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The Canonical Isomorphisms in the Yetter-Drinfeld Categories for Dual Quasi-Hopf Algebras

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Abstract: Hopf algebras, as a crucial generalization of groups, have a very symmetric structure and have been playing a prominent role in mathematical physics. In this paper, let H be a dual quasi-Hopf algebra which is a more general Hopf algebra structure. A. Balan firstly introduced the notion of right-right Yetter-Drinfeld modules over H and studied its Galois extension. As a continuation, the aim of this paper is to introduce more properties of Yetter-Drinfeld modules. First, we will describe all the other three kinds of Yetter-Drinfeld modules over H, and the monoidal and braided structure of the categories of Yetter-Drinfeld modules explicitly. Furthermore, we will prove that the category ${}^H_H\mathcal{YD}^{fd}$ of finite dimensional left-left Yetter-Drinfeld modules is rigid. Then we will compute explicitly the canonical isomorphisms in ${}^H_H\mathcal{YD}^{fd}$. Finally, as an application, we will rewrite the isomorphisms in the case of coquasitriangular dual quasi-Hopf algebra.

Keywords: dual quasi-Hopf algebra; Yetter-Drinfeld module; rigid braided monoidal category; canonical isomorphisms

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1. Introduction

Hopf algebras originated from the study of the homology of Lie groups and have a natural relationship with groups. Hopf algebras could be seen as an important generalization of groups. In fact, group algebra (a vector space with the basis being a group) is a class of significant Hopf algebra and plays a key role in the theory of Hopf algebra. Quasi-bialgebras and quasi-Hopf algebras have been introduced by Drinfeld in [1], in connection with the Knizhnik-Zamolodchikov system of partial differential equations, and have been used in several branches of mathematics and physics. In a quasi-bialgebra H, the comultiplication is not coassociative but is quasi-coassociative in the sense that the comultiplication is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$. Equivalently the representation category of H is not a strict monoidal category while the reassociation is not trivial. If we draw our attention to the category of co-representations of a coalgebra with non-associative multiplication, we get the concepts of dual quasi-bialgebra and dual quasi-Hopf algebra. These notions have been introduced by Majid in [2] to prove a Tannaka-Krein type theorem for quasi-Hopf algebras.

For a dual quasi-Hopf algebra H, the category of right H-comodules $\mathcal{M}^H$ is monoidal with the usual tensor product. The difference between a dual quasi-Hopf algebra and a Hopf algebra lies in the fact that the associativity of tensor product in the category $\mathcal{M}^H$ is not trivial but modified by an invertible element $\sigma \in (H \otimes H \otimes H)^*$. Consequently, the multiplication of H is no longer associative.

In [3], the left Yetter-Drinfeld module over quasi-Hopf algebras was first constructed by S. Majid with the help of the isomorphism between the category of Yetter-Drinfeld modules and the center of the representation category. Subsequently, Bulacu, Caenepeel, and Panaite in [4] introduced all kinds of Yetter-Drinfeld modules and showed that the category of finite dimensional Yetter-Drinfeld modules is rigid. Following the ideas of S.

Majid, in [5] Balan introduced the notion of left-left Yetter-Drinfeld modules over dual quasi-Hopf algebra and studied its Galois extension. Later on, Ardizzoni in [6] introduced another form of Yetter-Drinfeld module through the isomorphism between the category of Yetter-Drinfeld modules and the category of Hopf bimodules, and characterized as bosonizations the dual quasi-bialgebras with a projection onto a dual quasi-bialgebra. In [7], the authors introduced the left-left and right-left Yetter-Drinfeld modules and constructed a quantum cocommutative coalgebra in the category of Yetter-Drinfeld modules. In [8], the authors studied coquasitriangular pointed dual quasi-Hopf algebras and braided pointed tensor categories via the quiver approaches, and classified the Hopf quivers whose path coalgebras admit coquasitriangular dual quasi-Hopf algebras.

Motivated by these ideas, a natural question arises: what are the braided monoidal structures of the categories of left-left, left-right, and right-left Yetter-Drinfeld modules, and via the braidings of other properties of these categories, what could we obtain? In this paper, we will continue the study of the category of Yetter-Drinfeld modules over dual quasi-Hopf algebra H. Firstly, we will give the definition of left-right Yetter-Drinfeld modules and describe their braided monoidal structures explicitly. Moreover, we will show that the categories ${}^H_H\mathcal{YD}$ and ${}^H\mathcal{YD}_H$ are isomorphic, even in the situation where H is not finite-dimensional. Then we will show that the category ${}^H_H\mathcal{YD}^{fd}$ of finite-dimensional left-left Yetter-Drinfeld modules is rigid, and we compute explicitly the canonical isomorphisms in ${}^H_H\mathcal{YD}^{fd}$. Finally, as an application, we rewrite the isomorphisms in the case of coquasitriangular dual quasi-Hopf algebra.

This paper is organized as follows. In Section 2, we will review the basic results of dual quasi-Hopf algebras and monoidal categories. In Section 3, we will describe explicitly the braided monoidal structures of categories of all the three kinds of Yetter-Drinfeld modules over dual quasi-Hopf algebra H. In Section 4, we will show that the category ${}^H_H\mathcal{YD}^{fd}$ of finite dimensional left-left Yetter-Drinfeld modules is rigid and give the explicit forms of the left and right duals of any object. In any rigid braided monoidal category, there exist canonical isomorphisms $M \cong M^{**}$ and $(M \otimes N)^* \cong M^* \otimes N^*$ for any object M. In Section 5, we will pay attention to the computations of these isomorphisms in ${}^H_H\mathcal{YD}^{fd}$. In Section 6, as an application, we will recover the isomorphisms in the case of coquasitriangular dual quasi-Hopf algebra.

2. Preliminary

Throughout this article, let k be a fixed field. All algebras, coalgebras, linear spaces, etc. will be over k; unadorned $\otimes$ means $\otimes_k$.

2.1. Dual Quasi-Hopf Algebra

Recall from [9–11] that a dual quasi-bialgebra H is a coassociative coalgebra with comultiplication Δ and counit ε together with coalgebra morphisms $m_H : H \otimes H \to H$ (the multiplication, we write $m_H(h \otimes h') = hh'$) and $\eta_H : k \to H$ (the unit, we write $\eta_H(1) = 1$), and an invertible element $\sigma \in (H \otimes H \otimes H)^*$ (the reassociation), such that for all $a, b, c, d \in H$ the following relations hold

$$a_1(b_1c_1)\sigma(a_2, b_2, c_2) = \sigma(a_1, b_1, c_1)(a_2b_2)c_2, \tag{1}$$
$$1a = a1 = a, \tag{2}$$
$$\sigma(a_1, b_1, c_1d_1)\sigma(a_2b_2, c_2, d_2) = \sigma(b_1, c_1, d_1)\sigma(a_1, b_2c_2, d_2)\sigma(a_2, b_3, c_3), \tag{3}$$
$$\sigma(a, 1, b) = \varepsilon(a)\varepsilon(b). \tag{4}$$

H is called a dual quasi-Hopf algebra if, moreover, there exists an anti-morphism s of the coalgebra H and elements $\alpha, \beta \in H^*$ such that for all $h \in H$,

$$s(h_1)\alpha(h_2)h_3 = \alpha(h)1, \qquad h_1\beta(h_2)s(h_3) = \beta(h)1, \tag{5}$$
$$\sigma(h_1\beta(h_2), s(h_3), \alpha(h_4)h_5) = \sigma^{-1}(s(h_1), \alpha(h_2)h_3, \beta(h_4)s(h_5)) = \varepsilon(h). \tag{6}$$

Throughout this paper, we will always assume that s is a bijective. It follows from the axioms that $s(1) = 1$ and $\alpha(1)\beta(1) = 1$. Moreover (3) and (4) imply that

$$\sigma(1, a, b) = \sigma(a, b, 1) = \varepsilon(a)\varepsilon(b). \tag{7}$$

Together with a dual quasi-Hopf algebra $H = (H, m, 1, \Delta, \varepsilon, \sigma, s, \alpha, \beta)$, we also have H^{op}, H^{cop} and $H^{op,cop}$ as dual quasi-Hopf algebras. The dual quasi-Hopf structures are obtained by putting $\sigma_{cop} = \sigma^{-1}, \sigma_{op} = (\sigma^{-1})^{321}$, and $\sigma_{op,cop} = \sigma^{321}$. $s_{op} = s_{cop} = (s^{-1}_{op,cop}) = s^{-1}$, $\alpha_{cop} = \beta s^{-1}$, $\alpha_{op} = \alpha s^{-1}$, $\alpha_{op,cop} = \beta$, $\beta_{cop} = \alpha s^{-1}$, $\beta_{op} = \beta s^{-1}$, $\beta_{op,cop} = \alpha$. Here $\sigma^{321}(a, b, c) = \sigma(c, b, a)$.

We recall that an invertible element $F \in (H \otimes H)^*$ satisfying $F(1, a) = F(a, 1) = \varepsilon(a)$, induces a twisting transformation

$$a \cdot b = F(a_1, b_1)a_2b_2F^{-1}(a_3, b_3), \tag{8}$$

$$\sigma_F(a, b, c) = F(b_1, c_1)F(a_1, b_2c_2)\sigma(a_2, b_3, c_3)F^{-1}(a_3b_4, c_4)F^{-1}(a_4, b_5). \tag{9}$$

For a Hopf algebra, one knows that the antipode is an anti-algebra morphism, i.e., $s(ab) = s(b)s(a)$. For a dual quasi-Hopf algebra, this is true only up to a twist, namely, there exists a twist transformation $f \in (H \otimes H)^*$ such that for all $a, b \in H$,

$$f(a_1, b_1)s(a_2b_2)g(a_3, b_3) = s(b)s(a), \tag{10}$$

where g denotes the convolution inverse of f.

The element f can be computed explicitly. For all $a, b, c, d \in H$, set

$$\nu(a, b, c, d) = \sigma(a_1, b_1, c_1)\sigma^{-1}(a_2b_2, c_2, d),$$
$$\mu(a, b, c, d) = \sigma(a_1b_1, c_1, d)\sigma^{-1}(a_2, b_2, c_2).$$

Define elements $\lambda, \chi \in (H \otimes H)^*$ by

$$\lambda(a, b) = \nu(s(b_1), s(a_1), a_3, b_3)\alpha(a_2)\alpha(b_2),$$
$$\chi(a, b) = \mu(a_1, b_1, s(b_3), s(a_3))\beta(a_2)\beta(b_2).$$

Then f and g are given by the following formulae:

$$f(a, b) = \sigma^{-1}(s(b_1)s(a_1), a_3b_3, s(a_5b_5))\lambda(a_2, b_2)\beta(a_4b_4),$$
$$g(a, b) = \sigma^{-1}(s(a_1b_1), a_3b_3, s(b_5)s(a_5))\chi(a_4, b_4)\alpha(a_2b_2).$$

The elements λ, χ and the twist f fulfill the relations

$$f(a_1, b_1)\alpha(a_2b_2) = \lambda(a, b), \quad \beta(a_1b_1)g(a_2, b_2) = \chi(a, b), \tag{11}$$

$$\alpha(a) = \beta(s(a_2))f(s(a_1), a_3) = \beta(s^{-1}(a_2))f(s^{-1}(a_3), a_1), \tag{12}$$

$$\beta(a) = g(a_1, s(a_3))\alpha(s(a_2)) = \alpha(s^{-1}(a_2))g(a_3, s^{-1}(a_1)). \tag{13}$$

The corresponding reassociation is given by

$$\sigma_f(a, b, c) = \sigma(s(c), s(b), s(a)). \tag{14}$$

2.2. Coquasitriangular Dual Quasi-Hopf Algebra

Recall from [12] that a coquasitriangular dual quasi-Hopf algebra is a dual quasi-Hopf algebra H with an invertible element $\varphi \in (H \otimes H)^*$ satisfying

$$(1)\ \varphi(ab, c) = \sigma(c_1, a_1, b_1)\varphi(a_2, c_2)\sigma^{-1}(a_3, c_3, b_2)\varphi(b_3, c_4)\sigma(a_4, b_4, c_5),$$
$$(2)\ \varphi(a, bc) = \sigma^{-1}(b_1, c_1, a_1)\varphi(a_2, c_2)\sigma(b_2, a_3, c_3)\varphi(a_4, b_3)\sigma^{-1}(a_5, b_4, c_4),$$

(3) $\varphi(a_1, b_1)a_2b_2 = b_1a_1\varphi(a_2, b_2)$,
(4) $\varphi(1, a) = \varphi(a, 1) = \varepsilon(a)$,

for all $a, b, c \in H$.

Let (H, φ) be a coquasitriangular dual quasi-Hopf algebra. Define $u \in \mathrm{Hom}(H, k)$ by

$$u(a) = \sigma^{-1}(a_7, s(a_3), s^2(a_1))\beta(s(a_2))\varphi(a_6, s(a_4))\alpha(a_5),$$

for all $a \in H$. It is proved in [12] that u is invertible with the inverse given by

$$u^{-1}(a) = q^R(a_1, s^2(a_4))p^R(s^2(a_6), a_3)\varphi(s^2(a_5), a_2).$$

Moreover u satisfies the following identities (see [12]):

$$s^2(a_1)u(a_2) = u(a_1)a_2, \tag{15}$$
$$\alpha(s(a_1))u(a_2) = \varphi(a_3, s(a_1))\alpha(a_2), \tag{16}$$
$$f(b_1, a_1)\varphi(a_2, b_2) = \varphi(s(a_1), s(b_1))f(a_2, b_2), \tag{17}$$
$$u \circ s^2 = u. \tag{18}$$

2.3. Monoidal Categories and Center Construction

A monoidal category means a category $\mathcal{C}$ with objects U, V, W, etc., a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation consisting of functorial isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ satisfying a pentagon identity, and a compatible unit object I and associated functorial isomorphisms (the left and the right unit constraints, $l_V : V \cong V \otimes I$ and $r_V : V \cong I \otimes V$, respectively.) Now if $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories then, roughly speaking, we say that $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor if it respects the tensor products (in the sense that for any two objects $U, V \in \mathcal{C}$ there exists a functorial isomorphism $\Psi : F(U) \otimes F(V) \to F(U \otimes V)$ such that Ψ respects the associativity constraints), the unit object and the left and right unit constraints (for a complete Definition see [2]).

