.63), (\mathfrak{p}_5, 0.13, 0.53, 0.16)\},\\ \mathscr{H}(\neg\mathfrak{q}_3) = \{&(\mathfrak{p}_1, 0.23, 0.33, 0.01), (\mathfrak{p}_2, 0.25, 0.12, 0.57), (\mathfrak{p}_3, 0.43, 0.25, 0.30),\\ &(\mathfrak{p}_4, 0.19, 0.46, 0.25), (\mathfrak{p}_5, 0.12, 0.36, 0.19)\},\\ \mathscr{H}(\neg\mathfrak{q}_5) = \{&(\mathfrak{p}_1, 0.06, 0.18, 0.48), (\mathfrak{p}_2, 0.69, 0.13, 0.03), (\mathfrak{p}_3, 0.24, 0.06, 0.12),\\ &(\mathfrak{p}_4, 0.27, 0.22, 0.36), (\mathfrak{p}_5, 0.08, 0.41, 0.39)\}.\end{aligned}$$

For $\mathfrak{p}_i \in \mathscr{P}$, $\mathfrak{q}_j \in \mathscr{Q}$ and $\neg\mathfrak{q}_j \in \neg\mathscr{Q}$, Table 2 represents the above P_RFBSS $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ with entries

$$a_{ij} = \langle \mathscr{G}(\mathfrak{p}_i)(\mathfrak{q}_j), \mathscr{H}(\mathfrak{p}_i)(\neg\mathfrak{q}_j)\rangle.$$

Table 2. Tabular form of the P_RFBSS $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$.

Ω	$\mathfrak{q}_1, \neg\mathfrak{q}_1$	$\mathfrak{q}_3, \neg\mathfrak{q}_3$	$\mathfrak{q}_5, \neg\mathfrak{q}_5$
$\mathfrak{p}_1$	$\langle(0.13, 0.03, 0.43), (0.26, 0.36, 0.23)\rangle$	$\langle(0.44, 0.13, 0.02), (0.23, 0.33, 0.11)\rangle$	$\langle(0.68, 0.21, 0.01), (0.06, 0.18, 0.48)\rangle$
$\mathfrak{p}_2$	$\langle(0.23, 0.33, 0.19), (0.02, 0.43, 0.12)\rangle$	$\langle(0.45, 0.13, 0.32), (0.25, 0.12, 0.57)\rangle$	$\langle(0.11, 0.56, 0.23), (0.69, 0.13, 0.03)\rangle$
$\mathfrak{p}_3$	$\langle(0.53, 0.13, 0.23), (0.33, 0.23, 0.43)\rangle$	$\langle(0.13, 0.12, 0.11), (0.43, 0.25, 0.30)\rangle$	$\langle(0.44, 0.23, 0.22), (0.24, 0.06, 0.12)\rangle$
$\mathfrak{p}_4$	$\langle(0.63, 0.06, 0.13), (0.13, 0.02, 0.63)\rangle$	$\langle(0.46, 0.26, 0.13), (0.19, 0.46, 0.25)\rangle$	$\langle(0.56, 0.12, 0.26), (0.27, 0.22, 0.36)\rangle$
$\mathfrak{p}_5$	$\langle(0.73, 0.13, 0.12), (0.13, 0.53, 0.16)\rangle$	$\langle(0.11, 0.16, 0.39), (0.12, 0.36, 0.19)\rangle$	$\langle(0.78, 0.01, 0.12), (0.08, 0.41, 0.39)\rangle$

According to the committee's report, the candidate $\mathfrak{p}_5$ scores as the most intelligent $(\mathfrak{q}_1)$ relatively; however, the candidate $\mathfrak{p}_2$ is the least dull $(\neg\mathfrak{q}_1)$ among all. This represents the condition that a nearly below-average student is not necessarily a dull student. Similarly, $\mathfrak{p}_5$ obtains the highest agreement score for being a well-mannered student $(\mathfrak{q}_5)$; however, the relatively bigger neutral membership and smaller negative membership degree for $\mathfrak{p}_1$ (considering the same attribute) make the later candidate able to stand in a fair competition with the former one.

From Definition 5, the P_RFBSS $(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ is a union of two P_RFSSs $(\mathscr{G}, \mathscr{Q})$ and $(\mathscr{H}, \neg\mathscr{Q})$, shown in Tables 3 and 4 with entries $\mathscr{G}(\mathfrak{p}_i)(\mathfrak{q}_j)$ and $\mathscr{H}(\mathfrak{p}_i)(\neg\mathfrak{q}_j)$, respectively.

Table 3. P_RFSS $(\mathscr{G}, \mathscr{Q})$.

$(\mathscr{G}, \mathscr{Q})$	$\mathfrak{q}_1$	$\mathfrak{q}_3$	$\mathfrak{q}_5$
$\mathfrak{p}_1$	(0.13, 0.03, 0.43)	(0.44, 0.13, 0.02)	(0.68, 0.21, 0.01)
$\mathfrak{p}_2$	(0.23, 0.33, 0.19)	(0.45, 0.13, 0.32)	(0.11, 0.56, 0.23)
$\mathfrak{p}_3$	(0.53, 0.13, 0.23)	(0.13, 0.12, 0.11)	(0.44, 0.23, 0.22)
$\mathfrak{p}_4$	(0.63, 0.06, 0.13)	(0.46, 0.26, 0.13)	(0.56, 0.12, 0.26)
$\mathfrak{p}_5$	(0.73, 0.13, 0.12)	(0.11, 0.16, 0.39)	(0.78, 0.01, 0.12)

Table 4. P_RFSS $(\mathscr{H}, \neg\mathscr{Q})$.

$(\mathscr{H}, \neg\mathscr{Q})$	$\neg\mathfrak{q}_1$	$\neg\mathfrak{q}_3$	$\neg\mathfrak{q}_5$
$\mathfrak{p}_1$	(0.26, 0.36, 0.23)	(0.23, 0.33, 0.11)	(0.06, 0.18, 0.48)
$\mathfrak{p}_2$	(0.02, 0.43, 0.12)	(0.25, 0.12, 0.57)	(0.69, 0.13, 0.03)
$\mathfrak{p}_3$	(0.33, 0.23, 0.43)	(0.43, 0.25, 0.30)	(0.24, 0.06, 0.12)
$\mathfrak{p}_4$	(0.13, 0.02, 0.63)	(0.19, 0.46, 0.25)	(0.27, 0.22, 0.36)
$\mathfrak{p}_5$	(0.13, 0.53, 0.16)	(0.12, 0.36, 0.19)	(0.08, 0.41, 0.39)

The following discusses subset relations among P_RFBSSs and illustrates this concept via a numerical example.

Definition 6. *For a universal set $\mathscr{P}$ and two P_RFBSSs, $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ and $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ over $\mathscr{P}$, the set Ω_2 is called a picture fuzzy bipolar soft subset of Ω_1, denoted as $\Omega_2 \mathrel{\widetilde{\subseteq}} \Omega_1$, if:*

1. $\mathscr{Q}_2 \subseteq \mathscr{Q}_1$;

2. $\mathscr{G}_2(\mathfrak{q}) \subseteq \mathscr{G}_1(\mathfrak{q})$, *that is,* $\eta^+_{\mathscr{G}_2}(\mathfrak{q})(\mathfrak{p}) \leq \eta^+_{\mathscr{G}_1}(\mathfrak{q})(\mathfrak{p})$, $\eta^o_{\mathscr{G}_2}(\mathfrak{q})(\mathfrak{p}) \leq \eta^o_{\mathscr{G}_1}(\mathfrak{q})(\mathfrak{p})$ *and* $\eta^-_{\mathscr{G}_2}(\mathfrak{q})(\mathfrak{p}) \geq \eta^-_{\mathscr{G}_1}(\mathfrak{q})(\mathfrak{p})$, $\forall\ \mathfrak{q} \in \mathscr{Q}$;
3. $\mathscr{H}_1(\neg\mathfrak{q}) \subseteq \mathscr{H}_2(\neg\mathfrak{q})$, *that is,* $\zeta^+_{\mathscr{H}_1}(\neg\mathfrak{q})(\mathfrak{p}) \leq \zeta^+_{\mathscr{H}_2}(\neg\mathfrak{q})(\mathfrak{p})$, $\zeta^o_{\mathscr{H}_1}(\neg\mathfrak{q})(\mathfrak{p}) \leq \zeta^o_{\mathscr{H}_2}(\neg\mathfrak{q})(\mathfrak{p})$ *and* $\zeta^-_{\mathscr{H}_1}(\neg\mathfrak{q})(\mathfrak{p}) \geq \zeta^-_{\mathscr{H}_2}(\neg\mathfrak{q})(\mathfrak{p})$, $\forall\ \neg\mathfrak{q} \in \neg\mathscr{Q}$ *and* $\mathfrak{p} \in \mathscr{P}$.

Example 2. *Consider the* $\mathrm{P_R}$*FBSS* $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ *as taken in Example 1. Suppose, in the next academic session, another set of five new candidates needs to be evaluated for the merit scholarship award. This time, the committee decides to drop the manners criterion from the decision attributes. Hence, for* $\mathscr{Q}_1 = \{\mathfrak{q}_1 = \text{intelligent},\ \mathfrak{q}_3 = \text{regular}\} \subseteq \mathscr{R}$ *and* $\mathscr{Q}_1 = \{\neg\mathfrak{q}_1 = \text{dull},\ \neg\mathfrak{q}_3 = \text{not regular}\} \subseteq \neg\mathscr{R}$, *a new* $\mathrm{P_R}$*FBSS* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *is provided in Table 5.*

Table 5. Tabular form of the $\mathrm{P_R}$FBSS Ω_1.

Ω_1	$\mathfrak{q}_1, \neg\mathfrak{q}_1$	$\mathfrak{q}_3, \neg\mathfrak{q}_3$
$\mathfrak{p}_1$	$\langle(0.09, 0.01, 0.45), (0.27, 0.41, 0.11)\rangle$	$\langle(0.21, 0.12, 0.04), (0.34, 0.36, 0.09)\rangle$
$\mathfrak{p}_2$	$\langle(0.11, 0.09, 0.22), (0.07, 0.44, 0.02)\rangle$	$\langle(0.12, 0.11, 0.33), (0.36, 0.14, 0.20)\rangle$
$\mathfrak{p}_3$	$\langle(0.23, 0.02, 0.24), (0.35, 0.33, 0.12)\rangle$	$\langle(0.09, 0.02, 0.13), (0.44, 0.26, 0.29)\rangle$
$\mathfrak{p}_4$	$\langle(0.23, 0.03, 0.17), (0.15, 0.07, 0.02)\rangle$	$\langle(0.32, 0.23, 0.15), (0.32, 0.47, 0.02)\rangle$
$\mathfrak{p}_5$	$\langle(0.19, 0.11, 0.26), (0.14, 0.55, 0.06)\rangle$	$\langle(0.09, 0.14, 0.41), (0.22, 0.37, 0.13)\rangle$

It is clear from Definition 6 that $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg, \mathscr{Q}_1) \tilde{\subset} (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$. *This subset relation indicates that the newg students are not as intelligent and regular as their senior fellows.*

The concept of the equality relation between $\mathrm{P_R}$FBSSs is studied in the following definition:

Definition 7. *Two* $\mathrm{P_R}$*FBSSs* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *and* $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *over a universal set* $\mathscr{P}$ *are said to be equal if* $\Omega_1 \tilde{\subset} \Omega_2$ *and* $\Omega_2 \tilde{\subset} \Omega_1$. *The equality relation between them is denoted by* $\Omega_1 \triangleq \Omega_2$.

The following definition discusses the notion of a complement to any $\mathrm{P_R}$FBSS and elaborates it with an example.

Definition 8. *For an arbitrary* $\mathrm{P_R}$*FBSS* $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ *over the universal set* $\mathscr{P}$, *its complement* $\Omega^c = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q}) = (\mathscr{G}^c, \mathscr{H}^c, \mathscr{Q}, \neg\mathscr{Q})$ *is again a* $\mathrm{P_R}$*FBSS over* $\mathscr{P}$, *where:*

$$\mathscr{G}^c(\mathfrak{q}) = \{\langle \mathfrak{p}, \eta^-_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}), \eta^\circ_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p}), \eta^+_{\mathscr{G}}(\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P}\},$$
$$\mathscr{H}^c(\neg\mathfrak{q}) = \{\langle \mathfrak{p}, \zeta^-_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}), \zeta^\circ_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p}), \zeta^+_{\mathscr{H}}(\neg\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P}\},$$

for all $\mathfrak{q} \in \mathscr{Q}$ *and* $\neg\mathfrak{q} \in \neg\mathscr{Q}$.

Example 3. *Consider again the* $\mathrm{P_R}$*FBSS* $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$, *as taken in Example 1. Then, using Definition 8, its complement* $\Omega^c = (\mathscr{G}^c, Q^c, \mathscr{Q}, \neg\mathscr{Q})$ *is calculated and displayed in Table 6.*

Table 6. The complement of $\mathrm{P_R}$FBSS $(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$.

Ω^c	$\mathfrak{q}_1, \neg\mathfrak{q}_1$	$\mathfrak{q}_3, \neg\mathfrak{q}_3$	$\mathfrak{q}_5, \neg\mathfrak{q}_5$
$\mathfrak{p}_1$	$\langle(0.43, 0.03, 0.13), (0.23, 0.36, 0.26)\rangle$	$\langle(0.02, 0.13, 0.44), (0.01, 0.33, 0.23)\rangle$	$\langle(0.01, 0.21, 0.68), (0.48, 0.18, 0.06)\rangle$
$\mathfrak{p}_2$	$\langle(0.19, 0.33, 0.23), (0.12, 0.43, 0.02)\rangle$	$\langle(0.32, 0.13, 0.45), (0.57, 0.12, 0.25)\rangle$	$\langle(0.23, 0.56, 0.11), (0.03, 0.13, 0.69)\rangle$
$\mathfrak{p}_3$	$\langle(0.23, 0.13, 0.53), (0.43, 0.23, 0.33)\rangle$	$\langle(0.11, 0.12, 0.13), (0.30, 0.25, 0.43)\rangle$	$\langle(0.22, 0.23, 0.44), (0.12, 0.06, 0.24)\rangle$
$\mathfrak{p}_4$	$\langle(0.13, 0.06, 0.63), (0.63, 0.02, 0.13)\rangle$	$\langle(0.13, 0.26, 0.46), (0.25, 0.46, 0.19)\rangle$	$\langle(0.26, 0.12, 0.56), (0.36, 0.22, 0.27)\rangle$
$\mathfrak{p}_5$	$\langle(0.12, 0.13, 0.73), (0.16, 0.53, 0.13)\rangle$	$\langle(0.39, 0.16, 0.11), (0.19, 0.36, 0.12)\rangle$	$\langle(0.12, 0.01, 0.78), (0.39, 0.41, 0.08)\rangle$

Two utmost cases of $\mathrm{P_R}$FBSSs are investigated in the following definitions.

