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New Conditions for Testing the Oscillation of Third-Order Differential Equations with Distributed Arguments

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Abstract: In this paper, we consider a certain class of third-order nonlinear delay differential equations with distributed arguments. By the principle of comparison, we obtain the conditions for the nonexistence of positive decreasing solutions as well as, and by using the Riccati transformation technique, we obtain the conditions for the nonexistence of increasing solutions. Therefore, we get new sufficient criteria that ensure that every solution of the studied equation oscillates. Asymmetry plays an important role in describing the properties of solutions of differential equations. An example is given to illustrate the importance of our results.

Keywords: third-order differential equations; distributed arguments; oscillation criteria

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1. Introduction

The first kernel of differential equations and their associated asymmetric properties began to appear in the middle of the seventeenth century and continued to develop until it became an effective and indispensable tool for solving and explaining the various phenomena of life. Differential equations have become an integral part of most sciences because of their great impact on the progress of science and the support of its outputs. The qualitative theory is one of the most important theories that has been associated with the study of differential equations on a large scale, and one of its most important branches is the study of the qualitative properties of solutions to differential equations.

Lately, the study of the oscillation of solutions to differential equations has received great attention from scientists due to its multiple applications in various sciences, such as engineering, economics, and physics. Especially the medical field, for example, the red blood cell preservation model and the lung expansion model in patients with COVID-19, in addition to a model for diagnosing diabetes patients [1].

In recent decades, many studies have appeared interested in obtaining sufficient conditions to ensure the oscillation or non-oscillation of solutions of second-order (nonlinear and linear) neutral delay differential equations; see [2–4]. In addition to the references therein, few of these results were with distributed deviating arguments; for example, Sahiner and Wang in [5,6] established some Philos-type oscillation criteria for the equation

$$\left(\jmath(\imath)((\vartheta(\imath)+g(\imath)\vartheta(\imath-\varsigma)))'\right)'+\int_m^n \Omega(\imath,\xi)\vartheta(\varepsilon(\imath,\xi))\mathrm{d}\xi=0,$$

in canonical form. After a while, Xu et al. [7] were concerned with the oscillation problem for the following neutral delay.

Equation with continuous distributed

$$\left(\jmath(\imath)\varpi(\vartheta(\imath))((\vartheta(\imath)+g(\imath)\vartheta(\imath-\varsigma)\mathrm{d}\xi))'\right)'+\int_m^n \Omega(\imath,\xi)\vartheta(\varepsilon(\imath,\xi))\mathrm{d}\xi=0.$$

While the study of the oscillation of third-order differential equations is comparatively less if any, they exist, and most of them revolve around delay differential equations, for example, [8–13]. Over the previous few years, a number of results have appeared related to the study and development of the oscillation of solutions of third-order neutral differential equations with a continuously distributed delay [14–18].

Zhang et al. [19], by using different techniques, established some results that discuss the oscillation of solutions to the equation

$$\left(\jmath(\imath)\left(\vartheta(\imath)+\int_c^d g(\imath,\xi)\vartheta(\varsigma(\imath,\xi))\mathrm{d}\xi\right)''\right)'+\int_m^n \Omega(\imath,\xi)f(\vartheta(\varepsilon(\imath,\xi)))\mathrm{d}\xi=0, \tag{1}$$

with canonical form and with hypothesis

$$0\le p(t)\equiv\int_c^d p(t)\mathrm{d}s\le p<1 \tag{2}$$

On the other hand, Candan in [20] studied the following equation

$$\left(\jmath_1(\imath)\left[(\jmath_2(\imath)\varpi'(\imath))'\right]^{\ell}\right)'+\int_m^n \Omega_1(\imath,\xi)\vartheta(\imath-\xi)\mathrm{d}\xi+\int_m^n \Omega_2(\imath,\xi)\vartheta(\imath+\xi)\mathrm{d}\xi=0, \tag{3}$$

where

$$\varpi(\imath)=g_1(\imath)\vartheta(\imath-\varsigma_1)+g_2(\imath)\vartheta(\imath+\varsigma_2)$$

and used the condition $\ell\ge 1$ to ensure that the solutions of the Equation (3) are either oscillatory or converge to zero. Fu et al. [21] studied oscillation and asymptotic behavior of solutions to equation

$$\left(\jmath(\imath)\left[\left(\vartheta(\imath)+\int_c^d g(\imath,\xi)\vartheta(\varsigma(\imath,\xi))\mathrm{d}\xi\right)''\right]^{\ell}\right)'+\int_m^n \Omega(\imath,\xi)f(\vartheta(\varepsilon(\imath,\xi)))\mathrm{d}\xi=0,$$

with canonical form and under conditions from (2) $0<\ell<1$ holds. Wanga et al. [22] investigated a general third-order neutral delay differential equation

$$\left(\jmath_1(\imath)\left[\left(\jmath_2(\imath)(\vartheta(\imath)+g(\imath,\xi)\vartheta(\varsigma(\imath,\xi)))'\right)'\right]^{\ell}\right)'+\int_m^n \Omega(\imath,\xi)f(\vartheta(\varepsilon(\imath,\xi)))\mathrm{d}\xi=0,$$

and assumed that

$$0\le g(\imath)\le g_0<1$$

and

$$\int_{\imath_0}^{\infty}\jmath_1^{-1}(\imath)\mathrm{d}\xi=\infty \text{ and } \int_{\imath_0}^{\infty}\jmath_2^{-1}(\imath)\mathrm{d}\xi=\infty.$$

They presented some conditions that ensure that the solutions to the equation are either oscillatory or converge to zero. Further, Gaoa et al. [23] obtained some oscillation criteria for Equation (1), but in the case of noncanonical equations under condition (2). For more results for similar equations, see, for example, [24–34].

In this paper, we establish the oscillatory behavior of the solution of the following third-order neutral differential equation with a continuously distributed delay

$$\left(\jmath(\imath)\left[(\vartheta(\imath)+g(\imath)\vartheta(\varsigma(\imath)))''\right]^{\ell}\right)'+\int_m^n \Omega(\imath,\xi)\vartheta^{\ell}(\varepsilon(\imath,\xi))\mathrm{d}\xi=0, \tag{4}$$

For $\imath\ge\imath_0$, where $\ell>0$ is the ratio of odd positive integers. Throughout this paper, the following hypotheses are assumed to hold:

($\mathbf{I}_1$) $\jmath, g, \varsigma \in C(T_{\imath_0})$, $\Omega(\imath,\xi)$, $\varepsilon(\imath,\xi) \in C(T_{\imath_0} \times (m,n), \mathbb{R})$, where $[\imath_0,\infty) = T_{\imath_0}$ and $\jmath > 0$, $g(\imath) \leq g_0 < \infty$, $\Omega(\imath,\xi) \geq 0$ does not vanish identically such that

$$\int^{\infty} \jmath^{-1/\ell}(\xi)\mathrm{d}\xi = \infty;$$

($\mathbf{I}_2$) $\varsigma(\imath) < \imath$, $\varepsilon(\imath,\xi) < \imath$, $\varsigma'(\imath) \geq \varsigma_0 > 0$, $\lim_{\imath\to\infty}\varsigma(\imath) = \infty$, $\lim_{\imath\to\infty}\varepsilon(\imath,\xi) = \infty$ and $\varsigma \circ \varepsilon = \varepsilon \circ \varsigma$.

