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New Class Up and Down Pre-Invex Fuzzy Number Valued Mappings and Related Inequalities via Fuzzy Riemann Integrals

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Abstract: Numerous applications of the theory of convex and nonconvex mapping exist in the fields of applied mathematics and engineering. In this paper, we have defined a new class of nonconvex functions which is known as up and down pre-invex (pre-incave) fuzzy number valued mappings (F-N-$V{\cdot}M$s). The well-known fuzzy Hermite–Hadamard (HH)-type and related inequalities are taken into account in this work. We extend this mileage further using fuzzy Riemann integrals and the fuzzy number up and down pre-invexity. Additionally, by imposing some light restrictions on pre-invex (pre-incave) fuzzy number valued mappings, we have introduced two new significant classes of fuzzy number valued up and down pre-invexity (pre-incavity), which are referred to as lower up and down pre-invex (pre-incave) and upper up and down pre-invex (pre-incave) fuzzy number valued mappings. By using these definitions, we have amassed a large number of both established and novel exceptional situations that serve as implementations of the key findings. To support the validity of the fuzzy inclusion relations put out in this research, we also provide a few examples of fuzzy numbers valued up and down pre-invexity.

Keywords: fuzzy-number valued mapping; fuzzy Riemann integral; up and down pre-invex fuzzy number valued mapping; Hermite–Hadamard inequality; Hermite–Hadamard–Fejér inequality

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1. Introduction

A variety of scientific fields, including mathematical analysis, optimization, economics, finance, engineering, management science, and game theory, have greatly benefited from the active, interesting, and appealing field of convexity theory study. Numerous scholars study the idea of convex functions, attempting to broaden and generalize its various manifestations by using cutting-edge concepts and potent methods. Convexity theory offers a comprehensive framework for developing incredibly effective, fascinating, and potent numerical tools to approach and resolve a wide range of issues in both pure and applied sciences; see [1–4]. For more information, see [5–10] and the references there

Convexity has been developed, broadened, and extended in several sectors in recent years. Inequality theory has benefited greatly from the introduction of convex functions.

Numerous studies have shown strong connections between the theories of inequality and convex functions.

Functional analysis, physics, statistics theory, and optimization theory all benefit from integral inequality. Only a handful of the applications of inequalities in research [11–18] include statistical difficulties, probability, and numerical quadrature equations. Convex analysis and inequalities have developed into an alluring, captivating, and attention-grabbing topic for researchers and attention as a result of various refinements, variants, extensions, wide-ranging perspectives, and applications; the reader can refer to [19–26]. Kadakal and Iscan recently presented n-polynomial convex functions, which are an extension of convexity [27]. We refer the readers for more study; see [28–36] and the references therein.

A well-known particular instance of the harmonic mean is the power mean. In areas such as statistics, computer science, trigonometry, geometry, probability, finance, and electric circuit theory, it is frequently employed when average rates are sought. Since it helps to control the weights of each piece of data, the harmonic mean is the ideal statistic for rates and ratios. The harmonic convex set is defined by the harmonic mean. Shi [37] was the first to present the harmonic convex set in 2003. The harmonic and p-harmonic convex functions were introduced and explored for the first time by Anderson et al. [38] and Noor et al. [39], respectively. Awan et al. [40] introduced a new class known as n-polynomial harmonically convex function while maintaining their focus on refinements. For further study, see [41–53] and the references therein.

We learned that there is a certain class of function known as the exponential convex function and that there are many people working on this subject right now; see [54,55]. This information was motivated and encouraged by recent actions and research in the field of convex analysis. The convexity of the exponential type was described by Dragomir [56]. Dragomir's work was continued by Awan et al. [57], who investigated and looked at a brand-new family of exponentially convex functions. Kadakal et al. offered a fresh idea for exponential-type convexity; see [58]. The idea of n-polynomial harmonic exponential-type convex functions was recently suggested by Geo et al. [59]. In statistical learning, information sciences, data mining, stochastic optimization, and sequential prediction [27,60,61] and the references therein, the benefits and applications of exponential-type convexity are used.

Additionally, symmetry and inequality have a direct relationship with convexity. Because of their close association, whatever one we focus on may be applied to the other, demonstrating the important relationship between convexity and symmetry; see [62]. The traditional ideas of convexity have been successfully extended in a number of instances. Weir and Mond, for instance, developed the category of pre-invex functions. The concept of h-convexity, which also encompasses several other types of convex functions, was first presented by Varosanec [63]. Additionally, Varosanec has discovered a few h-convex function-related classical inequalities. Noor et al. [64] proposed the class of h-pre-invex functions and noted that additional classes of pre-invexity and classical convexity may be recaptured by taking into account various appropriate options for the real function. In a related study, they also developed a number of novel Hermite–Hadamard and Dragomir–Agarwal inequalities. The class of (h1, h2)-convex functions was first presented by Cristescu et al. [65], who also looked into it and covered some of its fundamental characteristics. Zhao et al. [66] developed the class of interval-valued h-convex functions and came up with some novel iterations of the Hermite–Hadamard inequality by drawing inspiration from interval analysis and convex analysis. For more information, see [67–73] and the references are therein.

In order to construct the HH inequalities for harmonic convex functions, Iscan [74] first developed the idea of a harmonic convex set. By defining the harmonic h-convex functions on the harmonic convex set and expanding the HH inequalities that Iscan [74] developed, Mihai [75] advanced the concept of harmonic convex functions.

Remember that fuzzy interval-valued functions are fuzzy-number-valued mappings. On the other hand, Nanda and Kar [76] were the first to introduce the idea of convex F-N-$V{\cdot}M$s. In order to offer new versions of HH and fractional type of inequalities, Khan et al. [77,78] presented h-convex F-N-$V{\cdot}M$s and (h1, h2)-convex F-N-$V{\cdot}M$s and obtained some by utilizing fuzzy Riemann Liouville Fractional Integrals and fuzzy Riemannian integrals, respectively. Similarly, using fuzzy order relations and fuzzy Riemann Liouville Fractional Integrals, Sana and Khan et al. [79] developed new iterations of fuzzy fractional HH inequalities for harmonically convex F-N-$V{\cdot}M$s. We direct the readers to [80–108] and the references therein for further information on extended convex functions, fuzzy intervals, and fuzzy integrals.

Motivated and inspired by ongoing research, we have developed a novel extension of *HH* inequalities for up and down pre-invex *F-N-V·M*s via fuzzy inclusion relation. We have developed new iterations of the *HH* inequalities exploiting fuzzy Riemann operators with the help of this class. In addition, we explored the applicability of our study in rare instances.

2. Preliminaries

Let $\mathcal{X}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$ and $€ \in \mathcal{X}_C$ be defined by

$$€ = [€_*,\ €^*] = \{\mathfrak{o} \in \mathbb{R} |\ €_* \leq \mathfrak{o} \leq €^*\},\ (€_*,\ €^* \in \mathbb{R}). \tag{1}$$

If $€_* = €^*$, then $€$ is said to be degenerate. In this article, all intervals will be non-degenerate intervals. If $€_* \geq 0$, then $[€_*,\ €^*]$ is called the positive interval. The set of all positive intervals is denoted by $\mathcal{X}_C^+$ and defined as $\mathcal{X}_C^+ = \{[€_*,\ €^*] : [€_*,\ €^*] \in \mathcal{X}_C \text{ and } €_* \geq 0\}$.

Let $\imath \in \mathbb{R}$ and $\imath \cdot €$ be defined by

$$\imath \cdot € = \begin{cases} [\imath €_*,\ \imath €^*] & \text{if } \imath > 0, \\ \{0\} & \text{if } \imath = 0, \\ [\imath €^*, \imath €_*] & \text{if } \imath < 0. \end{cases} \tag{2}$$

Then, the Minkowski difference $¥ - €$, addition $€ + ¥$ and $€ \times ¥$ for $€, ¥ \in \mathcal{X}_C$ are defined by

$$[¥_*,\ ¥^*] + [€_*,\ €^*] = [¥_* + €_*,\ \ ¥^* + €^*], \tag{3}$$

$$[¥_*,\ ¥^*] \times [€_*,\ €^*] = [\min\{_*€_*,\ ¥^*€_*,\ ¥_*€^*,\ ¥^*€^*\},\ \max\{¥_*€_*,\ ¥^*€_*,\ ¥_*€^*,\ ¥^*€^*\}] \tag{4}$$

$$[¥_*,\ ¥^*] - [€_*,\ €^*] = [¥_* - €^*,\ \ ¥^* - €_*]. \tag{5}$$

Remark 1. *(i) For given* $[¥_*,\ ¥^*],\ [€_*,\ €^*] \in \mathbb{R}_I$, *the relation* "$\supseteq_I$" *defined on* $\mathbb{R}_I$ *by*

$$[€_*,\ €^*] \supseteq_I [¥_*,\ ¥^*]\ if\ and\ only\ if\ €_* \leq ¥_*,\ \ ¥^* \leq €^*, \tag{6}$$

for all $[¥_*,\ ¥^*],\ [€_*,\ €^*] \in \mathbb{R}_I$, *it is a partial interval inclusion relation. The relation* $[€_*,\ €^*] \supseteq_I [¥_*,\ ¥^*]$ *is coincident to* $[€_*,\ €^*] \supseteq [¥_*,\ ¥^*]$ *on* $\mathbb{R}_I$. *It can be easily seen that* "$\supseteq_I$" *looks like "up and down" on the real line* $\mathbb{R}$, *so we refer to* "$\supseteq_I$" *as "up and down" (or "UD" order, in short) [92].*

(ii) For given $[¥_*,\ ¥^*],\ [€_*,\ €^*] \in \mathbb{R}_I$, *we say that* $[¥_*,\ ¥^*] \leq_I [€_*,\ €^*]$ *if and only if* $¥_* \leq €_*,\ ¥^* \leq €^*$, *it is a partial interval order relation. The relation* $[¥_*,\ ¥^*] \leq_I [€_*,\ €^*]$ *coincident to* $[¥_*,\ ¥^*] \leq [€_*,\ €^*]$ *on* $\mathbb{R}_I$. *It can be easily seen that* "$\leq_I$" *looks like "left and right" on the real line* $\mathbb{R}$, *so we call* "$\leq_I$" *is "left and right" (or "LR" order, in short) [91,92].*

For $[¥_*,\ ¥^*],\ [€_*,\ €^*] \in \mathcal{X}_C$, *the Hausdorff–Pompeiu distance between intervals* $[¥_*,\ ¥^*]$, *and* $[€_*,\ €^*]$*is defined by*

$$d_H([¥_*,\ ¥^*],\ [€_*,\ €^*]) = \max\{|¥_* - €_*|,\ |¥^* - €^*|\}. \tag{7}$$

It is a familiar fact that $(\mathcal{X}_C, d_H)$ *is a complete metric space; see [83,89,90].*

Definition 1. *([82,83]) A fuzzy subset L of* $\mathbb{R}$ *is distinguished by a mapping* $\widetilde{€} : \mathbb{R} \to [0,1]$ *called the membership mapping of L. That is, a fuzzy subset L of* $\mathbb{R}$ *is a mapping* $\widetilde{€} : \mathbb{R} \to [0,1]$. *So, for further study, we have chosen this notation. We appoint* $\mathbb{E}$ *to denote the set of all fuzzy subsets of* $\mathbb{R}$.