If H is a dual quasi-Hopf algebra, then the categories $\mathcal{M}^H$ and ${}^H\mathcal{M}$ are monoidal categories. The associative constraint on $\mathcal{M}^H$ is the following: for any $M, N, P \in \mathcal{M}^H$, and $m \in M, n \in N$, $a_{M,N,P} : (M \otimes N) \otimes P \to M \otimes (N \otimes P)$ is given by

$$a_{M,N,P}((m \otimes n) \otimes p) = \sigma(m_{(1)}, n_{(1)}, p_{(1)})m_{(0)} \otimes (n_{(0)} \otimes p_{(0)}).$$

On ${}^H\mathcal{M}$, the associative constraint is given by

$$a_{M,N,P}((m \otimes n) \otimes p) = \sigma^{-1}(m_{(-1)}, n_{(-1)}, p_{(-1)})m_{(0)} \otimes (n_{(0)} \otimes p_{(0)}).$$

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category, and $V \in \mathcal{C}$. $V^* \in \mathcal{C}$ is called the left dual of V, if there exist two morphisms $ev_V : V^* \otimes V \to I$ and $coev_V : I \to V \otimes V^*$ such that

$$(V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes V) = V,$$
$$(ev_V \otimes V^*) \circ a^{-1}_{V^*,V,V^*} \circ (V^* \otimes coev_V) = V^*.$$

${}^*V \in \mathcal{C}$ is called a right dual of V if there exist two morphisms $ev'_V : V \otimes {}^*V \to I$ and $coev'_V : I \to {}^*V \otimes V$ such that

$$({}^*V \otimes ev'_V) \circ a_{{}^*V,V,{}^*V} \circ (coev'_V \otimes {}^*V) = {}^*V,$$
$$(ev'_V \otimes V) \circ a^{-1}_{V,{}^*V,V} \circ (V \otimes coev'_V) = V.$$

$\mathcal{C}$ is called a rigid monoidal category if every object of $\mathcal{C}$ has a left and right dual. The category ${}^H\mathcal{M}_{fd}$ of finite dimensional modules over a dual quasi-Hopf algebra H is rigid.

For $V \in {}^H\mathcal{M}_{fd}$, $V^* = \mathrm{Hom}(V,k)$ with left coaction $\lambda(\varphi) = \langle \varphi, v_{i(0)} \rangle s^{-1}(v_{i(-1)}) \otimes v^i$. The evaluation and co-evaluation are given by

$$ev_V(\varphi \otimes v) = \beta(s^{-1}(v_{(-1)}))\varphi(v_{(0)}), \quad coev_V(1) = \alpha(s^{-1}(v_{i(-1)}))v_{i(0)} \otimes v^i.$$

where $\{v_i\}_i$ is a basis in V with dual basis $\{v^i\}_i$.

The right dual ${}^*V$ of V is the same dual vector space equipped with the left H-comodule structure given by $\lambda(\varphi) = \langle \varphi, v_{i(0)} \rangle s(v_{i(-1)}) \otimes v^i$ and

$$ev'_V(v \otimes \varphi) = \beta(v_{(-1)})\varphi(v_{(0)}), \quad coev'_V(1) = v^i \otimes \alpha(v_{i(-1)})v_{i(0)}.$$

For a braided monoidal category $\mathcal{C}$, let $\mathcal{C}^{in}$ be equal to $\mathcal{C}$ as a monoidal category, with the mirror-reversed braiding $\tilde{c}_{M,N} = c^{-1}_{M,N}$.

Following [3], the left weak center $\mathcal{W}_l(\mathcal{C})$ is the category with the objects $(V, s_{V,-})$, where $V \in \mathcal{C}$ and $s_{V,-} : V \otimes - \to - \otimes V$ is a family of natural transformations such that $s_{V,I} = id_V$ and for all $X, Y \in \mathcal{C}$,

$$(X \otimes s_{V,Y}) \circ a_{X,V,Y} \circ (s_{V,X} \otimes Y) = a_{X,Y,V} \circ s_{V,X\otimes Y} \circ a_{V,X,Y}.$$

A morphism between $(V, s_{V,-})$ and $(V', s_{V',-})$ consists of $\psi : V \to V'$ in $\mathcal{C}$ such that

$$(X \otimes \psi) \circ s_{V,X} = c_{V',X} \circ (\psi \otimes X).$$

$\mathcal{W}_l(\mathcal{C})$ is a prebraided monoidal category. The tensor product is

$$(V, s_{V,-}) \otimes (V', s_{V',-}) = (V \otimes V', s_{V\otimes V',-}),$$

with

$$s_{V\otimes V',X} = a_{X,V,V'} \circ (s_{V,X} \otimes V') \circ a^{-1}_{V,X,V'} \circ (V \otimes s_{V',X}) \circ a_{V,V',X},$$

and the unit is (I, id). The braiding s on $\mathcal{W}_l(\mathcal{C})$ is given by

$$c_{V,V'} = s_{V,V'} : (V, s_{V,-}) \otimes (V', s_{V',-}) \to (V', s_{V',-}) \otimes (V, s_{V,-}).$$

The center $\mathcal{Z}_l(\mathcal{C})$ is the full subcategory of $\mathcal{W}_l(\mathcal{C})$ consisting of objects $(V, s_{V,-})$ with $s_{V,-}$ a natural isomorphism. $\mathcal{Z}_l(\mathcal{C})$ is a braided monoidal category.

The right weak center $\mathcal{W}_r(\mathcal{C})$ is the category with the objects $(V, c_{-,V})$, where $V \in \mathcal{C}$ and $t_{-,V} : - \otimes V \to V \otimes -$ is a family of natural transforms such that $t_{I,V} = id_V$ and

$$a^{-1}_{V,X,Y} \circ t_{X\otimes Y,V} \circ a^{-1}_{X,Y,V} = (t_{X,V} \otimes Y) \circ a^{-1}_{X,V,Y} \circ (X \otimes t_{Y,V}),$$

for all $X, Y \in \mathcal{C}$. A morphism between $(V, t_{-,V})$ and $(V', t_{-,V'})$ consists of $\psi : V \to V'$ in $\mathcal{C}$ such that

$$(\psi \otimes X) \circ t_{X,V} = t_{X,V'} \circ (X \otimes \psi).$$

$\mathcal{W}_r(\mathcal{C})$ is a prebraided monoidal category. The unit is (I, id) and the tensor product is

$$(V, t_{-,V}) \otimes (V', t_{-,V'}) = (V \otimes V', (V, t_{-,V\otimes V'})),$$

with

$$t_{-,V\otimes V'} = a^{-1}_{V,V',X} \circ (V \otimes t_{X,V'}) \circ a_{V,X,V'} \circ (t_{X,V} \otimes V') \circ a^{-1}_{X,V,V'}.$$

The braiding d is given by

$$d_{V,V'} = t_{V,V'} : (V, s_{-,V}) \otimes (V', s_{-,V'}) \to (V', s_{-,V'}) \otimes (V, s_{-,V}).$$

The center $\mathcal{Z}_r(\mathcal{C})$ is the full subcategory of $\mathcal{W}_r(\mathcal{C})$ consisting of objects $(V, t_{-,V})$ with $t_{-,V}$ a natural isomorphism. $\mathcal{Z}_r(\mathcal{C})$ is a braided monoidal category.

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category. Then we have a second monoidal structure on $\mathcal{C}$, defined as

$$\overline{\mathcal{C}} = (\mathcal{C}, \overline{\otimes} = \otimes \circ \tau, I, \overline{a}, r, l),$$

where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}, (U, V) \mapsto (V, U)$ and $\overline{a}$ given by $\overline{a}_{U,V,W} = a^{-1}_{W,V,U}$.

If c is a braiding on $\mathcal{C}$, then $\overline{c}$, defined by $\overline{c}_{U,V} = c_{V,U}$ is a braiding on $\overline{\mathcal{C}}$.

It is obvious that

Proposition 1 ([4]). *Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category. Then*

$$\overline{\mathcal{W}_l(\mathcal{C})} \cong \mathcal{W}_r(\overline{\mathcal{C}}), \ \ \overline{\mathcal{W}_r(\mathcal{C})} \cong \mathcal{W}_l(\overline{\mathcal{C}}),$$

as the prebraided monoidal category, and

$$\overline{\mathcal{Z}_l(\mathcal{C})} \cong \mathcal{Z}_r(\overline{\mathcal{C}}), \ \ \overline{\mathcal{Z}_r(\mathcal{C})} \cong \mathcal{Z}_l(\overline{\mathcal{C}}),$$

as a braided monoidal category.

Definition 1 ([7]). *Let H be a dual quasi-bialgebra.*

(1) *A k-space M is called a left-left Yetter-Drinfeld module if M is a left H-comodule (denote the left coaction by $\lambda_M : M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$) and H acts on M from the left (denote the left action by $h \cdot m$) such that the following conditions hold:*

$$\begin{aligned} &\sigma(h_1, g_1, m_{(-1)})\sigma((h_2g_2 \cdot m_{(0)})_{(-1)}, h_3, g_3)(h_2g_2 \cdot m_{(0)})_{(0)} \\ &\quad = \sigma(h_1, (g_1 \cdot m)_{(-1)}, g_2)h_2 \cdot (g_1 \cdot m)_{(0)}, \quad (19)\\ &1_H \cdot m = m, \quad (20)\\ &h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)} = (h_1 \cdot m)_{(-1)}h_2 \otimes (h_1 \cdot m)_{(0)}, \quad (21)\end{aligned}$$

for all $h, g \in H$ and $m \in M$. The category of left-left Yetter-Drinfeld modules over H is denoted by ${}^H_H\mathcal{YD}$ with the morphisms being left H-linear and left H-colinear.

(2) *A right-left Yetter-Drinfeld module is a left H-comodule M together with a right H-action $\cdot$ on M such that for all $g, h \in H, m \in M$,*

$$\begin{aligned} &\sigma^{-1}(m_{(-1)}, h_1, g_1)\sigma^{-1}(h_3, g_3, (m_{(0)} \cdot h_2g_2)_{(-1)})(m_{(0)} \cdot h_2g_2)_{(0)} \\ &= \sigma^{-1}(h_2, (m \cdot h_1)_{(-1)}, g_1)(m \cdot h_1)_{(0)} \cdot g_2, \\ &m \cdot 1 = m, \\ &h_2(m \cdot h_1)_{(-1)} \otimes (m \cdot h_1)_{(0)} = m_{(-1)}h_1 \otimes m_{(0)} \cdot h_2. \end{aligned}$$

The category of right-left Yetter-Drinfeld modules over H is denoted by ${}^H\mathcal{YD}_H$ with the morphisms being right H-linear and left H-colinear.

3. Yetter-Drinfeld Modules over a Dual Quasi-Hopf Algebra

In this section, we will describe braided monoidal structures of the categories of Yetter-Drinfeld modules over dual quasi-Hopf algebra H and show that the categories ${}^H_H\mathcal{YD}$ and ${}^H\mathcal{YD}_H$ are isomorphic.

Let H be a dual quasi-Hopf algebra. Recall from [5], for all $a, b \in H$, define elements p^R, q^R, p^L, q^L in $(H \otimes H)^*$ by

$$p^R(a, b) = \sigma^{-1}(a, b_1, s(b_3))\beta(b_2), \quad q^R(a, b) = \sigma(a, b_3, s^{-1}(b_1))\alpha(s^{-1}(b_2)),$$
$$p^L(a, b) = \sigma(s^{-1}(a_3), a_1, b)\beta(s^{-1}(a_2)), \quad q^L(a, b) = \sigma^{-1}(s(a_1), a_3, b)\alpha(a_2).$$

Lemma 1. *Let H be a dual quasi-Hopf algebra. For all $a, b \in H$,*

$$p^R(a_1, b)a_2 = (a_1b_1)p^R(a_2, b_2)s(b_3), \quad q^R(a_2, b)a_1 = (a_2b_3)q^R(a_1, b_2)s^{-1}(b_1), \tag{22}$$
$$p^L(a, b_1)b_2 = s^{-1}(a_3)p^L(a_2, b_2)(a_1b_1), \quad q^L(a, b_2)b_1 = s(a_1)q^L(a_2, b_1)(a_3b_2), \tag{23}$$

and

$$q^R(a_1b_1, s(b_3))p^R(a_2, b_2) = \varepsilon(a)\varepsilon(b), \quad p^L(s(a_1), a_3b_2)q^L(a_2, b_1) = \varepsilon(a)\varepsilon(b), \tag{24}$$
$$q^L(s^{-1}(a_3), a_1b_1)p^L(a_2, b_2) = \varepsilon(a)\varepsilon(b), \; q^R(a_1, b_2)p^R(a_2b_3, s^{-1}(b_1)) = \varepsilon(a)\varepsilon(b). \tag{25}$$

Moreover we have the following formulae

$$\begin{aligned}(1)\quad & q^R(a_1, b_1)q^R(a_2b_2, c_1)\sigma^{-1}(a_3, b_3, c_2)\\ & = \sigma(a_2(b_4c_4), s^{-1}(c_1), s^{-1}(b_1))f(s^{-1}(c_2), s^{-1}(b_2))q^R(a_1, b_3c_3).\end{aligned} \tag{26}$$
$$\begin{aligned}(2)\quad & \sigma(a_1, b_1, c_1)p^R(a_2b_2, c_2)p^R(a_3, b_3)\\ & = \sigma^{-1}(a_1(b_1c_1), s(c_4), s(b_4))p^R(a_2, b_2c_2)g(b_3, c_3).\end{aligned} \tag{27}$$

Proof. The identities (24)–(26) come from [7]. Since H^{cop} is also a dual quasi-Hopf algebra, by (26) we could obtain (27). □

Proposition 2. *Let H be a dual quasi-Hopf algebra, $M \in {}^H M$, and $\cdot : H \otimes M \to M$ a k-linear map satisfying (19) and (20). Then (21) is equivalent to*

$$\begin{aligned}(h \cdot m)_{(-1)} \otimes (h \cdot m)_{(0)} =& q^R((h_1m_{(-1)})_1, s(h_5))(h_1m_{(-1)})_2 s(h_4)\\ & \otimes p^R((h_2 \cdot m_{(0)})_{(-1)}, h_3)(h_2 \cdot m_{(0)})_{(0)},\end{aligned} \tag{28}$$

for all $h \in H, m \in M$.