A solution to (4) means $\vartheta \in C([\imath_\vartheta,\infty), T_0)$ (where $T_0 = [0,\infty)$) with $\imath_\vartheta \geq \imath_0$, which satisfies the properties $\varpi(\imath) \in C^2([\imath_\vartheta,\infty), T_0)$, $\jmath(\varpi'')^\ell \in C^1([\imath_\vartheta,\infty), T_0)$ and satisfies (4) on $[\imath_\vartheta,\infty)$. We consider the nontrivial solutions of (4) existing on some half-line $[\imath_\vartheta,\infty)$ and satisfying the condition $\sup\{|\vartheta(\imath)| : \imath_\vartheta \leq \imath\} > 0$ for all $\imath_\vartheta \geq \imath_\vartheta$. Moreover, solution ϑ is called an oscillatory solution when it is neither positive nor negative eventually. Otherwise, it is called a nonoscillatory solution.

The purpose of this paper is to derive some new results on the oscillation of all solutions to (4). In contrast to the published results, which provide some almost oscillation criteria for Equation (4) (See, for example [19–22]). The results obtained can be applied in the case where $\varepsilon(\imath,\xi) \geq \imath$ and $g(\imath) \leq g_0 < \infty$. Therefore, the current results continue and extend the results mentioned in the previous literature.

Remark 1. *In this paper:*

(**a**) *We consider every inequality satisfied eventually. Thus, they are satisfied* $\forall \imath$ *large enough;*

(**b**) *Without loss of generality, we only deal with a solution* $\vartheta > 0$ *of (4) (a solution* $\vartheta < 0$ *is similar);*

(**c**) *We set*

$$\varpi(\imath) = \vartheta(\imath) + g(\imath)\vartheta(\varsigma(\imath)).$$

Definition 1. *Let* $\varpi(\imath) > 0$ *and* $\varpi''(\imath) > 0$ *and* $\left(\jmath(\imath)(\varpi''(\imath))^\ell\right)'$ *be nonpositive functions. Then*

(**i**) ***class*** Y_1 *means the set of all solutions* ϑ *of Equation (4) that satisfy* $\varpi'(\imath) > 0$;

(**ii**) ***class*** Y_2 *means the set of all solutions* ϑ *of Equation (4) that satisfy* $\varpi'(\imath) < 0$.

We present the following Lemma that will help us to prove our next results.

Lemma 1 ([35])**.** *Let* U *and* V *be nonnegative functions. Then*

$$(U+V)^\gamma \leq K(U^\gamma + V^\gamma), \tag{5}$$

where

$$K = 1 \text{ if } \gamma \in (0,1] \text{ and } K = 2^{\gamma-1} \text{ if } \gamma \in (1,\infty).$$

Lemma 2 ([36])**.** *Assume that* $\vartheta > 0$ *is a solution of*

$$\left(\jmath(\imath)\left[(\vartheta(\imath) + g(\imath)\vartheta(\varsigma(\imath)))''\right]^\ell\right)' + \Omega(\imath)\vartheta^\ell(\varepsilon(\imath)) = 0.$$

If

$$\int_{\imath_0}^{\infty}\int_{\xi}^{\infty}\left(\frac{1}{\jmath(\varsigma(u))}\int_u^\infty \widetilde{\Omega}(\imath,\xi)\right)\mathrm{d}u\mathrm{d}\xi = \infty, \tag{6}$$

then $\vartheta \in Y_2$ *and* $\lim_{\imath\to\infty}\vartheta(\imath) = 0$.

For conciseness, we provide the notes below:

$$\begin{aligned} \eta(\imath,\imath_1) &= \int_{\imath_1}^{\imath}\frac{1}{\jmath^{1/\ell}(\xi)},\ \eta_1(\imath,\imath_1) = \int_{\imath_1}^{\imath}\eta(\imath,\xi)\mathrm{d}\xi, \\ \widetilde{\Omega}(\imath) &:= \min\{\Omega(\imath,\xi), \Omega(\varsigma(\imath,\xi))\},\ \widetilde{\Omega}_1(\imath,\xi) := \int_m^n \widetilde{\Omega}(\imath,\xi)\mathrm{d}\xi. \end{aligned}$$

2. Nonexistence of Decreasing Solutions

In this section, we present conditions that ensure the nonexistence of positive decreasing solutions of (4).

Theorem 1. *If* $\exists$ *a function* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *satisfying* $\varepsilon(\imath, n) < \zeta(\imath)$ *and* $\varsigma^{-1}(\zeta(\imath)) < \imath$*, such that the equation*

$$v'(\imath) + \frac{k\varsigma_0}{K(\varsigma_0 + g_0^\ell)} \eta_1^\ell(\zeta(\imath), \varepsilon(\imath, n))\widetilde{\Omega}_1(\imath)v\Big(\varsigma^{-1}(\zeta(\imath))\Big) = 0 \tag{7}$$

is oscillatory, then $\Upsilon_2 = \emptyset$.

Proof. Assume $\vartheta > 0$ is a solution of (4). It is easy to see that

$$\begin{aligned} &\int_m^n \Omega(\imath, \xi)\vartheta^\ell(\varepsilon(\imath, \xi))\mathrm{d}\xi + \int_m^n \Omega(\varsigma(\imath, \xi))\vartheta^\beta(\varepsilon(\varsigma(\imath, \xi)))\mathrm{d}\xi \\ \geq\ &\int_m^n \widetilde{\Omega}(\imath, \xi)\Big(\vartheta^\ell(\varepsilon(\imath, \xi)) + \vartheta^\ell(\varepsilon(\varsigma(\imath, \xi)))\Big)\mathrm{d}\xi. \end{aligned} \tag{8}$$

From Lemma 1, we note that

$$\frac{1}{K}\varpi^\ell(\varepsilon(\imath, \xi)) \leq \Big(\vartheta^\ell(\varepsilon(\imath, \xi)) + g_0^\ell \vartheta^\ell(\varsigma(\varepsilon(\imath, \xi)))\Big).$$

In (8), we get

$$\int_m^n \widetilde{\Omega}(\imath, \xi)\Big(\vartheta^\ell(\varepsilon(\imath, \xi)) + \vartheta^\ell(\varepsilon(\varsigma(\imath, \xi)))\Big)\mathrm{d}\xi \geq \frac{1}{K}\int_m^n \widetilde{\Omega}(\imath, \xi)\varpi^\ell(\varepsilon(\imath, \xi))\mathrm{d}\xi. \tag{9}$$

Now, from (4), we obtain

$$\frac{1}{\varsigma'(\imath)}\Big(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\Big)' + \int_m^n \Omega(\varsigma(\imath, \xi))\vartheta^\beta(\varepsilon(\varsigma(\imath, \xi)))\mathrm{d}\xi \leq 0. \tag{10}$$