Let $\widetilde{€} \in \mathbb{E}$. *Then,* $\widetilde{€}$ *is known as a fuzzy number or fuzzy interval if the following properties are satisfied by* $\widetilde{€}$*:*

(1) $\widetilde{€}$ *is normal i.e., if there exists* $\mathfrak{o} \in \mathbb{R}$ *and* $\widetilde{€}(\mathfrak{o}) = 1$;

(2) $\widetilde{€}$ *should be upper semi-continuous on* $\mathbb{R}$ *if for given* $\mathfrak{o} \in \mathbb{R}$, *there exist* $\varepsilon > 0$ *and there exist* $\delta > 0$ *such that* $\widetilde{€}(\mathfrak{o}) - \widetilde{€}(s) < \varepsilon$ *for all* $s \in \mathbb{R}$ *with* $|\mathfrak{o} - s| < \delta$;

(3) $\widetilde{€}$ *should be fuzzy convex that is* $\widetilde{€}((1-\imath)\mathfrak{o} + \imath s) \geq min\left(\widetilde{€}(\mathfrak{o}), \widetilde{€}(s)\right)$, *for all* $\mathfrak{o}, s \in \mathbb{R}$, *and* $\imath \in [0,\ 1]$;

(4) $\widetilde{€}$ *should be compactly supported: that is, cl*$\left\{\mathfrak{o} \in \mathbb{R}\middle| \widetilde{€}(\mathfrak{o}) > 0\right\}$ *is compact.*

We appoint $\mathbb{E}_C$ *to denote the set of all fuzzy numbers of* $\mathbb{R}$.

Definition 2. *([82,83]) Given* $\widetilde{€} \in \mathbb{E}_C$, *the level sets or cut sets are given by* $\left[\widetilde{€}\right]^{\lambda} = \left\{\mathfrak{o} \in \mathbb{R}\middle| \widetilde{€}(\mathfrak{o}) \geq \lambda\right\}$ *for all* $\lambda \in [0,\ 1]$ *and by* $\left[\widetilde{€}\right]^{0} = \left\{\mathfrak{o} \in \mathbb{R}\middle| \widetilde{€}(\mathfrak{o}) > 0\right\}$. *These sets are known as* λ*-level sets or* λ*-cut sets of* $\widetilde{€}$.

Proposition 1. *([85]) Let* $\widetilde{€}, \widetilde{¥} \in \mathbb{E}_C$. *Then, relation "*$\leq_{\mathbb{F}}$*" given on* $\mathbb{E}_C$ *by*

$$\widetilde{€} \leq_{\mathbb{F}} \widetilde{¥}\ if\ and\ only\ if,\ \left[\widetilde{€}\right]^{\lambda} \leq_I \left[\widetilde{¥}\right]^{\lambda},\ for\ every\ \lambda \in [0,\ 1], \tag{8}$$

it is a left and right-order relation.

Proposition 2. *Let* $\widetilde{€}, \widetilde{¥} \in \mathbb{E}_C$. *Then, relation "*$\supseteq_{\mathbb{F}}$*" given on* $\mathbb{E}_C$ *by*

$$\widetilde{€} \supseteq_{\mathbb{F}} \widetilde{¥}\ when\ and\ only\ when,\ \left[\widetilde{€}\right]^{\lambda} \supseteq_I \left[\widetilde{¥}\right]^{\lambda},\ for\ every\ \lambda \in [0,\ 1], \tag{9}$$

it is an up and down-order relation on $\mathbb{E}_C$.

Proof. The proof follows directly from the up and down relation $\supseteq_I$ defined on $\mathcal{X}_C$.

Remember the approaching notions, which are offered in the literature. If $\widetilde{€}, \widetilde{¥} \in \mathbb{E}_C$ and $\imath \in \mathbb{R}$, then, for every $\lambda \in [0,\ 1]$, the arithmetic operations are defined by

$$\left[\widetilde{€} \oplus \widetilde{¥}\right]^{\lambda} = \left[\widetilde{€}\right]^{\lambda} + \left[\widetilde{¥}\right]^{\lambda}, \tag{10}$$

$$\left[\widetilde{€} \otimes \widetilde{¥}\right]^{\lambda} = \left[\widetilde{€}\right]^{\lambda} \times \left[\widetilde{¥}\right]^{\lambda}, \tag{11}$$

$$\left[\imath \odot \widetilde{€}\right]^{\lambda} = \imath . \left[\widetilde{€}\right]^{\lambda}. \tag{12}$$

These operations follow directly from Equations (4), (5), and (6), respectively. □

Theorem 1 . *([83]) The space* $\mathbb{E}_C$ *dealing with a supremum metric, i.e., for* $\widetilde{€}, \widetilde{¥} \in \mathbb{E}_C$

$$d_{\infty}\left(\widetilde{€}, \widetilde{¥}\right) = \sup_{0 \leq \lambda \leq 1} d_H\left(\left[\widetilde{€}\right]^{\lambda}, \left[\widetilde{¥}\right]^{\lambda}\right), \tag{13}$$

is a complete metric space, where H denotes the well-known Hausdorff metric on the space of intervals.

Riemann Integral Operators for the Interval- and Fuzzy-Number Valued Mappings

Now, we define and discuss some properties of fractional integral operators of interval- and fuzzy-number-valued mappings.

Theorem 2. *([83,84]) If $\mathfrak{G} : [\nu, \tau] \subset \mathbb{R} \to \mathcal{X}_C$ is an interval-valued mapping (I-V·M) satisfying that $\mathfrak{G}(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}),\ \mathfrak{G}^*(\mathfrak{o})]$, then $\mathfrak{G}$ is Aumann integrable (IA-integrable) over $[\nu, \tau]$ when and only when $\mathfrak{G}_*(\mathfrak{o})$ and $\mathfrak{G}^*(\mathfrak{o})$ both are integrable over $[\nu, \tau]$ such that*

$$(IA)\int_{\nu}^{\tau} \mathfrak{G}(\mathfrak{o})d\mathfrak{o} = \left[\int_{\nu}^{\tau} \mathfrak{G}_*(\mathfrak{o})d\mathfrak{o},\ \int_{\nu}^{\tau} \mathfrak{G}^*(\mathfrak{o})d\mathfrak{o}\right]. \tag{14}$$

Definition 3. *([91]) Let $\widetilde{\mathfrak{G}} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be called fuzzy-number valued mapping. Then, for every $\lambda \in [0,\ 1]$, as well as λ-levels, define the family of I-V·Ms $\mathfrak{G}_\lambda : \mathbb{I} \subset \mathbb{R} \to \mathcal{X}_C$ satisfying that $\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)]$ for every $\mathfrak{o} \in \mathbb{I}$. Here, for every $\lambda \in [0,\ 1]$, the endpoint real-valued mappings $\mathfrak{G}_*(\cdot, \lambda),\ \mathfrak{G}^*(\cdot, \lambda), : \mathbb{I} \to \mathbb{R}$ are called lower and upper mappings of $\mathfrak{G}_\lambda$.*

Definition 4. *([91]) Let $\widetilde{\mathfrak{G}} : \mathbb{I} \subset \mathbb{R} \to \mathbb{E}_C$ be a F-N-V·M. Then, $\widetilde{\mathfrak{G}}(\mathfrak{o})$ is said to be continuous at $\mathfrak{o} \in \mathbb{I}$, if for every $\lambda \in [0,\ 1]$, $\mathfrak{G}_\lambda(\mathfrak{o})$ is continuous when and only when, both endpoint mappings $\mathfrak{G}_*(\mathfrak{o}, \lambda)$, and $\mathfrak{G}^*(\mathfrak{o}, \lambda)$ are continuous at $\mathfrak{o} \in \mathbb{I}$.*

Definition 5. *([84]) Let $\widetilde{\mathfrak{G}} : [\nu,\ \tau] \subset \mathbb{R} \to \mathbb{E}_C$ be F-N-V·M. The fuzzy Aumann integral ((FA)-integral) of $\mathfrak{G}$ over $[\nu,\ \tau]$, denoted by $(FA)\int_{\nu}^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}$, is defined level-wise by*

$$\left[(FA)\int_{\nu}^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}\right]^{\lambda} = (IA)\int_{\nu}^{\tau} \mathfrak{G}_\lambda(\mathfrak{o})d\mathfrak{o} = \left\{\int_{\nu}^{\tau} \mathfrak{G}(\mathfrak{o}, \lambda)d\mathfrak{o} : \mathfrak{G}(\mathfrak{o}, \lambda) \in S(\mathfrak{G}_\lambda)\right\}, \tag{15}$$

where $S(\mathfrak{G}_\lambda) = \{\mathfrak{G}(\cdot, \lambda) \to \mathbb{R} : \mathfrak{G}(\cdot, \lambda)$ is integrable and $\mathfrak{G}(\mathfrak{o}, \lambda) \in \mathfrak{G}_\lambda(\mathfrak{o})\}$, for every $\lambda \in [0,\ 1]$. $\mathfrak{G}$ is (FA)-integrable over $[\nu,\ \tau]$ if $(FA)\int_{\nu}^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \in \mathbb{E}_C$.

Theorem 3. *([85]) Let $\widetilde{\mathfrak{G}} : [\nu,\ \tau] \subset \mathbb{R} \to \mathbb{E}_C$ be a F-N-V·M as well as λ-levels define the family of I-V·Ms $\mathfrak{G}_\lambda : [\nu,\ \tau] \subset \mathbb{R} \to \mathcal{X}_C$ satisfying that $\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)]$ for every $\mathfrak{o} \in [\nu,\ \tau]$ and for every $\lambda \in [0,\ 1]$. Then, $\widetilde{\mathfrak{G}}$ is (FA)-integrable over $[\nu,\ \tau]$ when and only when, $\mathfrak{G}_*(\mathfrak{o}, \lambda)$ and $\mathfrak{G}^*(\mathfrak{o}, \lambda)$ both are integrable over $[\nu,\ \tau]$. Moreover, if $\widetilde{\mathfrak{G}}$ is (FA)-integrable over $[\nu,\ \tau]$, then*

$$\left[(FA)\int_{\nu}^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}\right]^{\lambda} = \left[\int_{\nu}^{\tau} \mathfrak{G}_*(\mathfrak{o}, \lambda)d\mathfrak{o},\ \int_{\nu}^{\tau} \mathfrak{G}^*(\mathfrak{o}, \lambda)d\mathfrak{o}\right] = (IA)\int_{\nu}^{\tau} \mathfrak{G}_\lambda(\mathfrak{o})d\mathfrak{o}, \tag{16}$$

for every $\lambda \in [0,\ 1]$.

Breckner discussed the coming emerging idea of interval-valued convexity in [86].

Definition 6. *A interval valued mapping $\mathfrak{G} : \mathbb{I} = [\nu,\ \tau] \to \mathcal{X}_C$ is called convex inteval valued mapping if*

$$\mathfrak{G}(\imath\mathfrak{o} + (1-\imath)s) \supseteq \imath\mathfrak{G}(\mathfrak{o}) + (1-\imath)\mathfrak{G}(s), \tag{17}$$

for all $\mathfrak{o},\ s \in [\nu,\ \tau],\ \imath \in [0,\ 1]$, where $\mathcal{X}_C$ is the collection of all real valued intervals. If (17) is reversed, then $\mathfrak{G}$ is called concave.