Proof. The proof is similar to that of [5]. □

Example 1. *Let (H, φ) be a coquasitriangular dual quasi-Hopf algebra. Then any left H-comodule M is a left Yetter-Drinfeld module over H. Indeed for all $g \in H, m \in M$, define*

$$g \cdot m = \varphi(m_{(-1)}, g)m_{(0)}.$$

Then for the relation (19)

$$\begin{aligned}&\sigma(h_1, g_1, m_{(-1)})\sigma((h_2g_2 \cdot m_{(0)})_{(-1)}, h_3, g_3)(h_2g_2 \cdot m_{(0)})_{(0)}\\ &= \sigma(h_1, g_1, m_{(-1)1})\sigma(m_{(-1)3}, h_3, g_3)\varphi(m_{(-1)2}, h_2g_2)m_{(0)}\\ &= \sigma(h_1, g_1, m_{(-1)1})\sigma(m_{(-1)7}, h_6, g_6)\sigma^{-1}(h_2, g_2, m_{(-1)2})\varphi(m_{(-1)3}, g_3)\\ &\quad \sigma(h_3, m_{(-1)4}, g_4)\varphi(m_{(-1)5}, h_4)\sigma^{-1}(m_{(-1)6}, h_5, g_5)m_{(0)}\\ &= \varphi(m_{(-1)1}, g_1)\sigma(h_1, m_{(-1)2}, g_2)\varphi(m_{(-1)3}, h_2)m_{(0)}\\ &= \sigma(h_1, (g_1 \cdot m)_{(-1)}, g_2)h_2 \cdot (g_1 \cdot m)_{(0)}.\end{aligned}$$

And for the relation (21)

$$\begin{aligned}&h_1m_{(-1)} \otimes h_2 \cdot m_{(0)} = h_1m_{(-1)1} \otimes m_{(0)}\varphi(m_{(-1)2}, h_2)\\ &= \varphi(m_{(-1)1}, h_1)m_{(-1)2}h_2 \otimes m_{(0)} = (h_1 \cdot m)_{(-1)}h_2 \otimes (h_1 \cdot m)_{(0)}.\end{aligned}$$

Proposition 3 ([5])**.** *Let H be a dual quasi-bialgebra and ${}^H\mathcal{M}$ the category of left H-comodules. Then we have category isomorphism ${}^H_H\mathcal{YD} \cong \mathcal{W}_r({}^H\mathcal{M})$.*

The action of H on the tensor product $M \otimes N$ of two left-left Yetter-Drinfeld modules M and N is given by

$$h \cdot (m \otimes n) = \sigma(h_1, m_{(-1)}, n_{(-1)1})\sigma^{-1}((h_2 \cdot m_{(0)})_{(-1)1}, h_3, n_{(-1)2}) \\ \sigma((h_2 \cdot m_{(0)})_{(-1)2}, (h_4 \cdot n_{(0)})_{(-1)}, h_5)(h_2 \cdot m_{(0)})_{(0)} \otimes (h_4 \cdot n_{(0)})_{(0)},$$

for all $m \in M, n \in N$. The braiding is given by

$$c_{M,N}(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}.$$

Furthermore we have the following result.

Theorem 1. *Let H be a dual quasi-Hopf algebra. The braiding c is invertible with the inverse $c^{-1}_{N,M} : M \otimes N \to N \otimes M$ given by*

$$c^{-1}_{N,M}(m \otimes n) = q^L(s^{-1}(n_{(-1)6}), m_{(-1)1}n_{(-1)1})\sigma(s^{-1}(n_{(-1)5}), m_{(-1)2}, n_{(-1)2}) \\ p^R((s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)3}))n_{(0)} \otimes (s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(0)}.$$

Proof. For all $m \in M, n \in N$,

$$\begin{aligned}
&c^{-1}_{N,M}(c_{N,M}(n \otimes m)) = c^{-1}_{N,M}(n_{(-1)} \cdot m \otimes n_{(0)}) \\
&= q^L(s^{-1}(n_{(-1)7}), (n_{(-1)1} \cdot m)_{(-1)1}n_{(-1)2})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)1} \cdot m)_{(-1)2}, n_{(-1)3}) \\
&\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)})_{(0)} \\
&= q^L(s^{-1}(n_{(-1)7}), (n_{(-1)1} \cdot m)_{(-1)}n_{(-1)2})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)1} \cdot m)_{(0)(-1)}, n_{(-1)3}) \\
&\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)1} \cdot m)_{(0)(0)})_{(0)} \\
&\overset{(21)}{=} q^L(s^{-1}(n_{(-1)7}), n_{(-1)1}m_{(-1)})\sigma(s^{-1}(n_{(-1)6}), (n_{(-1)2} \cdot m_{(0)})_{(-1)}, n_{(-1)3}) \\
&\quad p^R((s^{-1}(n_{(-1)5}) \cdot (n_{(-1)2} \cdot m_{(0)})_{(0)})_{(-1)}, s^{-1}(n_{(-1)4}))n_{(0)} \otimes (s^{-1}(n_{(-1)5}) \cdot (n_{(-1)2} \cdot m_{(0)})_{(0)})_{(0)} \\
&\overset{(19)}{=} q^L(s^{-1}(n_{(-1)9}), n_{(-1)1}m_{(-1)1}) \\
&\quad \sigma(s^{-1}(n_{(-1)8}), n_{(-1)2}, m_{(-1)2})\sigma((s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(-1)}), s^{-1}(n_{(-1)6}), n_{(-1)4}) \\
&\quad p^R((s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(0)(-1)}, s^{-1}(n_{(-1)5}))n_{(0)} \otimes (s^{-1}(n_{(-1)7})n_{(-1)3} \cdot m_{(0)})_{(0)(0)} \\
&= q^L(s^{-1}(n_{(-1)7}), n_{(-1)1}m_{(-1)1})\sigma(s^{-1}(n_{(-1)6}), n_{(-1)2}, m_{(-1)2})\beta(s^{-1}(n_{(-1)4})) \\
&\quad n_{(0)} \otimes s^{-1}(n_{(-1)5})n_{(-1)3} \cdot m_{(0)} \\
&\overset{(5)}{=} q^L(s^{-1}(n_{(-1)5}), n_{(-1)1}m_{(-1)1})\sigma(s^{-1}(n_{(-1)4}), n_{(-1)2}, m_{(-1)2})\beta(s^{-1}(n_{(-1)3})) \\
&\quad n_{(0)} \otimes m_{(0)} \\
&= q^L(s^{-1}(n_{(-1)3}), n_{(-1)1}m_{(-1)1})p^L(n_{(-1)2}, m_{(-1)2})n_{(0)} \otimes m_{(0)} \\
&\overset{(25)}{=} n \otimes m.
\end{aligned}$$

That is, $c^{-1}_{N,M} \circ c_{N,M} = id_{N \otimes M}$. Similarly $c_{N,M} \circ c^{-1}_{N,M} = id_{M \otimes N}$. The proof is completed. □

We also introduce left-right Yetter-Drinfeld modules in the following definition.

Definition 2. *Let H be a dual quasi-Hopf algebra. A left-right Yetter-Drinfeld module is a right H-comodule M together with a left H-action $\cdot$ on M such that for all $g, h \in H, m \in M$,*

$$\sigma^{-1}((h_2g_2 \cdot m_{(0)})_{(1)}, h_1, g_1)\sigma^{-1}(h_3, g_3, m_{(1)})(h_2g_2 \cdot m_{(0)})_{(0)} \\ = \sigma^{-1}(h_2, (g_2 \cdot m)_{(1)}, g_1)h_1 \cdot (g_2 \cdot m)_{(0)},$$

$$1 \cdot m = m,$$
$$(h_2 \cdot m)_{(0)} \otimes (h_2 \cdot m)_{(1)} h_1 = h_1 \cdot m_{(0)} \otimes h_2 m_{(1)}.$$

The category of left-right Yetter-Drinfeld modules over H is denoted by ${}_H\mathcal{YD}^H$ with the morphisms being left H-linear and right H-colinear.

Theorem 2. *Let H be a dual quasi-bialgebra. Then we have the following category isomorphisms:*

$$\mathcal{W}_l({}^H\mathcal{M}) \cong {}^H\mathcal{YD}_H, \quad \mathcal{W}_r(\mathcal{M}^H) \cong {}_H\mathcal{YD}^H.$$

If H is a dual quasi-Hopf algebra, then these three weak centers are equal to the centers.

Proof. The proof is straightforward and left to the reader. □

- The prebraided monoidal structure on $\mathcal{W}_l({}^H\mathcal{M})$ induces a monoidal structure on ${}^H\mathcal{YD}_H$. We find that the action on $M \otimes N$ of two right-left Yetter-Drinfeld modules M and N are given by

$$\begin{aligned}(m \otimes n) \cdot h =& \sigma^{-1}(m_{(-1)1}, n_{(-1)}, h_1)\sigma(m_{(-1)2}, h_3, (n_{(0)} \cdot h_2)_{(-1)1}) \\ & \sigma^{-1}(h_5, (m_{(0)} \cdot h_4)_{(-1)}, (n_{(0)} \cdot h_2)_{(-1)2})(m_{(0)} \cdot h_4)_{(0)} \otimes (n_{(0)} \cdot h_2)_{(0)},\end{aligned}$$

for all $h \in H, m \in M$, and $n \in N$.

The braiding $d_{M,N} : M \otimes N \to N \otimes M$ is given by

$$d_{M,N}(m \otimes n) = n_{(0)} \otimes m \cdot n_{(-1)}.$$

In the case when H is a dual quasi-Hopf algebra, the inverse of $d_{M,N}$ is given by

$$\begin{aligned}d^{-1}_{M,N}(n \otimes m) =& q^R(n_{(-1)1} m_{(-1)1}, s(n_{(-1)6}))\sigma^{-1}(n_{(-1)2}, m_{(-1)2}, s(n_{(-1)5})) \\ & p^L(s(n_{(-1)3}), (m_{(0)} \cdot s(n_{(-1)4}))_{(-1)})(m_{(0)} \cdot s(n_{(-1)4}))_{(0)} \otimes n_{(0)}.\end{aligned}$$

- The prebraided monoidal structure on ${}_H\mathcal{YD}^H$: for $M, N \in {}_H\mathcal{YD}^H$, the action on $M \otimes N$ is given by

$$\begin{aligned}h \cdot (m \otimes n) =& \sigma^{-1}(h_5, m_{(1)}, n_{(1)2})\sigma((h_4 \cdot m_{(0)})_{(1)2}, h_3, n_{(1)1}) \\ & \sigma^{-1}((h_4 \cdot m_{(0)})_{(1)1}, (h_2 \cdot n_{(0)})_{(1)}, h_1)(h_4 \cdot m_{(0)})_{(0)} \otimes (h_2 \cdot n_{(0)})_{(0)},\end{aligned}$$

and

$$(m \otimes n)_{(0)} \otimes (m \otimes n)_{(1)} = m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)},$$

the braiding is the following:

$$t_{M,N}(m \otimes n) = m_{(1)} \cdot n \otimes m_{(0)}.$$

In the case when H is a dual quasi-Hopf algebra, the inverse of $t_{M,N}$ is given by

$$\begin{aligned}t^{-1}_{M,N}(n \otimes m) = & p^L(s(m_{(1)1}), n_{(1)2} m_{(1)6})\sigma^{-1}(s(m_{(1)2}), n_{(1)1}, m_{(1)5}) \\ & q^R((s(m_{(1)3}) \cdot n_{(0)})_{(1)}, s(m_{(1)4}))m_{(0)} \otimes (s(m_{(1)3}) \cdot n_{(0)})_{(0)}.\end{aligned}$$

Proposition 4. *We have an isomorphism of monoidal categories*

$$F : \overline{{}^{H^{op,cop}}\mathcal{M}} \to \mathcal{M}^H,$$

where F acts on objects and morphisms as identity, and the right H-coaction is given by $m_{[0]} \otimes m_{[1]} = m_{(-1)} \otimes m_{(0)}$. *Similarly, we have*

$$\overline{\mathcal{M}^{H^{op,cop}}} \to {}^{H}\mathcal{M}.$$

Proof. We only need to verify that F preserves the monoidal structure. For all objects $M, N \in {}^{H^{op,cop}}\mathcal{M}$,

$$(n \otimes m)_{(-1)} \otimes (n \otimes m)_{(0)} = m_{(-1)}n_{(-1)} \otimes n_{(0)} \otimes m_{(0)}.$$