Definition 9. *A P_RFBSS on a universal set $\mathscr{P}$ is called a relative null P_RFBSS, symbolized by $(\Phi, \mathfrak{V}, \mathscr{Q}, \neg\mathscr{Q})$, if $\Phi(\mathfrak{q})(\mathfrak{p}) = (0,0,1)$ and $\mathfrak{V}(\neg\mathfrak{q})(\mathfrak{p}) = (1,0,0)$ $\forall$ $\mathfrak{q} \in \mathscr{Q}$, $\neg\mathfrak{q} \in \neg\mathscr{Q}$, $\mathfrak{p} \in \mathscr{P}$.*

Similarly, a P_RFBSS on a universal set $\mathscr{P}$ is referred to as a relative absolute P_RFBSS, represented by $(\mathfrak{V}, \Phi, \mathscr{Q}, \neg\mathscr{Q})$, if $\mathfrak{V}(l)(\mathfrak{p}) = (1,0,0)$ and $\Phi(\neg l)(\mathfrak{p}) = (0,0,1)$ $\forall$ $q \in \mathscr{Q}$, $\neg\mathfrak{q} \in \neg\mathscr{Q}$, $\mathfrak{p} \in \mathscr{P}$.

The following two definitions provide the concepts of fundamental operations (AND and OR) for P_RFBSSs, which are further explained with the support of a numerical example:

Definition 10. *For any two P_RFBSSs $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ and $(\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ over a universal set $\mathscr{P}$, the AND operation between them, denoted by $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \bar{\wedge} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$, is defined as $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \bar{\wedge} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2) = (\Lambda, \Gamma, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2)$, where, for all $(\mathfrak{q}_i, \mathfrak{q}_j) \in \mathscr{Q}_1 \times \mathscr{Q}_2$, $(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2$ and $\mathfrak{p} \in \mathscr{P}$, the mappings $\Lambda : \mathscr{Q}_1 \times \mathscr{Q}_2 \to \mathscr{F}^{\mathscr{P}}$ and $\Gamma : \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2 \to \mathscr{F}^{\mathscr{P}}$ are given as:*

$$\begin{aligned}
\Lambda(\mathfrak{q}_i, \mathfrak{q}_j)(\mathfrak{p}) = &\big(\min(\eta^+_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^+_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p})), \min(\eta^o_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^o_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p})), \max(\eta^-_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^-_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p}))\big),\\
\Gamma(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j)(\mathfrak{p}) = &\big(\max(\zeta^+_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^+_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p})), \min(\zeta^o_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^o_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p})),\\
&\min(\zeta^-_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^-_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p}))\big).
\end{aligned}$$

Definition 11. *For any two P_RFBSSs $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ and $(\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ over a universal set $\mathscr{P}$, the OR operation among them, represented by $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \underline{\vee} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$, is defined as $(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \underline{\vee} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2) = (\Psi, \Upsilon, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2)$, where, for all $(\mathfrak{q}_i, \mathfrak{q}_j) \in \mathscr{Q}_1 \times \mathscr{Q}_2$, $(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2$ and $\mathfrak{p} \in \mathscr{P}$, the mappings $\Psi : \mathscr{Q}_1 \times \mathscr{Q}_2 \to \mathscr{F}^{\mathscr{P}}$ and $\Upsilon : \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2 \to \mathscr{F}^{\mathscr{P}}$ are provided by:*

$$\begin{aligned}
\Psi(\mathfrak{q}_i, \mathfrak{q}_j)(\mathfrak{p}) = &\big(\max(\eta^+_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^+_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p})), \min(\eta^o_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^o_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p})), \min(\eta^-_{\mathscr{G}_1}(\mathfrak{q}_i)(\mathfrak{p}), \eta^-_{\mathscr{G}_2}(\mathfrak{q}_j)(\mathfrak{p}))\big),\\
\Upsilon(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j)(\mathfrak{p}) = &\big(\min(\zeta^+_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^+_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p})), \min(\zeta^o_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^o_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p})),\\
&\max(\zeta^-_{\mathscr{H}_1}(\neg\mathfrak{q}_i)(\mathfrak{p}), \zeta^-_{\mathscr{H}_2}(\neg\mathfrak{q}_j)(\mathfrak{p}))\big).
\end{aligned}$$

Example 4. *Consider a company that manufactures four new laptops using different hardware and software combinations. Two tech experts are hired to check the quality of these new models according to certain attributes. Consider $\mathscr{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4\}$ as representing the new laptops and $\mathscr{R} = \{\mathfrak{q}_1 = \text{high speed}, \ \mathfrak{q}_2 = \text{good battery life}, \mathfrak{q}_3 = \text{good connectivity}\}$ as representing the set of favorable parameters. The corresponding not set of attributes is $\neg\mathscr{R} = \{\neg\mathfrak{q}_1 = \text{lagging}, \ \neg\mathfrak{q}_2 = \text{low battery life}, \ \neg\mathfrak{q}_3 = \text{distorted connectivity}\}$. Assume that the first expert considers $\mathscr{Q}_1 = \{\mathfrak{q}_1, \mathfrak{q}_2\} \subseteq \mathscr{R}$ and the second expert takes $\mathscr{Q}_2 = \{\mathfrak{q}_1, \mathfrak{q}_3\} \subseteq \mathscr{R}$ as their favorable set of parameters to compare the qualities of new laptops. Then, the estimations of experts are provided as P_RFBSSs Ω_1 and Ω_2, which are displayed in Tables 7 and 8, respectively.*

Table 7. Tabular form of the P_RFBSS Ω_1=$(\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$.

Ω_1	$\mathfrak{q}_1, \neg\mathfrak{q}_1$	$\mathfrak{q}_2, \neg\mathfrak{q}_2$
$\mathfrak{p}_1$	$\langle(0.02, 0.01, 0.19), (0.04, 0.26, 0.31)\rangle$	$\langle(0.11, 0.31, 0.02), (0.12, 0.16, 0.21)\rangle$
$\mathfrak{p}_2$	$\langle(0.03, 0.06, 0.39), (0.13, 0.01, 0.26)\rangle$	$\langle(0.12, 0.26, 0.31), (0.11, 0.12, 0.59)\rangle$
$\mathfrak{p}_3$	$\langle(0.26, 0.31, 0.16), (0.13, 0.30, 0.01)\rangle$	$\langle(0.13, 0.21, 0.61), (0.11, 0.22, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.81, 0.01, 0.02), (0.13, 0.16, 0.36)\rangle$	$\langle(0.06, 0.02, 0.11), (0.09, 0.16, 0.52)\rangle$

Table 8. Tabular form of the P_RFBSS Ω_2=$(\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$.

Ω_2	$\mathfrak{q}_1, \neg\mathfrak{q}_1$	$\mathfrak{q}_3, \neg\mathfrak{q}_3$
$\mathfrak{p}_1$	$\langle(0.01, 0.16, 0.23), (0.03, 0.11, 0.45)\rangle$	$\langle(0.16, 0.19, 0.23), (0.19, 0.29, 0.32)\rangle$
$\mathfrak{p}_2$	$\langle(0.11, 0.12, 0.13), (0.21, 0.32, 0.09)\rangle$	$\langle(0.13, 0.14, 0.03), (0.26, 0.16, 0.39)\rangle$
$\mathfrak{p}_3$	$\langle(0.23, 0.61, 0.14), (0.13, 0.24, 0.09)\rangle$	$\langle(0.34, 0.02, 0.09), (0.26, 0.02, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.11, 0.23, 0.19), (0.22, 0.12, 0.16)\rangle$	$\langle(0.04, 0.01, 0.03), (0.11, 0.12, 0.13)\rangle$

Then, the AND and OR operations between the P_RFBSSs Ω_1 and Ω_2 are shown in Tables 9 and 10, respectively.

Table 9. AND operation between Ω_1 and Ω_2.

$\Omega_1 \barwedge \Omega_2$	$(\mathfrak{q}_1, \mathfrak{q}_1), (\neg\mathfrak{q}_1, \neg\mathfrak{q}_1)$	$(\mathfrak{q}_1, \mathfrak{q}_3), (\neg\mathfrak{q}_1, \neg\mathfrak{q}_3)$
$\mathfrak{p}_1$	$\langle(0.01, 0.01, 0.23), (0.04, 0.11, 0.31)\rangle$	$\langle(0.02, 0.01, 0.23), (0.19, 0.26, 0.31)\rangle$
$\mathfrak{p}_2$	$\langle(0.03, 0.06, 0.39), (0.21, 0.01, 0.09)\rangle$	$\langle(0.03, 0.06, 0.39), (0.26, 0.01, 0.26)\rangle$
$\mathfrak{p}_3$	$\langle(0.23, 0.31, 0.16), (0.13, 0.24, 0.01)\rangle$	$\langle(0.26, 0.02, 0.16), (0.26, 0.02, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.11, 0.01, 0.19), (0.22, 0.12, 0.16)\rangle$	$\langle(0.04, 0.01, 0.03), (0.13, 0.12, 0.13)\rangle$
$\Omega_1 \barwedge \Omega_2$	$(\mathfrak{q}_2, \mathfrak{q}_1), (\neg\mathfrak{q}_2, \neg\mathfrak{q}_1)$	$(\mathfrak{q}_2, \mathfrak{q}_3), (\neg\mathfrak{q}_2, \neg\mathfrak{q}_3)$
$\mathfrak{p}_1$	$\langle(0.01, 0.16, 0.23), (0.12, 0.11, 0.21)\rangle$	$\langle(0.11, 0.19.0.23), (0.19, 0.16, 0.21)\rangle$
$\mathfrak{p}_2$	$\langle(0.11, 0.12, 0.31), (0.21, 0.12, 0.09)\rangle$	$\langle(0.12, 0.14, 0.31), (0.26, 0.12, 0.39)\rangle$
$\mathfrak{p}_3$	$\langle(0.13, 0.21, 0.61), (0.13, 0.22, 0.01)\rangle$	$\langle(0.13, 0.02, 0.61), (0.26, 0.02, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.06, 0.02, 0.19), (0.22, 0.12, 0.16)\rangle$	$\langle(0.04, 0.01, 0.11), (0.11, 0.12, 0.13)\rangle$

Table 10. OR operation between Ω_1 and Ω_2.

$\Omega_1 \veebar \Omega_2$	$(\mathfrak{q}_1, \mathfrak{q}_1), (\neg\mathfrak{q}_1, \neg\mathfrak{q}_1)$	$(\mathfrak{q}_1, \mathfrak{q}_3), (\neg\mathfrak{q}_1, \neg\mathfrak{q}_3)$
$\mathfrak{p}_1$	$\langle(0.02, 0.01, 0.19), (0.03, 0.11, 0.45)\rangle$	$\langle(0.16, 0.01, 0.19), (0.04, 0.26, 0.32)\rangle$
$\mathfrak{p}_2$	$\langle(0.11, 0.06, 0.13), (0.13, 0.01, 0.26)\rangle$	$\langle(0.13, 0.06, 0.03), (0.13, 0.01, 0.39)\rangle$
$\mathfrak{p}_3$	$\langle(0.26, 0.31, 0.14), (0.13, 0.24, 0.09)\rangle$	$\langle(0.34, 0.02, 0.09), (0.13, 0.02, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.81, 0.01, 0.02), (0.13, 0.12, 0.36)\rangle$	$\langle(0.81, 0.01, 0.02), (0.11, 0.12, 0.36)\rangle$
$\Omega_1 \veebar \Omega_2$	$(\mathfrak{q}_2, \mathfrak{q}_1), (\neg\mathfrak{q}_2, \neg\mathfrak{q}_1)$	$(\mathfrak{q}_2, \mathfrak{q}_3), (\neg\mathfrak{q}_2, \neg\mathfrak{q}_3)$
$\mathfrak{p}_1$	$\langle(0.11, 0.16, 0.02), (0.03, 0.11, 0.45)\rangle$	$\langle(0.16, 0.19, 0.02), (0.12, 0.16, 0.32)\rangle$
$\mathfrak{p}_2$	$\langle(0.12, 0.12, 0.13), (0.21, 0.12, 0.09)\rangle$	$\langle(0.13, 0.14, 0.03), (0.11, 0.12, 0.59)\rangle$
$\mathfrak{p}_3$	$\langle(0.23, 0.21, 0.14), (0.11, 0.22, 0.09)\rangle$	$\langle(0.34, 0.02, 0.09), (0.11, 0.02, 0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.11, 0.02, 0.11), (0.09, 0.12, 0.52)\rangle$	$\langle(0.06, 0.01, 0.03), (0.09, 0.12, 0.52)\rangle$

The following verifies the commutative, associative and distributive laws with respect to P_RFBSS AND and OR operations.

Proposition 1. *Let $\mathscr{P}$ be a universal set, and let $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$, $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ and $\Omega_3 = (\mathscr{G}_3, \mathscr{H}_3, \mathscr{Q}_3, \neg\mathscr{Q}_3)$ be three P_RFBSSs on $\mathscr{P}$. Then:*

1. $\Omega_1 \barwedge \Omega_2 = \Omega_2 \barwedge \Omega_1$;
2. $\Omega_1 \veebar \Omega_2 = \Omega_1 \veebar \Omega_1$;
3. $\Omega_1 \barwedge (\Omega_2 \barwedge \Omega_3) = (\Omega_1 \barwedge \Omega_2) \barwedge \Omega_3$;
4. $\Omega_1 \veebar (\Omega_2 \veebar \Omega_3) = (\Omega_1 \veebar \Omega_2) \veebar \Omega_3$;
5. $\Omega_1 \barwedge (\Omega_2 \veebar \Omega_3) = (\Omega_1 \barwedge \Omega_2) \veebar (\Omega_1 \barwedge \Omega_3)$;
6. $\Omega_1 \veebar (\Omega_2 \barwedge \Omega_3) = (\Omega_1 \veebar \Omega_2) \barwedge (\Omega_1 \veebar \Omega_3)$.

Proof.