Definition 7. *([76]) The F-N-V·M $\widetilde{\mathfrak{G}} : [\nu,\ \tau] \to \mathbb{E}_C$ is called convex F-N-V·M on $[\nu,\ \tau]$ if*

$$\widetilde{\mathfrak{G}}(\imath\mathfrak{o} + (1-\imath)s\ ) \leq_{\mathbb{F}} \imath \odot \widetilde{\mathfrak{G}}(\mathfrak{o}) \oplus (1-\imath) \odot \widetilde{\mathfrak{G}}(s), \tag{18}$$

for all $\mathfrak{o},\ s \in [\nu,\ \tau],\ \imath \in [0,\ 1]$, where $\widetilde{\mathfrak{G}}(\mathfrak{o}) \geq_{\mathbb{F}} \tilde{0}$ for all $\mathfrak{o} \in [\nu,\ \tau]$. If (18) is reversed then, $\widetilde{\mathfrak{G}}$ is called concave F-N-V·M on $[\nu,\ \tau]$. $\widetilde{\mathfrak{G}}$ is affine if and only if it is both convex and concave F-N-V·M.

Definition 8. *([92]) The F-N-V·M $\widetilde{\mathfrak{G}} : [\nu,\ \tau] \to \mathbb{E}_C$ is called up and down convex F-N-V·M on $[\nu,\ \tau]$ if*

$$\widetilde{\mathfrak{G}}(\imath\mathfrak{o} + (1-\imath)s) \supseteq_{\mathbb{F}} \imath \odot \widetilde{\mathfrak{G}}(\mathfrak{o}) \oplus (1-\imath) \odot \widetilde{\mathfrak{G}}(s), \tag{19}$$

for all $\mathfrak{o},\ s \in [\nu,\ \tau]$, $\imath \in [0,\ 1]$, *where* $\widetilde{\mathfrak{G}}(\mathfrak{o}) \geq_{\mathbb{F}} \widetilde{0}$ *for all* $\mathfrak{o} \in [\nu,\ \tau]$. *If (19) is reversed then,* $\widetilde{\mathfrak{G}}$ *is called up and down concave F-N-V·M on* $[\nu,\ \tau]$. $\widetilde{\mathfrak{G}}$ *is up and down affine F-N-V·M if and only if it is both up and down convex and up and down concave F-N-V·M.*

Theorem 4. *([92]) Let* $\widetilde{\mathfrak{G}} : [\nu, \tau] \to \mathbb{E}_C$ *be an F-N-V·M, whose* λ*-levels define the family of I-V·Ms* $\mathfrak{G}_\lambda : [\nu, \tau] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *are given by*

$$\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)], \tag{20}$$

for all $\mathfrak{o} \in [\nu, \tau]$ *and for all* $\lambda \in [0,\ 1]$. *Then,* $\widetilde{\mathfrak{G}}$ *is up and down convex F-N-V·M on* $[\nu, \tau]$ *if and only if, for all* $\lambda \in [0,\ 1]$, $\mathfrak{G}_*(\mathfrak{o},\ \lambda)$ *is a convex mapping and* $\mathfrak{G}^*(\mathfrak{o},\ \lambda)$ *is a concave mapping.*

3. Up and Down Fuzzy-Number Valued Mappings and Related Integral Inequalities

Definition 9. *The F-N-V·M* $\widetilde{\mathfrak{G}} : [\nu,\ \tau] \to \mathbb{E}_C$ *is called up and down pre-invex F-N-V·M on* $[\nu,\ \tau]$ *if*

$$\widetilde{\mathfrak{G}}(\mathfrak{o} + (1-\imath)\jmath(s,\ \mathfrak{o})) \supseteq_{\mathbb{F}} \imath \odot \widetilde{\mathfrak{G}}(\mathfrak{o}) \oplus (1-\imath) \odot \widetilde{\mathfrak{G}}(s), \tag{21}$$

for all $\mathfrak{o},\ s \in [\nu,\ \tau]$, $\imath \in [0,\ 1]$, *where* $\widetilde{\mathfrak{G}}(\mathfrak{o}) \geq_{\mathbb{F}} \widetilde{0}$ *for all* $\mathfrak{o} \in [\nu,\ \tau]$. *If (19) is reversed then,* $\widetilde{\mathfrak{G}}$ *is called up and down pre-incave F-N-V·M on* $[\nu,\ \tau]$. $\widetilde{\mathfrak{G}}$ *is up and down affine F-N-V·M if and only if it is both up and down pre-invex and up and down pre-incave F-N-V·M.*

Theorem 5. *Let* $\widetilde{\mathfrak{G}} : [\nu, \tau] \to \mathbb{E}_C$ *be an F-N-V·M, whose* λ*-levels define the family of I-V·Ms* $\mathfrak{G}_\lambda : [\nu, \tau] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *are given by*

$$\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)], \tag{22}$$

for all $\mathfrak{o} \in [\nu, \tau]$ *and for all* $\lambda \in [0,\ 1]$. *Then,* $\widetilde{\mathfrak{G}}$ *is up and down pre-invex F-N-V·M on* $[\nu, \tau]$ *if and only if, for all* $\lambda \in [0,\ 1]$, $\mathfrak{G}_*(\mathfrak{o},\ \lambda)$ *is a pre-invex mapping and* $\mathfrak{G}^*(\mathfrak{o},\ \lambda)$ *is a pre-incave mapping.*

Remark 2. *If* $\mathfrak{G}_*(\mathfrak{o}, \lambda) \neq \mathfrak{G}^*(\mathfrak{o}, \lambda)$ *and* $\lambda = 1$, *then we obtain the definition of a pre-invex interval-valued function; see [95]:*

$$\mathfrak{G}(\mathfrak{o} + (1-\imath)\jmath(s,\ \mathfrak{o})) \supseteq I\mathfrak{G}(\mathfrak{o}) + (1-\imath)\mathfrak{G}(s). \tag{23}$$

If $\mathfrak{G}_*(\mathfrak{o}, \lambda) = \mathfrak{G}^*(\mathfrak{o}, \lambda)$ *and* $\lambda = 1$, *then we obtain the classical definition of pre-invex functions.*

Now, we have obtained in the following some new definitions from the literature which will be helpful to investigate some classical and new results as special cases of the main results.

Definition 10. *Let* $\widetilde{\mathfrak{G}} : [\nu, \tau] \to \mathbb{E}_C$ *be a F-N-V·M, whose* λ*-levels define the family of I-V·Ms* $\mathfrak{G}_\lambda : [\nu, \tau] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ *and are given by*

$$\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)], \tag{24}$$

for all $\mathfrak{o} \in [\nu, \tau]$ *and for all* $\lambda \in [0,\ 1]$. *Then,* $\widetilde{\mathfrak{G}}$ *is lower up and down pre-invex (pre-incave) F-N-V·M on* $[\nu, \tau]$ *if and only if, for all* $\lambda \in [0,\ 1]$,

$$\mathfrak{G}_*(\mathfrak{o} + (1-\imath)\jmath(s,\ \mathfrak{o}),\ \lambda) \leq (\geq)\imath\mathfrak{G}_*(\mathfrak{o}, \lambda) + (1-\imath)\mathfrak{G}_*(s, \lambda),$$

and

$$\mathfrak{G}^*(\mathfrak{o} + (1-\imath)\jmath(s,\ \mathfrak{o}),\ \lambda) = \imath\mathfrak{G}^*(\mathfrak{o}, \lambda) + (1-\imath)\mathfrak{G}^*(s, \lambda).$$

Definition 11. *Let $\widetilde{\mathfrak{G}} : [\nu, \tau] \to \mathbb{E}_C$ be a F-N-V·M, whose λ-levels define the family of I-V·Ms $\mathfrak{G}_\lambda : [\nu, \tau] \to \mathcal{X}_C^+ \subset \mathcal{X}_C$ and are given by*

$$\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)], \tag{25}$$

for all $\mathfrak{o} \in [\nu, \tau]$ and for all $\lambda \in [0,\ 1]$. Then, $\widetilde{\mathfrak{G}}$ is upper up and down pre-invex (pre-incave) F-N-V·M on $[\nu, \tau]$ if and only if, for all $\lambda \in [0,\ 1]$,

$$\mathfrak{G}_*(\mathfrak{o} + (1 - \imath)\jmath(s,\ \mathfrak{o}),\ \lambda) = \imath\mathfrak{G}_*(\mathfrak{o}, \lambda) + (1 - \imath)\mathfrak{G}_*(s, \lambda),$$

and

$$\mathfrak{G}^*(\mathfrak{o} + (1 - \imath)\jmath(s,\ \mathfrak{o}),\ \lambda) \le (\ge)\imath\mathfrak{G}^*(\mathfrak{o}, \lambda) + (1 - \imath)\mathfrak{G}^*(s, \lambda).$$

Remark 3. *Both concepts "up and down pre-invex F-N-V·M" and classical "pre-invex F-N-V·M, see [97]" behave alike when $\mathfrak{G}$ is lower up and down pre-invex F-N-V·M.*

Both concepts "pre-invex interval-valued mapping" (see [95]) and "left and right pre-invex interval-valued mapping" (see [88]) are coincident when $\mathfrak{G}$ is lower up and down pre-invex *F-N-V·M* with $\lambda = 1$.

If we take $\jmath(s,\ \mathfrak{o}) = s - \mathfrak{o}$, then we acquire classical and new results from Definitions 7–9, Remarks 2 and 3, and Theorem 5; see [76,86,92,93].

Since $\jmath : K \times K \to \mathbb{R}$ is a bi-function, then we require the following condition to prove the upcoming results:

Condition C. *Let K be an invex set with respect to $\jmath$. For any $\nu,\ \tau \in K$ and $\imath \in [0,\ 1]$,*

$$\jmath(\tau, \nu + \imath\jmath(\tau, \nu)) = (1 - \imath)\jmath(\tau, \nu),$$

$$\jmath(\nu, \nu + \imath\jmath(\tau, \nu)) = -\imath\jmath(\tau, \nu).$$

From Condition C, it can be easily seen that when $\imath$ = 0, then $\jmath(\tau, \nu)$ = 0 if and only if, $\tau = \nu$, for all $\nu,\ \tau \in K$. For more useful details and the applications of Condition C, see [94,97].

Theorem 6. *(The fuzzy HH-type inequality for up and down pre-invex F-N-V·M). Suppose that $\widetilde{\mathfrak{G}} : [\nu,\ \nu + \jmath(\tau,\ \nu)] \to \mathbb{E}$ is an up and down pre-invex F-N-V·M along with the family of I-V·Ms $\mathfrak{G}_\lambda : [\nu,\ \nu + \jmath(\tau,\ \nu)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ as well as $\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda),\ \mathfrak{G}^*(\mathfrak{o}, \lambda)]$ for all $\mathfrak{o} \in [\nu,\ \nu + \jmath(\tau,\ \nu)]$ and for all $\lambda \in [0,\ 1]$. If $\jmath$ satisfies the Condition C and $\widetilde{\mathfrak{G}} \in \mathcal{FR}_{([\nu,\ \nu+\jmath(\tau,\nu)],\lambda)}$, then*

$$\widetilde{\mathfrak{G}}\left(\frac{2\nu + \jmath(\tau,\ \nu)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\jmath(\tau,\ \nu)} \odot (FA)\int_{\nu}^{\nu+\jmath(\tau,\ \nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\ \oplus\ \widetilde{\mathfrak{G}}(\tau)}{2}. \tag{26}$$

Proof. Let $\widetilde{\mathfrak{G}} : [\nu,\ \nu + \jmath(\tau,\ \nu)] \to \mathbb{E}$ be an up and down pre-invex *F-N-V·M*. Then, by hypothesis, we have

$$2 \odot \widetilde{\mathfrak{G}}\left(\frac{2\nu + \jmath(\tau,\ \nu)}{2}\right) \supseteq_{\mathbb{F}} \widetilde{\mathfrak{G}}(\nu + (1 - \imath)\jmath(\tau,\ \nu))\ \oplus\ \widetilde{\mathfrak{G}}(\nu + \imath\jmath(\tau,\ \nu)).$$