The associativity constraint $a_{P,N,M} : (P \otimes N) \otimes M \to P \otimes (N \otimes M)$ is defined as

$$a_{P,N,M}(p \otimes n \otimes m) = \sigma^{-1}(m_{(-1)}, n_{(-1)}, p_{(-1)})p_{(0)} \otimes n_{(0)} \otimes m_{(0)}.$$

As for the monoidal structure on $\overline{{}^{H^{op,cop}}\mathcal{M}}$, we have $M\overline{\otimes}N = N \otimes M$. Then

$$(m\overline{\otimes}n)_{(-1)} \otimes (m\overline{\otimes}n)_{(0)} = m_{(-1)}n_{(-1)} \otimes (m_{(0)}\overline{\otimes}n_{(0)}).$$

The associativity constraint $\overline{a}_{M,N,P} : (M\overline{\otimes}N)\overline{\otimes}P \to M\overline{\otimes}(N\overline{\otimes}P)$ is defined as

$$\overline{a}_{M,N,P}(m\overline{\otimes}n\overline{\otimes}p) = \sigma(m_{(-1)}, n_{(-1)}, p_{(-1)})m_{(0)}\overline{\otimes}n_{(0)}\overline{\otimes}p_{(0)}.$$

The proof is completed. □

Proposition 5. *Let H be a dual quasi-Hopf algebra. Then we have the following isomorphisms of braided monoidal categories:*

$$\mathcal{YD}_H^H \cong \overline{{}^{H^{op,cop}}_{H^{op,cop}}\mathcal{YD}}, \quad {}^{H}\mathcal{YD}_H \cong \overline{{}_{H^{op,cop}}\mathcal{YD}^{H^{op,cop}}}.$$

Proof. By Proposition 1 and Proposition 4, we obtain

$$\mathcal{YD}_H^H \cong \mathcal{Z}_l(\mathcal{M}^H) \cong \mathcal{Z}_l(\overline{{}^{H^{op,cop}}\mathcal{M}}) \cong \overline{\mathcal{Z}_r({}^{H^{op,cop}}\mathcal{M})} \cong \overline{{}^{H^{op,cop}}_{H^{op,cop}}\mathcal{YD}},$$

and

$${}^{H}\mathcal{YD}_H \cong \mathcal{Z}_l({}^{H}\mathcal{M}) \cong \mathcal{Z}_l(\overline{\mathcal{M}^{H^{op,cop}}}) \cong \overline{\mathcal{Z}_r(\mathcal{M}^{H^{op,cop}})} \cong \overline{{}_{H^{op,cop}}\mathcal{YD}^{H^{op,cop}}}.$$

□

Proposition 6 ([4]). *Let $\mathcal{C}$ be a monoidal category. Then we have a braided isomorphism of braided monoidal categories $T : \mathcal{Z}_l(\mathcal{C}) \to \mathcal{Z}_r(\mathcal{C})^{in}$, given by*

$$T(V, s_{V,-}) = (V, s_{V,-}^{-1}) \quad \text{and} \quad T(v) = v.$$

Of course, the conclusion holds for the right center. By this isomorphism, we have the following result.

Proposition 7. *Let H be a dual quasi-Hopf algebra, and ${}^{H}\mathcal{YD}_H^{in}$ the category ${}^{H}\mathcal{YD}_H$ with the braiding*

$$\tilde{c}_{M,N} = c_{M,N}^{-1}.$$

Then we have an isomorphism of braided monoidal categories

$$T : {}^{H}\mathcal{YD}_H^{in} \cong {}^{H}_{H}\mathcal{YD},$$

defined as follows. For $M \in {}^{H}\mathcal{YD}_H$, $T(M) = M$ as a left H-comodule; the left H-action is given by

$$h \triangleright m = q^R(h_1 m_{(-1)1}, s(h_6))\sigma^{-1}(h_2, m_{(-1)2}, s(h_5))$$

$$p^L(s(h_3),(m_{(0)}\cdot s(h_4))_{(-1)})(m_{(0)}\cdot s(h_4))_{(0)},$$

for all $h\in H, m\in M$, where $\cdot$ is the right action of H on M. The functor T sends a morphism to itself.

Proof. The functor T is just the composition of the isomorphisms

$$^H\mathcal{YD}_H^{in}\to \mathcal{Z}_l(^H\mathcal{M})^{in}\to \mathcal{Z}_r(^H\mathcal{M})\to {}^H_H\mathcal{YD}.$$

For $M\in {}^H\mathcal{YD}_H^{in}$, we compute the corresponding left-left Yetter-Drinfeld module structure on M is the following:

$$\begin{aligned}h\triangleright m &= (id\otimes\varepsilon)s^{-1}_{M,H}(h\otimes m)=(id\otimes\varepsilon)\tilde{c}_{M,H}(h\otimes m)=(id\otimes\varepsilon)d^{-1}_{M,H}(h\otimes m)\\ &= q^R(h_1m_{(-1)1},s(h_6))\sigma^{-1}(h_2,m_{(-1)2},s(h_5))p^L(s(h_3),(m_{(0)}\cdot s(h_4))_{(-1)})(m_{(0)}\cdot s(h_4))_{(0)},\end{aligned}$$

as claimed. □

In the same way, we have the following result.

Proposition 8. *Let H be a dual quasi-Hopf algebra. Then the categories $\mathcal{YD}_H^H$ and $_H\mathcal{YD}^{Hin}$ are isomorphic as braided monoidal categories.*

4. The Rigid Braided Category $^H_H\mathcal{YD}^{fd}$

It is well known that the category of finite dimensional Yetter-Drinfeld modules over a Hopf algebra with a bijective antipode is rigid. Since $^H\mathcal{M}_{fd}$ is rigid, the same result holds for the category of finite dimensional Yetter-Drinfeld modules over a dual quasi-Hopf algebra. In this section, we will give the explicit forms.

Proposition 9 ([4]). *Let $\mathcal{C}$ be a rigid monoidal category. Then the weak left (respectively right) center of $\mathcal{C}$ is a rigid braided monoidal category.*

For Example, for any object $(V,c_{-,V})\in\mathcal{Z}_r(\mathcal{C})$, $^*(V,c_{-,V})=(^*V,c_{-,^*V})$, with $c_{-,^*V}$ given by the following composition:

$$\begin{aligned}c_{X,^*V}: X\otimes{}^*V &\xrightarrow{coev'_V\otimes id_{X\otimes^*V}} (^*V\otimes V)\otimes(X\otimes{}^*V)\xrightarrow{a_{^*V,V,X\otimes^*V}} {}^*V\otimes(V\otimes(X\otimes{}^*V))\\ &\xrightarrow{id_{^*V}\otimes a^{-1}_{V,X,^*V}} {}^*V\otimes((V\otimes X)\otimes{}^*V)\xrightarrow{id_{^*V}\otimes c^{-1}_{V,X}\otimes id_{^*V}} {}^*V\otimes((X\otimes V)\otimes{}^*V)\\ &\xrightarrow{id_{^*V}\otimes a_{V,X,^*V}} {}^*V\otimes(X\otimes(V\otimes{}^*V))\xrightarrow{a^{-1}_{^*V,V,V\otimes^*V}} (^*V\otimes X)\otimes(V\otimes{}^*V)\\ &\xrightarrow{id_{^*V\otimes X}\otimes ev'_v} {}^*V\otimes X.\end{aligned}\tag{29}$$

Lemma 2. *Let H be a dual quasi-Hopf algebra. Then for all $a,b,c\in H$, the following relations hold:*

$$q^L(a_1,b_1c_1)\sigma(a_2,b_2,c_2)=q^L(a_2,b_1)\sigma^{-1}(s(a_1),a_3b_2,c),\tag{30}$$
$$p^R(s(a_1),a_3b_3)q^L(a_2,b_2)q^L(b_1,s(a_4b_4))=f(a,b).\tag{31}$$

Proof. By the Definition of q^L, it is easy to verify (30). Then for all $a,b\in H$,

$$\begin{aligned}&p^R(s(a_1),a_3b_3)q^L(a_2,b_2)q^L(b_1,s(a_4b_4))\\ &=\sigma^{-1}(s(a_1),a_3b_3,s(a_5b_5))\beta(a_4b_4)q^L(a_2,b_2)q^L(b_1,s(a_6b_6))\\ &\overset{(10),(11)}{=}\sigma^{-1}(s(a_1),a_3b_3,s(b_5)s(a_5))q^L(a_2,b_2)\chi(a_4,b_4)q^L(b_1,s(b_6)s(a_6))f(a_7,b_7)\\ &=\sigma^{-1}(s(a_1),a_5b_3,s(b_9)s(a_{10}))\sigma^{-1}(s(a_2),a_4,b_2)\alpha(a_3)\end{aligned}$$

$$\sigma(a_6b_4, s(b_8), s(a_9))\sigma^{-1}(a_7, b_5, s(b_7))\beta(a_8)\beta(b_6)q^L(b_1, s(b_{10})s(a_{11}))f(a_{12}, b_{11})$$
$$\overset{(3),(5)}{=} \sigma^{-1}(s(a_2), a_4, b_2)\sigma^{-1}(s(a_1), a_5b_3, s(b_9)s(a_{10}))\sigma^{-1}(a_6, b_4, s(b_8)s(a_9))$$
$$\alpha(a_3)\sigma(b_5, s(b_7), s(a_8))\beta(b_6)\beta(a_7)q^L(b_1, s(b_{10})s(a_{11}))f(a_{12}, b_{11})$$
$$\overset{(3),(5)}{=} \sigma^{-1}(s(a_1), a_3, b_2(s(b_6)s(a_6)))\alpha(a_2)\sigma(b_3, s(b_5), s(a_5))$$
$$\beta(b_4)\beta(a_4)q^L(b_1, s(b_7)s(a_7))f(a_8, b_8)$$
$$\overset{(1),(5)}{=} \sigma^{-1}(s(a_1), a_3, s(a_5))\alpha(a_2)\sigma(b_2, s(b_4), s(a_6))\beta(b_3)\beta(a_4)q^L(b_1, s(b_5)s(a_7))f(a_8, b_6)$$
$$\overset{(1),(5)}{=} \sigma(b_2, s(b_4), s(a_1))\beta(b_3)q^L(b_1, s(b_5)s(a_2))f(a_3, b_6)$$
$$= p^L(s(b_2), s(a_1))q^L(b_1, s(b_3)s(a_2))f(a_3, b_4)$$
$$\overset{(25)}{=} f(a, b),$$

as needed. The proof is completed. □

Theorem 3. *Let H be a dual quasi-Hopf algebra. Then ${}^H_H\mathcal{YD}^{fd}$ is a braided monoidal rigid category. For a finite-dimensional left-left Yetter-Drinfeld module M with basis $\{m_i\}_i$ and dual basis $\{m^i\}_i$, the left and right duals M^* and ${}^*M$ are equal to $Hom(M, k)$ as a vector space, with the following H-action and H-coaction:*

(1) For ${}^*M$,

$$\lambda_{{}^*M}(\varphi) = \langle\varphi, m_{i(0)}\rangle s(m_{i(-1)}) \otimes m^i, \tag{32}$$
$$\begin{aligned} h \cdot \varphi = & f(s^{-1}(h_3), m_{i(-1)})g((s^{-1}(h_2) \cdot m_{i(0)})_{(-1)}, s^{-1}(h_1)) \\ & \varphi((s^{-1}(h_2) \cdot m_{i(0)})_{(0)})m^i. \end{aligned} \tag{33}$$

(2) For M^*,

$$\lambda_{M^*}(\varphi') = \langle\varphi', m_{i(0)}\rangle s^{-1}(m_{i(-1)}) \otimes m^i, \tag{34}$$
$$\begin{aligned} h \cdot \varphi' = & f(s^{-1}(m_{i(-1)}), h_3)g(h_1, s^{-1}((s(h_2) \cdot m_{i(0)})_{(-1)})) \\ & \varphi'((s(h_2) \cdot m_{i(0)})_{(0)}))m^i, \end{aligned} \tag{35}$$

for all $h \in H, \varphi \in {}^*M, \varphi' \in M^*$.