1. Suppose that

$$\Omega_1 \barwedge \Omega_2 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \barwedge (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2) = (\Lambda, \Gamma, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg\mathscr{Q}_1 \times \neg\mathscr{Q}_2).$$

Then, by Definition 10, $\forall\, (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathscr{Q}_1 \times \mathscr{Q}_2$, we have:

$$\begin{aligned}\Lambda(\mathfrak{q}_i, \mathfrak{q}_j) =& \big(\min(\eta^{+}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{+}_{\mathscr{G}_2}(\mathfrak{q}_j)), \min(\eta^{o}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{o}_{\mathscr{G}_2}(\mathfrak{q}_j)), \max(\eta^{-}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{-}_{\mathscr{G}_2}(\mathfrak{q}_j))\big)\\ =& \big(\min(\eta^{+}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{+}_{\mathscr{G}_1}(\mathfrak{q}_i)), \min(\eta^{o}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{o}_{\mathscr{G}_1}(\mathfrak{q}_i)), \max(\eta^{-}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{-}_{\mathscr{G}_1}(\mathfrak{q}_i))\big)\\ =& \tilde{\Lambda}(\mathfrak{q}_j, \mathfrak{q}_i).\end{aligned}$$

Similarly, $\forall\ (\neg \mathfrak{q}_i, \neg \mathfrak{q}_j) \in \neg \mathscr{Q}_1 \times \neg \mathscr{Q}_2$, we have:

$$\begin{aligned}\Gamma(\neg \mathfrak{q}_i, \neg \mathfrak{q}_j) =& \big(\max(\eta^{+}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{+}_{\mathscr{G}_2}(\neg \mathfrak{q}_j)), \min(\eta^{o}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{o}_{\mathscr{G}_2}(\neg \mathfrak{q}_j)), \min(\eta^{-}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{-}_{\mathscr{G}_2}(\neg \mathfrak{q}_j))\big)\\ =& \big(\max(\eta^{+}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{+}_{\mathscr{G}_1}(\neg \mathfrak{q}_i)), \min(\eta^{o}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{o}_{\mathscr{G}_1}(\neg \mathfrak{q}_i)), \min(\eta^{-}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{-}_{\mathscr{G}_1}(\neg \mathfrak{q}_i))\big)\\ =& \tilde{\Gamma}(\neg \mathfrak{q}_j, \neg \mathfrak{q}_i).\end{aligned}$$

Such that, $\forall\ (\mathfrak{q}_j, \mathfrak{q}_i) \in \mathscr{Q}_2 \times \mathscr{Q}_1$ and $(\neg \mathfrak{q}_j, \neg \mathfrak{q}_i) \in \neg \mathscr{Q}_2 \times \neg \mathscr{Q}_1$,

$$(\tilde{\Lambda}, \tilde{\Gamma}, \mathscr{Q}_2 \times \mathscr{Q}_1, \neg \mathscr{Q}_2 \times \neg \mathscr{Q}_1) = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg \mathscr{Q}_2) \barwedge (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg \mathscr{Q}_1) = \Omega_2 \barwedge \Omega_1.$$

Hence, $\Omega_1 \barwedge \Omega_2 = \Omega_2 \barwedge \Omega_1$.

2. Suppose that

$$\Omega_1 \veebar \Omega_2 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg \mathscr{Q}_1) \veebar (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg \mathscr{Q}_2) = (\Psi, \Upsilon, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg \mathscr{Q}_1 \times \neg \mathscr{Q}_2).$$

Then, by Definition 11, $\forall\ (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathscr{Q}_1 \times \mathscr{Q}_2$, we have:

$$\begin{aligned}\Psi(\mathfrak{q}_i, \mathfrak{q}_j) =& \big(\max(\eta^{+}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{+}_{\mathscr{G}_2}(\mathfrak{q}_j)), \min(\eta^{o}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{o}_{\mathscr{G}_2}(\mathfrak{q}_j)), \min(\eta^{-}_{\mathscr{G}_1}(\mathfrak{q}_i), \eta^{-}_{\mathscr{G}_2}(\mathfrak{q}_j))\big)\\ =& \big(\max(\eta^{+}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{+}_{\mathscr{G}_1}(\mathfrak{q}_i)), \min(\eta^{o}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{o}_{\mathscr{G}_1}(\mathfrak{q}_i)), \min(\eta^{-}_{\mathscr{G}_2}(\mathfrak{q}_j), \eta^{-}_{\mathscr{G}_1}(\mathfrak{q}_i))\big)\\ =& \tilde{\Psi}(\mathfrak{q}_j, \mathfrak{q}_i).\end{aligned}$$

Similarly, $\forall\ (\neg \mathfrak{q}_i, \neg \mathfrak{q}_j) \in \neg \mathscr{Q}_1 \times \neg \mathscr{Q}_2$, we have:

$$\begin{aligned}\Upsilon(\neg \mathfrak{q}_i, \neg \mathfrak{q}_j) =& \big(\min(\eta^{+}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{+}_{\mathscr{G}_2}(\neg \mathfrak{q}_j)), \min(\eta^{o}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{o}_{\mathscr{G}_2}(\neg \mathfrak{q}_j)), \max(\eta^{-}_{\mathscr{G}_1}(\neg \mathfrak{q}_i), \eta^{-}_{\mathscr{G}_2}(\neg \mathfrak{q}_j))\big)\\ =& \big(\min(\eta^{+}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{+}_{\mathscr{G}_1}(\neg \mathfrak{q}_i)), \min(\eta^{o}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{o}_{\mathscr{G}_1}(\neg \mathfrak{q}_i)), \max(\eta^{-}_{\mathscr{G}_2}(\neg \mathfrak{q}_j), \eta^{-}_{\mathscr{G}_1}(\neg \mathfrak{q}_i))\big)\\ =& \tilde{\Upsilon}(\neg \mathfrak{q}_j, \neg \mathfrak{q}_i).\end{aligned}$$

Such that, $\forall\ (\mathfrak{q}_j, \mathfrak{q}_i) \in \mathscr{Q}_2 \times \mathscr{Q}_1$ and $(\neg \mathfrak{q}_j, \neg \mathfrak{q}_i) \in \neg \mathscr{Q}_2 \times \neg \mathscr{Q}_1$,

$$(\tilde{\Psi}, \tilde{\Upsilon}, \mathscr{Q}_2 \times \mathscr{Q}_1, \neg \mathscr{Q}_2 \times \neg \mathscr{Q}_1) = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg \mathscr{Q}_2) \veebar (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg \mathscr{Q}_1) = \Omega_2 \veebar \Omega_1.$$

Hence, $\Omega_1 \veebar \Omega_2 = \Omega_2 \veebar \Omega_1$.

The remaining parts (3–6) can be verified with similar arguments. □

In the following, we verify the famous De Morgan's laws for $\mathrm{P_R}$FBSS AND and OR operations.

Proposition 2. *Let $\mathscr{P}$ be a universal set, and let $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg \mathscr{Q}_1)$ and $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg \mathscr{Q}_2)$ be two $\mathrm{P_R}$FBSSs on $\mathscr{P}$. Then:*

1. $(\Omega_1 \barwedge \Omega_2)^c = (\Omega_1)^c \veebar (\Omega_2)^c$;
2. $(\Omega_1 \veebar \Omega_2)^c = (\Omega_1)^c \barwedge (\Omega_2)^c$.

Proof.

1. From Definitions 8 and 10,

$$\begin{aligned}(\Omega_1 \barwedge \Omega_2)^c =& \big((\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg \mathscr{Q}_1) \barwedge (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg \mathscr{Q}_2)\big)^c\\ =& (\Lambda, \Gamma, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg \mathscr{Q}_1 \times \neg \mathscr{Q}_2)^c\\ =& (\Lambda^c, \Gamma^c, \mathscr{Q}_1 \times \mathscr{Q}_2, \neg \mathscr{Q}_1 \times \neg \mathscr{Q}_2).\end{aligned}$$

where $\forall\ (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have

$$\begin{aligned}
\Lambda^c(\mathfrak{q}_i, \mathfrak{q}_j) =& \big(\min(\eta^+_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\mathfrak{q}_j)), \max(\eta^-_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\mathfrak{q}_j))\big)^c \\
=& \big(\max(\eta^-_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^+_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\mathfrak{q}_j))\big) \\
=& \big(\max(\eta^+_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}^c_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}^c_2}(\mathfrak{q}_j)), \min(\eta^-_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}^c_2}(\mathfrak{q}_j))\big) \\
=& \bar{\Lambda}(\mathfrak{q}_i, \mathfrak{q}_j).
\end{aligned}$$

Similarly, $\forall\ (\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2$, we have:

$$\begin{aligned}
\Gamma^c(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) =& \big(\max(\eta^+_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^-_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\neg\mathfrak{q}_j))\big)^c \\
=& \big(\min(\eta^-_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \max(\eta^+_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\neg\mathfrak{q}_j))\big) \\
=& \big(\min(\eta^+_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j)), \max(\eta^-_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j))\big) \\
=& \bar{\Gamma}(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j).
\end{aligned}$$

Such that, $\forall\ (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2$,

$$\begin{aligned}
(\bar{\Lambda}, \bar{\Gamma}, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2) =& (\mathcal{G}^c_1, \mathcal{H}^c_1, \mathcal{Q}_1, \neg\mathcal{Q}_1) \veebar (\mathcal{G}^c_2, \mathcal{H}^c_2, \mathcal{Q}_2, \neg\mathcal{Q}_2) \\
=& (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg\mathcal{Q}_1)^c \veebar (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg\mathcal{Q}_2)^c \\
=& (\Omega_1)^c \veebar (\Omega_2)^c.
\end{aligned}$$

Hence, $(\Omega_1 \barwedge \Omega_2)^c = (\Omega_1)^c \veebar (\Omega_2)^c$.

2. From Definitions 8 and 11,

$$\begin{aligned}
(\Omega_1 \veebar \Omega_2)^c =& \big((\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg\mathcal{Q}_1) \veebar (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg\mathcal{Q}_2)\big)^c \\
=& (\Psi, \Upsilon, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2)^c \\
=& (\Psi^c, \Upsilon^c, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2).
\end{aligned}$$

where $\forall\ (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$, we have:

$$\begin{aligned}
\Psi^c(\mathfrak{q}_i, \mathfrak{q}_j) =& \big(\max(\eta^+_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^-_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\mathfrak{q}_j))\big)^c \\
=& \big(\min(\eta^-_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\mathfrak{q}_j)), \max(\eta^+_{\mathcal{G}_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\mathfrak{q}_j))\big) \\
=& \big(\min(\eta^+_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^+_{\mathcal{G}^c_2}(\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^o_{\mathcal{G}^c_2}(\mathfrak{q}_j)), \max(\eta^-_{\mathcal{G}^c_1}(\mathfrak{q}_i), \eta^-_{\mathcal{G}^c_2}(\mathfrak{q}_j))\big) \\
=& \bar{\Psi}(\mathfrak{q}_i, \mathfrak{q}_j).
\end{aligned}$$

Similarly, $\forall\ (\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2$, we have:

$$\begin{aligned}
\Upsilon^c(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) =& \big(\min(\eta^+_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \max(\eta^-_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\neg\mathfrak{q}_j))\big)^c \\
=& \big(\max(\eta^-_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}_2}(\neg\mathfrak{q}_j)), \min(\eta^+_{\mathcal{G}_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}_2}(\neg\mathfrak{q}_j))\big) \\
=& \big(\max(\eta^+_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^+_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j)), \min(\eta^o_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^o_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j)), \min(\eta^-_{\mathcal{G}^c_1}(\neg\mathfrak{q}_i), \eta^-_{\mathcal{G}^c_2}(\neg\mathfrak{q}_j))\big) \\
=& \bar{\Upsilon}(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j).
\end{aligned}$$

Such that, $\forall\ (\mathfrak{q}_i, \mathfrak{q}_j) \in \mathcal{Q}_1 \times \mathcal{Q}_2$ and $(\neg\mathfrak{q}_i, \neg\mathfrak{q}_j) \in \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2$,

$$\begin{aligned}
(\bar{\Psi}, \bar{\Upsilon}, \mathcal{Q}_1 \times \mathcal{Q}_2, \neg\mathcal{Q}_1 \times \neg\mathcal{Q}_2) =& (\mathcal{G}^c_1, \mathcal{H}^c_1, \mathcal{Q}_1, \neg\mathcal{Q}_1) \barwedge (\mathcal{G}^c_2, \mathcal{H}^c_2, \mathcal{Q}_2, \neg\mathcal{Q}_2) \\
=& (\mathcal{G}_1, \mathcal{H}_1, \mathcal{Q}_1, \neg\mathcal{Q}_1)^c \barwedge (\mathcal{G}_2, \mathcal{H}_2, \mathcal{Q}_2, \neg\mathcal{Q}_2)^c \\
=& (\Omega_1)^c \barwedge (\Omega_2)^c.
\end{aligned}$$

Hence, $(\Omega_1 \veebar \Omega_2)^c = (\Omega_1)^c \barwedge (\Omega_2)^c$.

□

The next four definitions give the notions of extended (restricted) union and intersection, respectively, along with illustrative numerical examples.

Definition 12. *For any two P_RFBSSs $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ and $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ over a universal set $\mathscr{P}$, their extended union, represented by $\Omega_1 \Cup_{\mathcal{E}} \Omega_2$, is again a P_RFBSS $((\mathscr{G}_1 \Cup \mathscr{G}_2), (\mathscr{H}_1 \Cap \mathscr{H}_2), \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2)$ on $\mathscr{P}$, which is defined as follows:*

$$(\mathscr{G}_1 \Cup \mathscr{G}_2)(\mathfrak{q}) = \begin{cases} \mathscr{G}_1(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1(\mathfrak{q}) \underline{\cup} \mathscr{G}_2(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2. \end{cases}$$

$$(\mathscr{H}_1 \Cap \mathscr{H}_2)(\neg\mathfrak{q}) = \begin{cases} \mathscr{H}_1(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) - (\neg\mathscr{Q}_2), \\ \mathscr{H}_2(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_2) - (\neg\mathscr{Q}_1), \\ \mathscr{H}_1(\neg\mathfrak{q}) \overline{\cap} \mathscr{H}_2(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) \cap (\neg\mathscr{Q}_2). \end{cases}$$

where

$$\begin{aligned} \mathscr{G}_1(\mathfrak{q}) \underline{\cup} \mathscr{G}_2(\mathfrak{q}) =& \big\{ \langle \mathfrak{p}, \max\big(\eta^+_{\mathscr{G}_1}, \eta^+_{\mathscr{G}_2}\big)(\mathfrak{q})(\mathfrak{p}), \min\big(\eta^o_{\mathscr{G}_1}, \eta^o_{\mathscr{G}_2}\big)(\mathfrak{q})(\mathfrak{p}), \\ & \min\big(\eta^-_{\mathscr{G}_1}, \eta^-_{\mathscr{G}_2}\big)(\mathfrak{q})(\mathfrak{p}) \rangle \mid \mathfrak{p} \in \mathscr{P},\ \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2 \big\}, \\ \mathscr{H}_1(\neg\mathfrak{q}) \overline{\cap} \mathscr{H}_2(\neg\mathfrak{q}) =& \big\{ \langle \mathfrak{p}, \min\big(\zeta^+_{\mathscr{H}_1}, \zeta^+_{\mathscr{H}_2}\big)(\neg\mathfrak{q})(\mathfrak{p}), \min\big(\zeta^o_{\mathscr{H}_1}, \zeta^o_{\mathscr{H}_2}\big)(\neg\mathfrak{q})(\mathfrak{p}), \\ & \max\big(\zeta^-_{\mathscr{H}_1}, \zeta^-_{\mathscr{H}_2}\big)(\neg\mathfrak{q})(\mathfrak{p}) \rangle \mid \mathfrak{p} \in \mathscr{P},\ \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2 \big\}. \end{aligned}$$

Definition 13. *For any two P_RFBSSs $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ and $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ over a universal set $\mathscr{P}$, their restricted union denoted by $\Omega_1 \Cup_{\mathcal{R}} \Omega_2$ is a P_RFBSS $((\mathscr{G}_1 \Cup \mathscr{G}_2), (\mathscr{H}_1 \Cap \mathscr{H}_2), \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2)$ on $\mathscr{P}$, such that $\forall\ \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2 \neq \emptyset$ and $\neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2 \neq \emptyset$,*

$$(\mathscr{G}_1 \Cup \mathscr{G}_2)(\mathfrak{q}) = \mathscr{G}_1(\mathfrak{q}) \underline{\cup} \mathscr{G}_2(\mathfrak{q}) \quad \textit{and} \quad (\mathscr{H}_1 \Cap \mathscr{H}_2)(\neg\mathfrak{q}) = \mathscr{H}_1(\neg\mathfrak{q}) \overline{\cap} \mathscr{H}_2(\neg\mathfrak{q}).$$

Proposition 3. *Let $\mathscr{P}$ be a universal set and let $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$, $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ and $\Omega_3 = (\mathscr{G}_3, \mathscr{H}_3, \mathscr{Q}_3, \neg\mathscr{Q}_3)$ be three P_RFBSSs on $\mathscr{P}$. Then:*

1. $\Omega_1 \Cup_{\mathcal{E}} \Omega_2 = \Omega_2 \Cup_{\mathcal{E}} \Omega_1$;
2. $\Omega_1 \Cup_{\mathcal{R}} \Omega_2 = \Omega_1 \Cup_{\mathcal{R}} \Omega_1$;
3. $\Omega_1 \Cup_{\mathcal{E}} (\Omega_2 \Cup_{\mathcal{E}} \Omega_3) = (\Omega_1 \Cup_{\mathcal{E}} \Omega_2) \Cup_{\mathcal{E}} \Omega_3$;
4. $\Omega_1 \Cup_{\mathcal{R}} (\Omega_2 \Cup_{\mathcal{R}} \Omega_3) = (\Omega_1 \Cup_{\mathcal{R}} \Omega_2) \Cup_{\mathcal{R}} \Omega_3$.