Therefore, for every $\lambda \in [0,\ 1]$, we have

$$\begin{aligned} 2\mathfrak{G}_*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right) &\le \mathfrak{G}_*(\nu + (1 - \imath)\jmath(\tau,\ \nu),\ \lambda) + \mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \\ 2\mathfrak{G}^*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right) &\ge \mathfrak{G}^*(\nu + (1 - \imath)\jmath(\tau,\ \nu), \lambda) + \mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda). \end{aligned}$$

Then

$$2\int_0^1 \mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right)d\imath \le \int_0^1 \mathfrak{G}_*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)d\imath + \int_0^1 \mathfrak{G}_*(\nu+\imath\jmath(\tau,\nu),\lambda)d\imath,$$
$$2\int_0^1 \mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right)d\imath \ge \int_0^1 \mathfrak{G}^*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)d\imath + \int_0^1 \mathfrak{G}^*(\nu+\imath\jmath(\tau,\nu),\lambda)d\imath.$$

It follows that

$$\mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right) \le \frac{1}{\jmath(\tau,\nu)}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}_*(\mathfrak{o},\lambda)d\mathfrak{o},$$
$$\mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right) \ge \frac{2}{\jmath(\tau,\nu)}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}^*(\mathfrak{o},\lambda)d\mathfrak{o}.$$

That is

$$\left[\mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right),\mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\nu)}{2},\lambda\right)\right] \supseteq_I \frac{1}{\jmath(\tau,\nu)}\left[\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}_*(\mathfrak{o},\lambda)d\mathfrak{o},\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}^*(\mathfrak{o},\lambda)d\mathfrak{o}\right].$$

Thus,

$$\widetilde{\mathfrak{G}}\left(\frac{2\nu+\jmath(\tau,\nu)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\jmath(\tau,\nu)} \odot (FA)\int_\nu^{\nu+\jmath(\tau,\nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o}. \tag{27}$$

In a similar way as above, we have

$$\frac{1}{\jmath(\tau,\nu)} \odot (FA)\int_\nu^{\nu+\jmath(\tau,\nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)}{2}. \tag{28}$$

Combining (27) and (28), we have

$$\widetilde{\mathfrak{G}}\left(\frac{2\nu+\jmath(\tau,\nu)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\jmath(\tau,\nu)} \odot (FA)\int_\nu^{\nu+\jmath(\tau,\nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)}{2}.$$

This completes the proof. □

Remark 4. *If $\jmath(\tau,\nu) = \tau-\nu$, then Theorem 6 reduces to the result for convex F-N-V·M, see [93]:*

$$\widetilde{\mathfrak{G}}\left(\frac{\nu+\tau}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\tau-\nu} \odot (FA)\int_\nu^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)}{2}. \tag{29}$$

Let $\mathfrak{G}$ be a lower up and down pre-invex F-N-V·M. Then, we achieve the following inequality from Theorem 6; see [97]:

$$\widetilde{\mathfrak{G}}\left(\frac{2\nu+\jmath(\tau,\nu)}{2}\right) \le_{\mathbb{F}} \frac{1}{\jmath(\tau,\nu)} \odot (FA)\int_\nu^{\nu+\jmath(\tau,\nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \le_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)}{2}. \tag{30}$$

If lower up and down pre-invex F-N-V·M with $\jmath(\tau,\nu) = \tau-\nu$, then Theorem 6 reduces to the result for convex F-N-V·M, see [77]:

$$\widetilde{\mathfrak{G}}\left(\frac{\nu+\tau}{2}\right) \le_{\mathbb{F}} \frac{1}{\tau-\nu} \odot (FA)\int_\nu^{\tau} \widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \le_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)}{2}. \tag{31}$$

If $\mathfrak{G}_*(\mathfrak{o},\lambda) = \mathfrak{G}^*(\mathfrak{o},\lambda)$ with $\lambda = 1$, then Theorem 6 reduces to the result for pre-invex mapping; see [94]:

$$\mathfrak{G}\left(\frac{2\nu+\jmath(\tau,\nu)}{2}\right) \le \frac{1}{\jmath(\tau,\nu)}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}(\mathfrak{o})d\mathfrak{o} \le [\mathfrak{G}(\nu)+\mathfrak{G}(\tau)]\int_0^1 \imath d\imath. \tag{32}$$

If $\mathfrak{G}_*(\mathfrak{o},\ \lambda) = \mathfrak{G}^*(\mathfrak{o},\ \lambda)$ *with* $\jmath(\tau,\nu) = \tau - \nu$ *and* $\lambda = 1$, *then Theorem 6 reduces to the result for classical convex mapping:*

$$\mathfrak{G}\left(\frac{\nu+\tau}{2}\right) \leq \frac{1}{\tau-\nu}\int_{\nu}^{\tau}\mathfrak{G}(\mathfrak{o})d\mathfrak{o} \leq \frac{\mathfrak{G}(\nu)+\mathfrak{G}(\tau)}{2}. \tag{33}$$

Example 1. *Let* $\mathfrak{o} \in [2,\ 2+\jmath(3,\ 2)]$, *and the F-N-V·M* $\widetilde{\mathfrak{G}} : [\nu,\ \nu+\jmath(\tau,\ \nu)] = [2, 2+\ \jmath(3,\ 2)] \to \mathbb{E}_C$, *which is defined by*

$$\widetilde{\mathfrak{G}}(\mathfrak{o})(\theta) = \begin{cases} \frac{\theta-2+\mathfrak{o}^{\frac{1}{2}}}{1-\mathfrak{o}^{\frac{1}{2}}} & \theta \in \left[2-\mathfrak{o}^{\frac{1}{2}},\ 3\right] \\ \frac{2+\mathfrak{o}^{\frac{1}{2}}-\theta}{\mathfrak{o}^{\frac{1}{2}}-1} & \theta \in \left(3,\ 2+\mathfrak{o}^{\frac{1}{2}}\right] \\ 0 & otherwise, \end{cases} \tag{34}$$

Then, for each $\lambda \in [0,\ 1]$, *we have* $\mathfrak{G}_\lambda(\mathfrak{o}) = \left[(1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda, (1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda\right]$. *Since left and right end point mappings* $\mathfrak{G}_*(\mathfrak{o},\lambda) = (1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda$, *and* $\mathfrak{G}^*(\mathfrak{o},\ \lambda) = (1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda$ *are pre-invex and pre-incave mappings with* $\jmath(\tau,\nu) = \tau-\nu$ *for each* $\lambda \in [0,\ 1]$, *respectively, then* $\mathfrak{G}(\mathfrak{o})$ *is up and down pre-invex F-N-V·M with* $\jmath(\tau,\nu) = \tau-\nu$. *We clearly see that* $\mathfrak{G} \in L([\tau,\ \nu], \mathbb{E}_C)$ *and*

$$\mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right) = \mathfrak{G}_*\left(\frac{5}{2},\ \lambda\right) = (1-\lambda)\frac{4-\sqrt{10}}{2}+3\lambda$$

$$\mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right) = \mathfrak{G}^*\left(\frac{5}{2},\ \lambda\right) = (1+\lambda)\frac{4+\sqrt{10}}{2}+3\lambda$$

Note that

$$\frac{1}{\jmath(\tau,\ \nu)}\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}_*(\mathfrak{o},\ \lambda)d\mathfrak{o} = \int_2^3\left((1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda\right)d\mathfrak{o} = \frac{843}{2000}(1-\lambda)+3\lambda,$$

$$\frac{1}{\jmath(\tau,\ \nu)}\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}^*(\mathfrak{o},\ \lambda)d\mathfrak{o} = \int_2^3\left((1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+3\lambda\right)d\mathfrak{o} = \frac{179}{50}(1+\lambda)+3\lambda,$$

and

$$\frac{\mathfrak{G}_*(\tau,\lambda)+\mathfrak{G}_*(\nu,\ \lambda)}{2} = (1-\lambda)\left(\frac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\lambda$$

$$\frac{\mathfrak{G}^*(\tau,\lambda)+\mathfrak{G}^*(\nu,\ \lambda)}{2} = (1+\lambda)\left(\frac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\lambda\ .$$

Therefore,

$$\left[(1-\lambda)\tfrac{4-\sqrt{10}}{2}+3\lambda, (1+\lambda)\tfrac{4+\sqrt{10}}{2}+3\lambda\right] \supseteq_I \left[\tfrac{843}{2000}(1-\lambda)+3\lambda, \tfrac{179}{50}(1+\lambda)+3\lambda\right]$$
$$\supseteq_I \left[(1-\lambda)\left(\tfrac{4-\sqrt{2}-\sqrt{3}}{2}\right)+3\lambda, (1+\lambda)\left(\tfrac{4+\sqrt{2}+\sqrt{3}}{2}\right)+3\lambda\right]$$

Hence,

$$\widetilde{\mathfrak{G}}\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\jmath(\tau,\ \nu)} \odot (FR)\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\widetilde{\mathfrak{G}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\mathfrak{G}}(\tau)\ \oplus\ \widetilde{\mathfrak{G}}(\nu)}{2}$$

and Theorem 6 is verified.

The next results, which are linked with the well-known Fejér–Hermite–Hadamard-type inequalities, will be obtained using symmetric mappings of one-variable forms.

Theorem 7. *Suppose that* $\mathfrak{G} : [\nu,\ \nu + \jmath(\tau,\ \nu)] \to \mathbb{E}$ *is an up and down pre-invex F-N-V·M along with a family of I-V·Ms* $\mathfrak{G}_\lambda : [\nu,\ \nu + \jmath(\tau,\ \nu)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ *as well as* $\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o}, \lambda), \mathfrak{G}^*(\mathfrak{o}, \lambda)]$ *for all* $\mathfrak{o} \in [\nu,\ \nu + \jmath(\tau,\ \nu)]$ *and for all* $\lambda \in [0,\ 1]$. *If* $\jmath$ *satisfies Condition C and* $\mathfrak{G} \in \mathcal{FR}_{([\nu,\ \nu+\jmath(\tau,\ \nu)],\ \lambda)}$, *and* $\mathfrak{V} : [\nu,\ \nu + \jmath(\tau,\ \nu)] \to \mathbb{R}$, $\mathfrak{V}(\mathfrak{o}) \geq 0$, *symmetric with respect to* $\nu + \frac{1}{2}\jmath(\tau,\ \nu)$, *then*

$$\frac{1}{\jmath(\tau,\ \nu)} \odot (FA) \int_\nu^{\nu+\jmath(\tau,\ \nu)} \widetilde{\mathfrak{G}}(\mathfrak{o}) \odot \mathfrak{V}(\mathfrak{o}) d\mathfrak{o} \supseteq_{\mathbb{F}} \left[\widetilde{\mathfrak{G}}(\nu)\ \oplus\ \mathfrak{G}(\tau)\right] \odot \int_0^1 \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath. \tag{35}$$

Proof. Let $\widetilde{\mathfrak{G}}$ be an up and down pre-invex *F-N-V·M*. Then, for each $\lambda \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) \\
&\quad \leq (\imath\mathfrak{G}_*(\nu,\ \lambda) + (1-\imath)\mathfrak{G}_*(\tau,\ \lambda))\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)), \\
&\mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) \\
&\quad \geq (\imath\mathfrak{G}^*(\nu,\ \lambda) + (1-\imath)\mathfrak{G}^*(\tau,\ \lambda))\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)).
\end{aligned} \tag{36}$$