Proof. The left H-coaction on ${}^*M$ viewed as an object in ${}^H_H\mathcal{YD}$ is the same as the left H-coaction on ${}^*M$ viewed as an object in ${}^H\mathcal{M}$. Now we compute the left H-action using (29). For all $h \in H, \varphi \in {}^*M$,

$$h \cdot \varphi = (id \otimes \varepsilon)c_{H,{}^*M}(h \otimes \varphi)$$
$$= \sigma^{-1}(m^i_{(-1)1}, m_{i(-1)2}, h_1 s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)8})\sigma(m_{i(-1)3}, h_2, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)7})$$
$$\alpha(m_{i(-1)1})q^L(s^{-1}(h_8), m_{i(-1)4}h_3)\sigma(s^{-1}(h_7), m_{i(-1)5}, h_4)p^R((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_5))$$
$$\sigma^{-1}(h_9, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)2}, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)6}))\beta((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)4})$$
$$\sigma(m^i_{(-1)2}, h_{10}, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)3}s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)5}))\varphi((s^{-1}(h_6) \cdot m_{i(0)})_{(0)})m^i_{(0)}$$
$$\overset{(5),(32)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)4}))\sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)3}))$$
$$q^L(s^{-1}(h_8), m_{i(-1)3}h_3)\sigma(s^{-1}(h_7), m_{i(-1)4}, h_4)p^R((s^{-1}(h_6) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_5))$$
$$p^R(h_9, (s^{-1}(h_6) \cdot m_{i(0)})_{(-1)2})\varphi((s^{-1}(h_6) \cdot m_{i(0)})_{(0)})m^i$$
$$\overset{(30)}{=} q^L(m_{i(-1)1}, h_1 s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)4}))\sigma(m_{i(-1)2}, h_2, s((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)3}))$$
$$q^L(s^{-1}(h_7), m_{i(-1)3})\sigma^{-1}(h_8, s^{-1}(h_6)m_{i(-1)4}, h_3)p^R((s^{-1}(h_5) \cdot m_{i(0)})_{(-1)1}, s^{-1}(h_4))$$

$$
\begin{aligned}
&p^R(h_9,(s^{-1}(h_5)\cdot m_{i(0)})_{(-1)2})\varphi((s^{-1}(h_5)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(20)}{=} q^L(m_{i(-1)1},h_1s((s^{-1}(h_6)\cdot m_{i(0)})_{(-1)5}))\sigma(m_{i(-1)2},h_2,s((s^{-1}(h_6)\cdot m_{i(0)})_{(-1)4}))\\
&q^L(s^{-1}(h_7),m_{i(-1)3})\sigma^{-1}(h_8,(s^{-1}(h_6)\cdot m_{i(0)})_{(-1)1}s^{-1}(h_5),h_3)p^R((s^{-1}(h_6)\cdot m_{i(0)})_{(-1)2},s^{-1}(h_4))\\
&p^R(h_9,(s^{-1}(h_6)\cdot m_{i(0)})_{(-1)3})\varphi((s^{-1}(h_6)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(30)}{=} q^L(m_{i(-1)1},h_1s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)5}))\sigma(m_{i(-1)2},h_2,s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)4}))\\
&q^L(s^{-1}(h_6),m_{i(-1)3})\sigma(h_7,(s^{-1}(h_5)\cdot m_{i(0)})_{(-1)1},s^{-1}(h_4))p^R(h_8(s^{-1}(h_5)\cdot m_{i(0)})_{(-1)2},s^{-1}(h_3))\\
&p^R(h_9,(s^{-1}(h_5)\cdot m_{i(0)})_{(-1)3})\varphi((s^{-1}(h_5)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(27)}{=} q^L(m_{i(-1)1},h_1s((s^{-1}(h_7)\cdot m_{i(0)})_{(-1)6}))\sigma(m_{i(-1)2},h_2,s((s^{-1}(h_7)\cdot m_{i(0)})_{(-1)5}))\\
&q^L(s^{-1}(h_8),m_{i(-1)3})\sigma^{-1}(h_9((s^{-1}(h_7)\cdot m_{i(0)})_{(-1)1}s^{-1}(h_6)),h_3,s((s^{-1}(h_7)\cdot m_{i(0)})_{(-1)4}))\\
&p^R(h_{10},(s^{-1}(h_7)\cdot m_{i(0)})_{(-1)2}s^{-1}(h_5))g((s^{-1}(h_7)\cdot m_{i(0)})_{(-1)3},s^{-1}(h_4))\varphi((s^{-1}(h_7)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(21)}{=} q^L(m_{i(-1)1},h_1s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)4}))\sigma(m_{i(-1)2},h_2,s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)3}))\\
&q^L(s^{-1}(h_8),m_{i(-1)3})\sigma^{-1}(h_9(s^{-1}(h_7)m_{i(-1)4}),h_3,s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)2}))\\
&p^R(h_{10},(s^{-1}(h_6)m_{i(-1)5}))g((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)1},s^{-1}(h_4))\varphi((s^{-1}(h_5)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(23)}{=} q^L(m_{i(-1)1},h_1s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)4}))\sigma(m_{i(-1)2},h_2,s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)3}))\\
&q^L(s^{-1}(h_7),m_{i(-1)4})\sigma^{-1}(m_{i(-1)3},h_3,s((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)2}))\\
&p^R(h_8,(s^{-1}(h_6)m_{i(-1)5}))g((s^{-1}(h_5)\cdot m_{i(0)})_{(-1)1},s^{-1}(h_4))\varphi((s^{-1}(h_5)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(10)}{=} q^L(m_{i(-1)1},s(s^{-1}(h_3)m_{i(-1)4})g((s^{-1}(h_2)\cdot m_{i(0)})_{(-1)},s^{-1}(h_1))\\
&q^L(s^{-1}(h_5),m_{i(-1)2})p^R(h_6,s^{-1}(h_4)m_{i(-1)3})\varphi((s^{-1}(h_2)\cdot m_{i(0)})_{(0)})m^i\\
&\overset{(31)}{=} f(s^{-1}(h_3),m_{i(-1)})g((s^{-1}(h_2)\cdot m_{i(0)})_{(-1)},s^{-1}(h_1))\varphi((s^{-1}(h_2)\cdot m_{i(0)})_{(0)})m^i,
\end{aligned}
$$

as claimed. The structure on M^* can be computed similarly. The proof is completed. □

5. The Canonical Isomorphisms in ${}^H_H\mathcal{YD}^{fd}$

If $\mathcal{C}$ is a rigid braided monoidal category, then for any objects $M, N \in \mathcal{C}$, there exist two canonical isomorphisms in $\mathcal{C}$

$$M \cong M^{**}, \quad (M \otimes N)^* \cong M^* \otimes N^*.$$

In this section, we aim to give the explicit forms of the above isomorphisms in the particular case $\mathcal{C} = {}^H_H\mathcal{YD}^{fd}$.

Let $\mathcal{C}$ be a rigid monoidal category and objects $M, N \in \mathcal{C}$ and $\nu : M \to N$ is a morphism in $\mathcal{C}$. Following [13] we can define the transposes of ν as the compositions:

$$
\begin{aligned}
\nu^* : N^* &\xrightarrow{id\otimes coev} N^* \otimes (M \otimes M^*) \xrightarrow{id\otimes\nu\otimes id} N^* \otimes (N \otimes M^*)\\
&\xrightarrow{a^{-1}_{N^*,N,M^*}} (N^* \otimes N) \otimes M^* \xrightarrow{ev\otimes id} M^*,\\
{}^*\nu : {}^*N &\xrightarrow{coev\otimes id} ({}^*M \otimes M) \otimes {}^*N \xrightarrow{id\otimes\nu\otimes id} ({}^*M \otimes N) \otimes {}^*N\\
&\xrightarrow{a_{{}^*M,N,{}^*N}} {}^*M \otimes (N \otimes {}^*N) \xrightarrow{id\otimes ev} {}^*M.
\end{aligned}
$$

From [4] we have two isomorphisms $\theta_M : M \cong {}^*(M^*)$ and $\theta'_M : M \cong ({}^*M)^*$. Both isomorphisms are natural in M. θ_M and its inverse could be described explicitly as follows:

$$\theta_M : M \xrightarrow{id\otimes coev} M \otimes ({}^*M \otimes ({}^*M)^*) \xrightarrow{a^{-1}} (M \otimes {}^*M) \otimes ({}^*M)^* \xrightarrow{ev\otimes id} ({}^*M)^*;$$

$$\theta_M^{-1}:({}^*M)^* \xrightarrow{id\otimes coev'} ({}^*M)^*\otimes({}^*M\otimes M) \xrightarrow{a^{-1}} (({}^*M)^*\otimes{}^*M)\otimes M \xrightarrow{ev} M.$$

We also have a natural isomorphism $\Theta_M : M^* \to {}^*M$, which can be described as follows, see [14] for details.

$$\begin{aligned}\Theta_M :&M^* \xrightarrow{id\otimes coev'} M^*\otimes({}^*M\otimes M) \xrightarrow{a^{-1}} (M^*\otimes{}^*M)\otimes M\\ &\xrightarrow{c_{M^*,{}^*M}\otimes id} ({}^*M\otimes M^*)\otimes M \xrightarrow{a} {}^*M\otimes(M^*\otimes M) \xrightarrow{id\otimes ev} {}^*M;\\ \Theta_M^{-1} :&{}^*M \xrightarrow{coev\otimes id} (M\otimes M^*)\otimes{}^*M \xrightarrow{a} M\otimes(M^*\otimes{}^*M)\\ &\xrightarrow{id\otimes c^{-1}_{M^*,{}^*M}} M\otimes({}^*M\otimes M^*) \xrightarrow{a^{-1}} (M\otimes{}^*M)\otimes M^* \xrightarrow{ev'\otimes id} M^*.\end{aligned}$$

Thus the functors $(-)^*$ and ${}^*(-)$ are naturally isomorphic, and we conclude that

$$M^{**} = (M^*)^* \cong {}^*(M^*) \cong M \cong ({}^*M)^* \cong {}^*({}^*M) = {}^{**}M.$$

Now we will apply these results to the particular case when $\mathcal{C} = {}^H_H\mathcal{YD}^{fd}$.

(1) For all $n^* \in N^*$,

$$\begin{aligned}\nu^*(n^*) =& \sum_i \sigma(n^*_{(-1)}, m_{i(-1)2}, m^i_{(-1)})n^*_{(0)}(\nu(m_{i(0)}))\alpha(s^{-1}(m_{i(-1)1}))\beta(s^{-1}(m_{i(-1)3}))m^i_{(0)}\\ \overset{(34)}{=}& \sum_i \sigma(s^{-1}(m_{i(-1)5}), m_{i(-1)3}, s^{-1}(m_{i(-1)1}))\alpha(s^{-1}(m_{i(-1)2}))\beta(s^{-1}(m_{i(-1)4}))\\ & n^*(\nu(m_{i(0)}))m^i\\ =& \sum_i n^*(\nu(m_i))m^i = n^*\circ\nu,\end{aligned}$$

where $\{m_i\}$ is a basis of M and $\{m^i\}$ its dual basis. By a similar computation, we have

$${}^*\nu({}^*n) = {}^*n\circ\nu.$$

(2) For θ_M we have

$$\begin{aligned}\theta_M(m) &= \sigma(m_{(-1)}, f_{i(-1)2}, f^i_{(-1)})\alpha(s^{-1}(f_{i(-1)1}))\beta(m_{(0)(-1)})f_{i(0)}(m_{(0)(0)})f^i_{(0)}\\ &= \sigma(m_{(-1)1}, s(m_{j(-1)})_2, f^i_{(-1)})\alpha(s^{-1}(s(m_{j(-1)})_1)\beta(m_{(-1)2})m^j(m_{(0)})f_i(m_{j(0)})f^i_{(0)}\\ &= \sigma(m_{(-1)1}, s(m_{(-1)3}), f^i_{(-1)})\alpha(m_{(-1)4})\beta(m_{(-1)2})f_i(m_{(0)})f^i_{(0)}\\ &= \sigma(m_{(-1)1}, s(m_{(-1)3}), s^{-1}(m^k_{(-1)}))\alpha(m_{(-1)4})\beta(m_{(-1)2})m^k_{(0)}(m_{(0)})m^{k*}\\ &= \sigma(m_{(-1)1}, s(m_{(-1)3}), m_{(-1)5}))\alpha(m_{(-1)4})\beta(m_{(-1)2})m^k(m_{(0)})m^{k*}\\ &= m^k(m)m^{k*},\end{aligned}$$

where $\{f_i\}$ is a basis of M with dual basis $\{f^i\}$ in M^*, and $\{m^k\}$ are bases of ${}^*M$, and m^{k*} is the image of m_k under the canonical map $M \to M^{**}$. Moreover the morphism θ'_M is defined by the same formula as θ_M. The maps θ_M^{-1} is given by

$$\theta_M^{-1}(({}^*m)^*) = \langle({}^*m)^*, m^i\rangle m_i.$$

Moreover the morphisms θ'_M and θ'^{-1}_M are defined the same as θ_M and θ_M^{-1}, respectively.
(2) As to Θ_M, for all $m^* \in M^*$, we have

$$\begin{aligned}\Theta_M(m^*) =& \sigma(m^*_{(-1)1}, m^i_{(-1)}, m_{i(-1)2})\sigma^{-1}((m^*_{(-1)2}\cdot m^i_{(0)})_{(-1)}, m^*_{(-1)3}, m_{i(-1)3})\\ & \alpha(m_{i(-1)1})\beta(s^{-1}(m_{i(-1)4})\langle m^*_{(0)}, m_{i(0)}\rangle(m^*_{(-1)2}\cdot m^i_{(0)})_{(0)}\end{aligned}$$

