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Mittag-Leffler Type Stability of Delay Generalized Proportional Caputo Fractional Differential Equations: Cases of Non-Instantaneous Impulses, Instantaneous Impulses and without Impulses

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Abstract: In this paper, nonlinear differential equations with a generalized proportional Caputo fractional derivative and finite delay are studied in this paper. The eventual presence of impulses in the equations is considered, and the statement of initial value problems in three cases is defined: namely non-instantaneous impulses, instantaneous impulses and no impulses. The relations between these three cases are discussed. Additionally, some stability properties are investigated. We apply the Mittag–Leffler function which plays a vital role and which gives well-known bounds on the norm of the solutions. The symmetry of this function about a line and the bounds is a property that plays an important role in stability. Several sufficient conditions are presented via appropriate new comparison results and the modified Razumikhin method. The results generalize several known results in the literature.

Keywords: generalized proportional fractional derivatives; delays; non-instantaneous impulses; instantaneous impulses; Mittag–Leffler stability; Razumikhin method; Lyapunov functions

MSC: 34A34; 34K45; 34A08; 34D20

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1. Introduction

Fractional calculus in real world phenomena is very applicable because of some typical properties such as memory. Various types of kernels in fractional integrals and fractional derivatives are applied (for example, in [1,2] the fourth-order time-fractional integro-differential equation with various types of kernels are studied numerically). A very general type of kernel was studied in [3] and called a general fractional integral/derivative. These general fractional integrals and derivatives were systematically studied by Y. Luchko [4,5] in appropriate function spaces in the framework of fractional calculus. Luchko also studied some qualitative properties of solutions of various types of differential equations with general fractional derivatives (see, [5]). In this paper, we focus on stability for a particular kernel (to be described in Section 3). Stability properties for fractional differential equations were studied by many authors (see, for example, [6,7]). As mentioned in [8], the generalized energy of a system does not have to decay exponentially for the system to be stable in the sense of Lyapunov, and recently the Mittag–Leffler stability and the fractional Lyapunov direct method were introduced for various types of fractional differential equations (see, for example, [9–12]) and applied in fractional models ([13–17]).

Many real processes are characterized by rapid changes in their state, and they are adequately modeled by differential equations with impulses. The acting time of these changes could be short relative to the duration of the whole process and they could be modeled as *instantaneous impulses* (see, for example, the classical book for ordinary differential equations [18] and the cited references therein). In some processes, the duration of changes might not be negligible, i.e., they start at arbitrary fixed points and remain active on finite time intervals. These types of changes could be modeled by *non-instantaneous impulses* (see, the overview given in the book [19]).

Even though fractional derivatives have memory, often various types of delays are involved in the fractional differential equations to represent some dynamics of the corresponding processes. When one studies fractional differential equations with delays and any type of impulse, there are a number of technical and theoretical difficulties.

In this paper, we study nonlinear differential equations with finite delay and with a generalized proportional Caputo fractional derivative. We consider three main cases: the case when there are non-instantaneous impulses in the equation, the case when there are instantaneous impulses in the equation and the case without any impulses. In all of these cases, we set up the initial value problem and we discuss the relation between them. The appropriate Mittag–Leffler type stability is defined, and several sufficient conditions are obtained. Our study is based on the Razumikhin method and its appropriate modifications. Some of the obtained results are generalizations of results known in the literature for the case of Caputo fractional differential equations.

Our contributions in this paper include:

1. The statement of the initial value problem for nonlinear systems of generalized proportional Caputo fractional differential equations with finite delays, and we consider three cases:
   - With non-instantaneous impulses;
   - With instantaneous impulses;
   - Without impulses.
2. An appropriate interpretation and connection between the three cases are provided.
3. Generalized proportional Mittag–Leffler stability of the three types of systems is defined.
4. The appropriate modifications of the Razumikhin method are applied in the three cases.
5. Some extensions of the comparison principle are provided.
6. Sufficient conditions for the Mittag–Leffler-type stability are obtained.

The paper is organized as follows. In Section 2, we recall some basic definitions about generalized proportional fractional integrals and Caputo-type derivatives, and some basic results are presented. In Section 3, we discuss the statements of fractional order delay systems in our three cases, and the relationships between them is provided. In Section 4, in the three cases, the generalized proportional Mittag–Leffler stability is defined, some comparison results are proved and several sufficient conditions are obtained with the help of appropriate modifications of the Razumikhin method.

2. Preliminary Notes and Results

We will give some basic notations used in this paper.