And

$$\begin{aligned}
&\mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu)) \\
&\quad \leq ((1-\imath)\mathfrak{G}_*(\nu,\ \lambda) + \imath\mathfrak{G}_*(\tau,\ \lambda))\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu)), \\
&\mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu)) \\
&\quad \geq ((1-\imath)\mathfrak{G}^*(\nu,\ \lambda) + \imath\mathfrak{G}^*(\tau,\ \lambda))\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu)).
\end{aligned} \tag{37}$$

After adding (36) and (37), and integrating over $[0,\ 1]$, we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath \\
&\quad + \int_0^1 \mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath \\
&\leq \int_0^1 \begin{bmatrix} \mathfrak{G}_*(\nu,\ \lambda)\{\imath\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) + (1-\imath)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))\} \\ +\mathfrak{G}_*(\tau,\ \lambda)\{(1-\imath)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) + \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))\} \end{bmatrix} d\imath, \\
&\int_0^1 \mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath \\
&\quad + \int_0^1 \mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath \\
&\geq \int_0^1 \begin{bmatrix} \mathfrak{G}^*(\nu,\ \lambda)\{\imath\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) + (1-\imath)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))\} \\ +\mathfrak{G}^*(\tau,\ \lambda)\{(1-\imath)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu)) + \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))\} \end{bmatrix} d\imath. \\
&= 2\mathfrak{G}_*(\nu,\ \lambda) \int_0^1 \imath\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath + 2\mathfrak{G}_*(\tau,\ \lambda) \int_0^1 \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath, \\
&= 2\mathfrak{G}^*(\nu,\ \lambda) \int_0^1 \imath\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath + 2\mathfrak{G}^*(\tau,\ \lambda) \int_0^1 \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath.
\end{aligned}$$

Since $\mathfrak{V}$ is symmetric, then

$$\begin{aligned}
&= 2[\mathfrak{G}_*(\nu,\ \lambda) + \mathfrak{G}_*(\tau,\ \lambda)] \int_0^1 \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath, \\
&= 2[\mathfrak{G}^*(\nu,\ \lambda) + \mathfrak{G}^*(\tau,\ \lambda)] \int_0^1 \imath\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath.
\end{aligned} \tag{38}$$

Since

$$\begin{aligned}
&\int_0^1 \mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath \\
&= \int_0^1 \mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath \\
&= \frac{1}{\jmath(\tau,\ \nu)} \int_\nu^{\nu+\jmath(\tau,\ \nu)} \mathfrak{G}_*(\mathfrak{o}, \lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}, \\
&\int_0^1 \mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\mathfrak{V}(\nu + \imath\jmath(\tau,\ \nu))d\imath \\
&= \int_0^1 \mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\nu),\ \lambda)\mathfrak{V}(\nu + (1-\imath)\jmath(\tau,\nu))d\imath \\
&= \frac{1}{\jmath(\tau,\ \nu)} \int_\nu^{\nu+\jmath(\tau,\ \nu)} \mathfrak{G}^*(\mathfrak{o},\ \lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}.
\end{aligned} \tag{39}$$

From (39), we have

$$\frac{1}{\jmath(\tau,\nu)}\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}\leq[\mathfrak{G}_*(\nu,\lambda)+\mathfrak{G}_*(\tau,\lambda)]\int_0^1 \imath\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath,$$
$$\frac{1}{\jmath(\tau,\nu)}\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}\geq[\mathfrak{G}^*(\nu,\lambda)+\mathfrak{G}^*(\tau,\lambda)]\int_0^1 \imath\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath,$$

that is

$$\begin{aligned}&\left[\frac{1}{\jmath(\tau,\nu)}\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o},\ \frac{1}{\jmath(\tau,\nu)}\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}\right]\\ &\supseteq_I[\mathfrak{G}_*(\nu,\lambda)+\mathfrak{G}_*(\tau,\lambda),\ \mathfrak{G}^*(\nu,\lambda)+\mathfrak{G}^*(\tau,\lambda)]\int_0^1 \imath\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath,\end{aligned}$$

Hence

$$\frac{1}{\jmath(\tau,\nu)}\odot(FA)\int_{\nu}^{\nu+\jmath(\tau,\nu)}\widetilde{\mathfrak{G}}(\mathfrak{o})\odot\mathfrak{V}(\mathfrak{o})d\mathfrak{o}\supseteq_{\mathbb{F}}\left[\widetilde{\mathfrak{G}}(\nu)\oplus\widetilde{\mathfrak{G}}(\tau)\right]\odot\int_0^1 \imath\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath.$$

Next, we construct the first *HH*-Fejér inequality for up and down pre-invex *F-N-V·M*, which generalizes first *HH*-Fejér inequalities for up and down pre-invex mapping; see [94,96]. □

Theorem 8. *Suppose that* $\widetilde{\mathfrak{G}}:[\nu,\nu+\jmath(\tau,\nu)]\to\mathbb{E}$ *are two up and down pre-invex F-N-V·Ms along with* $\nu<\nu+\jmath(\tau,\nu)$ *and family of I-V·Ms* $\mathfrak{G}_\lambda:[\nu,\nu+\jmath(\tau,\nu)]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ *as well as* $\mathfrak{G}_\lambda(\mathfrak{o})=[\mathfrak{G}_*(\mathfrak{o},\lambda),\mathfrak{G}^*(\mathfrak{o},\lambda)]$ *for all* $\mathfrak{o}\in[\nu,\nu+\jmath(\tau,\nu)]$ *and for all* $\lambda\in[0,1]$. *If* $\widetilde{\mathfrak{G}}\in\mathcal{FR}_{([\nu,\nu+\jmath(\tau,\nu)],\lambda)}$ *and* $\mathfrak{V}:[\nu,\nu+\jmath(\tau,\nu)]\to\mathbb{R}$, $\mathfrak{V}(\mathfrak{o})\geq 0$, *symmetric with respect to* $\nu+\frac{1}{2}\jmath(\tau,\nu)$, *and* $\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}>0$, *and Condition C for* $\jmath$, *then*

$$\widetilde{\mathfrak{G}}\left(\nu+\frac{1}{2}\jmath(\tau,\nu)\right)\supseteq_{\mathbb{F}}\frac{1}{\int_{\nu}^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}}\odot(FA)\int_{\nu}^{\nu+\jmath(\tau,\nu)}\widetilde{\mathfrak{G}}(\mathfrak{o})\odot\mathfrak{V}(\mathfrak{o})d\mathfrak{o}. \tag{40}$$

Proof . Using Condition C, we can write

$$\nu+\frac{1}{2}\jmath(\tau,\nu)=\nu+\imath\jmath(\tau,\nu)+\frac{1}{2}\jmath(\nu+(1-\imath)\jmath(\tau,\nu),\ \nu+\imath\jmath(\tau,\nu))$$

Since $\mathfrak{G}$ is an up and down pre-invex, then for $\lambda\in[0,1]$, we have

$$\begin{aligned}&\mathfrak{G}_*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right)\\ &=\mathfrak{G}_*\left(\nu+\imath\jmath(\tau,\nu)+\tfrac{1}{2}\jmath(\ \nu+(1-\imath)\jmath(\tau,\nu),\ \nu+\imath\jmath(\tau,\nu)\ ),\lambda\right)\\ &\leq\tfrac{1}{2}(\mathfrak{G}_*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)+\mathfrak{G}_*(\nu+\imath\jmath(\tau,\nu),\lambda)),\\ &\mathfrak{G}^*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right)\\ &=\mathfrak{G}^*\left(\nu+\imath\jmath(\tau,\nu)+\tfrac{1}{2}\jmath(\ \nu+(1-\imath)\jmath(\tau,\nu),\ \nu+\imath\jmath(\tau,\nu)\ ),\lambda\right)\\ &\geq(\mathfrak{G}^*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)+\mathfrak{G}^*(\nu+\imath\jmath(\tau,\nu),\lambda)).\end{aligned} \tag{41}$$

By multiplying (41) by $\mathfrak{V}(\nu+(1-\imath)\jmath(\tau,\nu))=\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))$ and integrating it by $\imath$ over $[0,1]$, we obtain

$$\begin{aligned}&\mathfrak{G}_*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right)\int_0^1\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\\ &\leq\tfrac{1}{2}\left(\begin{array}{c}\int_0^1\mathfrak{G}_*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+(1-\imath)\jmath(\tau,\nu))d\imath\\ +\int_0^1\mathfrak{G}_*(\nu+\imath\jmath(\tau,\nu),\lambda)d\imath\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\end{array}\right),\\ &\mathfrak{G}^*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right)\int_0^1\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\\ &\geq\tfrac{1}{2}\left(\begin{array}{c}\int_0^1\mathfrak{G}^*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+(1-\imath)\jmath(\tau,\nu))d\imath\\ +\int_0^1\mathfrak{G}^*(\nu+\imath\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\end{array}\right).\end{aligned} \tag{42}$$

Since

$$\begin{aligned}
&\int_0^1 \mathfrak{G}_*(\nu+\imath\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\\
&=\int_0^1 \mathfrak{G}_*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+(1-\imath)\jmath(\tau,\nu))d\imath\\
&=\frac{1}{\jmath(\tau,\nu)}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o},\\
&\int_0^1 \mathfrak{G}^*(\nu+\imath\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+\imath\jmath(\tau,\nu))d\imath\\
&=\int_0^1 \mathfrak{G}^*(\nu+(1-\imath)\jmath(\tau,\nu),\lambda)\mathfrak{V}(\nu+(1-\imath)\jmath(\tau,\nu))d\imath\\
&=\frac{1}{\jmath(\tau,\nu)}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}.
\end{aligned} \tag{43}$$

From (43), we have

$$\begin{aligned}
\mathfrak{G}_*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right) &\le \frac{1}{\int_\nu^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o},\\
\mathfrak{G}^*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right) &\ge \frac{1}{\int_\nu^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}}\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}.
\end{aligned}$$

From this, we have

$$\begin{aligned}
&\left[\mathfrak{G}_*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right),\ \mathfrak{G}^*\left(\nu+\tfrac{1}{2}\jmath(\tau,\nu),\lambda\right)\right]\\
\supseteq_I &\frac{1}{\int_\nu^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}}\left[\int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o},\ \int_\nu^{\nu+\jmath(\tau,\nu)} \mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{V}(\mathfrak{o})d\mathfrak{o}\right],
\end{aligned} \tag{44}$$

From (44), we have

$$\widetilde{\mathfrak{G}}\left(\nu+\frac{1}{2}\jmath(\tau,\nu)\right) \supseteq_{\mathbb{F}} \frac{1}{\int_\nu^{\nu+\jmath(\tau,\nu)}\mathfrak{V}(\mathfrak{o})d\mathfrak{o}} \odot\ (FA)\int_\nu^{\nu+\jmath(\tau,\nu)} \widetilde{\mathfrak{G}}(\mathfrak{o})\odot\mathfrak{V}(\mathfrak{o})d\mathfrak{o}.$$

This completes the proof. □

Remark 5. *If $\jmath(\tau,\nu)=\tau-\nu$, then the inequalities in Theorems 7 and 8 reduce for up and down convex F-N-V·Ms; see [93].*

If $\mathfrak{G}_*(\nu,\lambda)=\mathfrak{G}^*(\nu,\lambda)$ with $\lambda=1$, then Theorems 7 and 8 reduces to classical first and second HH-Fejér inequality for pre-invex mapping; see [94].

If $\mathfrak{G}_*(\nu,\lambda)=\mathfrak{G}^*(\nu,\lambda)$ with $\lambda=1$ and $\jmath(\tau,\nu)=\tau-\nu$ then Theorems 7 and 8 reduce to classical first and second HH-Fejér inequality for convex mapping; see [96].