Proof.

1. From Definition 12:

$$\begin{aligned} \Omega_1 \Cup_{\mathcal{E}} \Omega_2 =& (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \Cup_{\mathcal{E}} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2) \\ =& (\mathscr{G}_1 \Cup \mathscr{G}_2, \mathscr{H}_1 \Cap \mathscr{H}_2, \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2). \end{aligned}$$

Such that $\forall\ \mathfrak{q} \in \mathscr{Q}_1 \cup \mathscr{Q}_2$:

$$\mathscr{G}_1 \Cup \mathscr{G}_2 = \begin{cases} \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1 \underline{\cup} \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2. \end{cases} = \begin{cases} \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2 \underline{\cup} \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 \cap \mathscr{Q}_1. \end{cases} = \mathscr{G}_2 \Cup \mathscr{G}_1.$$

Similarly, $\forall\ \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2$,

$$\mathscr{H}_1 \Cap \mathscr{H}_2 = \begin{cases} \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 - \neg\mathscr{Q}_2, \\ \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 - \neg\mathscr{Q}_1, \\ \mathscr{H}_1 \overline{\cap} \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2. \end{cases} = \begin{cases} \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 - \neg\mathscr{Q}_1, \\ \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 - \neg\mathscr{Q}_2, \\ \mathscr{H}_2 \overline{\cap} \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 \cap \neg\mathscr{Q}_1. \end{cases} = \mathscr{H}_2 \Cap \mathscr{H}_1.$$

This implies that

$$(\mathscr{G}_1 \Cup \mathscr{G}_2, \mathscr{H}_1 \Cap \mathscr{H}_2, \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2) = (\mathscr{G}_2 \Cup \mathscr{G}_1, \mathscr{H}_2 \Cap \mathscr{H}_1, \mathscr{Q}_2 \cup \mathscr{Q}_1, \neg\mathscr{Q}_2 \cup \neg\mathscr{Q}_1).$$

Hence, $\Omega_1 \Cup_{\mathcal{E}} \Omega_2 = \Omega_2 \Cup_{\mathcal{E}} \Omega_1$.

2. From Definition 13:

$$\begin{aligned}\Omega_1 \Cup_{\mathcal{R}} \Omega_2 =& (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \Cup_{\mathcal{R}} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)\\ =& (\mathscr{G}_1 \Cup \mathscr{G}_2, \mathscr{H}_1 \Cap \mathscr{H}_2, \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2).\end{aligned}$$

Such that $\forall\ \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2$,

$$\mathscr{G}_1 \Cup \mathscr{G}_2 = \mathscr{G}_1 \underline{\cup} \mathscr{G}_2 = \mathscr{G}_2 \underline{\cup} \mathscr{G}_1 = \mathscr{G}_2 \Cup \mathscr{G}_1,$$

and $\forall\ \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2$,

$$\mathscr{H}_1 \Cap \mathscr{H}_2 = \mathscr{H}_1 \overline{\cap}\, \mathscr{H}_2 = \mathscr{H}_2 \overline{\cap}\, \mathscr{H}_1 = \mathscr{H}_2 \Cap \mathscr{H}_1.$$

This implies that

$$(\mathscr{G}_1 \Cup \mathscr{G}_2, \mathscr{H}_1 \Cap \mathscr{H}_2, \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2) = (\mathscr{G}_2 \Cup \mathscr{G}_1, \mathscr{H}_2 \Cap \mathscr{H}_1, \mathscr{Q}_2 \cap \mathscr{Q}_1, \neg\mathscr{Q}_2 \cap \neg\mathscr{Q}_1).$$

Hence, $\Omega_1 \Cup_{\mathcal{R}} \Omega_2 = \Omega_2 \Cup_{\mathcal{R}} \Omega_1$.

The remaining parts (3 and 4) can be proven similarly. □

Definition 14. *For any two* P_R*FBSSs* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *and* $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *over a universal set* $\mathscr{P}$*, their extended intersection, represented by* $\Omega_1 \Cap_{\mathcal{E}} \Omega_2$*, is again a* P_R*FBSS* $((\mathscr{G}_1 \Cap \mathscr{G}_2), (\mathscr{H}_1 \Cup \mathscr{H}_2), \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2)$ *on* $\mathscr{P}$*, which is defined as:*

$$(\mathscr{G}_1 \Cap \mathscr{G}_2)(\mathfrak{q}) = \begin{cases} \mathscr{G}_1(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2,\\ \mathscr{G}_2(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1,\\ \mathscr{G}_1(\mathfrak{q}) \overline{\cap} \mathscr{G}_2(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2.\end{cases}$$

$$(\mathscr{H}_1 \Cup \mathscr{H}_2)(\neg\mathfrak{q}) = \begin{cases} \mathscr{H}_1(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) - (\neg\mathscr{Q}_2),\\ \mathscr{H}_2(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_2) - (\neg\mathscr{Q}_1),\\ \mathscr{H}_1(\neg\mathfrak{q}) \underline{\cup} \mathscr{H}_2(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) \cap (\neg\mathscr{Q}_2).\end{cases}$$

where

$$\begin{aligned}\mathscr{G}_1(\mathfrak{q}) \overline{\cap} \mathscr{G}_2(\mathfrak{q}) =& \{\langle \mathfrak{p}, \min(\eta^+_{\mathscr{G}_1}, \eta^+_{\mathscr{G}_2})(\mathfrak{q})(\mathfrak{p}), \min(\eta^o_{\mathscr{G}_1}, \eta^o_{\mathscr{G}_2})(\mathfrak{q})(\mathfrak{p}),\\ & \max(\eta^-_{\mathscr{G}_1}, \eta^-_{\mathscr{G}_2})(\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P},\ \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2\},\\ \mathscr{H}_1(\neg\mathfrak{q}) \underline{\cup} \mathscr{H}_2(\neg\mathfrak{q}) =& \{\langle \mathfrak{p}, \max(\zeta^+_{\mathscr{H}_1}, \zeta^+_{\mathscr{H}_2})(\neg\mathfrak{q})(\mathfrak{p}), \min(\zeta^o_{\mathscr{H}_1}, \zeta^o_{\mathscr{H}_2})(\neg\mathfrak{q})(\mathfrak{p}),\\ & \min(\zeta^-_{\mathscr{H}_1}, \zeta^-_{\mathscr{H}_2})(\neg\mathfrak{q})(\mathfrak{p})\rangle \mid \mathfrak{p} \in \mathscr{P},\ \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2\}.\end{aligned}$$

Definition 15. *For any two* P_R*FBSSs* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *and* $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *over a universal set* $\mathscr{P}$*, their restricted intersection, denoted by* $\Omega_1 \Cap_{\mathcal{R}} \Omega_2$*, is a* P_R*FBSS* $((\mathscr{G}_1 \Cap \mathscr{G}_2), (\mathscr{H}_1 \Cup \mathscr{H}_2), \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2)$ *on* $\mathscr{P}$*, such that* $\forall\ \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2 \neq \emptyset$ *and* $\neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2 \neq \emptyset$*,*

$$(\mathscr{G}_1 \Cap \mathscr{G}_2)(\mathfrak{q}) = \mathscr{G}_1(\mathfrak{q}) \,\overline{\cap}\, \mathscr{G}_2(\mathfrak{q}) \quad \textit{and} \quad (\mathscr{H}_1 \Cup \mathscr{H}_2)(\neg\mathfrak{q}) = \mathscr{H}_1(\neg\mathfrak{q}) \,\underline{\cup}\, \mathscr{H}_2(\neg\mathfrak{q}).$$

Example 5. *Reconsider the* P_R*FBSSs* Ω_1 *and* Ω_2 *in Example 4. Suppose the company wants to analyze the laptops again, both by taking the maximum scores from the two reports to shortlist the best laptops and by taking the minimum scores from the two reports to shortlist the worst ones. Consequently, the extended union* $\Omega_1 \Cup_{\mathcal{E}} \Omega_2$ *and the extended intersection* $\Omega_1 \Cap_{\mathcal{E}} \Omega_2$ *are represented by Tables 11 and 12, respectively.*

Table 11. Extended union between Ω_1 and Ω_2.

$\Omega_1 \Cup_{\mathcal{E}} \Omega_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\langle(0.02,0.01,0.19),(0.03,0.11,0.45)\rangle$	$\langle(0.11,0.31,0.02),(0.12,0.16,0.21)\rangle$	$\langle(0.16,0.19,0.23),(0.19,0.29,0.32)\rangle$
$\mathfrak{p}_2$	$\langle(0.11,0.06,0.13),(0.13,0.01,0.26)\rangle$	$\langle(0.12,0.26,0.31),(0.11,0.12,0.59)\rangle$	$\langle(0.13,0.14,0.03),(0.26,0.16,0.39)\rangle$
$\mathfrak{p}_3$	$\langle(0.26,0.31,0.14),(0.13,0.24,0.09)\rangle$	$\langle(0.13,0.21,0.61),(0.11,0.22,0.01)\rangle$	$\langle(0.34,0.02,0.09),(0.26,0.02,0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.81,0.01,0.02),(0.13,0.12,0.36)\rangle$	$\langle(0.06,0.02,0.11),(0.09,0.16,0.52)\rangle$	$\langle(0.04,0.01,0.03),(0.11,0.12,0.13)\rangle$

Table 12. Extended intersection between Ω_1 and Ω_2.

$\Omega_1 \Cap_{\mathcal{E}} \Omega_2$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$
$\mathfrak{p}_1$	$\langle(0.01,0.01,0.23),(0.04,0.11,0.31)\rangle$	$\langle(0.11,0.31,0.02),(0.12,0.16,0.21)\rangle$	$\langle(0.16,0.19,0.23),(0.19,0.29,0.32)\rangle$
$\mathfrak{p}_2$	$\langle(0.03,0.06,0.39),(0.21,0.01,0.09)\rangle$	$\langle(0.12,0.26,0.31),(0.11,0.12,0.59)\rangle$	$\langle(0.13,0.14,0.03),(0.26,0.16,0.39)\rangle$
$\mathfrak{p}_3$	$\langle(0.23,0.31,0.16),(0.13,0.24,0.01)\rangle$	$\langle(0.13,0.21,0.61),(0.11,0.22,0.01)\rangle$	$\langle(0.34,0.02,0.09),(0.26,0.02,0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.11,0.01,0.19),(0.22,0.12,0.16)\rangle$	$\langle(0.06,0.02,0.11),(0.09,0.16,0.52)\rangle$	$\langle(0.04,0.01,0.03),(0.11,0.12,0.13)\rangle$

These analyses indicate the properties of laptops by considering the maximum and minimum grades using the reports of both experts. However, the combined analyses come out to yield reports with same scores for the attributes not considered mutually. This further asks for more filtered analyses considering only the attributes considered by both of the experts. To deal with this, Tables 13 and 14 provide the restricted union $\Omega_1 \Cup_{\mathcal{R}} \Omega_2$ and restricted intersection $\Omega_1 \Cap_{\mathcal{R}} \Omega_2$, respectively.

Table 13. Restricted union between Ω_1 and Ω_2.

$\Omega_1 \Cup_{\mathcal{R}} \Omega_2$	$\mathfrak{q}_1$
$\mathfrak{p}_1$	$\langle(0.02,0.01,0.19),(0.03,0.11,0.45)\rangle$
$\mathfrak{p}_2$	$\langle(0.11,0.06,0.13),(0.13,0.01,0.26)\rangle$
$\mathfrak{p}_3$	$\langle(0.26,0.31,0.14),(0.13,0.24,0.09)\rangle$
$\mathfrak{p}_4$	$\langle(0.81,0.01,0.02),(0.13,0.12,0.36)\rangle$

Table 14. Restricted intersection between Ω_1 and Ω_2.

$\Omega_1 \Cap_{\mathcal{R}} \Omega_2$	$\mathfrak{q}_1$
$\mathfrak{p}_1$	$\langle(0.01,0.01,0.23),(0.04,0.11,0.31)\rangle$
$\mathfrak{p}_2$	$\langle(0.03,0.06,0.39),(0.21,0.01,0.09)\rangle$
$\mathfrak{p}_3$	$\langle(0.23,0.31,0.16),(0.13,0.24,0.01)\rangle$
$\mathfrak{p}_4$	$\langle(0.11,0.01,0.19),(0.22,0.12,0.16)\rangle$

The following shows the commutative and associative properties of extended and restricted P_RFBSS intersection.

Proposition 4. *Let $\mathscr{P}$ be a universal set and let $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$, $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ and $\Omega_3 = (\mathscr{G}_3, \mathscr{H}_3, \mathscr{Q}_3, \neg\mathscr{Q}_3)$ be three P_RFBSSs on $\mathscr{P}$. Then:*

1. $\Omega_1 \Cap_{\mathcal{E}} \Omega_2 = \Omega_2 \Cap_{\mathcal{E}} \Omega_1$;
2. $\Omega_1 \Cap_{\mathcal{R}} \Omega_2 = \Omega_1 \Cap_{\mathcal{R}} \Omega_1$;
3. $\Omega_1 \Cap_{\mathcal{E}} (\Omega_2 \Cap_{\mathcal{E}} \Omega_3) = (\Omega_1 \Cap_{\mathcal{E}} \Omega_2) \Cap_{\mathcal{E}} \Omega_3$;
4. $\Omega_1 \Cap_{\mathcal{R}} (\Omega_2 \Cap_{\mathcal{R}} \Omega_3) = (\Omega_1 \Cap_{\mathcal{R}} \Omega_2) \Cap_{\mathcal{R}} \Omega_3$.