Let $u : [0, b] \to \mathbb{R}^n$, $b > 0$, $b \le \infty$ and $\tau \in (0, b)$. Then, we will use the following notations $u(\tau) = u(\tau - 0) = \lim_{t\uparrow\tau} u(t)$ and $u(\tau + 0) = \lim_{t\downarrow\tau} u(t)$.

Let $r > 0$ be a given number and consider the set $E = \{\phi : [-r, 0] \to \mathbb{R}^n$ is continuous everywhere except at a finite number of points $\tau_j \in (-r, 0) :\ \phi(\tau_j - 0) = \phi(\tau_j), \phi(\tau_j + 0) < \infty\}$ with a norm $||\phi||_0 = \sup_{s\in[-r,0]} ||\phi(s)||$, where $||.||$ is a norm in $\mathbb{R}^n$.

Let two sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ be given such that $0 < s_{i-1} \le t_i < s_i < t_{i+1}$, $i = 1, 2, \dots$, and $\lim_{k\to\infty} s_k = \infty$. Denote $t_0 = 0$.

Let $J \subset [0, \infty)$ be a given interval. Consider the following classes of functions:

$$\begin{aligned} NPC(J, \mathbb{R}^n) = \{u : J \to \mathbb{R}^n :\ & u \in C[J \cap \big(\bigcup \cup_{k=0}^{\infty}(t_k, s_k]\big), \mathbb{R}^n] : \\ & u(s_k) = u(s_k - 0) = \lim_{t\uparrow s_k} u(t) < \infty, \\ & u(s_k + 0) = \lim_{t\downarrow s_k} u(t) < \infty,\ \ k :\ s_k \in J\}, \end{aligned}$$

and

$$\begin{aligned} PC(J,\mathbb{R}^n) = \{v : J \to \mathbb{R}^n :\ & v \in C[J \cap \big([0,\infty)/\{t_k\}_{k=1}^{\infty}\big),\mathbb{R}^n] : \\ & v(t_k) = v(t_k - 0) = \lim_{t \uparrow t_k} v(t) < \infty, \\ & v(t_k + 0) = \lim_{t \downarrow t_k} v(t) < \infty,\ k :\ t_k \in J\}, \end{aligned}$$

We will give a brief overview of the literature on fractional integrals and derivatives with general kernels. In [4], Luchko described what was known in the literature on general fractional integrals (GFI) and general fractional derivatives (GFD) and studied GFI and GFD with the Sonine kernel. In [5], Luchko studied some analytical properties of initial-value problems for single and multi-term fractional differential equations with GFD with a Sonine kernel that possess integrable singularities of power function-type at the point zero. Luchko introduced the set of Sonine kernels $\mathbb{S}_{-1}$ and he considered GFI with a kernel $\kappa \in \mathbb{S}_{-1}$ (Definition 3.2 [5]):

$$(\mathbb{I}_{(\kappa)} f)(t) = \int_0^t \kappa(t-\tau) f(\tau) d\tau,\ \ t > 0, \tag{1}$$

GFD of Riemann–Liouville type (Definition 3.3 [5]):

$$(\mathbb{D}_{(\kappa)} f)(t) = \frac{d}{dt}\int_0^t \kappa(t-\tau) f(\tau) d\tau,\ \ t > 0, \tag{2}$$

and GFD of Caputo-type (Definition 3.3 [5]):

$$({}_*\mathbb{D}_{(\kappa)} f)(t) = (\mathbb{D}_{(\kappa)} f)(t) - f(0)\kappa(t),\ \ t > 0. \tag{3}$$

In [5], the first fundamental theorem of fractional calculus for the GFD (Theorem 3.1 [5]) and the second fundamental theorem of FC for the GFD (Theorem 3.2 [5]) are proved. Additionally, an explicit form of the solution of the initial value problem (IVP) for the linear fractional differential equation with Caputo type GFD is obtained. This formula significantly depends on the kernel $\kappa \in \mathbb{S}_{-1}$. Since the main goal of this paper is the study of fractional generalization of exponential stability, i.e., so-called Mittag–Leffler-type of stability, we will use a spacial type of the kernel $\kappa \in \mathbb{S}_{-1}$:

$$\kappa(t;\alpha,\rho) = \frac{\rho^{\alpha-1}\ t^{-\alpha}}{\Gamma(1-\alpha)} e^{\frac{\rho-1}{\rho}t} \in \mathbb{S}_{-1},\ \alpha \in (0,1), \rho \in (0,1],\ t \geq 0. \tag{4}$$