Example 2. *We consider the F-N-V·M $\mathfrak{G}:[0,\ \jmath(2,0)]\to\mathbb{E}_C$ defined by,*

$$\mathfrak{G}(\mathfrak{o})(\theta)=\begin{cases}\frac{\theta-2+\mathfrak{o}^{\frac{1}{2}}}{\frac{3}{2}-2-\mathfrak{o}^{\frac{1}{2}}} & \theta\in\left[2-\mathfrak{o}^{\frac{1}{2}},\ \frac{3}{2}\right]\\ \frac{2+\mathfrak{o}^{\frac{1}{2}}-\theta}{2+\mathfrak{o}^{\frac{1}{2}}-\frac{3}{2}} & \theta\in\left(\frac{3}{2},\ 2+\mathfrak{o}^{\frac{1}{2}}\right]\\ 0 & otherwise,\end{cases} \tag{45}$$

Then, for each $\lambda\in[0,\ 1]$, we have $\mathfrak{G}_\lambda(\mathfrak{o})=\left[(1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda,(1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda\right]$. Since end point mappings $\mathfrak{G}_*(\mathfrak{o},\lambda)$, and $\mathfrak{G}^*(\mathfrak{o},\lambda)$ are pre-invex and pre-incave mappings with $\jmath(\tau,\nu)=\tau-\nu$, respectively, for each $\lambda\in[0,\ 1]$, then $\mathfrak{G}(\mathfrak{o})$ is up and down pre-invex F-N-V·M. If

$$\mathfrak{V}(\mathfrak{o})=\begin{cases}\sqrt{\mathfrak{o}}, & \sigma\in[0,1],\\ \sqrt{2-\mathfrak{o}}, & \sigma\in(1,\ 2],\end{cases} \tag{46}$$

then $\mathfrak{V}(2-\mathfrak{o})=\mathfrak{V}(\mathfrak{o})\ge 0$, for all $\mathfrak{o}\in[0,\ 2]$. Since $\mathfrak{G}_*(\mathfrak{o},\lambda)=(1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda$ and $\mathfrak{G}^*(\mathfrak{o},\lambda)=(1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda$. Now, we compute the following:

$$\begin{aligned}
&\frac{1}{\jmath(\tau,\nu)}\int_{\tau}^{\nu+\jmath(\tau,\nu)}[\mathfrak{G}_*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}=\frac{1}{2}\int_0^2[\mathfrak{G}_*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}\\
&=\frac{1}{2}\int_0^1[\mathfrak{G}_*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}+\frac{1}{2}\int_1^2\mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{B}(\mathfrak{o})d\mathfrak{o},\\
&=\frac{1}{2}\int_0^1\left[(1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda\right](\sqrt{\mathfrak{o}})d\mathfrak{o}+\frac{1}{2}\int_1^2\left[(1-\lambda)\left(2-\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda\right](\sqrt{2-\mathfrak{o}})d\mathfrak{o}\\
&=\frac{1}{4}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\lambda\left[\frac{\pi}{8}-\frac{1}{12}\right],\\
&\frac{1}{\jmath(\tau,\nu)}\int_{\tau}^{\nu+\jmath(\tau,\nu)}[\mathfrak{G}^*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}=\frac{1}{2}\int_0^2[\mathfrak{G}^*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}\\
&=\frac{1}{2}\int_0^1[\mathfrak{G}^*(\mathfrak{o},\lambda)]\mathfrak{B}(\mathfrak{o})d\mathfrak{o}+\frac{1}{2}\int_1^2\mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{B}(\mathfrak{o})d\mathfrak{o},\\
&=\frac{1}{2}\int_0^1\left[(1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda\right](\sqrt{\mathfrak{o}})d\mathfrak{o}+\frac{1}{2}\int_1^2\left[(1+\lambda)\left(2+\mathfrak{o}^{\frac{1}{2}}\right)+\frac{3}{2}\lambda\right](\sqrt{2-\mathfrak{o}})d\mathfrak{o}\\
&=\frac{1}{4}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\lambda\left[\frac{\pi}{8}+\frac{31}{12}\right].
\end{aligned}\tag{47}$$

And

$$\begin{aligned}
&[\mathfrak{G}_*(\tau,\ \lambda)+\mathfrak{G}_*(\nu,\ \lambda)]\int_0^1\ \imath\mathfrak{B}(\nu+\imath\jmath(\tau,\ \nu))\ d\imath\\
&=\left[4(1-\lambda)-\sqrt{2}(1-\lambda)+3\lambda\right]\left[\int_0^{\frac{1}{2}}\imath\sqrt{2\imath}d\imath+\int_{\frac{1}{2}}^1\imath\sqrt{2(1-\imath)}d\imath\right]\\
&=\frac{1}{3}\left(4(1-\lambda)-\sqrt{2}(1-\lambda)+3\lambda\right),\\
&[\mathfrak{G}^*(\tau,\ \lambda)+\mathfrak{G}^*(\nu,\ \lambda)]\int_0^1\imath\mathfrak{B}(\nu+\imath\jmath(\tau,\ \nu))d\imath\\
&=\left[4(1+\lambda)+\sqrt{2}(1+\lambda)+3\lambda\right]\left[\int_0^{\frac{1}{2}}\imath\sqrt{2\imath}d\imath+\int_{\frac{1}{2}}^1\imath\sqrt{2(1-\imath)}d\imath\right]\\
&=\frac{1}{3}\left(4(1+\lambda)+\sqrt{2}(1+\lambda)+3\lambda\right).
\end{aligned}\tag{48}$$

From (47) and (48), we have

$$\begin{aligned}
&\left[\frac{1}{4}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\lambda\left[\frac{\pi}{4}-\frac{7}{6}\right],\ \frac{1}{4}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\lambda\left[\frac{\pi}{4}+\frac{25}{6}\right]\right]\\
&\supseteq_I\left[\frac{1}{3}\left(4(1-\lambda)-\sqrt{2}(1-\lambda)+3\lambda\right),\ \frac{1}{3}\left(4(1+\lambda)+\sqrt{2}(1+\lambda)+3\lambda\right)\right],\ \textit{for all}\ \lambda\in[0,\ 1].
\end{aligned}$$

Hence, Theorem 7 is verified.
For Theorem 8, we have

$$\begin{aligned}
\mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right)&=\mathfrak{G}_*(1,\ \lambda)=\frac{2+\lambda}{2},\\
\mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\ \lambda\right)&=\mathfrak{G}^*(1,\ \lambda)=\frac{3(2+3\lambda)}{2},
\end{aligned}\tag{49}$$

$$\begin{aligned}
&\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{B}(\mathfrak{o})d\mathfrak{o}=\int_0^1\sqrt{\mathfrak{o}}d\mathfrak{o}+\int_1^2\sqrt{2-\mathfrak{o}}d\mathfrak{o}=\frac{4}{3},\\
&\frac{1}{\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{B}(\mathfrak{o})d\mathfrak{o}}\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}_*(\mathfrak{o},\lambda)\mathfrak{B}(\mathfrak{o})d\mathfrak{o}=\frac{3}{8}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\frac{3\lambda}{2}\left[\frac{\pi}{8}-\frac{1}{12}\right],\\
&\frac{1}{\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{B}(\mathfrak{o})d\mathfrak{o}}\int_{\tau}^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}^*(\mathfrak{o},\lambda)\mathfrak{B}(\mathfrak{o})d\mathfrak{o}=\frac{3}{8}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\frac{3\lambda}{2}\left[\frac{\pi}{8}+\frac{31}{12}\right].
\end{aligned}\tag{50}$$

From (49) and (50), we have

$$\left[\frac{2+\lambda}{2},\ \frac{3(2+3\lambda)}{2}\right]\supseteq_I\left[\frac{3}{8}\left[\frac{13}{3}-\frac{\pi}{2}\right]+\frac{3\lambda}{2}\left[\frac{\pi}{8}-\frac{1}{12}\right],\ \frac{3}{8}\left[\frac{19}{3}+\frac{\pi}{2}\right]+\frac{3\lambda}{2}\left[\frac{\pi}{8}+\frac{31}{12}\right]\right].$$

Hence, Theorem 8 has been verified.

Further, we also offer the fuzzy integral relations of the product of two up and down pre-invex F-N-V·Ms.

Theorem 9. *Suppose that* $\widetilde{\mathfrak{G}},\widetilde{\mathfrak{E}}:[\nu,\ \nu+\jmath(\tau,\ \nu)]\to\mathbb{E}$ *are two up and down pre-invex F-N-V·Ms along with the family of I-V·Ms* $\mathfrak{G}_\lambda,\ \mathfrak{E}_\lambda:[\nu,\ \nu+\jmath(\tau,\ \nu)]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ *as well as*

$\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o},\lambda),\ \mathfrak{G}^*(\mathfrak{o},\lambda)]$ *and* $\mathfrak{E}_\lambda(\mathfrak{o}) = [\mathfrak{E}_*(\mathfrak{o},\lambda),\ \mathfrak{E}^*(\mathfrak{o},\lambda)]$ *for all* $\mathfrak{o} \in [\nu,\ \nu + \jmath(\tau,\ \nu)]$ *and for all* $\lambda \in [0,\ 1]$. *If* $\jmath$ *satisfy Condition C and* $\widetilde{\mathfrak{G}}(\mathfrak{o}) \otimes \widetilde{\mathfrak{E}}(\mathfrak{o}) \in \mathcal{FR}_{([\nu,\ \nu+\jmath(\tau,\ \nu)],\ \lambda)}$, *then*

$$\frac{1}{\jmath(\tau,\ \nu)} \odot (FA)\int_{\nu}^{\nu+\jmath(\tau,\ \nu)} \widetilde{\mathfrak{G}}(\mathfrak{o}) \otimes \widetilde{\mathfrak{E}}(\mathfrak{o})d\mathfrak{o} \supseteq_{\mathbb{F}} \frac{\widetilde{\Delta}(\nu,\tau)}{3} \oplus \frac{\widetilde{\nabla}(\nu,\tau)}{6}, \tag{51}$$

where $\widetilde{\Delta}(\nu,\tau) = \widetilde{\mathfrak{G}}(\nu) \otimes \widetilde{\mathfrak{E}}(\nu) \oplus \widetilde{\mathfrak{G}}(\tau) \otimes \widetilde{\mathfrak{E}}(\tau)$, $\widetilde{\nabla}(\nu,\tau) = \widetilde{\mathfrak{G}}(\nu) \otimes \widetilde{\mathfrak{E}}(\tau) \oplus \widetilde{\mathfrak{G}}(\tau) \otimes \widetilde{\mathfrak{E}}(\nu)$, *and* $\Delta_\lambda(\nu,\tau) = [\Delta_*((\nu,\tau),\ \lambda),\ \Delta^*((\nu,\tau),\ \lambda)]$ *and* $\nabla_\lambda(\nu,\tau) = [\nabla_*((\nu,\tau),\ \lambda),\ \nabla^*((\nu,\tau),\ \lambda)]$.