Proof.

1. From Definition 14:

$$\begin{aligned}\Omega_1 \Cap_{\mathcal{E}} \Omega_2 =& (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \Cap_{\mathcal{E}} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)\\ =& (\mathscr{G}_1 \Cap \mathscr{G}_2, \mathscr{H}_1 \Cup \mathscr{H}_2, \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2).\end{aligned}$$

Such that $\forall\, \mathfrak{q} \in \mathscr{Q}_1 \cup \mathscr{Q}_2$,

$$\mathscr{G}_1 \Cap \mathscr{G}_2 = \begin{cases} \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1 \overline{\sqcap} \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2. \end{cases} = \begin{cases} \mathscr{G}_2 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2 \overline{\sqcap} \mathscr{G}_1 & \text{if } \mathfrak{q} \in \mathscr{Q}_2 \cap \mathscr{Q}_1. \end{cases} = \mathscr{G}_2 \Cap \mathscr{G}_1.$$

Similarly, $\forall\, \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2$,

$$\mathscr{H}_1 \Cup \mathscr{H}_2 = \begin{cases} \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 - \neg\mathscr{Q}_2, \\ \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 - \neg\mathscr{Q}_1, \\ \mathscr{H}_1 \underline{\sqcup} \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2. \end{cases} = \begin{cases} \mathscr{H}_2 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 - \neg\mathscr{Q}_1, \\ \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_1 - \neg\mathscr{Q}_2, \\ \mathscr{H}_2 \underline{\sqcup} \mathscr{H}_1 & \text{if } \neg\mathfrak{q} \in \neg\mathscr{Q}_2 \cap \neg\mathscr{Q}_1. \end{cases} = \mathscr{H}_2 \Cup \mathscr{H}_1.$$

This implies that

$$(\mathscr{G}_1 \Cap \mathscr{G}_2, \mathscr{H}_1 \Cup \mathscr{H}_2, \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2) = (\mathscr{G}_2 \Cap \mathscr{G}_1, \mathscr{H}_2 \Cup \mathscr{H}_1, \mathscr{Q}_2 \cup \mathscr{Q}_1, \neg\mathscr{Q}_2 \cup \neg\mathscr{Q}_1).$$

Hence, $\Omega_1 \Cap_{\mathcal{E}} \Omega_2 = \Omega_2 \Cap_{\mathcal{E}} \Omega_1$.

2. From Definition 15:

$$\begin{aligned} \Omega_1 \Cap_{\mathcal{R}} \Omega_2 =& (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1) \Cap_{\mathcal{R}} (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2) \\ =& (\mathscr{G}_1 \Cap \mathscr{G}_2, \mathscr{H}_1 \Cup \mathscr{H}_2, \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2). \end{aligned}$$

Such that $\forall\, \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2$,

$$\mathscr{G}_1 \Cap \mathscr{G}_2 = \mathscr{G}_1 \overline{\sqcap} \mathscr{G}_2 = \mathscr{G}_2 \overline{\sqcap} \mathscr{G}_1 = \mathscr{G}_2 \Cap \mathscr{G}_1,$$

and $\forall\, \neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2$,

$$\mathscr{H}_1 \Cup \mathscr{H}_2 = \mathscr{H}_1 \underline{\sqcup} \mathscr{H}_2 = \mathscr{H}_2 \underline{\sqcup} \mathscr{H}_1 = \mathscr{H}_2 \Cup \mathscr{H}_1.$$

This implies that

$$(\mathscr{G}_1 \Cap \mathscr{G}_2, \mathscr{H}_1 \Cup \mathscr{H}_2, \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2) = (\mathscr{G}_2 \Cap \mathscr{G}_1, \mathscr{H}_2 \Cup \mathscr{H}_1, \mathscr{Q}_2 \cap \mathscr{Q}_1, \neg\mathscr{Q}_2 \cap \neg\mathscr{Q}_1).$$

Hence, $\Omega_1 \Cap_{\mathcal{R}} \Omega_2 = \Omega_2 \Cap_{\mathcal{R}} \Omega_1$.

The remaining parts (3 and 4) can be proven in the same way as the above parts. □

Proposition 5. *Let* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$, $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *and* $\Omega_3 = (\mathscr{G}_3, \mathscr{H}_3, \mathscr{Q}_3, \neg\mathscr{Q}_3)$ *be three* $\mathrm{P_R}$*FBSSs on the universe* $\mathscr{P}$*. Then:*

1. $\Omega_1 \Cup_{\mathcal{E}} (\Omega_2 \Cap_{\mathcal{E}} \Omega_3) = (\Omega_1 \Cup_{\mathcal{E}} \Omega_2) \Cap_{\mathcal{E}} (\Omega_1 \Cup_{\mathcal{E}} \Omega_3)$;
2. $\Omega_1 \Cup_{\mathcal{R}} (\Omega_2 \Cap_{\mathcal{R}} \Omega_3) = (\Omega_1 \Cup_{\mathcal{R}} \Omega_2) \Cap_{\mathcal{R}} (\Omega_1 \Cup_{\mathcal{R}} \Omega_3)$;
3. $\Omega_1 \Cap_{\mathcal{E}} (\Omega_2 \Cup_{\mathcal{E}} \Omega_3) = (\Omega_1 \Cap_{\mathcal{E}} \Omega_2) \Cup_{\mathcal{E}} (\Omega_1 \Cap_{\mathcal{E}} \Omega_3)$;
4. $\Omega_1 \Cap_{\mathcal{R}} (\Omega_2 \Cup_{\mathcal{R}} \Omega_3) = (\Omega_1 \Cap_{\mathcal{R}} \Omega_2) \Cup_{\mathcal{R}} (\Omega_1 \Cap_{\mathcal{R}} \Omega_3)$.

Proof. The proof can be easily deduced from the arguments used in proofs of Propositions 3 and 4. □

The following lemma discusses the concepts of minimal and maximal $\mathrm{P_R}$FBSSs.

Lemma 1. *Let* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *and* $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *be two* $\mathrm{P_R}$*FBSSs on a universe* $\mathscr{P}$*. Then:*

1. $\Omega_1 \Cup_{\mathcal{E}} \Omega_2$ *is the minimal* $\mathrm{P_R}$*FBSS over* $\mathscr{P}$ *containing both* Ω_1 *and* Ω_2*.*
2. $\Omega_1 \Cap_{\mathcal{R}} \Omega_2$ *is the maximal* $\mathrm{P_R}$*FBSS over* $\mathscr{P}$ *that is contained in both* Ω_1 *and* Ω_2*.*

Proof. It is directly followed by Definitions 12 and 15. □

Theorem 1. *Let* $\Omega_1 = (\mathscr{G}_1, \mathscr{H}_1, \mathscr{Q}_1, \neg\mathscr{Q}_1)$ *and* $\Omega_2 = (\mathscr{G}_2, \mathscr{H}_2, \mathscr{Q}_2, \neg\mathscr{Q}_2)$ *be two* $\mathrm{P_R}$*FBSSs on the universe* $\mathscr{P}$*. Then:*

1. $(\Omega_1 \Cup_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \Cap_{\mathcal{E}} (\Omega_2)^c$;
2. $(\Omega_1 \Cup_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \Cap_{\mathcal{R}} (\Omega_2)^c$;
3. $(\Omega_1 \Cap_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \Cup_{\mathcal{E}} (\Omega_2)^c$;
4. $(\Omega_1 \Cap_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \Cup_{\mathcal{R}} (\Omega_2)^c$.

Proof. 1. By Definitions 8 and 12, we obtain

$$(\Omega_1 \cup_{\mathcal{E}} \Omega_2)^c = ((\mathscr{G}_1 \Cup \mathscr{G}_2)^c, (\mathscr{H}_1 \Cap \mathscr{H}_2)^c, \mathscr{Q}_1 \cup \mathscr{Q}_2, \neg\mathscr{Q}_1 \cup \neg\mathscr{Q}_2),$$

where:

$$(\mathscr{G}_1 \Cup \mathscr{G}_2)^c(\mathfrak{q}) = \begin{cases} \mathscr{G}_1^c(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 - \mathscr{Q}_2, \\ \mathscr{G}_2^c(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_2 - \mathscr{Q}_1, \\ \mathscr{G}_1^c(\mathfrak{q}) \overline{\sqcap} \mathscr{G}_2^c(\mathfrak{q}), & \text{if } \mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2. \end{cases}$$
$$= (\mathscr{G}_1^c \Cap \mathscr{G}_2^c)(\mathfrak{q}), \text{ by Definition 14, and}$$

$$(\mathscr{H}_1 \Cap \mathscr{H}_2)^c(\neg\mathfrak{q}) = \begin{cases} \mathscr{H}_1^c(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) - (\neg\mathscr{Q}_2), \\ \mathscr{H}_2^c(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_2) - (\neg\mathscr{Q}_1), \\ \mathscr{H}_1^c(\neg\mathfrak{q}) \underline{\sqcup} \mathscr{H}_2^c(\neg\mathfrak{q}), & \text{if } \neg\mathfrak{q} \in (\neg\mathscr{Q}_1) \cap (\neg\mathscr{Q}_2). \end{cases}$$
$$= (\mathscr{H}_1^c \Cup \mathscr{H}_2^2)(\neg\mathfrak{q}), \text{ by Definition 14.}$$

Hence, $(\Omega_1 \Cup_{\mathcal{E}} \Omega_2)^c = (\Omega_1)^c \Cap_{\mathcal{E}} (\Omega_2)^c$.

2. By Definitions 8 and 13, we obtain:

$$(\Omega_1 \cup_{\mathcal{R}} \Omega_2)^c = ((\mathscr{G}_1 \Cup \mathscr{G}_2)^c, (\mathscr{H}_1 \Cap \mathscr{H}_2)^c, \mathscr{Q}_1 \cap \mathscr{Q}_2, \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2),$$

where, for all $\mathfrak{q} \in \mathscr{Q}_1 \cap \mathscr{Q}_2$,

$$\begin{aligned}(\mathscr{G}_1 \Cup \mathscr{G}_2)^c(\mathfrak{q}) &= \mathscr{G}_1^c(\mathfrak{q}) \overline{\sqcap} \mathscr{G}_2^c(\mathfrak{q}) \\ &= (\mathscr{G}_1^c \Cap \mathscr{G}_2^c)(\mathfrak{q}), \text{ by Definition 15.}\end{aligned}$$

Similarly, for all $\neg\mathfrak{q} \in \neg\mathscr{Q}_1 \cap \neg\mathscr{Q}_2$,

$$\begin{aligned}(\mathscr{H}_1 \Cap \mathscr{H}_2)^c(\neg\mathfrak{q}) &= \mathscr{H}_1^c(\neg\mathfrak{q}) \underline{\sqcup} \mathscr{H}_2^c(\neg\mathfrak{q}) \\ &= (\mathscr{H}_1^c \Cup \mathscr{H}_2^2)(\neg\mathfrak{q}), \text{ by Definition 15.}\end{aligned}$$

Hence, $(\Omega_1 \Cup_{\mathcal{R}} \Omega_2)^c = (\Omega_1)^c \Cap_{\mathcal{R}} (\Omega_2)^c$.

The remaining parts (3 and 4) can be easily followed using similar arguments. □

4. Application to MCDM

This section first discusses an algorithm under the initiated $\mathrm{P_R}$FBSSs, and then implements it on a daily-life MCDM problem to rank available alternatives subject to the governing criterion. The following Algorithm 1 is used to choose the best alternative or to find the rankings of alternatives when the estimations are given in the form of a $\mathrm{P_R}$FBSS.

Algorithm 1: Selection of best alternative under $\mathrm{P_R}$FBSS environment.

(1) **Input:**

(i) a universe $\mathscr{P}$ having n alternatives,
(ii) a set $\mathscr{R}$ of decision-attributes,
(iii) a not set $\neg\mathscr{R}$ of decision-attributes opposite to those in $\mathscr{R}$,
(iv) a $\mathrm{P_R}$FBSS $\Omega = (\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ in tabular form based on the decision-maker's opinions where $\mathscr{Q} \subseteq \mathscr{R}$, and $\neg\mathscr{Q} \subseteq \neg\mathscr{R}$.

(2) Construct the score-value table using Definition 3 by finding the scores of all the PFNs $(\mathscr{F}_{ij})$ regarding favorable set of attributes $\mathscr{Q} \subseteq \mathscr{R}$, which are given in the associated PFSS $(\mathscr{G}, \mathscr{Q})$. Then compute the object-wise sum (i.e., $\sum_i s(\mathscr{F}_{ij})$) of score values in the last column of this table.

(3) Construct the score-value-table using Definition 3 by finding the scores of all the PFNs $(\neg\mathscr{F}_{ij})$ regarding unfavorable set of attributes $\neg\mathscr{Q} \subseteq \neg\mathscr{R}$, which are given in the associated PFSS $(\mathscr{H}, \neg\mathscr{Q})$. Then compute the object-wise sum (i.e., $\sum_i s(\neg\mathscr{F}_{ij})$) of score values in the last column of this table.

(4) Calculate the final scores for all the objects using formula
$\mathfrak{s}_i = \sum_i s(\mathscr{F}_{ij}) - \sum_i s(\neg\mathscr{F}_{ij})$.

(5) Find k for which $\mathfrak{s}_k = \max(\mathfrak{s}_i)$.

Output: The last step declares $\mathfrak{p}_k$ as the most appropriate alternative. In case of multiple values of k, any of the alternative $\mathfrak{p}_k$ with same score can be considered as an optimal alternative.

To better understand the processing of Algorithm 1, readers may refer to flowchart diagram (see Figure 2).

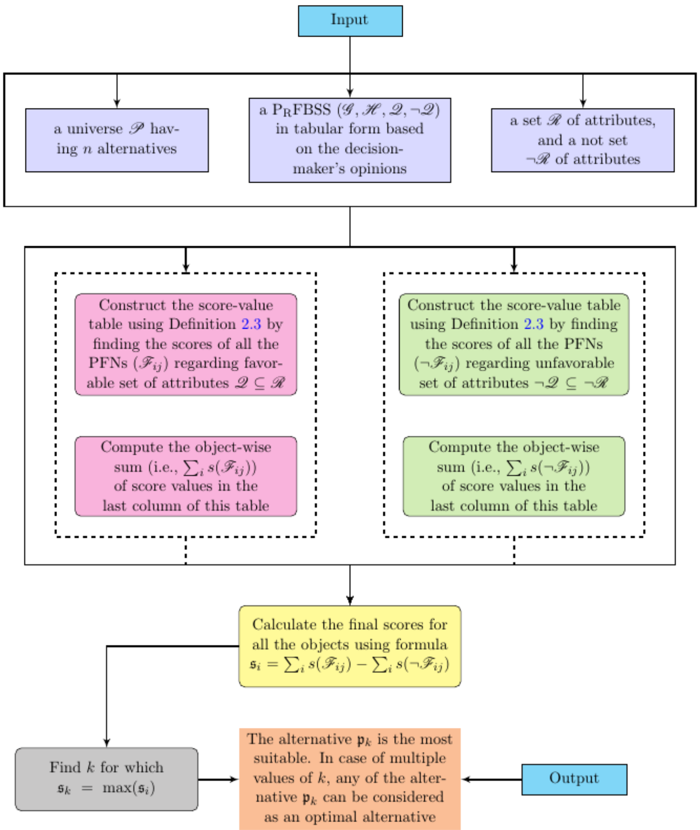

Figure 2. Graphical structure of Algorithm 1.