Combining (4) with (10), then using inequality (9), we have

$$\begin{aligned} 0 \ \geq\ & \Big(\jmath(\imath)(\varpi''(\imath))^\ell\Big)' + \int_m^n \Omega(\imath, \xi)\vartheta^\ell(\varepsilon(\imath, \xi))\mathrm{d}\xi + \frac{g_0^\ell}{\varsigma_0}\Big(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\Big)' \\ & + g_0^\ell \int_m^n \Omega(\varsigma(\imath, \xi))\vartheta^\ell(\varepsilon(\varsigma(\imath, \xi)))\mathrm{d}\xi. \end{aligned}$$

That is

$$\left(\jmath(\imath)(\varpi''(\imath))^\ell + \frac{g_0^\ell}{\varsigma_0}\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\right)' + \frac{1}{K}\int_m^n \widetilde{\Omega}(\imath, \xi)\varpi^\ell(\varepsilon(\imath, \xi))\mathrm{d}\xi \leq 0. \tag{11}$$

Thus

$$\left(\jmath(\imath)(\varpi''(\imath))^\ell + \frac{g_0^\ell}{\varsigma_0}\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\right)' + \frac{1}{K}\varpi^\ell(\varepsilon(\imath, n))\widetilde{\Omega}_1(\imath) \leq 0. \tag{12}$$

Since $\Big(\jmath(\imath)(\varpi''(\imath))^\ell\Big)' \leq 0$, we find

$$\jmath(\imath)(\varpi''(\imath))^\ell \leq \jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell,$$

and hence

$$\jmath(\imath)(\varpi''(\imath)) + \frac{g_0^\ell}{\varsigma_0}\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell \leq \left(1 + \frac{g_0^\ell}{\varsigma_0}\right)\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell.$$

Now, set

$$v(\imath) := \jmath(\imath)\left(\varpi''(\imath)\right)^{\ell} + \frac{g_0^{\ell}}{\varsigma_0}\jmath(\varsigma(\imath))\left(\varpi''(\varsigma(\imath))\right)^{\ell}. \tag{13}$$

That is

$$\frac{\varsigma_0}{\varsigma_0 + g_0^{\ell}} v(\imath) \leq \jmath(\varsigma(\imath))\left(\varpi''(\varsigma(\imath))\right)^{\ell}$$

or

$$\frac{\varsigma_0}{\varsigma_0 + g_0^{\ell}} v\left(\varsigma^{-1}(\zeta(\imath))\right) \leq \jmath(\zeta(\imath))\left(\varpi''(\zeta(\imath))\right)^{\ell}. \tag{14}$$

On the other hand, since $\jmath(\imath)(\varpi''(\imath))^{\ell}$ is nonincreasing, we note that

$$\begin{aligned} -\varpi'(u) &\geq \int_u^v \frac{1}{\jmath^{1/\ell}(\xi)} \jmath^{1/\ell}(\xi)\varpi''(\xi)\mathrm{d}\xi \\ &\geq \eta(v,u)\jmath^{1/\ell}(v)\varpi''(v) \end{aligned}$$

and

$$\varpi(u) \geq \jmath^{1/\ell}(v)\varpi''(v)\eta_1(v,u).$$

Thus,

$$\varpi^{\ell}(\varepsilon(\imath,n)) \geq \left(\jmath^{1/\ell}(\zeta(\imath))\varpi''(\zeta(\imath))\right)^{\ell} \eta_1^{\ell}(\zeta(\imath),\varepsilon(\imath,n)). \tag{15}$$

Using (14) and (15) in (12), it is easy to note that v is a positive solution of

$$v'(\imath) + \frac{\varsigma_0 \eta_1^{\ell}(\zeta(\imath),\varepsilon(\imath,n))}{K\left(\varsigma_0 + g_0^{\ell}\right)} \widetilde{\Omega}_1(\imath) v\left(\varsigma^{-1}(\zeta(\imath))\right) \leq 0.$$

From (Theorem 1 in [2]), the associated Equation (7) also has a positive solution. The proof is complete. □

Corollary 1. *If $\exists$ a function $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ and $\varepsilon(\imath,n) < \zeta(\imath)$ and $\varsigma^{-1}(\zeta(\imath)) < \imath$, such that*

$$\liminf_{\imath\to\infty} \int_{\varsigma^{-1}(\zeta(\imath))}^{\imath} \eta_1^{\ell}(\zeta(\xi),\varepsilon(\imath,n))\widetilde{\Omega}_1(\xi)\mathrm{d}\xi > \frac{K\left(\varsigma_0 + g_0^{\ell}\right)}{\varsigma_0 e} \tag{16}$$

or

$$\int_{\imath}^{\imath+\varsigma^{-1}(\zeta(\imath))} \eta_1^{\ell}(\zeta(\xi),\varepsilon(\imath,n))\widetilde{\Omega}_1(\xi)\mathrm{d}\xi > 0, \ \text{for } \imath \geq \imath_0 \tag{17}$$

and

$$\int_{\imath_0}^{\infty} \eta_1^{\ell}(\zeta(\imath),\varepsilon(\imath,n))\widetilde{\Omega}(\imath) \ln\left(e\int_{\imath}^{\imath+\varsigma^{-1}(\zeta(\imath))} \frac{\varsigma_0}{K\left(\varsigma_0 + g_0^{\ell}\right)} \eta_1^{\ell}(\zeta(\xi),\varepsilon(\imath,n))\widetilde{\Omega}_1(\xi)\mathrm{d}\xi\right)\mathrm{d}\imath = \infty, \tag{18}$$

then $Y_2 = \emptyset$.

Proof. In view of [13,37], condition (16) or conditions (17) and (16) imply the oscillation of (7). □

3. Nonexistence of Increasing Solutions

Now in this section, we present conditions that ensure the nonexistence of positive increasing solutions of (4).

Theorem 2. *Assume that $\exists$ a function $\wp \in C^1(T_{\imath_0}, (0,\infty))$ and $\varepsilon(\imath) \leq \varsigma(\imath)$, such that*

$$\limsup_{\imath\to\infty} \int_{\imath_2}^{\imath} \left[\frac{1}{K}\wp(\imath)\widetilde{\Omega}_1(\imath) - \frac{\varsigma_0 + g_0^{\ell}}{4\ell\varsigma_0} \frac{(\wp'(\imath))^2}{\wp(\imath)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1}\eta(\varepsilon(\imath,m),\imath_1)\varepsilon'(\imath,m)}\right]\mathrm{d}\xi = \infty, \tag{19}$$

for $\imath_1 \geq \imath_0, \imath_2 \geq \imath_1$. *Then* $Y_1 = \emptyset$.