Example 3. *Let* $[\tau,\nu] = [0,2]$, *and the F-N-V·Ms* $\mathfrak{G}, \mathfrak{E} : [\tau,\ \nu] = [0,2] \to \mathbb{E}_C$, *which is defined by*

$$\widetilde{\mathfrak{G}}(\mathfrak{o})(\theta) = \begin{cases} \frac{\theta}{\mathfrak{o}} & \theta \in [0,\ \mathfrak{o}] \\ \frac{2\mathfrak{o}-\theta}{\mathfrak{o}} & \theta \in (\mathfrak{o},\ 2\mathfrak{o}] \\ 0 & otherwise, \end{cases}$$

$$\widetilde{\mathfrak{E}}(\mathfrak{o})(\theta) = \begin{cases} \frac{\theta-\mathfrak{o}}{2-\mathfrak{o}} & \theta \in [\mathfrak{o},\ 2] \\ \frac{8-e^{\mathfrak{o}}-\theta}{8-e^{\mathfrak{o}}-2} & \theta \in (2,\ 8-e^{\mathfrak{o}}] \\ 0 & otherwise. \end{cases}$$

Then, for each $\lambda \in [0,\ 1]$, *we have* $\mathfrak{G}_\lambda(\mathfrak{o}) = [\lambda\mathfrak{o}, (2-\lambda)\mathfrak{o}]$ *and* $\mathfrak{E}_\lambda(\mathfrak{o}) = [(1-\lambda)\mathfrak{o} + 2\lambda,\ (1-\lambda)(8-e^{\mathfrak{o}}) + 2\lambda]$. *Since left and right end point mappings* $\mathfrak{G}_*(\mathfrak{o},\lambda) = \lambda\mathfrak{o}$, $\mathfrak{G}^*(\mathfrak{o},\ \lambda) = (2-\lambda)\mathfrak{o}$, $\mathfrak{E}_*(\mathfrak{o},\lambda) = (1-\lambda)\mathfrak{o} + 2\lambda$ *and* $\mathfrak{E}^*(\mathfrak{o},\ \lambda) = (1-\lambda)(8-e^{\mathfrak{o}}) + 2\lambda$ *are pre-invex and pre-incave mappings with* $|(\tau,\nu) = \tau - \nu$ *for each* $\lambda \in [0,\ 1]$, *respectively, then* $\mathfrak{G}(\mathfrak{o})$ *and* $\mathfrak{E}(\mathfrak{o})$ *both are up and down pre-invex F-N-V·Ms with* $|(\tau,\nu) = \tau - \nu$. *We clearly see that* $\mathfrak{G}(\mathfrak{o}) \otimes \mathfrak{E}(\mathfrak{o}) \in L([\tau,\ \nu], \mathbb{E}_C)$ *and*

$$\frac{1}{\jmath(\tau,\ \nu)} \int_{\tau}^{\nu+\jmath(\tau,\ \nu)} \mathfrak{G}_*(\mathfrak{o},\lambda) \times \mathfrak{E}_*(\mathfrak{o},\lambda)d\mathfrak{o} = \frac{1}{2}\int_0^2 \left(\lambda(1-\lambda)\mathfrak{o}^2 + 2\lambda^2\mathfrak{o}\right)d\mathfrak{o} = \frac{2}{3}\lambda(2+\lambda),$$
$$\frac{1}{\jmath(\tau,\ \nu)} \int_{\tau}^{\nu+\jmath(\tau,\ \nu)} \mathfrak{G}^*(\mathfrak{o},\lambda) \times \mathfrak{E}^*(\mathfrak{o},\lambda)d\mathfrak{o} = \frac{1}{2}\int_0^2 ((1-\lambda)(2-\lambda)\mathfrak{o}(8-e^{\mathfrak{o}}) + 2\lambda(2-\lambda)\mathfrak{o})d\mathfrak{o} \approx \frac{(2-\lambda)}{2}\left(\frac{1903}{250} - \frac{903}{250}\lambda\right).$$

Note that

$$\Delta_*(\tau,\nu) = [\mathfrak{G}_*(\tau) \times \mathfrak{E}_*(\tau) + \mathfrak{G}_*(\nu) \times \mathfrak{E}_*(\nu)] = 4\lambda$$

$$\Delta^*(\tau,\nu) = [\mathfrak{G}^*(\tau) \times \mathfrak{E}^*(\tau) + \mathfrak{G}^*(\nu) \times \mathfrak{E}^*(\nu)] = 2(2-\lambda)\left[(1-\lambda)\left(8-e^2\right) + 2\lambda\right],$$

$$\nabla_*(\tau,\nu) = [\mathfrak{G}_*(\tau) \times \mathfrak{E}_*(\nu) + \mathfrak{G}_*(\nu) \times \mathfrak{E}_*(\tau)] = 4\lambda^2,$$

$$\nabla_*(\tau,\nu) = [\mathfrak{G}^*(\tau) \times \mathfrak{E}^*(\nu) + \mathfrak{G}^*(\nu) \times \mathfrak{E}^*(\tau)] = 2(2-\lambda)(7-5\lambda).$$

Therefore, we have

$$\begin{aligned} &\tfrac{1}{3}\Delta_\lambda((\tau,\nu),\ \lambda) + \tfrac{1}{6}\nabla_\lambda((\tau,\nu),\ \lambda) \\ &= \tfrac{1}{3}\left[4\lambda, 2(2-\lambda)\left[(1-\lambda)\left(8-e^2\right) + 2\lambda\right]\right] + \tfrac{1}{3}\left[2\lambda^2, (2-\lambda)(7-5\lambda)\right] \\ &= \tfrac{1}{3}\left[2\lambda(2+\lambda), (2-\lambda)\left[2(1-\lambda)\left(8-e^2\right) - \lambda + 7\right]\right]. \end{aligned}$$

It follows that

$$\left[\frac{2}{3}\lambda(1+2\lambda), \frac{(2-\lambda)}{2}\left(\frac{1903}{250} - \frac{903}{250}\lambda\right)\right] \supseteq_I \frac{1}{3}\left[2\lambda(2+\lambda), (2-\lambda)\left[2(1-\lambda)\left(8-e^2\right) - \lambda + 7\right]\right]$$

and Theorem 9 has been demonstrated.

Theorem 10. *Suppose that* $\mathfrak{G}, \widetilde{\mathfrak{E}} : [\nu,\ \nu + \jmath(\tau,\ \nu)] \to \mathbb{E}$ *are two up and down pre-invex F-N-V·Ms along with the family of I-V·Ms* $\mathfrak{G}_\lambda,\ \mathfrak{E}_\lambda : [\nu,\ \nu + \jmath(\tau,\ \nu)] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ as well as $\mathfrak{G}_\lambda(\mathfrak{o}) = [\mathfrak{G}_*(\mathfrak{o},\lambda),\ \mathfrak{G}^*(\mathfrak{o},\lambda)]$ *and* $\mathfrak{E}_\lambda(\mathfrak{o}) = [\mathfrak{E}_*(\mathfrak{o},\lambda),\ \mathfrak{E}^*(\mathfrak{o},\lambda)]$ *for all* $\mathfrak{o} \in [\nu,\ \nu + \jmath(\tau,\ \nu)]$ *and for all* $\lambda \in [0,\ 1]$. *If* $\jmath$ *satisfy Condition C and* $\widetilde{\mathfrak{G}}(\mathfrak{o}) \widetilde{\times} \widetilde{\mathfrak{E}}(\mathfrak{o}) \in \mathcal{FR}_{([\nu,\ \nu+\jmath(\tau,\ \nu)],\ \lambda)}$, *then*

$$2 \odot \widetilde{\mathfrak{G}}\left(\frac{2\nu + \jmath(\tau,\ \nu)}{2}\right) \otimes \widetilde{\mathfrak{E}}\left(\frac{2\nu + \jmath(\tau,\ \nu)}{2}\right) \supseteq_{\mathbb{F}} \frac{1}{\imath(\tau,\ \nu)} \odot\ (FA) \int_{\nu}^{\nu+\jmath(\tau,\ \nu)} \widetilde{\mathfrak{G}}(\mathfrak{o}) \otimes \widetilde{\mathfrak{E}}(\mathfrak{o}) d\mathfrak{o}\ \oplus\ \frac{\widetilde{\Delta}(\nu,\tau)}{6} \oplus \frac{\widetilde{\nabla}(\nu,\tau)}{3}, \quad (52)$$

where $\widetilde{\Delta}(\nu,\tau) = \widetilde{\mathfrak{G}}(\nu) \otimes \widetilde{\mathfrak{E}}(\nu)\ \oplus\ \widetilde{\mathfrak{G}}(\tau) \otimes \widetilde{\mathfrak{E}}(\tau)$, $\widetilde{\nabla}(\nu,\tau) = \widetilde{\mathfrak{G}}(\nu) \otimes \widetilde{\mathfrak{E}}(\tau)\ \oplus\ \widetilde{\mathfrak{G}}(\tau) \otimes \widetilde{\mathfrak{E}}(\nu)$, *and* $\Delta_\lambda(\nu,\tau) = [\Delta_*((\nu,\tau),\ \lambda),\ \Delta^*((\nu,\tau),\ \lambda)]$ *and* $\nabla_\lambda(\nu,\tau) = [\nabla_*((\nu,\tau),\ \lambda),\ \nabla^*((\nu,\tau),\ \lambda)]$.

Proof. Using Condition C, we can write

$$\nu + \frac{1}{2}\jmath(\tau,\ \nu) = \nu + \imath\jmath(\tau,\ \nu) + \frac{1}{2}\jmath(\nu + (1-\imath)\jmath(\tau,\nu),\ \nu + \imath\jmath(\tau,\nu)).$$

By hypothesis, for each $\lambda \in [0,\ 1]$, we have

$$\begin{aligned}
&\mathfrak{G}_*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right) \times \mathfrak{E}_*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\\
&\mathfrak{G}^*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right) \times \mathfrak{E}^*\left(\tfrac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\\
&= \mathfrak{G}_*\left(\nu + \imath\jmath(\tau,\ \nu) + \frac{1}{2}\jmath\begin{pmatrix}\nu + (1-\imath)\jmath(\tau,\nu),\\ \nu + \imath\jmath(\tau,\nu)\end{pmatrix},\lambda\right)\\
&\times \mathfrak{E}_*\left(\nu + \imath\jmath(\tau,\ \nu) + \frac{1}{2}\jmath\begin{pmatrix}\nu + (1-\imath)\jmath(\tau,\nu),\\ \nu + \imath\jmath(\tau,\nu)\end{pmatrix},\lambda\right)\\
&= \mathfrak{G}^*\left(\nu + \imath\jmath(\tau,\ \nu) + \frac{1}{2}\jmath\begin{pmatrix}\nu + (1-\imath)\jmath(\tau,\nu),\\ \nu + \imath\jmath(\tau,\nu)\end{pmatrix},\lambda\right)\\
&\times \mathfrak{E}^*\left(\nu + \imath\jmath(\tau,\ \nu) + \frac{1}{2}\jmath\begin{pmatrix}\nu + (1-\imath)\jmath(\tau,\nu),\\ \nu + \imath\jmath(\tau,\nu)\end{pmatrix},\lambda\right)\\
&\le \frac{1}{4}\begin{bmatrix}\mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}\\
&+ \frac{1}{4}\begin{bmatrix}\mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix},\\
&\ge \frac{1}{4}\begin{bmatrix}\mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}\\
&+ \frac{1}{4}\begin{bmatrix}\mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix},\\
&\le \frac{1}{4}\begin{bmatrix}\mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}\\
&+ \frac{1}{4}\begin{bmatrix}(\imath\mathfrak{G}_*(\nu,\ \lambda) + (1-\imath)\mathfrak{G}_*(\tau,\ \lambda)) \times \begin{pmatrix}(1-\imath)\mathfrak{E}_*(\nu,\ \lambda)\\ +\imath\mathfrak{E}_*(\tau,\ \lambda)\end{pmatrix}\\ +((1-\imath)\mathfrak{G}_*(\nu,\ \lambda) + \imath\mathfrak{G}_*(\tau,\ \lambda)) \times \begin{pmatrix}\imath\mathfrak{E}_*(\nu,\ \lambda)+\\ (1-\imath)\mathfrak{E}_*(\tau,\ \lambda)\end{pmatrix}\end{bmatrix},\\
&\ge \frac{1}{4}\begin{bmatrix}\mathfrak{G}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}^*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}\\
&+ \frac{1}{4}\begin{bmatrix}(\imath\mathfrak{G}^*(\nu,\ \lambda) + (1-\imath)\mathfrak{G}^*(\tau,\ \lambda)) \times \begin{pmatrix}(1-\imath)\mathfrak{E}^*(\nu,\ \lambda)\\ +\imath\mathfrak{E}^*(\tau,\ \lambda)\end{pmatrix}\\ +((1-\imath)\mathfrak{G}^*(\nu,\ \lambda) + \imath\mathfrak{G}^*(\tau,\ \lambda)) \times \begin{pmatrix}\imath\mathfrak{E}^*(\nu,\ \lambda)+\\ (1-\imath)\mathfrak{E}^*(\tau,\ \lambda)\end{pmatrix}\end{bmatrix},\\
&= \frac{1}{4}\begin{bmatrix}\mathfrak{G}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + (1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda) \times \mathfrak{E}_*(\nu + \imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}
\end{aligned}$$