Application: Selection of Best Graphic Designer for a Studio

Graphic design is one of the largest industries due to its significance in marketing, advertisements, showcasing, modeling, movies, and so on. Nowadays, the graphic design industry needs to grow and glow in all disciplines ranging from arts to sciences to keep up with the ever-evolving digital tools and to stay competitive with the digital platforms in showcasing and implement marketing strategies, advertising, and designing. Graphic design is mainly based on the customer's needs and the graphic designer's particular tools to achieve the required goals. A graphic designer communicates ideas and information through visual concepts by turning their ideas into digital images and presentations. Using different graphic design software such as Adobe Photoshop, Adobe Illustrator, Corel Draw, Blender, Crello, etc., they create immersive visuals that inspire audiences. Graphic design being a huge field, which includes designing everything from posters to logos, packaging designs, brand identity designs, etc., graphic designers are employed all over the world in many places based on their experience, expertise and interests relevant to the specific areas they excel in. Today, the graphic designing market stands globally at a massive total of approximately USD 43.4 billion, with the industry growing at a rate of about 2.5% each year. The growth of the graphic design industry has been at 0.6% annually since 2017, but a massive growth of 3.7% is expected in 2022 (https://www.ibisworld.com) (accesssed on 3 October 2022).

Among the many different uses of graphic designing, digital marketing and advertisement are the most important. Companies and businesses are switching over to digital marketing strategies because they are much more cost-efficient than traditional methods of advertising. Graphic design plays a key role in digital marketing because it is such a visual industry. Having a strong and appealing design for a company's website, social media pages and other marketing channels can help the company to connect with its audience and increase its brand loyalty. Digital marketing is all about reaching potential customers on their terms. Customers are constantly online and are looking for businesses that cater to their needs easily and conveniently. Graphic design is an essential aspect of digital marketing because it is one of the most visual forms of marketing. It uses images, visuals and other images to promote products or services and reach customers. The visual appeal of the designs in digital marketing campaigns is extremely important because it can make or break a brand. A business loses out on potential customers if the representative website and social media pages are cluttered, messy or unclear. However, if the website and social media pages are extremely clean, organized and visually appealing, one will be able to reach a wider audience and convert more customers. However, choosing the most appropriate graphic designer for the business is not a piece of cake. Graphic designers must be chosen that will perform for the recruiter in the best way.

Consider a multinational online marketing company that needs to fill in the vacancy of a graphic designer to help their clients reach their customers effectively. Suppose $\mathscr{P} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_{15}\}$ is the set of fifteen candidates who applied for the vacancy. Now, the head of the graphic design studio of the company collects personalized demo tests from each candidate by giving them a model task to promote a certain product through a catchy video advertisement in a way that keeps the audience engaged and interested within prescribed time limits, followed by critical comments and discussion with the candidates about their creation. Based on these tests, the head of the studio decides to pick out the best candidate considering multiple criteria. Suppose $\mathscr{R} = \{\mathfrak{q}_1, \mathfrak{q}_2, \ldots \mathfrak{q}_7\}$ is the set of favorable criteria considered by the selector, where, for $\mathfrak{q} = 1, 2, \ldots, 7$, the criterion $\mathfrak{q}_j$ can serve as active listening, good communication skills, time management, improvement based on critical comments, patience, co-operation and effective storytelling. Accordingly, the considered not-set of parameters is $\neg\mathscr{R} = \{\neg\mathfrak{q}_1 = \text{impassivity}, \neg\mathfrak{q}_2 = \text{bad communication skills}, \neg\mathfrak{q}_3 = \text{disorganization}, \neg\mathfrak{q}_4 = \text{stubborness}, \neg\mathfrak{q}_5 = \text{impatience}, \neg\mathfrak{q}_6 = \text{noncooperation}, \neg\mathfrak{q}_7 = \text{impatience}\}$. After a brief discussion among the members of the selection committee appointed by the studio, the head of the studio decides to evaluate each candidate under a favorable subset of parameters $\mathscr{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{q}_5, \mathfrak{q}_7\}$ of $\mathscr{R}$.

According to the estimations of the committee members about the applicants, a $\mathrm{P_R}$FBSS $(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$ is constructed which describes the qualities of the candidates (see Table 15).

Table 15. Tabular form of the $\mathrm{P_R}$FBSS $(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$.

$(\mathscr{G}, \mathscr{H}, \mathscr{Q}, \neg\mathscr{Q})$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$	$\mathfrak{q}_5$	$\mathfrak{q}_7$
$\mathfrak{p}_1$	$\langle(0.01,0.09,0.32),(0.06,0.19,0.36)\rangle$	$\langle(0.92,0.02,0.01),(0.01,0.09,0.02)\rangle$	$\langle(0.16,0.17,0.19),(0.17,0.18,0.19)\rangle$	$\langle(0.01,0.03,0.11),(0.02,0.13,0.06)\rangle$	$\langle(0.01,0.02,0.03),(0.07,0.69,0.01)\rangle$
$\mathfrak{p}_2$	$\langle(0.21,0.01,0.11),(0.13,0.81,0.02)\rangle$	$\langle(0.16,0.01,0.02),(0.11,0.01,0.02)\rangle$	$\langle(0.39,0.11,0.26),(0.11,0.01,0.02)\rangle$	$\langle(0.32,0.21,0.01),(0.02,0.01,0.02)\rangle$	$\langle(0.02,0.03,0.01),(0.02,0.11,0.16)\rangle$
$\mathfrak{p}_3$	$\langle(0.11,0.19,0.66),(0.72,0.11,0.01)\rangle$	$\langle(0.16,0.26,0.16),(0.36,0.19,0.02)\rangle$	$\langle(0.11,0.12,0.33),(0.11,0.23,0.39)\rangle$	$\langle(0.56,0.01,0.19),(0.11,0.13,0.19)\rangle$	$\langle(0.16,0.16,0.13),(0.14,0.19,0.23)\rangle$
$\mathfrak{p}_4$	$\langle(0.13,0.11,0.12),(0.23,0.16,0.02)\rangle$	$\langle(0.27,0.01,0.09),(0.69,0.01,0.13)\rangle$	$\langle(0.12,0.23,0.34),(0.13,0.23,0.16)\rangle$	$\langle(0.26,0.31,0.01),(0.21,0.41,0.01)\rangle$	$\langle(0.03,0.16,0.19),(0.17,0.11,0.21)\rangle$
$\mathfrak{p}_5$	$\langle(0.34,0.23,0.21),(0.16,0.01,0.02)\rangle$	$\langle(0.16,0.26,0.31),(0.21,0.23,0.33)\rangle$	$\langle(0.16,0.21,0.23),(0.39,0.01,0.11)\rangle$	$\langle(0.13,0.23,0.23),(0.13,0.01,0.01)\rangle$	$\langle(0.21,0.31,0.02),(0.69,0.01,0.02)\rangle$
$\mathfrak{p}_6$	$\langle(0.39,0.19,0.06),(0.21,0.22,0.12)\rangle$	$\langle(0.34,0.11,0.01),(0.12,0.11,0.19)\rangle$	$\langle(0.09,0.02,0.06),(0.09,0.13,0.26)\rangle$	$\langle(0.11,0.39,0.03),(0.01,0.02,0.96)\rangle$	$\langle(0.01,0.79,0.01),(0.01,0.01,0.02)\rangle$
$\mathfrak{p}_7$	$\langle(0.13,0.19,0.26),(0.19,0.01,0.23)\rangle$	$\langle(0.26,0.13,0.21),(0.12,0.33,0.15)\rangle$	$\langle(0.16,0.13,0.45),(0.17,0.01,0.16)\rangle$	$\langle(0.86,0.01,0.01),(0.13,0.73,0.01)\rangle$	$\langle(0.06,0.09,0.03),(0.08,0.07,0.08)\rangle$
$\mathfrak{p}_8$	$\langle(0.19,0.29,0.30),(0.12,0.07,0.08)\rangle$	$\langle(0.11,0.12,0.33),(0.23,0.11,0.09)\rangle$	$\langle(0.62,0.13,0.11),(0.02,0.31,0.33)\rangle$	$\langle(0.26,0.02,0.03),(0.19,0.29,0.32)\rangle$	$\langle(0.81,0.01,0.02),(0.73,0.01,0.01)\rangle$
$\mathfrak{p}_9$	$\langle(0.11,0.22,0.33),(0.33,0.22,0.11)\rangle$	$\langle(0.19,0.23,0.16),(0.11,0.21,0.13)\rangle$	$\langle(0.16,0.23,0.17),(0.11,0.13,0.33)\rangle$	$\langle(0.69,0.01,0.02),(0.03,0.01,0.01)\rangle$	$\langle(0.01,0.02,0.04),(0.06,0.09,0.23)\rangle$
$\mathfrak{p}_{10}$	$\langle(0.16,0.23,0.31),(0.16,0.34,0.21)\rangle$	$\langle(0.16,0.23,0.02),(0.20,0.21,0.31)\rangle$	$\langle(0.16,0.20,0.01),(0.01,0.16,0.02)\rangle$	$\langle(0.01,0.01,0.02),(0.39,0.46,0.02)\rangle$	$\langle(0.11,0.13,0.19),(0.23,0.11,0.16)\rangle$
$\mathfrak{p}_{11}$	$\langle(0.23,0.31,0.11),(0.23,0.16,0.19)\rangle$	$\langle(0.56,0.16,0.16),(0.16,0.32,0.41)\rangle$	$\langle(0.11,0.01,0.01),(0.01,0.13,0.39)\rangle$	$\langle(0.11,0.23,0.26),(0.13,0.12,0.11)\rangle$	$\langle(0.26,0.13,0.16),(0.91,0.02,0.01)\rangle$
$\mathfrak{p}_{12}$	$\langle(0.16,0.02,0.21),(0.11,0.22,0.31)\rangle$	$\langle(0.11,0.16,0.23),(0.16,0.31,0.22)\rangle$	$\langle(0.11,0.21,0.30),(0.11,0.21,0.31)\rangle$	$\langle(0.23,0.11,0.01),(0.01,0.02,0.07)\rangle$	$\langle(0.11,0.12,0.13),(0.13,0.16,0.23)\rangle$
$\mathfrak{p}_{13}$	$\langle(0.16,0.11,0.23),(0.13,0.11,0.12)\rangle$	$\langle(0.16,0.16,0.02),(0.13,0.71,0.02)\rangle$	$\langle(0.19,0.03,0.13),(0.36,0.19,0.11)\rangle$	$\langle(0.11,0.13,0.23),(0.16,0.19,0.16)\rangle$	$\langle(0.26,0.31,0.32),(0.21,0.22,0.23)\rangle$
$\mathfrak{p}_{14}$	$\langle(0.01,0.06,0.70),(0.16,0.23,0.21)\rangle$	$\langle(0.16,0.01,0.11),(0.12,0.01,0.30)\rangle$	$\langle(0.11,0.20,0.30),(0.12,0.30,0.24)\rangle$	$\langle(0.23,0.45,0.01),(0.01,0.02,0.06)\rangle$	$\langle(0.41,0.23,0.33),(0.56,0.11,0.23)\rangle$
$\mathfrak{p}_{15}$	$\langle(0.13,0.23,0.11),(0.21,0.09,0.06)\rangle$	$\langle(0.59,0.23,0.11),(0.02,0.19,0.59)\rangle$	$\langle(0.12,0.13,0.16),(0.13,0.12,0.01)\rangle$	$\langle(0.11,0.12,0.31),(0.13,0.23,0.27)\rangle$	$\langle(0.11,0.23,0.11),(0.69,0.01,0.02)\rangle$

The committee decides to use Algorithm 1 to find the best designer based on the $\mathrm{P_R}$FBSS report. Using Definition 3, the score values of all $\mathrm{P_R}$FNs in both $\mathrm{P_R}$FSSs $(\mathscr{G}, \mathscr{Q})$ and $(\mathscr{H}, \neg\mathscr{Q})$ are computed and then shown in Tables 16 and 17, respectively.

Table 16. Score-values table for $\mathrm{P_R}$FSS $(\mathscr{G}, \mathscr{Q})$.

$(\mathscr{G}, \mathscr{Q})$	$\mathfrak{q}_1$	$\mathfrak{q}_2$	$\mathfrak{q}_3$	$\mathfrak{q}_5$	$\mathfrak{q}_7$	$\sum s(\mathscr{F})$
$\mathfrak{p}_1$	0.3275	1.4100	0.5225	0.4475	0.4900	3.1975
$\mathfrak{p}_2$	0.6525	0.6475	0.7325	0.7625	0.5075	3.3025
$\mathfrak{p}_3$	0.2325	0.5150	0.4150	0.9625	0.5550	2.6800
$\mathfrak{p}_4$	0.5425	0.7225	0.3925	0.6775	0.3950	2.7300
$\mathfrak{p}_5$	0.6775	0.4400	0.4425	0.4575	0.6225	2.6400
$\mathfrak{p}_6$	0.8125	0.8075	0.5550	0.4975	0.3075	2.9800
$\mathfrak{p}_7$	0.4525	0.6225	0.4025	1.3525	0.5225	3.3525
$\mathfrak{p}_8$	0.4675	0.4150	1.0325	0.7400	1.2975	3.9525
$\mathfrak{p}_9$	0.3900	0.5525	0.5175	1.1775	0.4850	3.1225
$\mathfrak{p}_{10}$	0.4475	0.5925	0.6050	0.4975	0.4825	2.6250
$\mathfrak{p}_{11}$	0.5975	0.9400	0.6025	0.4225	0.6475	3.2100
$\mathfrak{p}_{12}$	0.5500	0.4550	0.4075	0.6975	0.5150	2.6250
$\mathfrak{p}_{13}$	0.5175	0.6100	0.6175	0.4625	0.5225	2.7300
$\mathfrak{p}_{14}$	0.1450	0.6025	0.4100	0.6125	0.6875	2.4575
$\mathfrak{p}_{15}$	0.5175	0.9775	0.5075	0.4250	0.4975	2.9250

Table 17. Score-values table for $\mathrm{P_RFSS}$ $(\mathscr{H}, \neg\mathscr{Q})$.