Proof. Assume that ϑ is a positive solution to (4) and $\varpi \in Y_1$. From (11), and since $\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)' \leq 0$, we obtain

$$\left(\jmath(\imath)(\varpi''(\imath))^{\ell} + \frac{g_0^{\ell}}{\varsigma_0}\jmath(\imath)(\varpi''(\varsigma(\imath)))^{\ell}\right)' \leq -\frac{1}{K}\varpi^{\ell}(\varepsilon(\imath,m))\widetilde{\Omega}(\imath). \tag{20}$$

and

$$\begin{aligned}\varpi'(\imath) &\geq \int_{\imath_1}^{\imath} \frac{1}{\jmath^{1/\ell}(\xi)}\jmath^{1/\ell}(\xi)\varpi''(\xi)\mathrm{d}\xi \\ &\geq \eta(\imath,\imath_1)\jmath^{1/\ell}(\imath)\varpi''(\imath).\end{aligned} \tag{21}$$

Further,

$$\frac{\varpi(\imath)}{\jmath^{1/\ell}(\imath)} \geq \varpi''(\imath)\eta_1(\imath,\imath_1). \tag{22}$$

From (21) and (22), we have

$$\begin{aligned}\frac{\varpi'(\varepsilon(\imath))}{\varpi^{1-\ell}(\varepsilon(\imath))\jmath(\imath)(\varpi''(\imath))^{\ell}} &\geq \frac{\jmath(\varepsilon(\imath))(\varpi''(\varepsilon(\imath)))^{\ell}}{(\varpi''(\imath))^{\ell}\jmath(\imath)}\eta(\varepsilon(\imath),\imath_1)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1} \\ &\geq \eta(\varepsilon(\imath),\imath_1)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1}.\end{aligned} \tag{23}$$

Since $\varepsilon(\imath) \leq \varsigma(\imath)$,we get

$$\begin{aligned}\frac{\varpi^{\ell-1}(\varepsilon(\imath))\varpi'(\varepsilon(\imath))}{\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}} &\geq \frac{\jmath(\varepsilon(\imath))(\varpi''(\varepsilon(\imath)))^{\ell}}{\jmath(\varsigma(\imath))(\varpi''(\varepsilon))^{\ell}}\eta(\varepsilon(\imath),\imath_1)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1} \\ &\geq \eta(\varepsilon(\imath),\imath_1)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1}.\end{aligned} \tag{24}$$

We define the two functions

$$\omega(\imath) = \wp(\imath)\frac{\jmath(\imath)(\varpi''(\imath))^{\ell}}{\varpi^{\ell}(\varepsilon(\imath,m))} > 0 \tag{25}$$

and

$$v(\imath) = \wp(\imath)\frac{\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}}{\varpi^{\ell}(\varepsilon(\imath,m))} > 0. \tag{26}$$

Differentiating (25) and (26), respectively, then using (23) and (24), we obtain

$$\begin{aligned}\omega'(\imath) &= \wp'(\imath)\frac{\jmath(\imath)(\varpi''(\imath))^{\ell}}{\varpi^{\ell}(\varepsilon(\imath,m))} + \wp(\imath)\frac{\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)'}{\varpi^{\ell}(\varepsilon(\imath,m))} \\ &\quad -\frac{\ell\wp(\imath)\jmath(\imath)(\varpi''(\imath))^{\ell}\varpi^{\ell-1}(\varepsilon(\imath,m))\varpi'(\varepsilon(\imath,m))\varepsilon'(\imath)}{\varpi^{2\ell}(\varepsilon(\imath,m))}\end{aligned} \tag{27}$$

$$\begin{aligned}&\leq \frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath) + \frac{\wp(\imath)\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)'}{\varpi^{\ell}(\varepsilon(\imath,m))} \\ &\quad -\frac{\ell\varepsilon'(\imath)}{\wp(\imath)}\eta(\varepsilon(\imath,m),\imath_1)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}\omega^2(\imath).\end{aligned} \tag{28}$$

and

$$\begin{aligned} v'(\imath) &= \wp'(\imath)\frac{\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}}{\varpi^{\ell}(\varepsilon(\imath,m))}+\wp(\imath)\frac{\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'}{\varpi^{\ell}(\varepsilon(\imath,m))} \\ &\quad -\frac{\ell\wp(\imath)\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\varpi^{\ell-1}(\varepsilon(\imath,m))\varpi'(\varepsilon(\imath,m))\varepsilon'(\imath)}{\varpi^{2\ell}(\varepsilon(\imath,m))} \qquad (29) \\ &\le \wp'(\imath)\frac{v(\imath)}{\wp(\imath)}+\wp(\imath)\frac{\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'}{\varpi^{\ell}(\varepsilon(\imath,m))} \\ &\quad -\frac{\ell\varepsilon'(\imath)}{\wp(\imath)}\eta(\varepsilon(\imath,m),\imath_1)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}v^2(\imath). \qquad (30) \end{aligned}$$

Using inequalities (28) and (30), implies

$$\begin{aligned} \omega'(\imath)+\frac{\wp_0^{\ell}v'(\imath)}{\varsigma_0} &\le \frac{\wp(\imath)\left(\left((\varpi''(\imath))^{\ell}\jmath(\imath)\right)'+\frac{\wp_0^{\ell}}{\varsigma_0}\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'\right)}{\varpi^{\ell}(\varepsilon(\imath,m))} \\ &\quad +\frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath)-\frac{\ell\varepsilon'(\imath)}{\wp(\imath)}\eta(\varepsilon(\imath,m),\imath_1)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}\omega^2(\imath) \\ &\quad +\frac{\wp_0^{\ell}}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath)-\frac{\ell\varepsilon'(\imath)}{\wp(\imath)}\eta(\varepsilon(\imath,m),\imath_1)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}v^2(\imath)\right). \qquad (31) \end{aligned}$$

Applying (20), (31), and

$$Bu-Au^2\le\frac{n^2}{4A},\ \ m>0,$$

we have

$$\begin{aligned} \omega'(\imath)+\frac{\wp_0^{\ell}}{\varsigma_0}v'(\imath) &\le -\frac{\wp(\imath)}{K}\widetilde{\Omega}_1(\imath) \\ &\quad +\frac{\left(1+\frac{g_0^{\ell}}{\varsigma_0}\right)}{4\ell}\frac{(\wp'(\imath))^2}{\wp(\imath)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}\eta(\varepsilon(\imath,m),\imath_1)\varepsilon'(\imath)}. \qquad (32) \end{aligned}$$

Integrating (32) from $\imath_2$ to $\imath$, we get

$$\int_{\imath_2}^{\imath}\left[\frac{\wp(\imath)}{K}\widetilde{\Omega}_1(\imath)-\frac{\left(1+\frac{g_0^{\ell}}{\varsigma_0}\right)}{4\ell}\frac{(\wp'(\imath))^2}{\wp(\imath)(\eta_1(\varepsilon(\imath,m),\imath_1))^{\ell-1}\eta(\varepsilon(\imath,m),\imath_1)\varepsilon'(\imath)}\right]\mathrm{d}\xi\le\omega(\imath_2)+\frac{\wp_0^{\ell}}{\varsigma_0}v(\imath_2).$$

This contradicts (19). The proof is complete. □

Theorem 3. *Assume that* $\exists$ *a function* $\wp\in C^1(T_{\imath_0},(0,\infty))$ *and* $\varepsilon(\imath)\ge\varsigma(\imath)$, *such that*

$$\limsup_{\imath\to\infty}\int_{\imath_2}^{\imath}\left[\frac{\wp(\xi)}{K}\widetilde{\Omega}_1(\xi)-\frac{\left(\varsigma_0+g_0^{\ell}\right)}{\varsigma_0(\ell+1)^{\ell+1}}\frac{(\wp'(\xi))^{\ell+1}}{(\wp(\xi)\eta(\varsigma(\imath,m),\imath_1)\varsigma'(\imath,m))^{\ell}}\right]\mathrm{d}\xi=\infty, \qquad (33)$$

for $\imath_1\ge\imath_0,\imath_2\ge\imath_1$. *Then* $\Upsilon_1=\varnothing$.