$$+\frac{1}{2}\begin{bmatrix}\left\{\imath^2+(1-\imath)^2\right\}\nabla_*((\nu,\tau),\ \lambda)\\ +\{\imath(1-\imath)+(1-\imath)\jmath\}\Delta_*((\nu,\tau),\ \lambda)\end{bmatrix},$$

$$=\frac{1}{4}\begin{bmatrix}\mathfrak{G}^*(\nu+(1-\imath)\jmath(\tau,\ \nu),\ \lambda)\times\mathfrak{E}^*(\nu+(1-\imath)\jmath(\tau,\ \nu),\ \lambda)\\ +\mathfrak{G}^*(\nu+\imath\jmath(\tau,\ \nu),\ \lambda)\times\mathfrak{E}^*(\nu+\imath\jmath(\tau,\ \nu),\ \lambda)\end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}\left\{\imath^2+(1-\imath)^2\right\}\nabla^*((\nu,\tau),\ \lambda)\\ +\{\imath(1-\imath)+(1-\imath)\imath\}\Delta^*((\nu,\tau),\ \lambda)\end{bmatrix}.$$

Integrating over $[0,\ 1]$, we have

$$\begin{gathered}2\ \mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\times\mathfrak{E}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\\ \leq\frac{1}{\jmath(\tau,\ \nu)}\int_\nu^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}_*(\mathfrak{o},\lambda)\times\mathfrak{E}_*(\mathfrak{o},\lambda)d\mathfrak{o}\\ +\frac{\Delta_*((\nu,\tau),\ \lambda)}{6}+\frac{\nabla_*((\nu,\tau),\ \lambda)}{3},\\ 2\ \mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\times\mathfrak{E}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\\ \geq\frac{1}{\jmath(\tau,\ \nu)}\int_\nu^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}^*(\mathfrak{o},\lambda)\times\mathfrak{E}^*(\mathfrak{o},\lambda)d\mathfrak{o}\\ +\frac{\Delta^*((\nu,\tau),\ \lambda)}{6}+\frac{\nabla^*((\nu,\tau),\ \lambda)}{3},\end{gathered}$$

from which, we have

$$\begin{gathered}2\left[\mathfrak{G}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\times\mathfrak{E}_*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right),\ \mathfrak{G}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\times\mathfrak{E}^*\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2},\lambda\right)\right]\\ \supseteq_I\frac{1}{\jmath(\tau,\ \nu)}\left[\int_\nu^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}_*(\mathfrak{o},\lambda)\times\mathfrak{E}_*(\mathfrak{o},\lambda)d\mathfrak{o},\ \int_\nu^{\nu+\jmath(\tau,\ \nu)}\mathfrak{G}^*(\mathfrak{o},\lambda)\times\mathfrak{E}^*(\mathfrak{o},\lambda)d\mathfrak{o}\right]\\ +\left[\frac{\Delta_*((\nu,\tau),\ \lambda)}{6},\ \frac{\Delta^*((\nu,\tau),\ \lambda)}{6}\right]+\left[\frac{\nabla_*((\nu,\tau),\ \lambda)}{3},\ \frac{\nabla^*((\nu,\tau),\ \lambda)}{3}\right],\end{gathered}$$

that is

$$\begin{gathered}2\odot\ \widetilde{\mathfrak{G}}\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2}\right)\otimes\widetilde{\mathfrak{E}}\left(\frac{2\nu+\jmath(\tau,\ \nu)}{2}\right)\supseteq_{\mathbb{F}}\frac{1}{\jmath(\tau,\ \nu)}\ \odot(FA)\int_\nu^{\nu+\jmath(\tau,\ \nu)}\widetilde{\mathfrak{G}}(\mathfrak{o})\otimes\widetilde{\mathfrak{E}}(\mathfrak{o})d\mathfrak{o}\ \oplus\ \frac{\widetilde{\Delta}(\nu,\tau)}{6}\\ \oplus\frac{\widetilde{\nabla}(\nu,\tau)}{32\tau d\tau lued\ functions\ ions\ with}.\end{gathered}$$

This completes the proof. □

Example 4. *We consider the F-N-V·Ms* $\widetilde{\mathfrak{G}},\ \widetilde{\mathfrak{E}}:[\tau,\ \nu]=[0,\ \jmath(2,\ 0)]\to\mathbb{F}_C(\mathbb{R})$. *Then, for each* $\lambda\in[0,\ 1]$, *we have* $\mathfrak{G}_\lambda(\mathfrak{o})=[\lambda\mathfrak{o},(2-\lambda)\mathfrak{o}]$ *and* $\mathfrak{E}_\lambda(\mathfrak{o})=[(1-\lambda)\mathfrak{o}+2\lambda,(1-\lambda)(8-e^{\mathfrak{o}})+2\lambda]$, *as in Example 3, then* $\widetilde{\mathfrak{G}}$ *and* $\widetilde{\mathfrak{E}}$ *are pre-invex and pre-incave functions with* $\jmath(\tau,\nu)=\tau-\nu$. *We have* $\mathfrak{G}_*(\mathfrak{o},\lambda)=\lambda\mathfrak{o}$, $\mathfrak{G}^*(\mathfrak{o},\ \lambda)=(2-\lambda)\mathfrak{o}$ *and* $\mathfrak{E}_*(\mathfrak{o},\lambda)=(1-\lambda)\mathfrak{o}+2\lambda$, $\mathfrak{E}^*(\mathfrak{o},\ \lambda)=(1-\lambda)(8-e^{\mathfrak{o}})+2\lambda$, *then*

$$\begin{gathered}2\ \mathfrak{G}_*\left(\tfrac{\tau+\nu}{2},\lambda\right)\times\mathfrak{E}_*\left(\tfrac{\tau+\nu}{2},\lambda\right)=2\lambda(1+\lambda),\\ 2\ \mathfrak{G}^*\left(\tfrac{\tau+\nu}{2},\lambda\right)\times\mathfrak{E}^*\left(\tfrac{\tau+\nu}{2},\lambda\right)=2\left[16-20\lambda+6\lambda^2+\left(2-3\lambda+\lambda^2\right)e\right],\end{gathered}$$

$$\frac{1}{\nu-\tau}\int_\tau^\nu\mathfrak{G}_*(\mathfrak{o},\lambda)\times\mathfrak{E}_*(\mathfrak{o},\lambda)d\mathfrak{o}=\frac{1}{2}\int_0^2\left(\lambda(1-\lambda)\mathfrak{o}^2+2\lambda^2\mathfrak{o}\right)d\mathfrak{o}=\frac{4}{3}\lambda(3-\lambda),$$

$$\frac{1}{\nu-\tau}\int_\tau^\nu\mathfrak{G}^*(\mathfrak{o},\lambda)\times\mathfrak{E}^*(\mathfrak{o},\lambda)d\mathfrak{o}=\frac{1}{2}\int_0^2((1-\lambda)(2-\lambda)\mathfrak{o}(8-e^{\mathfrak{o}})+2\lambda(2-\lambda)\mathfrak{o})d\mathfrak{o}\approx\frac{(2-\lambda)}{2}\left(\frac{1903}{250}-\frac{903}{250}\lambda\right).$$

$$\Delta_*(\tau,\nu)=[\mathfrak{G}_*(\tau)\times\mathfrak{E}_*(\tau)+\mathfrak{G}_*(\nu)\times\mathfrak{E}_*(\nu)]=4\lambda$$

$$\Delta^*(\tau,\nu)=[\mathfrak{G}^*(\tau)\times\mathfrak{E}^*(\tau)+\mathfrak{G}^*(\nu)\times\mathfrak{E}^*(\nu)]=2(2-\lambda)\left[(1-\lambda)\left(8-e^2\right)+2\lambda\right],$$

$$\nabla_*(\tau,\nu) = [\mathfrak{G}_*(\tau) \times \mathfrak{E}_*(\nu) + \mathfrak{G}_*(\nu) \times \mathfrak{E}_*(\tau)] = 4\lambda^2,$$

$$\nabla_*(\tau,\nu) = [\mathfrak{G}^*(\tau) \times \mathfrak{E}^*(\nu) + \mathfrak{G}^*(\nu) \times \mathfrak{E}^*(\tau)] = 2(2-\lambda)(7-5\lambda).$$

Therefore, we have

$$\begin{aligned} &\tfrac{1}{6}\Delta_\lambda((\tau,\nu),\ \lambda) + \tfrac{1}{3}\nabla_\lambda((\tau,\nu),\ \lambda) \\ &= \tfrac{1}{3}\left[2\lambda, (2-\lambda)\left[(1-\lambda)\left(8-e^2\right)+2\lambda\right]\right] + \tfrac{2}{3}\left[2\lambda^2, (2-\lambda)(7-5\lambda)\right] \\ &= \tfrac{1}{3}\left[2\lambda(1+2\lambda), (2-\lambda)\left[(1-\lambda)\left(8-e^2\right)-8\lambda+14\right]\right]. \end{aligned}$$

It follows that

$$\begin{aligned} 2\left[\lambda(1+\lambda), \left[16-20\lambda+6\lambda^2+\left(2-3\lambda+\lambda^2\right)e\right]\right] \supseteq_I &\left[\tfrac{2}{3}\lambda(2+\lambda), \tfrac{(2-\lambda)}{2}\left(\tfrac{1903}{250}-\tfrac{903}{250}\lambda\right)\right] \\ &+\tfrac{1}{3}\left[2\lambda(1+2\lambda), (2-\lambda)\left[(1-\lambda)\left(8-e^2\right)-8\lambda+14\right]\right], \end{aligned}$$

and Theorem 10 has been demonstrated.

4. Conclusions

In this study, we have discussed the notion of fuzzy number valued up and down pre-invex functions as an extension of convex functions. Hermite–Hadamard-type fuzzy inclusions for up and down convex *F-N-V·M*s have been constructed. In addition, some new Hermite–Hadamard-type fuzzy inclusions are used to study the product of two up and down convex *F-N-V·M*s. Some exceptional cases are also obtained. Moreover, some useful examples are presented to study the validity of our main results. To extend the findings of this study, other fuzzy number valued up and down convex functions on the coordinates can be applied. In the future, we can investigate Fejér–Hermite–Hadamard-type inequalities for fuzzy number valued up and down pre-invex mappings by using fuzzy number valued fractional and Riemann integrals on coordinates. The ideas and conclusions offered in this article are intended to motivate readers to conduct more study.

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