$(\mathscr{H}, \neg\mathscr{Q})$	$\neg\mathfrak{q}_1$	$\neg\mathfrak{q}_2$	$\neg\mathfrak{q}_3$	$\neg\mathfrak{q}_5$	$\neg\mathfrak{q}_7$	$\sum s(\neg\mathscr{F})$
$\mathfrak{p}_1$	0.3325	0.4775	0.5300	0.4575	0.3925	2.1900
$\mathfrak{p}_2$	0.4175	0.5975	0.5975	0.5075	0.4625	2.5825
$\mathfrak{p}_3$	1.1875	0.8025	0.3575	0.4825	0.4775	3.3075
$\mathfrak{p}_4$	0.6800	1.1225	0.4925	0.6025	0.5375	3.4350
$\mathfrak{p}_5$	0.6475	0.4875	0.8325	0.6225	1.1775	3.7675
$\mathfrak{p}_6$	0.5950	0.4975	0.4275	0.0250	0.4975	2.0425
$\mathfrak{p}_7$	0.5725	0.4625	0.5875	0.4425	0.5225	2.5875
$\mathfrak{p}_8$	0.5625	0.6575	0.2775	0.4575	1.2000	3.1550
$\mathfrak{p}_9$	0.7200	0.4925	0.4125	0.5225	0.4225	2.5695
$\mathfrak{p}_{10}$	0.4700	0.4925	0.4600	0.7650	0.6225	2.8100
$\mathfrak{p}_{11}$	0.5950	0.3750	0.2825	0.4950	1.4000	3.1475
$\mathfrak{p}_{12}$	0.4000	0.4725	0.4025	0.4700	0.4750	2.2200
$\mathfrak{p}_{13}$	0.5425	0.4425	0.7575	0.5325	0.5400	2.8150
$\mathfrak{p}_{14}$	0.4475	0.4675	0.4250	0.4750	0.9175	2.7325
$\mathfrak{p}_{15}$	0.6575	0.1775	0.5950	0.4375	0.1775	2.0450

Finally, Table 18 provides the final scores for each candidate.

Table 18. Final scores table.

$\mathscr{P}$	$\sum s(\mathscr{F})$	$\sum s(\neg\mathscr{F})$	$\mathfrak{s} = \sum s(\mathscr{F}) - \sum s(\neg\mathscr{F})$
$\mathfrak{p}_1$	3.1975	2.1900	1.0075
$\mathfrak{p}_2$	3.3025	2.5825	0.7200
$\mathfrak{p}_3$	2.6800	3.3075	−0.6275
$\mathfrak{p}_4$	2.7300	3.4350	−0.7050
$\mathfrak{p}_5$	2.6400	3.7675	−1.1275
$\mathfrak{p}_6$	2.9800	2.0425	0.9375
$\mathfrak{p}_7$	3.3525	2.5875	0.7650
$\mathfrak{p}_8$	3.9525	3.1550	0.7975
$\mathfrak{p}_9$	3.1225	2.5695	0.5530
$\mathfrak{p}_{10}$	2.6250	2.8100	−0.1850
$\mathfrak{p}_{11}$	3.2100	3.1475	0.0625
$\mathfrak{p}_{12}$	2.6250	2.2200	0.4050
$\mathfrak{p}_{13}$	2.7300	2.8150	−0.0850
$\mathfrak{p}_{14}$	2.4575	2.7325	−0.2750
$\mathfrak{p}_{15}$	2.9250	2.0450	0.8800

Clearly, the candidate $\mathfrak{p}_1$ aced the selection procedure, scoring higher than all other candidates. Therefore, the head of the studio decides to hire candidate $\mathfrak{p}_1$ for the post of graphic designer.

5. Comparison

In this section, we discuss the comparative analysis of the initiated model with some previous models, qualitatively and quantitatively. Furthermore, this section highlights the overall advantages and limitations of the proposed model.

5.1. Advantages

The previously existing models, namely, $\mathrm{P_RFSs}$, $\mathrm{B_RSSs}$ and $\mathrm{P_RFSSs}$, prove to be very powerful in their existing domains; however, they all have limitations. For instance, models such as $\mathrm{P_RFSSs}$ are unable to differentiate between the symmetrically opposing sets of decision attributes or criteria (bipolarity of decision attributes), whereas $\mathrm{B_RSSs}$ fails to deal with complicated uncertainties requiring the consideration of degrees of satisfaction, dissatisfaction and neutrality. The initiated $\mathrm{P_RFBSSs}$ as a generalization of these structures allows the consideration of positive, negative and neutral membership degrees while differentiating the decision attributes with respect to their bipolarities. In this way, the initiated model not only overcomes the limitations in the previous tools but also offers modeling and handling of decision-making problems from a relatively larger domain as compared to its predecessors. In short, P_RFBSSs are stronger, more efficient, more applicable and more generalizable than the previous models, including P_RFSs, B_RFSs, fuzzy B_RSSs and P_RFSSs.

5.2. Comparison

The developed hybrid model (P_RFBSSs) is the combination of two efficient models, namely, P_RFSs [20] and B_RSSs [51]. The reason for this development comes from the limitations of these models, i.e., P_RFSs cannot handle the bipolarity of decision attributes, whereas B_RSSs cannot handle uncertainties efficiently. Being a generalization of these models, the proposed P_RFBSS model overcomes these limitations efficiently. To show the supremacy of P_RFBSSs, the problem discussed in the previous section is again solved using fuzzy B_RSSs [52] and P_RFSSs [28]. The comparison of the final scores and the final ranking given by these three models is given in Tables 19 and 20, respectively. From this comparison, it is clear that fuzzy B_RSSs failed to make a distinction between $\mathfrak{p}_1$ and $\mathfrak{p}_7$. This is because of the information lost in the form of negative and neutral membership degrees. Similarly, P_RFSSs failed to differentiate between the ranks of $\mathfrak{p}_4$, $\mathfrak{p}_{13}$ and $\mathfrak{p}_{10}$, $\mathfrak{p}_{12}$. Furthermore, the optimal decision offered by P_RFSSs does not comply with the decisions offered by the other two models. The reason for this variation in results is the inability of P_RFSSs to consider the bipolarity of decision attributes. On the other hand, the proposed P_RFBSSs successfully provides a ranking of each alternative at a different rank from the other alternatives by considering the maximum conditions. In a similar way, models such as B_RSSs, P_RFSs, intuitionistic fuzzy soft sets, etc., will fail to provide a solution as efficient as P_RFBSSs. This proves the efficiency of the proposed model over the previous models. For a pictorial representation of the discussed comparison, readers may refer to Figure 3. In addition, Table 21 presents a comparative qualitative analysis between the applicability and properties of the proposed model and previous models.

Table 19. Quantitative comparison of the developed model with different existing models.

Models	Fuzzy B_RSSs [52]	P_RFSSs [28]	Proposed P_RFBSSs
$\mathfrak{p}_1$	0.78	3.1975	1.0075
$\mathfrak{p}_2$	0.71	3.3025	0.7200
$\mathfrak{p}_3$	−0.34	2.6800	−0.6275
$\mathfrak{p}_4$	−0.65	2.7300	−0.7050
$\mathfrak{p}_5$	−0.58	2.6400	−1.1275
$\mathfrak{p}_6$	0.50	2.9800	0.9375
$\mathfrak{p}_7$	0.78	3.3525	0.7650
$\mathfrak{p}_8$	0.70	3.9525	0.7975
$\mathfrak{p}_9$	0.52	3.1225	0.5530
$\mathfrak{p}_{10}$	−0.39	2.6250	−0.1850
$\mathfrak{p}_{11}$	−0.17	3.2100	0.0625
$\mathfrak{p}_{12}$	0.20	2.6250	0.4050
$\mathfrak{p}_{13}$	−0.11	2.7300	−0.0850
$\mathfrak{p}_{14}$	−0.05	2.4575	−0.2750
$\mathfrak{p}_{15}$	−0.12	2.9250	0.8800

Table 20. Comparison among the ranking of the presented model with different existing models.

Hybrid Models	Rankings
Fuzzy B_RSSs [52]	$\mathfrak{p}_1 = \mathfrak{p}_7 > \mathfrak{p}_2 > \mathfrak{p}_8 > \mathfrak{p}_9 > \mathfrak{p}_6 > \mathfrak{p}_{12} > \mathfrak{p}_{14} > \mathfrak{p}_{13} > \mathfrak{p}_{15} > \mathfrak{p}_{11} > \mathfrak{p}_3 > \mathfrak{p}_{10} > \mathfrak{p}_4 > \mathfrak{p}_5$
P_RFSSs [28]	$\mathfrak{p}_8 > \mathfrak{p}_7 > \mathfrak{p}_2 > \mathfrak{p}_{11} > \mathfrak{p}_1 > \mathfrak{p}_9 > \mathfrak{p}_6 > \mathfrak{p}_{15} > \mathfrak{p}_4 = \mathfrak{p}_{13} > \mathfrak{p}_3 > \mathfrak{p}_5 > \mathfrak{p}_{10} = \mathfrak{p}_{12} > \mathfrak{p}_{14}$
Proposed P_RFBSSs	$\mathfrak{p}_1 > \mathfrak{p}_6 > \mathfrak{p}_{15} > \mathfrak{p}_8 > \mathfrak{p}_7 > \mathfrak{p}_2 > \mathfrak{p}_9 > \mathfrak{p}_{12} > \mathfrak{p}_{11} > \mathfrak{p}_{13} > \mathfrak{p}_{10} > \mathfrak{p}_{14} > \mathfrak{p}_3 > \mathfrak{p}_4 > \mathfrak{p}_5$

Table 21. Qualitative comparison between decision models.

Decision Model	Advantages	Limitations
Fuzzy set [1]	Handles uncertainties in data sets depicting partial truths by using fuzzy memberships ranging from 0 to 1.	Does not consider a membership degree for disagreement.
Intuitionistic fuzzy set [5]	Uses fuzzy membership and non-membership degrees with their mutual sum bounded by unity.	Does not consider the impartiality or neutral opinion.
Picture fuzzy set [18–20]	Uses positive, neutral and negative membership degrees with their mutual sum bounded by unity.	Fails to handle information affected by multiple decision attributes or parameters.
Soft set [11]	Allows decision making with multiple decision attributes by considering parameterized families of sets together.	Cannot handle uncertain problems dealing with partial truths in the data set.
Fuzzy soft set [14]	Allows decision making with multiple decision attributes by considering parameterized families of fuzzy sets.	Fails to depict a measure of disagreement in the parameterized data set.
Intuitionistic Fuzzy soft set [15]	Deals with intuitionistic fuzzy information in soft environments.	Does not consider the neutral membership degrees.
q-Rung orthopair fuzzy soft set [16]	Extends the membership and non-membership degrees by generalizing the rank of IFS memberships.	Does not consider the neutrality of the information.
Interval-valued q-rung orthopair fuzzy soft set [17]	Increases flexibility by using fuzzy intervals instead of discrete fuzzy memberships in q-rung orthopair fuzzy information.	Fails to consider any measures of neutral membership concerning the information.
Picture fuzzy soft set [28]	Handles picture fuzzy information under the effect of multiple decision attributes.	Cannot depict the bipolarity of decision attributes.
q-Rung picture fuzzy set, q-Rung picture linguistic numbers [46]	Generalizes the sum-restriction condition for picture fuzzy sets by combining with q-rung orthopair fuzzy sets, and uses q-rung picture linguistic Heronian mean operators for decision making.	Fails to differentiate between the bipolar sets of decision attributes.
Picture fuzzy N-soft set [45]	Handles multinary data in picture fuzzy soft environment.	Cannot depict the bipolarity of decision attributes.
Complex picture fuzzy N-soft set [47]	Uses complex picture fuzzy numbers to interpret multinary data in N-soft environment.	Fails to depict bipolarity of decision attributes.
Bipolar soft set [51]	Handles the bipolarity of decision attributes by considering two symmetrically opposite attribute sets.	Fails to handle uncertain information in the data set.
Fuzzy bipolar soft set [52]	Deals with bipolar soft information in fuzzy environment.	Fails to consider non-membership degrees in bipolar soft environment.
q-Rung othopair fuzzy bipolar soft set [55]	Deals with q-rung orthopair fuzzy information in bipolar soft environment.	Cannot give a measure for neutrality in the decision makers opinions.
Proposed Picture fuzzy bipolar soft set	Allows the handling of bipolar soft information in picture fuzzy environment.	Free from all the limitations discussed above.

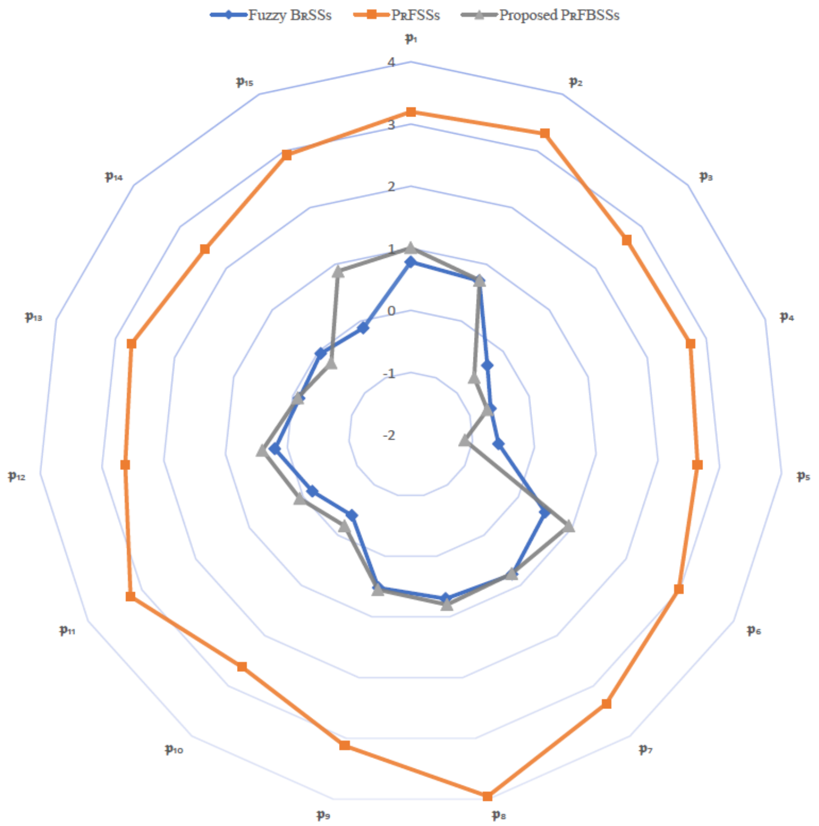

Figure 3. Comparison among the initiated P_RFBSS model and existing Fuzzy BSS [52] and P_RFSS [28] models.

5.3. Limitations

Like the gaps discussed in the previous literature, our proposed model also has some limitations which need to be addressed in the future. One of these limitations comes from the membership sum constraint of the P_RFSs, which restricts the proposed model to work in situations where the sum of positive, negative and neutral membership degrees exceeds unity. Another problem is the limitation of the proposed model and algorithm to a single expert. This makes the model less efficient for group decision-making scenarios.