Proof. Let $\vartheta>0$ be a solution of (4) and $\varpi\in\Upsilon_1$. By (21), we obtain

$$\varpi'(\varsigma(\imath,m))\ge\eta(\varsigma(\imath,m),\imath_1)\jmath^{1/\ell}(\varsigma(\imath,m))(\varpi''(\varsigma(\imath,m))) \qquad (34)$$

or

$$\varpi'(\varsigma(\imath,m))\ge\eta(\varsigma(\imath,m),\imath_1)\jmath^{1/\ell}(\imath)(\varpi''(\imath)). \qquad (35)$$

We define the two positive functions

$$\omega(\imath) = \wp(\imath)\frac{\jmath(\imath)(\varpi''(\imath))^{\ell}}{\varpi^{\ell}(\varsigma(\imath,m))} \tag{36}$$

and

$$v(\imath) = \wp(\imath)\frac{\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}}{\varpi^{\ell}(\varsigma(\imath,m))}. \tag{37}$$

From (36) and by using (35), we have

$$\begin{aligned}\omega'(\imath) &= \wp'(\imath)\frac{\jmath(\imath)(\varpi''(\imath))^{\ell}}{\varpi^{\ell}(\varsigma(\imath,m))} + \wp(\imath)\frac{\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)'}{\varpi^{\ell}(\varsigma(\imath,m))}\\ &\quad -\frac{\ell\wp(\imath)\jmath(\imath)(\varpi''(\imath))^{\ell}\varpi^{\ell-1}(\varsigma(\imath,m))\varpi'(\varsigma(\imath,m))\varsigma'(\imath,m)}{\varpi^{2\ell}(\varsigma(\imath,m))} \qquad (38)\\ &\le \frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath) + \frac{\wp(\imath)\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)'}{\varpi^{\ell}(\varsigma(\imath,m))}\\ &\quad -\frac{\ell\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)\omega^{\frac{\ell+1}{\ell}}(\imath). \qquad (39)\end{aligned}$$

Further, using (37) and (34) yields

$$\begin{aligned}v'(\imath) &= \wp'(\imath)\frac{\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}}{\varpi^{\ell}(\varsigma(\imath,m))} + \wp(\imath)\frac{\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'}{\varpi^{\ell}(\varsigma(\imath,m))}\\ &\quad -\frac{\ell\theta(\imath)\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\varpi^{\ell-1}(\varsigma(\imath,m))\varpi'(\varsigma(\imath,m))\varsigma'(\imath,m)}{\varpi^{2\ell}(\varsigma(\imath,m))} \qquad (40)\\ &\le \frac{\wp'(\imath)}{\wp(\imath)}v(\imath) + \frac{\wp(\imath)\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'}{\varpi^{\ell}(\varsigma(\imath,m))}\\ &\quad -\frac{\alpha\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath). \qquad (41)\end{aligned}$$

Using (39) and (41) and taking into account (11) and $\varepsilon(\imath,m) \le \varsigma(\imath,m)$, we find

$$\begin{aligned}\omega'(\imath) + \frac{\wp_0^{\ell}v'(\imath)}{\varsigma_0} &\le \wp(\imath)\frac{\left(\jmath(\imath)(\varpi''(\imath))^{\ell}\right)' + \frac{\wp_0^{\ell}}{\varsigma_0}\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^{\ell}\right)'}{\varpi^{\ell}(\varsigma(\imath,m))}\\ &\quad +\frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath) - \frac{\ell\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)\omega^{\frac{\ell+1}{\ell}}(\imath)\\ &\quad +\frac{\wp_0^{\ell}}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath) - \frac{\ell\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath)\right)\\ &\le -\frac{\wp(\imath)}{K}\widetilde{\Omega}_1(\imath)\\ &\quad +\frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath) - \frac{\ell\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)\omega^{\frac{\ell+1}{\ell}}(\imath)\\ &\quad +\frac{\wp_0^{\ell}}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath) - \frac{\ell\varsigma'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varsigma(\imath,m),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath)\right). \qquad (42)\end{aligned}$$

Applying the inequality

$$Bu - Au^{\frac{\ell+1}{\ell}} \leq \frac{\ell^{\ell}}{(\ell+1)^{\ell+1}} \frac{B^{\ell+1}}{A^{\ell}}, \quad m > 0. \tag{43}$$

Hence, we have

$$\begin{aligned} \omega'(\imath) + \frac{\wp_0^{\ell} v'(\imath)}{\varsigma_0} \quad \leq \quad & -\frac{\wp(\imath)}{K} \widetilde{\Omega}_1(\imath) \\ & + \frac{1}{(\ell+1)^{\ell+1}} \frac{(\wp'(\imath))^{1+\ell}}{(\wp(\imath)\eta(\varsigma(\imath,m),\imath_1)\varsigma'(\imath,m))^{\ell}} \\ & + \frac{\frac{\wp_0^{\ell}}{\varsigma_0}}{(\ell+1)^{\ell+1}} \frac{(\wp'(\imath))^{1+\ell}}{(\wp(\imath)\eta(\varsigma(\imath,m),\imath_1)\varsigma'(\imath,m))^{\ell}}. \end{aligned} \tag{44}$$

Integrating (44) from $\imath_2$ to $\imath$, this leads us to

$$\int_{\imath_2}^{\imath} \left[\frac{\wp(\imath)}{K} \widetilde{\Omega}_1(\imath) - \frac{\varsigma_0 + g_0^{\ell}}{(\ell+1)^{\ell+1}\varsigma_0} \frac{(\wp'(\imath))^{1+\ell}}{(\wp(\imath)\eta(\varsigma(\imath,m),\imath_1)\varsigma'(\imath,m))^{\ell}} \right] \mathrm{d}\xi \leq \omega(\imath_2) + \frac{\wp_0^{\ell}}{\varsigma_0} v(\imath_2).$$

This contradicts (33). The proof is complete. □

Theorem 4. *Assume that $\exists$ a function $\wp \in C^1(T_{\imath_0}, (0,\infty))$ and $\varepsilon(\imath) \geq \varsigma(\imath)$, such that*

$$\limsup_{\imath \to \infty} \int_{\imath_2}^{\imath} \left[\frac{\wp(\xi)}{K} \widetilde{\Omega}_1(\xi) - \frac{\left(1 + \frac{g_0^{\ell}}{\varsigma_0}\right)}{4\ell} \frac{(\wp'(\xi))^2}{\wp(\xi)(\eta_1(\varsigma(\imath,m),\imath_1))^{\ell-1}\eta(\varsigma(\imath,m),\imath_1)\varsigma'(\imath,m)} \right] \mathrm{d}\xi = \infty, \tag{45}$$

for $\imath_1 \geq \imath_0$, $\imath_2 \geq \imath_1$. Then $Y_1 = \emptyset$.