$$
\begin{aligned}
&= \sigma(s^{-1}(m_{i(-1)7}), m^i_{(-1)}, m_{i(-1)2})\sigma^{-1}(s^{-1}(m_{i(-1)6}) \cdot m^i_{(0)})_{(-1)}, s^{-1}(m_{i(-1)5}), m_{i(-1)3}) \\
&\alpha(m_{i(-1)1})\beta(s^{-1}(m_{i(-1)4})\langle m^*, m_{i(0)}\rangle((s^{-1}(m_{i(-1)6}) \cdot m^i_{(0)})_{(0)} \\
&= \sigma(s^{-1}(m_{i(-1)5}), m^i_{(-1)}, m_{i(-1)2})p^R((s^{-1}(m_{i(-1)4}) \cdot m^i_{(0)})_{(-1)}, s^{-1}(m_{i(-1)3})) \\
&\alpha(m_{i(-1)1})\langle m^*, m_{i(0)}\rangle((s^{-1}(m_{i(-1)4}) \cdot m^i_{(0)})_{(0)} \\
&= \sigma(s^{-1}(m_{i(-1)5}), s(m_{j(-1)}), m_{i(-1)2})p^R((s^{-1}(m_{i(-1)4}) \cdot m^j)_{(-1)}, s^{-1}(m_{i(-1)3})) \\
&\alpha(m_{i(-1)1})\langle m^*, m_{i(0)}\rangle\langle m^i, m_{j(0)}\rangle((s^{-1}(m_{i(-1)4}) \cdot m^j)_{(0)} \\
&= \sigma(s^{-1}(m_{j(-1)6}), s(m_{j(-1)1}), m_{j(-1)3})p^R((s^{-1}(m_{j(-1)5}) \cdot m^j)_{(-1)}, s^{-1}(m_{j(-1)4})) \\
&\alpha(m_{j(-1)2})\langle m^*, m_{j(0)}\rangle((s^{-1}(m_{j(-1)5}) \cdot m^j)_{(0)} \\
&= q^R(s^{-1}(m_{j(-1)4}), s(m_{j(-1)1}))p^R((s^{-1}(m_{j(-1)3}) \cdot m^j)_{(-1)}, s^{-1}(m_{j(-1)2})) \\
&\langle m^*, m_{j(0)}\rangle((s^{-1}(m_{j(-1)5}) \cdot m^j)_{(0)} \\
&= q^R(s^{-1}(m_{j(-1)6}), s(m_{j(-1)1}))p^R(m^i_{(-1)}, s^{-1}(m_{j(-1)2}))f(s^{-2}(m_{j(-1)3}), m_{i(-1)}) \\
&\langle m^*, m_{j(0)}\rangle g((s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(-1)}, s^{-2}(m_{j(-1)5}))\langle m^j, (s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(0)}\rangle m^i_{(0)} \\
&= q^R(s^{-1}(m_{j(-1)6}), s(m_{j(-1)1}))p^R(s(m_{k(-1)}), s^{-1}(m_{j(-1)2}))f(s^{-2}(m_{j(-1)3}), m_{i(-1)}) \\
&\langle m^*, m_{j(0)}\rangle g((s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(-1)}, s^{-2}(m_{j(-1)5}))\langle m^j, s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(0)}\rangle\langle m^i, m_{k(0)}\rangle m^k \\
&= q^R(s^{-1}(m_{j(-1)6}), s(m_{j(-1)1}))p^R(s(m_{i(-1)1}), s^{-1}(m_{j(-1)2}))f(s^{-2}(m_{j(-1)3}), m_{i(-1)2}) \\
&\langle m^*, m_{j(0)}\rangle g((s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(-1)}, s^{-2}(m_{j(-1)5}))\langle m^j, (s^{-2}(m_{j(-1)4}) \cdot m_{i(0)})_{(0)}\rangle m^i.
\end{aligned}
$$

By a similar computation, the inverse map $\Theta_M^{-1} : {}^*M \to M^*$ is given by

$$\Theta_M^{-1}({}^*m) = V(m_{i(-1)}, s(m_{j(-1)3}))\beta(m_{j(-1)1})\langle {}^*m, m_{j(0)}\rangle\langle m^j, s(m_{j(-1)2}) \cdot m_{i(0)}\rangle m^i,$$

where $V = (p^R * f) \circ (s^{-1} \otimes s^{-1})$. Thus we obtain the following result.

Proposition 10. *Let H be a dual quasi-Hopf algebra and $M, N \in {}^H_H\mathcal{YD}^{fd}$. Then ${}^r\Gamma_M = \theta'^{-1}_M \circ \Theta_{M^*} : M^{**} \to M$ is an isomorphism of Yetter-Drinfeld modules. Explicitly, ${}^r\Gamma_M$ is given by*

$$
\begin{aligned}
{}^r\Gamma_M(m^{**}) =& q^R(s^{-2}(m_{i(-1)1}), m_{i(-1)4})p^R((s^{-2}(m_{i(-1)2}) \cdot m_{i(0)})_{(-1)}, s^{-2}(m_{i(-1)3})) \\
&\langle m^{**}, m^i\rangle(s^{-2}(m_{i(-1)2}) \cdot m_{i(0)})_{(0)},
\end{aligned}
$$

for all $m^{**} \in M^{**}$. The inverse of ${}^r\Gamma_M$ is given by ${}^r\Gamma_M^{-1} = \Theta^{-1}_{M^*} \circ \theta'_M$, that is,

$$
\begin{aligned}
{}^r\Gamma_M^{-1}(m) =& V(s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)2}), m_{(-1)1})\alpha(m_{(-1)4}) \\
&g(m_{(-1)2}, s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)1}))\langle m^i, (s(m_{(-1)3}) \cdot m_{(0)})_{(0)}\rangle m^{i*}.
\end{aligned}
$$

Similarly ${}^l\Gamma_M = \theta_M^{-1} \circ \Theta^{-1}_{{}^*M} : {}^{**}M \to M$ is an isomorphism of Yetter-Drinfeld modules. Explicitly we have

$$
\begin{aligned}
{}^l\Gamma_M({}^{**}m) =& \alpha(m_{i(-1)3})p^R((s(m_{i(-1)2}) \cdot m_{i(0)})_{(-1)}, s(m_{i(-1)1})) \\
&\langle {}^{**}m, m^i\rangle(s(m_{i(-1)2}) \cdot m_{i(0)})_{(0)},
\end{aligned}
$$

for all ${}^{**}m \in {}^{**}M$. The inverse of ${}^l\Gamma_M$ is given by

$$
\begin{aligned}
{}^l\Gamma_M^{-1}(m) =& q^R(m_{(-1)1}, s^2(m_{(-1)6}))p^R(s^2((s^{-2}(m_{(-1)3}) \cdot m_{(0)})_{(-1)2})), m_{(-1)5}) \\
&\mathcal{F}^{-1}((s^{-2}(m_{(-1)3}) \cdot m_{(0)})_{(-1)1}, s^{-2}(m_{(-1)4}))\mathcal{F}(m_{(-1)7}, s^{-2}(m_{(-1)2})) \\
&\langle m^i, (s^{-2}(m_{(-1)3}) \cdot m_{(0)})_{(0)}\rangle m^{i*},
\end{aligned}
$$

for all $m \in M$, where $\mathcal{F}(a,b) = g(s(b_1), s(a_1))f(a_2, b_2)$.

Proof. For all $m^{**} \in M^{**}$,

$$
\begin{aligned}
&{}^r\Gamma_M(m^{**}) \\
&= q^R(s^{-1}(m^j_{(-1)6}), s(m^j_{(-1)1}))p^R(s(m^i_{(-1)1}), s^{-1}(m^j_{(-1)2}))f(s^{-2}(m^j_{(-1)3}), m^i_{(-1)2}) \\
&\langle m^{**}, m^j_{(0)}\rangle g((s^{-2}(m^j_{(-1)4}) \cdot m^i_{(0)})_{(-1)}, s^{-2}(m^j_{(-1)5}))\langle (s^{-2}(m^j_{(-1)4}) \cdot m^i_{(0)})_{(0)}, m_j\rangle m_i \\
&= q^R(s^{-2}(m_{l(-1)1}), m_{l(-1)8})p^R(m_{k(-1)2}, s^{-2}(m_{l(-1)7}))f(s^{-3}(m_{l(-1)6}), s^{-1}(m_{k(-1)1})) \\
&\langle m^{**}, m^l\rangle g(s^{-1}(m_{l(-1)9}), s^{-3}(m_{l(-1)2}))\langle m^i, m_{l(0)}\rangle\langle m^k, (s^{-2}(m_{l(-1)4}) \cdot m_{i(0)})_{(0)})\rangle \\
&f(s^{-1}(m_{i(-1)}), s^{-3}(m_{l(-1)3}))g(s^{-3}(m_{l(-1)5}), s^{-1}(s^{-2}(m_{l(-1)4}) \cdot m_{i(0)})_{(-1)}))m_{k(0)} \\
&= q^R(s^{-2}(m_{l(-1)1}), m_{l(-1)8})p^R(m_{k(-1)2}, s^{-2}(m_{l(-1)7}))f(s^{-3}(m_{l(-1)6}), s^{-1}(m_{k(-1)1})) \\
&\langle m^{**}, m^l\rangle g(s^{-1}(m_{l(-1)9}), s^{-3}(m_{l(-1)2}))\langle m^k, (s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(0)})\rangle \\
&f(s^{-1}(m_{l(-1)10}), s^{-3}(m_{l(-1)3}))g(s^{-3}(m_{l(-1)5}), s^{-1}(s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(-1)}))m_{k(0)} \\
&= q^R(s^{-2}(m_{l(-1)1}), m_{l(-1)8})p^R((s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(-1)3}, s^{-2}(m_{l(-1)7})) \\
&f(s^{-3}(m_{l(-1)6}), s^{-1}((s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(-1)2}))\langle m^{**}, m^l\rangle g(s^{-1}(m_{l(-1)9}), s^{-3}(m_{l(-1)2})) \\
&f(s^{-1}(m_{l(-1)10}), s^{-3}(m_{l(-1)3}))g(s^{-3}(m_{l(-1)5}), s^{-1}(s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(-1)1})) \\
&(s^{-2}(m_{l(-1)4}) \cdot m_{l(0)})_{(0)} \\
&= q^R(s^{-2}(m_{l(-1)1}), m_{l(-1)4})p^R((s^{-2}(m_{l(-1)2}) \cdot m_{l(0)})_{(-1)}, s^{-2}(m_{l(-1)3})) \\
&\langle m^{**}, m^l\rangle (s^{-2}(m_{l(-1)2}) \cdot m_{l(0)})_{(0)}.
\end{aligned}
$$

And

$$
\begin{aligned}
&{}^r\Gamma_M^{-1}(m) = V(m^i_{(-1)}, s(m^j_{(-1)3}))\beta(m^j_{(-1)1})\langle m^{k*}, m^j_{(0)}\rangle\langle m^{j*}, s(m^j_{(-1)2}) \cdot m^i_{(0)}\rangle m^k(m)m^{i*} \\
&= V(s^{-1}(m_{l(-1)}), m_{p(-1)1})\beta(s^{-1}(m_{p(-1)3}))\langle m^{k*}, m^p\rangle\langle m^j, m_{p(0)}\rangle \\
&\langle m^{j*}, m_{p(-1)2} \cdot m^l\rangle\langle m^k, m\rangle\langle m^i, m_{l(0)}\rangle m^{i*} \\
&= V(s^{-1}(m_{l(-1)}), m_{(-1)1})\beta(s^{-1}(m_{(-1)3}))\langle m_{(-1)2} \cdot m^l, m_{(0)}\rangle\langle m^i, m_{l(0)}\rangle m^{i*} \\
&= V(s^{-1}(m_{l(-1)}), m_{(-1)1})\beta(s^{-1}(m_{(-1)5}))\langle m^i, m_{l(0)}\rangle f(s^{-1}(m_{(-1)6}), m_{(-1)4}) \\
&g(m_{(-1)2}, s^{-1}(s(m_{(-1)3}) \cdot m_{(0)})_{(-1)})\langle m^l, (s(m_{(-1)3}) \cdot m_{(0)})_{(0)}\rangle m^{i*} \\
&= V(s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)2}), m_{(-1)1})\underline{\beta(s^{-1}(m_{(-1)5}))}\langle m^i, (s(m_{(-1)3}) \cdot m_{(0)})_{(0)}\rangle \\
&\underline{f(s^{-1}(m_{(-1)6}), m_{(-1)4})}g(m_{(-1)2}, s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)1}))m^{i*} \\
&= V(s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)2}), m_{(-1)1})\alpha(m_{(-1)4})\langle m^i, (s(m_{(-1)3}) \cdot m_{(0)})_{(0)}\rangle \\
&g(m_{(-1)2}, s^{-1}((s(m_{(-1)3}) \cdot m_{(0)})_{(-1)1}))m^{i*}.
\end{aligned}
$$

Similarly, we could obtain ${}^l\gamma_M$ and ${}^l\gamma_M^{-1}$ and the details are left to the reader. The proof is completed. □

Let $\mathcal{C}$ be a rigid monoidal category. For any objects $M, N \in \mathcal{C}$, there exists two isomorphisms

$$
\begin{aligned}
&{}^*\phi_{N,M} : {}^*N \otimes {}^*M \to {}^*(M \otimes N), \\
&\phi^*_{N,M} : N^* \otimes M^* \to (M \otimes N)^*.
\end{aligned}
$$

where ${}^*\phi_{N,M}$ is the composition

$$
\begin{aligned}
{}^*N\otimes{}^*M &\xrightarrow{coev'_{M\otimes N}\otimes id_{{}^*N\otimes{}^*M}} [{}^*(M\otimes N)\otimes(M\otimes N)]\otimes({}^*N\otimes{}^*M)\\
&\xrightarrow{a_{{}^*(M\otimes N),M\otimes N,{}^*N\otimes{}^*M}} {}^*(M\otimes N)\otimes[(M\otimes N)\otimes({}^*N\otimes{}^*M)]\\
&\xrightarrow{id_{{}^*(M\otimes N)}\otimes a^{-1}_{M\otimes N,{}^*N,{}^*M}} {}^*(M\otimes N)\otimes[((M\otimes N)\otimes{}^*N)\otimes{}^*M]\\
&\xrightarrow{id_{{}^*(M\otimes N)}\otimes a_{M,N,{}^*N}\otimes id_{{}^*M}} {}^*(M\otimes N)\otimes[(M\otimes(N\otimes{}^*N))\otimes{}^*M]\\
&\xrightarrow{id_{{}^*(M\otimes N)}\otimes id_M\otimes ev'_N\otimes id_{{}^*M}} {}^*(M\otimes N)\otimes[M\otimes{}^*M]\\
&\xrightarrow{id_{{}^*(M\otimes N)}\otimes ev'_M} {}^*(M\otimes N),
\end{aligned}
$$

with the inverse ${}^*\phi^{-1}_{N,M}$ given by the composition

$$
\begin{aligned}
{}^*(M\otimes N) &\xrightarrow{coev'_N\otimes d_{{}^*(M\otimes N)}} ({}^*N\otimes N)\otimes{}^*(M\otimes N)\\
&\xrightarrow{a_{{}^*N,N,{}^*(M\otimes N)}} {}^*N\otimes[N\otimes{}^*(M\otimes N)]\\
&\xrightarrow{id_{{}^*N}\otimes coev'_M\otimes id_N\otimes id_{{}^*(M\otimes N)}} {}^*N\otimes[(({}^*M\otimes M)\otimes N)\otimes{}^*(M\otimes N)]\\
&\xrightarrow{id_{{}^*N}\otimes a_{{}^*M,M,N}\otimes id_{{}^*(M\otimes N)}} {}^*N\otimes[({}^*M\otimes(M\otimes N))\otimes{}^*(M\otimes N)]\\
&\xrightarrow{id_{{}^*N}\otimes a_{{}^*M,M\otimes N,{}^*(M\otimes N)}} {}^*N\otimes[{}^*M\otimes((M\otimes N)\otimes{}^*(M\otimes N))]\\
&\xrightarrow{id_{{}^*N}\otimes id_{{}^*M}\otimes ev'_{M\otimes N}} {}^*N\otimes{}^*M.
\end{aligned}
$$

Moreover if $\mathcal{C}$ is braided, then we have the following isomorphism

$$ {}^*\sigma_{M,N} = {}^*\phi_{N,M}\circ c^{-1}_{{}^*N,{}^*M} : {}^*M\otimes{}^*N \to {}^*(M\otimes N). $$

Before proceeding we need the following Lemma.