6. Conclusions and Future Directions

Many complicated decision problems in society are affected by several criteria. For instance, job recruitment for various designations need an effective investigation of multiple attributes of the candidate. Consequently, decision-making problems often require consideration of several criteria affecting the decisions in one place. As a helping hand, multi-criteria decision-making (MCDM) tools offer efficient solutions while considering multiple decision criteria affecting the decision-making process. Among these tools, picture fuzzy soft sets (P_RFSSs) allow solutions to complicated uncertain situations considering the degrees of agreement, disagreement and neutrality in a picture fuzzy soft environment. However, P_RFSSs cannot handle the bipolarity of criteria and are ineffective in problems concerning bipolar criteria. On the other hand, bipolar soft sets (B_RSSs) can easily deal with bipolarity of criterion. However, unlike P_RFSSs, B_RSSs fail to consider uncertainties in the information efficiently. To fill these gaps, we have presented a new hybrid MCDM model called picture fuzzy bipolar soft sets (P_RFBSSs) by the fusion of P_RFSs and B_RSSs. The proposed model is capable of handling picture fuzzy information under two symmetrically opposing sets of decision attributes. Moreover, we have investigated some novel properties of the proposed P_RFBSS model such as sub-set, super-set, equality, complement, relative null and absolute P_RFBSSs, extended union and intersection and restricted union and intersection. Furthermore, two fundamental operations of P_RFBSSs, namely, AND and OR operations are studied. The commutative, associative and distributive properties along with De Morgan's laws about these initiated notions and operations are presented and verified. An application regarding the recruitment of a graphic designer for a studio is discussed under the novel P_RFBSSs. The best designer is selected using the proposed algorithm to show the effective applicability of the initiated model in different real-life complex situations involving uncertainties in P_RFBSS environments. Finally, concerning both qualitative and quantitative perspectives, we have studied a detailed comparative analysis of the proposed model with certain existing models.

One limitation of the model is its inability to discuss problems where the sum of positive, neutral and negative membership degrees exceeds unity. Another limitation is the restriction of decision making to a single expert's opinions. For future works, one can extend our initiated model to

- Rough picture fuzzy bipolar soft sets;
- Picture fuzzy bipolar soft expert sets;
- Picture fuzzy bipolar soft aggregation operators;
- q-Rung picture fuzzy bipolar soft sets;
- Interval-valued picture fuzzy bipolar soft sets;
- Rough picture fuzzy bipolar soft expert sets.

Author Contributions: Conceptualization, G.A., M.Z.U.A. and Q.X.; methodology, G.A. and M.Z.U.A.; validation, G.A., M.Z.U.A. and F.M.O.T.; formal analysis, G.A. and M.Z.U.A.; investigation, G.A., M.Z.U.A. and Q.X.; data curation, G.A. and M.Z.U.A.; writing—original draft preparation, M.Z.U.A., Q.X. and F.M.O.T.; writing—review and editing, G.A., M.Z.U.A. and F.M.O.T.; visualization, G.A. and M.Z.U.A.; supervision, G.A. and Q.X.; project administration, Q.X. and F.M.O.T.; funding acquisition, F.M.O.T. All authors have read and agreed to the published version of the manuscript.

Funding: This work was funded by the King Saud University, Riyadh, Saudi Arabia under Researchers Supporting Project number (RSP2022R440).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
2. Lin, S.S.; Shen, S.L.; Zhou, A.; Xu, Y.S. Risk assessment and management of excavation system based on fuzzy set theory and machine learning methods. *Autom. Constr.* **2021**, *122*, 103490. [CrossRef]
3. Gholamizadeh, K.; Zarei, E.; Omidvar, M.; Yazdi, M. Fuzzy sets theory and human reliability: Review, applications, and contributions. In: Yazdi, M. (eds) Linguistic Methods Under Fuzzy Information in System Safety and Reliability Analysis. *Stud. Fuzziness Soft Comput.* **2022**, *414*, 91–137.
4. Xiao, F. EFMCDM: Evidential fuzzy multicriteria decision making based on belief entropy. *IEEE Trans. Fuzzy Syst.* **2019**, *28*, 1477–1491. [CrossRef]
5. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
6. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
7. Xiao, F. A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems. *IEEE Trans. Syst. Man Cybern. Syst.* **2019**, *51*, 3980–3992. [CrossRef]
8. Wang, Z.; Xiao, F.; Ding, W. Interval-valued intuitionistic fuzzy jenson-shannon divergence and its application in multi-attribute decision making. *Appl. Intell.* **2022**, *52*, 16168–16184. [CrossRef]
9. Yu, D.; Sheng, L.; Xu, Z. Analysis of evolutionary process in intuitionistic fuzzy set theory: A dynamic perspective. *Inf. Sci.* **2022**, *601*, 175–188. [CrossRef]
10. Wang, Z.; Xiao, F.; Cao, Z. Uncertainty measurements for Pythagorean fuzzy set and their applications in multiple-criteria decision making. *Soft Comput.* **2022**, *26*, 9937–9952. [CrossRef]

11. Molodtsov, D.A. Soft set theory- First results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
12. Ali, M.I.; Feng, F.; Liu, X.Y.; Min, W.K.; Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
13. Roy, A.R.; Maji, P.K. A fuzzy soft set theoretic approach to decision making problems. *J. Comput. Appl. Math.* **2007**, *203*, 412–418. [CrossRef]
14. Maji, P.K.; Biswas, R.; Roy, A.R. Fuzzy soft sets. *J. Fuzzy Math.* **2001**, *9*, 589–602.
15. Maji, P. K.; Biswas, R.; Roy, A.R. Intuitionistic fuzzy soft sets. *J. Fuzzy Math.* **2001**, *9*, 677–692.
16. Hussain, A.; Ali, M.I.; Mahmood T.; Munir, M. q-Rung orthopair fuzzy soft average aggregation operators and their application in multi-criteria decision making. *Int. J. Intell. Syst.* **2020**, *35*, 571–599. [CrossRef]
17. Ali, G.; Afzal, M.; Asif, M.; Shazad, A. Attribute reduction approaches under interval-valued q-rung orthopair fuzzy soft framework. *Appl. Intell.* **2022**, *52*, 8975–9000. [CrossRef]
18. Cuong, B.C. Picture fuzzy sets-first results. part 1. In *Seminar Neuro-Fuzzy Systems with Applications*; Tech. Rep.; Institute of Mathematics: Hanoi, Vietnam, 2013.
19. Cuong, B.C. Picture fuzzy sets-first results. part 2. In *Seminar Neuro-Fuzzy Systems with Applications*; Tech. Rep.; Institute of Mathematics: Hanoi, Vietnam, 2013.
20. Cuong, B.C.; Kreinovich, V. Picture fuzzy sets. *J. Comput. Sci. Cybern.* **2014**, *30*, 409–420.
21. Ganie, A.H.; Singh, S.; Bhatia, P.K. Some new correlation coefficients of picture fuzzy sets with applications. *Neural Comput. Appl.* **2020**, *32*, 12609–12625. [CrossRef]
22. Tchier, F.; Ali, G.; Gulzar, M.; Pamučar, D.; Ghorai, G. A New Group Decision-Making Technique under Picture Fuzzy Soft Expert Information. *Entropy* **2021**, *23*, 1176. [CrossRef]
23. Saraji, M.K.; Streimikiene, D. Evaluating the circular supply chain adoption in manufacturing sectors: A picture fuzzy approach. *Technol. Soc.* **2022**, *70*, 102050. [CrossRef]
24. Rong, Y.; Liu, Y.; Pei, Z. A novel multiple attribute decision-making approach for evaluation of emergency management schemes under picture fuzzy environment. *Int. J. Mach. Learn. Cybern.* **2022**, *13*, 633–661. [CrossRef]
25. Simic, V.; Karagoz, S.; Deveci, M.; Aydin, N. Picture fuzzy extension of the CODAS method for multi-criteria vehicle shredding facility location. *Expert Syst. Appl.* **2021**, *175*, 114644. [CrossRef]
26. Wang, L.; Peng, J.J.; Wang, J.Q. A multi-criteria decision-making framework for risk ranking of energy performance contracting project under picture fuzzy environment. *J. Clean. Prod.* **2018**, *191*, 105–118. [CrossRef]
27. Cuong, B.C.; Pham, V.H. Some Fuzzy Logic Operators for Picture Fuzzy Sets. In Proceedings of the Seventh International Conference on Knowledge and Systems Engineering (KSE), Ho Chi Minh City, Vietnam, 8–10 October 2015; pp. 132–137. [CrossRef]
28. Yang, Y.; Liang, C.; Ji, S.; Liu, T. Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making. *J. Intell. Fuzzy Syst.* **2015**, *29*, 1711–1722. [CrossRef]
29. Singh, P. Correlation coefficients for picture fuzzy sets. *J. Intell. Fuzzy Syst.* **2015**, *28*, 591–604. [CrossRef]
30. Singh, S.; Ganie, A.H. On a new picture fuzzy correlation coefficient with its applications to pattern recognition and identification of an investment sector. *Comput. Appl. Math.* **2022**, *41*, 1–35. [CrossRef]
31. Ganie, A.H.; Singh, S. A picture fuzzy similarity measure based on direct operations and novel multi-attribute decision-making method. *Neural Comput. Appl.* **2021**, *33*, 9199–9219. [CrossRef]
32. Wei, G. Some similarity measures for picture fuzzy sets and their applications. *Iran. J. Fuzzy Syst.* **2018**, *15*, 77–89.
33. Ganie, A.H.; Singh, S. An innovative picture fuzzy distance measure and novel multi-attribute decision-making method. *Complex Intell. Syst.* **2021**, *7*, 781–805. [CrossRef]
34. Sindhu, M.S.; Rashid, T.; Kashif, A. An approach to select the investment based on bipolar picture fuzzy sets. *Int. J. Fuzzy Syst.* **2021**, *23*, 2335–2347. [CrossRef]
35. Akram, M.; Khan, A.; Ahmad, U.; Alcantud, J.C.R.; Al-Shamiri, M.M.A. A new group decision-making framework based on 2-tuple linguistic complex q-rung picture fuzzy sets. *Math. Biosci. Eng.* **2022**, *19*, 11281–11323. [CrossRef] [PubMed]
36. Senapati, T. Approaches to multi-attribute decision-making based on picture fuzzy Aczel–Alsina average aggregation operators. *Comput. Appl. Math.* **2022**, *41*, 1–19. [CrossRef]
37. He, S.; Wang, Y. Evaluating new energy vehicles by picture fuzzy sets based on sentiment analysis from online reviews. *Artif. Intell. Rev.* **2022**. [CrossRef]
38. Yildirim, B.F.; Yıldırım, S.K. Evaluating the satisfaction level of citizens in municipality services by using picture fuzzy VIKOR method: 2014–2019 period analysis. *Decis. Making: Appl. Manag. Eng.* **2022**, *5*, 50–66. [CrossRef]
39. Van Pham, H.; Khoa, N.D.; Bui, T.T.H.; Giang, N.T.H.; Moore, P. Applied picture fuzzy sets for group decision-support in the evaluation of pedagogic systems. *Int. J. Math. Eng. Manag. Sci.* **2022**, *7*, 243.
40. Akram, M.; Shahzadi, G.; Alcantud, J.C.R. Multi-attribute decision-making with q-rung picture fuzzy information. *Granul. Comput.* **2022**, *7*, 197–215. [CrossRef]
41. Akram, M. Multi-criteria decision-making methods based on q-rung picture fuzzy information. *J. Intell. Fuzzy Syst.* **2021**, *40*, 10017–10042. [CrossRef]
42. Haktanır, E.; Kahraman, C. A novel picture fuzzy CRITIC & REGIME methodology: Wearable health technology application. *Eng. Appl. Artif. Intell.* **2022**, *113*, 104942.

43. Shit, C.; Ghorai, G.; Xin, Q.; Gulzar, M. Harmonic aggregation operator with trapezoidal picture fuzzy numbers and its application in a multiple-attribute decision-making problem. *Symmetry* **2022**, *14*, 135. [CrossRef]
44. Karamti, H.; Sindhu, M.S.; Ahsan, M.; Siddique, I.; Mekawy, I.; El-Wahed Khalifa, H.A. A novel multiple-criteria decision-making approach based on picture fuzzy sets. *J. Funct. Spaces* **2022**, *2022*, 2537513. [CrossRef]
45. Rehman, U.U.; Mahmood, T. Picture fuzzy N-soft sets and their applications in decision-making problems. *Fuzzy Inf. Eng.* **2021**, *13*, 335–367. [CrossRef]
46. Li, L.; Zhang, R.; Wang, J.; Shang, X.; Bai, K. A novel approach to multi-attribute group decision-making with q-rung picture linguistic information. *Symmetry* **2018**, *10*, 172. [CrossRef]
47. Mahmood, T.; Ahmmad, J. Complex picture fuzzy N-soft sets and their decision-making algorithm. *Soft Comput.* **2021**, *25*, 13657–13678. [CrossRef]
48. Gundogdu, F.K. Picture Fuzzy Linear Assignment Method and Its Application to Selection of Pest House Location. In *International Conference on Intelligent and Fuzzy Systems*; Springer: Cham, Switzerland, 2020; pp. 101–109.
49. Akram, M.; Habib, A.; Alcantud, J.C.R. An optimization study based on Dijkstra algorithm for a network with trapezoidal picture fuzzy numbers. *Neural Comput. Appl.* **2021**, *33*, 1329–1342. [CrossRef]
50. Shoaib, M.; Mahmood, W.; Xin, Q.; Tchier, F. Certain operations on picture fuzzy graph with application. *Symmetry* **2021**, *13*, 2400. [CrossRef]
51. Shabir, M.; Naz, M. On bipolar soft sets. *arXiv* **2013**, arXiv:1303.1344.
52. Naz, M.; Shabir, M. On fuzzy bipolar soft sets, their algebraic structures and applications. *J. Intell. Fuzzy Syst.* **2014**, *26*, 1645–1656. [CrossRef]
53. Ali, G.; Ansari, M.N. Multiattribute decision-making under Fermatean fuzzy bipolar soft framework. *Granul. Comput.* **2022**, *7*, 337–352. [CrossRef]
54. Malik, N.; Shabir, M. Rough fuzzy bipolar soft sets and application in decision-making problems. *Soft Comput.* **2019**, *23*, 1603–1614. [CrossRef]
55. Ali, G.; Alolaiyan, H.; Pamučar, D.; Asif, M.; Lateef, N. A novel MADM framework under q-rung orthopair fuzzy bipolar soft sets. *Mathematics* **2021**, *9*, 2163. [CrossRef]
56. Akram, M.; Ali, G.; Shabir, M. A hybrid decision-making framework using rough mF bipolar soft environment. *Granul. Comput.* **2021**, *6*, 539–555. [CrossRef]
57. Ali, G.; Abidin, M.Z.U.; Xin, Q.; Tawfiq, F.M. Ranking of downstream fish passage designs for a hydroelectric project under spherical fuzzy bipolar soft framework. *Symmetry* **2022**, *14*, 2141. [CrossRef]