Proof. By using (38) and (40), in a similar way as in proof of Theorem 2, we get condition (45). □

4. Philos-Type

Let

$$O_0 = \{(\imath,\xi) : \imath > \xi \geq \imath_0\} \text{ and } O = \{(\imath,\xi) : \imath \geq \xi \geq \imath_0\}.$$

Moreover, the function $L \in C(O, \mathbb{R})$. If

$$L(\imath,\imath) = 0, \imath \geq \imath_0; \;\; L(\imath,\xi) > 0, \; (\imath,\xi) \in O_0;$$

and L has a nonpositive continuous partial derivative with respect to the second variable in O_0. Then L is said to belong to the class P. For brevity, let us assume that

$$\begin{aligned} W_1(\imath,\xi) \quad : \quad & = \frac{1}{K} L(\imath,\xi)\wp(\imath)\widetilde{\Omega}_1(\imath) - \frac{\left(1 + \frac{g_0^{\ell}}{\varsigma_0}\right)}{(\ell+1)^{\ell+1}} \frac{(h(\imath,\xi))^{\ell+1}}{(\wp(\xi)\eta(\varepsilon(\imath,m),\imath_1)\varepsilon'(\imath,m))^{\ell}}, \\ W_2(\imath,\xi) \quad : \quad & = \frac{1}{K} L(\imath,\xi)\wp(\imath)\widetilde{\Omega}_1(\imath) - \frac{\left(1 + \frac{g_0^{\ell}}{\varsigma_0}\right)}{(\ell+1)^{\ell-1}} \frac{(h(\imath,\xi))^2}{\wp(\imath)(\eta_1(\varepsilon(\imath),\imath_1))^{\ell-1}\eta(\varepsilon(\imath),\imath_1)\varsigma'(\imath)}, \\ W_3(\imath,\xi) \quad : \quad & = \frac{1}{K} L(\imath,\xi)\wp(\imath)\widetilde{\Omega}_1(\imath) - \frac{\left(1 + \frac{g_0^{\ell}}{\varsigma_0}\right)}{(1+\ell)^{-1+\ell}} \frac{(h(\imath,\xi))^{\ell+1}}{(\wp(\imath)\eta(\varsigma(\imath),\imath_1)\varsigma'(\imath))^{\ell}} \end{aligned}$$

and

$$W_4(\imath,\xi):=\frac{1}{K}L(\imath,\xi)\wp(\imath)\widetilde{\Omega}_1(\imath)-\frac{\left(1+\frac{g_0^\ell}{\varsigma_0}\right)}{4\ell}\frac{(h(\imath,\xi))^2}{\wp(\imath)(\eta_1(\varsigma(\imath),\imath_1))^{\ell-1}\eta(\varsigma(\imath),\imath_1)\varsigma'(\imath)}.$$

Theorem 5. *Let $\varepsilon(\imath)\le\varsigma(\imath)$ and $\varepsilon'(\imath)>0$. Moreover, suppose that $\exists$ a function $\wp\in C^1([\imath_0,\infty),(0,\infty))$, for all sufficiently large $\imath_1\ge\imath_0$, $\exists$ a $\imath_2>\imath_1$ and $L\in P$ such that*

$$-\frac{\partial}{\partial\xi}L(\imath,\xi)-L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}=h(\imath,\xi)(L(\imath,\xi))^{\frac{\ell}{\ell+1}}\frac{1}{\wp(\xi)},\ (\imath,\xi)\in O_0, \tag{46}$$

and

$$\limsup_{\imath\to\infty}\frac{1}{L(\imath,\imath_2)}\int_{\imath_2}^{\imath}W_1(\imath,\xi)\mathrm{d}\xi=\infty. \tag{47}$$

Then $\Upsilon_1=\emptyset$.

Proof. Suppose that ϑ is a positive solution of (4). By (21), we obtain

$$\begin{aligned}\varpi'(\varepsilon(\imath)) &\ge \eta(\varepsilon(\imath),\imath_1)\jmath^{1/\ell}(\varepsilon(\imath))(\varpi''(\varepsilon(\imath)))\\ &\ge \eta(\varepsilon(\imath),\imath_1)\jmath^{1/\ell}(\imath)\varpi''(\imath)\end{aligned} \tag{48}$$

and by $\varepsilon(\imath)\le\varsigma(\imath)$, we get

$$\varpi'(\varepsilon(\imath))\ge\eta(\varsigma(\imath),\imath_1)\jmath^{1/\ell}(\varsigma(\imath))\varpi''(\varsigma(\imath)). \tag{49}$$

We define ω and v as in Theorem 2 to get (27) and (29) then by (48), we get

$$\begin{aligned}w'(\imath) &= \wp'(\imath)\frac{w(\imath)}{\wp(\imath)}+\wp(\imath)\frac{\left(\jmath(\imath)(\varpi''(\imath))^\ell\right)'}{\varpi^\ell(\varepsilon(\imath,m))}\\ &\quad -\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath,m),\imath_1)w^{\frac{\ell+1}{\ell}}(\imath).\end{aligned} \tag{50}$$

By (49), we get

$$\begin{aligned}v'(\imath) &\le \wp'(\imath)\frac{v(\imath)}{\wp(\imath)}+\wp(\imath)\frac{\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\right)'}{\varpi^\ell(\varepsilon(\imath,m))}\\ &\quad -\frac{\ell\varepsilon'(\imath)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath).\end{aligned} \tag{51}$$