Lemma 3. *Let H be a dual quasi-Hopf algebra. The following relations hold:*

$$(1)\ \sigma(s(b_4),s^2(b_2),s(a_1))f(a_2,s(b_1))\beta(s(b_3)) = q^R(a_1,s(b_2))g(a_2s(b_1),b_3), \tag{36}$$

$$
\begin{aligned}
(2)\ &\sigma(s^{-2}(b_1),s^{-1}(b_3),s^{-1}(a_1))f(s^{-1}(b_4),s^{-1}(a_2))\beta(s^{-2}(b_2))\\
&= q^R(a_1,b_2)g(s^{-1}(b_1),s^{-1}(a_2b_3)),
\end{aligned} \tag{37}
$$

$$(3)\ f(s(a_1),a_4b_3)p^R(s(b_1),s(a_2))f(a_3,b_2) = q^L(a,b), \tag{38}$$

$$(4)\ f(a_1,b_1)f(a_2b_2,s(b_4))p^R(a_3,b_3) = q^L(s(b),s(a)), \tag{39}$$

$$
\begin{aligned}
(5)\ &q^R(a_1,s(b_5))g(a_2s(b_4),b_6)q^L(s(b_8),s(a_4s(b_2))s(c_2))\sigma(s(b_7),s(a_3s(b_3)),s(c_1))\\
&= f(c_1,a_1)f(c_2a_2,s(b_3))\sigma^{-1}(c_3,a_3,s(b_2))g(c_4,a_4s(b_1)),
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
(6)\ &q^R(a_1,s(b_4))g(s^{-1}(b_5),s^{-1}(a_2s(b_3)))\sigma(s^{-1}(b_6),s^{-1}(a_3s(b_2)),s(c_1))\\
&q^L(s^{-1}(b_7),s^{-1}(a_4s(b_1))s(c_2))\\
&= f(b_3,s^{-1}(a_1)s(c_1))\sigma^{-1}(s^2(c_3),a_3,s(b_2))f(s^{-1}(a_2),s(c_2))g(s^{-1}(a_4s(b_1)),s(c_3)).
\end{aligned} \tag{41}
$$

Proof. We only prove (36), (38) and (40), and the rest can be proved similarly.

(1) By the relations (9) and (14), for all $a,b,c\in H$,

$$\beta(b_1)f(b_2,c_1)f(a_1,b_3c_2)\sigma(a_2,b_4,c_3)g(a_3b_5,c_4) = \beta(b_1)\sigma(s(c),s(b_2),s(a))f(a_2,b_3).$$

Hence

$$\begin{aligned}&\beta(s(b_5))f(s(b_4),b_6)f(a_1,s(b_3)b_7)\sigma(a_2,s(b_2),b_8)g(a_3s(b_1),b_9)\\&=\beta(s(b_3))\sigma(s(b_4),s^2(b_2),s(a_1))f(a_2,s(b_1)).\end{aligned}$$

Using the relation (12), we have

$$\beta(s(b_3))\sigma(s(b_4),s^2(b_2),s(a_1))f(a_2,s(b_1))=q^R(a_1,s(b_2))g(a_2s(b_1),b_3).$$

(2) Similarly we could obtain the relation (38).
(3) For all $a,b,c\in H$,

$$\begin{aligned}&q^R(a_1,s(b_5))g(a_2s(b_4),b_6)q^L(s(b_8),s(a_4s(b_2))s(c_2))\\&\quad\sigma(s(b_7),s(a_3s(b_3)),s(c_1))f(c_3,a_5s(b_1))\\&\overset{(36)}{=}\beta(s(b_6))\sigma(s(b_7),s^2(b_5),s(a_1))\underline{f(a_2,s(b_4))}q^L(s(b_9),s(a_4s(b_2))s(c_2))\\&\sigma(s(b_8),\underline{s(a_3s(b_3))},s(c_1))f(c_3,a_5s(b_1))\\&\overset{(10)}{=}\beta(s(b_6))\underline{\sigma(s(b_7),s^2(b_5),s(a_1))}f(a_3,s(b_3))q^L(s(b_9),s(a_4s(b_2))s(c_2))\\&\underline{\sigma(s(b_8),s^2(b_4)s(a_2),s(c_1))}f(c_3,a_5s(b_1))\\&\overset{(3),(5)}{=}\beta(s(b_6))\sigma^{-1}(s^2(b_4),s(a_2),s(c_2))f(a_3,s(b_3))\sigma(s(b_7),s^2(b_5),s(a_1)s(c_1))\\&f(c_4,a_5s(b_1))q^L(s(b_8),s(a_4s(b_2))s(c_3))\\&=p^L(s^2(b_5),s(a_1)s(c_1))\sigma^{-1}(s^2(b_3),s(a_3),s(c_3))f(a_4,s(b_2))\\&f(c_4,a_5s(b_1))q^L(s(b_6),s^2(b_4)(s(a_2)s(c_2)))\\&\overset{(25)}{=}\sigma^{-1}(s^2(b_3),s(a_1),s(c_1))f(a_2,s(b_2))f(c_2,a_3s(b_1))\\&\overset{(9),(14)}{=}f(c_1,a_1)f(c_2a_2,s(b_2))\sigma^{-1}(c_3,a_3,s(b_1)).\end{aligned}$$

That is

$$\begin{aligned}&q^R(a_1,s(b_5))g(a_2s(b_4),b_6)q^L(s(b_8),s(a_4s(b_2))s(c_2))\sigma(s(b_7),s(a_3s(b_3)),s(c_1))\\&=f(c_1,a_1)f(c_2a_2,s(b_3))\sigma^{-1}(c_3,a_3,s(b_2))g(c_4,a_4s(b_1)).\end{aligned}$$

The proof is completed. □

Proposition 11. *Let H be a dual quasi-Hopf algebra and $M,N\in{}^H_H\mathcal{YD}^{fd}$. Denote $\{m_i\}_{i=1}^s$ and $\{m^i\}_{i=1}^s$ the dual bases in M and ${}^*M$, and $\{n_j\}_{j=1}^t$ and $\{n^j\}_{j=1}^t$ dual bases in N and ${}^*N$. Define the map ${}^*\sigma_{N,M}:{}^*M\otimes{}^*N\to{}^*(M\otimes N)$ by*

$$\begin{aligned}&{}^*\sigma_{N,M}({}^*m\otimes{}^*n)(m\otimes n)\\&=p^R(s(m_{(-1)1}n_{(-1)1}),n_{(-1)8})\sigma(s^{-1}(n_{(-1)6}),m_{(-1)3},n_{(-1)3})\\&f(s^{-1}(n_{(-1)7}),m_{(-1)2}n_{(-1)2})p^R((s^{-1}(n_{(-1)5})\cdot m_{(0)})_{(-1)},s^{-1}(n_{(-1)4}))\\&\langle{}^*n,n_{(0)}\rangle\langle{}^*m,(s^{-1}(n_{(-1)5})\cdot m_{(0)})_{(0)}\rangle.\end{aligned}\tag{42}$$

Then ${}^*\sigma_{N,M}$ is an isomorphism in ${}^H_H\mathcal{YD}$. The inverse of ${}^*\sigma_{N,M}$ is given by

$${}^*\sigma^{-1}_{N,M}(\mu)(m\otimes n)=f(m_{(-1)1},n_{(-1)})\langle\mu,n_{(-1)2}\cdot m_{(0)}\otimes n_{(0)}\rangle.\tag{43}$$

In a similar way, the isomorphism $\sigma^*_{M,N}:M^*\otimes N^*\to(M\otimes N)^*$ is given by

$$\sigma^*_{M,N}(m^*\otimes n^*)(m\otimes n)=(\phi^*_{N,M}\circ c^{-1}_{N^*,M^*})(m^*\otimes n^*)(m\otimes n)$$

$$
\begin{aligned}
&= p^R(s^{-1}(m_{(-1)1}n_{(-1)1}), s^{-2}(n_{(-1)8}))\sigma(s^{-1}(n_{(-1)6}), m_{(-1)3}, n_{(-1)3}) \\
&f(s^{-1}(m_{(-1)2}n_{(-1)2}), s^{-2}(n_{(-1)7}))p^R((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)4})) \\
&\langle n^*, n_{(0)}\rangle\langle m^*, s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(0)}\rangle,
\end{aligned} \tag{44}
$$

with its inverse

$$
\sigma^{*-1}_{N,M}(\mu)(m \otimes n) = f(s^{-1}(m_{(-1)}), s^{-1}(n_{(-1)1}))\langle \mu, n_{(-1)2} \cdot m_{(0)} \otimes n_{(0)}\rangle. \tag{45}
$$

Proof. For all ${}^*m \in {}^*M, {}^*n \in N^*$ and $m \in M, n \in N$,

$$
\begin{aligned}
&{}^*\phi_{N,M}({}^*n \otimes {}^*m)(m \otimes n) \\
&= \alpha(m_{(-1)2}n_{(-1)2})\sigma^{-1}(s(m_{(-1)1}n_{(-1)1}), m_{(-1)3}n_{(-1)3}, s(n_{(-1)9})s(m_{(-1)8})) \\
&\underline{\sigma(m_{(-1)4}n_{(-1)4}, s(n_{(-1)8}), s(m_{(-1)7}))\sigma^{-1}(m_{(-1)5}, n_{(-1)5}, s(n_{(-1)7}))\beta(n_{(-1)6})\beta(m_{(-1)6})} \\
&\langle {}^*n, n_{(0)}\rangle\langle {}^*m, m_{(0)}\rangle \\
&= \alpha(m_{(-1)2}n_{(-1)2})\sigma^{-1}(s(m_{(-1)1}n_{(-1)1}), m_{(-1)3}n_{(-1)3}, s(n_{(-1)5})s(m_{(-1)5})) \\
&\chi(m_{(-1)4}, n_{(-1)4})\langle {}^*n, n_{(0)}\rangle\langle {}^*m, m_{(0)}\rangle \\
&\overset{(10)}{=} q^L(m_{(-1)1}n_{(-1)1}, s(m_{(-1)4}n_{(-1)4})) \\
&g(m_{(-1)5}, n_{(-1)5})\underline{f(m_{(-1)3}, n_{(-1)3})\chi(m_{(-1)2}, n_{(-1)2})}\langle {}^*n, n_{(0)}\rangle\langle {}^*m, m_{(0)}\rangle \\
&= q^L(m_{(-1)1}n_{(-1)1}, s(m_{(-1)3}n_{(-1)3})) \\
&g(m_{(-1)4}, n_{(-1)4})\beta(m_{(-1)2}n_{(-1)2})\langle {}^*n, n_{(0)}\rangle\langle {}^*m, m_{(0)}\rangle \\
&\overset{(6)}{=} g(m_{(-1)}, n_{(-1)})\langle {}^*n, n_{(0)}\rangle\langle {}^*m, m_{(0)}\rangle.
\end{aligned}
$$

Similarly we obtain

$$
{}^*\phi^{-1}_{N,M}(\mu)(n \otimes m) = f(m_{(-1)}, n_{(-1)})\langle \mu, m_{(0)} \otimes n_{(0)}\rangle,
$$

for all $\mu \in {}^*(M \otimes N)$. Denote ${}^*\phi^{-1}_{N,M}(\mu) = \varphi^k \otimes \psi^k \in {}^*N \otimes {}^*M$, then

$$
\langle \varphi^k, n\rangle\langle \psi^k, m\rangle = f(m_{(-1)}, n_{(-1)})\langle \mu, m_{(0)} \otimes n_{(0)}\rangle. \tag{46}
$$