Using inequalities (50) and (51), we get

$$\begin{aligned}\omega'(\imath)+\frac{\wp_0^\ell}{\varsigma_0}v'(\imath) &\le \wp(\imath)\frac{\left(\jmath(\imath)(\varpi''(\imath))^\ell\right)'+\frac{\wp_0^\ell}{\varsigma_0}\left(\jmath(\varsigma(\imath))(\varpi''(\varsigma(\imath)))^\ell\right)'}{\varpi^\ell(\varepsilon(\imath,m))}\\ &\quad +\frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath)-\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath,m),\imath_1)w^{\frac{\ell+1}{\ell}}(\imath)\\ &\quad +\frac{\wp_0^\ell}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath)-\frac{\alpha\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath,m),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath)\right)\\ &\le -\frac{\wp(\imath)}{K}\widetilde{\Omega}_1(\imath)\\ &\quad +\frac{\wp'(\imath)}{\wp(\imath)}\omega(\imath)-\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath,m),\imath_1)w^{\frac{\ell+1}{\ell}}(\imath)\\ &\quad +\frac{\wp_0^\ell}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath)-\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath,m),\imath_1)v^{\frac{\ell+1}{\ell}}(\imath)\right).\end{aligned}$$

or

$$\begin{aligned}
\frac{\wp(\imath)}{K}\widetilde{\Omega}(\imath) \quad \leq \quad & -\omega'(\imath) - \frac{\wp_0^{\ell} v'(\imath)}{\varsigma_0} \\
& + \frac{\wp'(\imath)\omega(\imath)}{\wp(\imath)} - \frac{\ell\varepsilon'(\imath)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath), \imath_1) w^{\frac{\ell+1}{\ell}}(\imath) \\
& + \frac{\wp_0^{\ell}}{\varsigma_0}\left(\frac{\wp'(\imath)}{\wp(\imath)}v(\imath) - \frac{\ell\varepsilon'(\imath)}{\wp^{\frac{1}{\ell}}(\imath)}\eta(\varepsilon(\imath), \imath_1) v^{\frac{\ell+1}{\ell}}(\imath)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp(\xi)}{K}\widetilde{\Omega}_1(\xi)\mathrm{d}\xi \quad \leq \quad & -\int_{\imath_2}^{\imath} L(\imath,\xi)\omega'(\xi)\mathrm{d}\xi - \int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp_0^{\ell}}{\varsigma_0}v'(\xi)\mathrm{d}\xi \\
& + \int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}\omega(\xi)\mathrm{d}\xi \\
& - \int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\xi)}\eta(\varepsilon(\imath,m), \imath_1) w^{\frac{\ell+1}{\ell}}(\xi)\mathrm{d}\xi \\
& + \frac{\wp_0^{\ell}}{\varsigma_0}\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}v(\xi)\mathrm{d}\xi \\
& - \frac{\wp_0^{\ell}}{\varsigma_0}\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\xi)}\eta(\varepsilon(\imath,m), \imath_1) v^{\frac{\ell+1}{\ell}}(\xi)\mathrm{d}\xi.
\end{aligned}$$

This implies

$$\begin{aligned}
\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp(\xi)}{K}\widetilde{\Omega}_1(\xi)\mathrm{d}\xi \quad \leq \quad & L(\imath,\xi)\omega(\imath_2)\mathrm{d}\xi - \int_{\imath_2}^{\imath}\left(-\frac{\partial}{\partial\xi}L(\imath,\xi) - L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}\right)\omega(\xi)\mathrm{d}\xi \\
& - \int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\xi)}\eta(\varepsilon(\imath,m), \imath_1) w^{\frac{\ell+1}{\ell}}(\xi)\mathrm{d}\xi + L(\imath,\xi)\frac{\wp_0^{\ell}}{\varsigma_0}v(\imath_2) \\
& - \frac{\wp_0^{\ell}}{\varsigma_0}\int_{\imath_2}^{\imath}\left(-\frac{\partial}{\partial\xi}L(\imath,\xi) - L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}\right)v(\xi)\mathrm{d}\xi \\
& - \frac{\wp_0^{\ell}}{\varsigma_0}\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\ell\varepsilon'(\imath,m)}{\wp^{\frac{1}{\ell}}(\xi)}\eta(\varepsilon(\imath,m), \imath_1) v^{\frac{\ell+1}{\ell}}(\xi)\mathrm{d}\xi.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\imath_2}^{\imath} L(\imath,\xi)\frac{\wp(\xi)}{K}\widetilde{\Omega}_1(\xi)\mathrm{d}\xi \quad \leq \quad & L(\imath,\xi)\omega(\imath_2)\mathrm{d}\xi + L(\imath,\xi)\frac{\wp_0^{\ell}}{\varsigma_0}v(\imath_2) \\
& - \int_{\imath_2}^{\imath}\left(\frac{h(\imath,\xi)(L(\imath,\xi))^{\frac{\ell+1}{\ell}}\omega(\xi)}{\wp(\xi)}\right. \\
& \left. - \frac{\ell\varepsilon'(\imath,m)\eta(\varepsilon(\imath,m), \imath_1)L(\imath,\xi)w^{\frac{\ell+1}{\ell}}(\xi)}{\wp^{\frac{1}{\ell}}(\xi)}\right)\mathrm{d}\xi \\
& - \frac{\wp_0^{\ell}}{\varsigma_0}\int_{\imath_2}^{\imath}\left(\frac{h(\imath,\xi)(L(\imath,\xi))^{\frac{\ell+1}{\ell}}v(\xi)}{\wp(\xi)}\right. \\
& \left. - \frac{\ell\varepsilon'(\imath,m)\eta(\varepsilon(\imath,m), \imath_1)L(\imath,\xi)v^{\frac{\ell+1}{\ell}}(\xi)}{\wp^{\frac{1}{\ell}}(\xi)}\right)\mathrm{d}\xi.
\end{aligned}$$

Using (43), we have

$$\frac{1}{L(\imath,\xi)}\int_{\imath_2}^{\imath}\left(L(\imath,\xi)\frac{\wp(\xi)}{K}\widetilde{\Omega}_1(\xi)-\frac{\left(1+\frac{\wp_0^{\ell}}{\varsigma_0}\right)}{(\ell+1)^{\ell+1}}\frac{(h(\imath,\xi))^{(\ell+1)}}{(\wp(\xi)\eta(\varepsilon(\imath,m),\imath_1)\varepsilon'(\imath,m))^{\ell}}\right)\mathrm{d}\xi$$
$$\leq \quad \omega(\imath_2)\mathrm{d}\xi+\frac{\wp_0^{\ell}}{\varsigma_0}v(\imath_2).$$

The proof is complete. □

Theorem 6. *Let $\varepsilon(\imath)\leq\varsigma(\imath)$ and $\varepsilon'(\imath)>0$. Moreover, assume that $\exists$ a function $\wp\in C^1([\imath_0,\infty)$ $(0,\infty))$ and $L\in P$, for all sufficiently large $\imath_1\geq\imath_0$, $\exists$ a $\imath_2>\imath_1$ such that*

$$-\frac{\partial}{\partial\xi}L(\imath,\xi)-L(\imath,\xi)\frac{\wp'(\xi)}{\wp(\xi)}=h(\imath,\xi)(L(\imath,\xi))^{\frac{1}{2}}\frac{1}{\wp(\xi)},\ (\imath,\xi)\in O_0, \tag{52}$$

and

$$\limsup_{\imath\to\infty}\frac{1}{L(\imath,\imath_2)}\int_{\imath_2}^{\imath}W_2(\imath,\xi)\mathrm{d}\xi=\infty. \tag{53}$$

Then $Y_1=\emptyset$.

Proof. By using (31) in Theorem 2, in a similar way to the proof of Theorem 5, we get condition (53). □

Theorem 7. *Let $\varepsilon(\imath)\geq\varsigma(\imath)$ and $\varepsilon'(\imath)>0$. Moreover, assume that $L\in P$ and $\exists$ a function $\wp\in C^1([\imath_0,\infty),(0,\infty))$, for all sufficiently large $\imath_1\geq\imath_0$, $\exists$ a $\imath_2>\imath_1$ such that (46) hold and*

$$\limsup_{\imath\to\infty}\frac{1}{L(\imath,\imath_2)}\int_{\imath_2}^{\imath}W_3(\imath,\xi)\mathrm{d}\xi=\infty. \tag{54}$$

Then $Y_1=\emptyset$.

Proof. By using (46) in Theorem 3, similar to the proof of Theorem 5, we get condition (54). □

Theorem 8. *Let $\varepsilon(\imath)\geq\varsigma(\imath)$ and $\varepsilon'(\imath)>0$. Moreover, assume that $L\in P$ and $\exists$ a function $\wp\in C^1([\imath_0,\infty),(0,\infty))$, for all sufficiently large $\imath_1\geq\imath_0$, $\exists$ a $\imath_2>\imath_1$ such that (52) hold and*

$$\limsup_{\imath\to\infty}\frac{1}{L(\imath,\imath_2)}\int_{\imath_2}^{\imath}W_4(\imath,\xi)\mathrm{d}\xi=\infty. \tag{55}$$

Then $Y_1=\emptyset$.

Proof. By using Theorem 4, similar to the proof of Theorem 5, we conclude (54). □

5. Oscillation Criteria

Now, combining Corollary 1 with Theorems 2–4 and combining Corollary 1 with Theorems 5–8, respectively, it is easy to get new oscillation criteria for (4).

Theorem 9. *Assume that $\exists$ a functions $\zeta(\imath)\in C(T_{\imath_0},T_0)$ and $\wp\in C^1(T_{\imath_0},(0,\infty))$, and $\varepsilon(\imath,n)<\zeta(\imath)$, $\varsigma^{-1}(\zeta(\imath))<\imath$ and $\varepsilon(\imath)\leq\varsigma(\imath)$ such that (16) or (17) and (18) hold, if (19) is satisfied, then (4) is oscillatory.*

Theorem 10. *Assume that $\exists$ a functions $\zeta(\imath)\in C(T_{\imath_0},T_0)$ and $\wp\in C^1(T_{\imath_0},(0,\infty))$,and $\varepsilon(\imath,n)<\zeta(\imath)$, $\varsigma^{-1}(\zeta(\imath))<\imath$ and $\varepsilon(\imath)\geq\varsigma(\imath)$, such that (16) or (17) and (18) hold, if (33) is satisfied, then (4) is oscillatory.*

Theorem 11. *Assume that* $\exists$ *a function* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *and* $\wp \in C^1(T_{\imath_0}, (0,\infty))$, *and* $\varepsilon(\imath, n) < \zeta(\imath)$, $\varsigma^{-1}(\zeta(\imath)) < \imath$ *and* $\varepsilon(\imath) \geq \varsigma(\imath)$ *such that (16) or (17) and (18) hold, if (45) is satisfied, then (4) is oscillatory.*

Moreover, combining Corollary 1 with Theorems 5–8, respectively, we present the following Theorems:

Theorem 12. *Assume that* $\exists$ *a functions* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *and* $\wp \in C^1([\imath_0,\infty),(0,\infty))$ *and* $\varepsilon(\imath, n) < \zeta(\imath)$ *and* $\imath > \varsigma^{-1}(\zeta(\imath))$, $\varepsilon(\imath) \leq \varsigma(\imath)$, $\varepsilon'(\imath) > 0$ *such that (16) or (17) and (18) hold, if and* $L \in P$ *such that (46) and (47) is satisfied, then (4) is oscillatory.*

Theorem 13. *Assume that* $\exists$ *a functions* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *and* $\wp \in C^1([\imath_0,\infty),(0,\infty))$ *and* $\varepsilon(\imath, n) < \zeta(\imath)$ *and* $\imath > \varsigma^{-1}(\zeta(\imath))$, $\varepsilon(\imath) \leq \varsigma(\imath)$, $\varepsilon'(\imath) > 0$ *such that (16) or (17) and (18) hold, if* $L \in P$ *such that (52) and (53) is satisfied, then (4) is oscillatory.*

Theorem 14. *Assume that* $\exists$ *a functions* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *and* $\wp \in C^1([\imath_0,\infty),(0,\infty))$ *and* $\varepsilon(\imath, n) < \zeta(\imath)$ *and* $\imath > \varsigma^{-1}(\zeta(\imath))$, $\varepsilon(\imath) \geq \varsigma(\imath)$, $\varepsilon'(\imath) > 0$ *such that (16) or (17) and (18) hold, if* $L \in P$ *such that (46) and (54) is satisfied, then (4) is oscillatory.*

Theorem 15. *Assume that* $\exists$ *a functions* $\zeta(\imath) \in C(T_{\imath_0}, T_0)$ *and* $\wp \in C^1([\imath_0,\infty),(0,\infty))$ *and* $\varepsilon(\imath, n) < \zeta(\imath)$ *and* $\imath > \varsigma^{-1}(\zeta(\imath))$, $\varepsilon(\imath) \geq \varsigma(\imath)$, $\varepsilon'(\imath) > 0$ *such that (16) or (17) and (18) hold, if* $L \in P$ *such that (52) and (55) is satisfied, then (4) is oscillatory.*

Example 1. *Consider the third-order neutral differential equation*

$$\left(\imath\left[\left(\vartheta(\imath) + g_0(\imath)\vartheta\left(\frac{\imath}{2}\right)\right)''\right]^3\right)' + \frac{\lambda}{\imath^6}\vartheta^3\left(\frac{\varepsilon\imath}{2}\right) = 0, \quad \lambda > 0. \tag{56}$$

Setting $\wp = \imath^5$, *by Theorem 4, we obtain*

$$\lambda > \frac{\left(\frac{25}{2}\right)^2 2^5}{3}\left(1 + 2\wp_0^3\right) \tag{57}$$

or by Theorem 3, we obtain

$$\lambda > \frac{5^4 2^5}{4^3}\left(1 + 2\wp_0^3\right). \tag{58}$$

Thus, by condition (6) in Lemma 2, we see that (56) is almost oscillatory if (57) or (58) hold. For this example, condition (58) is better than condition (57).

6. Conclusions

This paper presents new criteria for the oscillation behavior of third-order neutral differential Equation (4) with continuously distributed delay in a form that is essentially new and of a high degree of generality. We obtained different conditions that guarantee the nonexistence of positive decreasing solutions by using the comparison technique with first-order delay equations. We also obtained conditions that guarantee the nonexistence of positive increasing solutions by using the Riccati transformation and integral averaging method. Therefore, we concluded with criteria that ensure the oscillation of all solutions to Equation (4).

It would be interesting to study (4) without restriction $\varsigma \circ \varepsilon = \varepsilon \circ \varsigma$ and in cases where $p(\imath)$ is an oscillatory function.

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