For all ${}^*m \in {}^*M, {}^*n \in N^*$ and $m \in M, n \in N$, we compute

$$
\begin{aligned}
&{}^*\sigma_{N,M}({}^*m \otimes {}^*n)(m \otimes n) = ({}^*\phi_{N,M} \circ c^{-1}_{{}^*N,{}^*M})({}^*m \otimes {}^*n)(m \otimes n) \\
&= q^L(n_{(-1)2}, s((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)3})s(n_{(-1)9})) \\
&p^R(s(m_{(-1)2}), n_{(-1)7})\sigma(n_{(-1)3}, s((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)2}), s(n_{(-1)8})) \\
&f(s^{-1}(n_{(-1)6}), m_{(-1)3})g((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)1}, s^{-1}(n_{(-1)4}))g(m_{(-1)1}, n_{(-1)1}) \\
&\langle {}^*n, n_{(0)}\rangle\langle {}^*m, (s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(0)}\rangle \\
&\overset{(28)}{=} \underline{q^R((s^{-1}(n_{(-1)14})m_{(-1)4}, n_{(-1)5})g((s^{-1}(n_{(-1)13})m_{(-1)5})n_{(-1)6}, s^{-1}(n_{(-1)4}))} \\
&\underline{q^L(n_{(-1)2}, s(((s^{-1}(n_{(-1)11})m_{(-1)7})n_{(-1)8}))s(n_{(-1)18}))} \\
&\underline{\sigma(n_{(-1)3}, s(((s^{-1}(n_{(-1)12})m_{(-1)6})n_{(-1)7})), s(n_{(-1)17}))}f(s^{-1}(n_{(-1)15}), m_{(-1)3}) \\
&g(m_{(-1)1}, n_{(-1)1})p^R((s^{-1}(n_{(-1)10}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)9}))p^R(s(m_{(-1)2}), n_{(-1)16}) \\
&\langle {}^*n, n_{(0)}\rangle\langle {}^*m, (s^{-1}(n_{(-1)10}) \cdot m_{(0)})_{(0)}\rangle \\
&\overset{(40)}{=} \underline{f(n_{(-1)13}, s^{-1}(n_{(-1)10})m_{(-1)4})}f(n_{(-1)14}(s^{-1}(n_{(-1)9})m_{(-1)5}), n_{(-1)2}) \\
&\sigma^{-1}(n_{(-1)15}, s^{-1}(n_{(-1)8})m_{(-1)6}, n_{(-1)3})g(n_{(-1)16}, (s^{-1}(n_{(-1)7})m_{(-1)7})n_{(-1)4})
\end{aligned}
$$

$$
\begin{aligned}
& g(m_{(-1)1}, n_{(-1)1}) p^R((s^{-1}(n_{(-1)6}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)5})) \underline{p^R(s(m_{(-1)2}), n_{(-1)12})} \\
& \underline{f(s^{-1}(n_{(-1)11}), m_{(-1)3})} \langle {}^*n, n_{(0)} \rangle \langle {}^*m, (s^{-1}(n_{(-1)6}) \cdot m_{(0)})_{(0)} \rangle \\
& \overset{(38)}{=} \underline{q^L(s^{-1}(n_{(-1)10}), m_{(-1)2}) f(n_{(-1)11}(s^{-1}(n_{(-1)9}) m_{(-1)3}), n_{(-1)2})} \\
& \sigma^{-1}(n_{(-1)12}, s^{-1}(n_{(-1)8}) m_{(-1)4}, n_{(-1)3}) g(n_{(-1)13}, (s^{-1}(n_{(-1)7}) m_{(-1)5}) n_{(-1)4}) \\
& g(m_{(-1)1}, n_{(-1)1}) p^R((s^{-1}(n_{(-1)6}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)5})) \\
& \langle {}^*n, n_{(0)} \rangle \langle {}^*m, (s^{-1}(n_{(-1)6}) \cdot m_{(0)})_{(0)} \rangle \\
& \overset{(23)}{=} q^L(s^{-1}(n_{(-1)7}), m_{(-1)1}) \sigma^{-1}(n_{(-1)8}, s^{-1}(n_{(-1)6}) m_{(-1)2}, n_{(-1)1}) \\
& g(n_{(-1)9}, (s^{-1}(n_{(-1)5}) m_{(-1)3}) n_{(-1)2}) p^R((s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)3})) \\
& \langle {}^*n, n_{(0)} \rangle \langle {}^*m, (s^{-1}(n_{(-1)4}) \cdot m_{(0)})_{(0)} \rangle \\
& = \underline{q^L(s^{-1}(n_{(-1)8}), m_{(-1)1} n_{(-1)1})} \sigma(s^{-1}(n_{(-1)6}), m_{(-1)3}, n_{(-1)3}) \\
& \underline{g(n_{(-1)9}, s^{-1}(n_{(-1)7})(m_{(-1)2} n_{(-1)2}))} p^R((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)4})) \\
& \langle {}^*n, n_{(0)} \rangle \langle {}^*m, (s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(0)} \rangle \\
& \overset{(38)}{=} p^R(s(m_{(-1)1} n_{(-1)1}), n_{(-1)8}) \sigma(s^{-1}(n_{(-1)6}), m_{(-1)3}, n_{(-1)3}) \\
& f(s^{-1}(n_{(-1)7}), m_{(-1)2} n_{(-1)2}) p^R((s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(-1)}, s^{-1}(n_{(-1)4})) \\
& \langle {}^*n, n_{(0)} \rangle \langle {}^*m, (s^{-1}(n_{(-1)5}) \cdot m_{(0)})_{(0)} \rangle.
\end{aligned}
$$

Obviously the inverse of ${}^*\sigma^{-1}_{N,M}$ is ${}^*\sigma^{-1}_{N,M} = c_{{}^*N,{}^*M} \circ {}^*\phi^{-1}_{N,M}$. For all $\mu \in {}^*(M \otimes N)$, $m \in M, n \in N$, we compute

$$
\begin{aligned}
& {}^*\sigma^{-1}_{N,M}(\mu)(m \otimes n) \\
& = (c_{{}^*N,{}^*M} \circ {}^*\phi^{-1}_{N,M})(\mu)(m \otimes n) \\
& = f(m_{(-1)1}, n_{(-1)}) g((n_{(-1)2} \cdot m_{(0)})_{(-1)}, n_{(-1)3}) \langle \psi^k, (n_{(-1)2} \cdot m_{(0)})_{(0)} \rangle \langle \varphi^k, n_{(0)} \rangle \\
& \overset{(46)}{=} f(n_{(-1)1}, m_{(-1)}) \langle \mu, n_{(-1)2} \cdot m_{(0)} \otimes n_{(0)} \rangle.
\end{aligned}
$$

By similar computations, we could obtain the identities (44) and (45). The proof is completed. □

6. Application

Let (H, φ) be a coquasitriangular dual quasi-Hopf algebra. Just as shown in Example 1, any left H-comodule is a left Yetter-Drinfeld module. In this section, we will rewrite the canonical isomorphisms.

As an immediate consequence of Proposition 10, we have

Proposition 12. *Let (H, φ) be a coquasitriangular dual quasi-Hopf algebra and M a finite-dimensional left H-comodule. Then $M \cong M^{**}$ and $M \cong {}^{**}M$ as left H-comodules.*

Proof. We have seen that M is an object in ${}^H_H\mathcal{YD}$, so $M \cong M^{**}$ and $M \cong {}^{**}M$ as left Yetter-Drinfeld modules. Thus $M \cong M^{**}$ and $M \cong {}^{**}M$ as left H-comodules. By a direct computation, we have that ${}^r\Gamma_M : M^{**} \to M$ is given by

$$
{}^r\Gamma_M(m^{**}) = u^{-1}(m_{i(-1)}) \langle m^{**}, m^i \rangle m_{i(0)},
$$

with the inverse

$$
{}^r\Gamma^{-1}_M(m) = V(s^{-1}(m_{(-1)7}), m_{(-1)1}) \varphi(m_{(-1)5}, s(m_{(-1)3})) \alpha(m_{(-1)4})
$$

$$
\begin{aligned}
& g(m_{(-1)2}, s^{-1}(m_{(-1)6}))\langle m^i, m_{(0)}\rangle m^{i*} \\
&= p^R(s^{-2}(m_{(-1)9}), s^{-1}(m_{(-1)1}))f(s^{-2}(m_{(-1)8}), s^{-1}(m_{(-1)2})) \\
&\quad \varphi(m_{(-1)6}, s(m_{(-1)4}))\alpha(m_{(-1)5})g(m_{(-1)3}, s^{-1}(m_{(-1)7}))\langle m^i, m_{(0)}\rangle m^{i*} \\
&\overset{(17),(13)}{=} p^R(s^{-2}(m_{(-1)7}), s^{-1}(m_{(-1)1}))f(s^{-2}(m_{(-1)6}), s^{-1}(m_{(-1)2})) \\
&\quad \varphi(s^{-1}(m_{(-1)5}), m_{(-1)3})\beta(s^{-1}(m_{(-1)4}))\langle m^i, m_{(0)}\rangle m^{i*} \\
&\overset{(17),(12)}{=} p^R(s^{-2}(m_{(-1)5}), s^{-1}(m_{(-1)1}))\varphi(s^{-2}(m_{(-1)4}), s^{-1}(m_{(-1)2})) \\
&\quad \alpha(s^{-2}(m_{(-1)3}))\langle m^i, m_{(0)}\rangle m^{i*} \\
&\overset{(15),(16)}{=} p^R(m_{(-1)3}, s^{-1}(m_{(-1)1}))\alpha(s^{-1}(m_{(-1)2}))u(s^{-2}(m_{(-1)4}))\langle m^i, m_{(0)}\rangle m^{i*} \\
&= u(m_{(-1)})\langle m^i, m_{(0)}\rangle m^{i*}.
\end{aligned}
$$

After similar computations, we obtain that ${}^l\Gamma_M : {}^{**}M \to M$ is given by

$$
{}^l\Gamma_M({}^{**}m) = u(m_{i(-1)})\langle {}^{**}m, m^i\rangle m_{i(0)},
$$

with its inverse

$$
{}^l\Gamma_M^{-1}(m) = u^{-1}(m_{(-1)})\langle m^i, m_{(0)}\rangle m^{i*}.
$$

□

Proposition 13. *Let (H, φ) be a coquasitriangular dual quasi-Hopf algebra and M, N two finite-dimensional left H-comodules. Then $M^* \otimes N^* \cong (M \otimes N)^*$ and ${}^*M \otimes {}^*N \cong {}^*(M \otimes N)$ as left H-comodules.*

Proof. The result is a direct consequence of Proposition 11. Now we will give these isomorphisms explicitly.

$$
{}^*\sigma_{N,M}^{-1}(\mu)(m \otimes n) = f(m_{(-1)1}, n_{(-1)1})\varphi(m_{(-1)2}, n_{(-1)2})\langle \mu, m_{(0)} \otimes n_{(0)}\rangle,
$$

and

$$
{}^*\sigma_{N,M}({}^*m \otimes {}^*n)(m \otimes n) = \varphi^{-1}(m_{(-1)1}, n_{(-1)1})g(m_{(-1)2}, n_{(-1)2})\langle {}^*m, m_{(0)}\rangle\langle {}^*n, n_{(0)}\rangle.
$$

Similarly we have

$$
\begin{aligned}
\sigma_{N,M}^{*-1}(\mu)(m \otimes n) &= f(s^{-1}(m_{(-1)1}), s^{-1}(n_{(-1)1}))\varphi(m_{(-1)2}, n_{(-1)2})\langle \mu, m_{(0)} \otimes n_{(0)}\rangle \\
&\overset{(17)}{=} \varphi(s^{-1}(m_{(-1)1}), s^{-1}(n_{(-1)1}))f(s^{-1}(n_{(-1)2}), s^{-1}(m_{(-1)2}))\langle \mu, m_{(0)} \otimes n_{(0)}\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sigma_{N,M}^{*}(m^* \otimes n^*)(m \otimes n) \\
&= \varphi^{-1}(s^{-1}(m_{(-1)2}), s^{-1}(n_{(-1)2}))g(s^{-1}(n_{(-1)1}), s^{-1}(m_{(-1)1}))\langle m^*, m_{(0)}\rangle\langle n^*, n_{(0)}\rangle.
\end{aligned}
$$

The proof is completed. □

7. Conclusions

Yang-Baxter equation (or star-triangle relation) is a consistency equation that was first introduced in the field of statistical mechanics, and it takes its name from the independent work of C. N. Yang from 1968, and R. J. Baxter from 1971. In mathematical physics, one of the most classic problems is to find the solutions to the Yang-Baxter equation. Braided monoidal categories have been playing an essential role since they could supply such solutions. Hence the attention of mathematicians was naturally drawn to the construction of braided monoidal categories. V. Drinfeld developed an elegant theory that the category of Yetter-Drifeld modules over any Hopf algebra turns out to be a braided monoidal category, thus supplying solutions to the Yang-Baxter equation. Since then, the idea was extended to a more general Hopf algebra structure. In this paper, we mainly focus on the properties of the Yetter-Drifeld category over dual quasi-Hopf algebras. Concretely, we firstly describe explicitly the braided monoidal structures of three kinds of Yetter-Drifeld categories; then prove that the subcategory ${}_H^H\mathcal{YD}^{fd}$ of finite dimensional Yetter-modules is rigid, and for any object, M, give the Yetter-Drinfeld module structures on M^* and ${}^*M$; finally, compute the canonical isomorphisms in ${}_H^H\mathcal{YD}^{fd}$, and present an application in coquasitriangular dual quasi-Hopf algebras case.

The results obtained in our paper indeed enrich the theory of the Yetter-Drinfeld category and could lay the foundation for further research on dual quasi-Hopf algebras, for example, the constructions of the category of Yetter-Drinfeld-Long bimodules and Drinfeld double of dual quasi-Hopf algebra. Moreover, The results could also be applied to the research on the theory of category, especially to the monoidal category, braided category, fusion category, and even to the construction of more complicated crossed group category.

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