Then, the definitions of GFI and GFD given by (1)–(3) are reduced:

$$\begin{aligned} &(\mathcal{I}^{1-\alpha,\rho} f)(t) = (\mathbb{I}_{(\kappa(t,1-\alpha,\rho))} f)(t) = \int_0^t \frac{\rho^{-\alpha}\ (t-s)^{\alpha-1}}{\Gamma(\alpha)} e^{\frac{\rho-1}{\rho}(t-s)} f(s) ds, \\ &\qquad \alpha > 0,\ \ \rho \in (0,1], \\ &({}^{RL}\mathcal{D}^{\alpha,\rho} f)(t) = (\mathbb{D}_{(\kappa(t;\alpha,\rho))} f)(t) \\ &\qquad = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} e^{\frac{\rho-1}{\rho}(t-s)} f(s) ds, \quad \alpha \in (0,1),\ \rho \in (0,1], \\ &({}^{C}\mathcal{D}^{\alpha,\rho} f)(t) = ({}_*\mathbb{D}_{(\kappa(t;\alpha,\rho))} f)(t) \\ &\qquad = \frac{\rho^{\alpha-1}}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha} e^{\frac{\rho-1}{\rho}(t-s)} f(s) ds - f(0)\frac{\rho^{\alpha-1}\ t^{-\alpha}}{\Gamma(1-\alpha)} e^{\frac{\rho-1}{\rho}t}, \\ &\qquad \text{for}\ \ t > 0,\ \ \alpha \in (0,1),\ \ \rho \in (0,1]. \end{aligned} \tag{5}$$

Remark 1. *The fractional integral* $(\mathcal{I}^{1-\alpha,\rho} f)(t)$*, the fractional derivatives* $({}^{RL}\mathcal{D}^{\alpha,\rho} f)(t)$ *and* $({}^{C}\mathcal{D}^{\alpha,\rho} f)(t)$ *are called generalized proportional fractional integral, generalized proportional Rieman–*

Liouville fractional integral and generalized proportional Caputo fractional derive, respectively, and they are studied in [20,21].

Remark 2. *(see Remark 3.2 [20]) If $\alpha \in (0,1)$ and $\rho \in (0,1]$ then the relation $({}_a^C\mathcal{D}^{\alpha,\rho} e^{\frac{\rho-1}{\rho}(.)})(t) = 0$ for $t > a$ holds. At the same time $({}_a^C\mathcal{D}^{\alpha,\rho} K)(t) \neq 0$ for $K \in \mathbb{R},\ K \neq 0$.*

We recall some results about generalized proportional Caputo fractional derivatives and their applications in differential equations, which will be applied in the main result in the paper.

Lemma 1. *(Proposition 5.2 [20]) For $\rho \in (0,1]$ and $\alpha \in (0,1)$ we have*

$$({}_a^C\mathcal{D}^{\alpha,\rho}(e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1})(t) = \frac{\rho^{\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1-\alpha}, \quad \beta > 0.$$

Lemma 2. *(Lemma 3.2 [22]) Let $u \in C^1([a,b],\mathbb{R})$ with $a,b \in \mathbb{R},\ b \leq \infty$ (if $b = \infty$ then the interval is half open), and $q \in (0,1),\ \rho \in (0,1]$ be two reals. Then,*

$$({}_a^C\mathcal{D}^{\alpha,\rho} u^2)(t) \leq 2u(t)({}_a^C\mathcal{D}^{\alpha,\rho} u)(t), \quad t \in (a,b].$$

Lemma 3. *(Lemma 5 [23]) Let $u \in C([t_0, T, \mathbb{R}),\ T > t_0$, and there exists a point $t^* \in (t_0, T]$ such that $u(t^*) = 0$, and $u(t) < 0$, for $t_0 \leq t < t^*$. Then, if the generalized proportional Caputo fractional derivative of u exists for $t = t^*$, then the inequality $(\ {}_{t_0}^{c}\mathcal{D}^{\alpha,\rho} u)(t)|_{t=t^*} > 0$ holds.*

Lemma 4. *(Example 5.7 [20]) The scalar linear generalized proportional Caputo fractional initial value problem*

$$({}_a^C\mathcal{D}^{\alpha,\rho} u)(t) = \lambda u(t), \ \ u(a) = u_0, \ \ \alpha \in (0,1), \ \rho \in (0,1]$$

has a solution

$$u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha}(\lambda(\frac{t-a}{\rho})^{\alpha}), \ \ t > a,$$

where $\lambda \in \mathbb{R}$, $E_{\alpha}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\alpha+1)}$ is the Mittag–Leffler function of one parameter.

Lemma 5. *Let $\alpha \in (0,1)$ and $\rho \in (0,1]$. Then*

$$({}_a^C\mathcal{D}^{\alpha,\rho}\left(e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha}\left(\lambda\left(\frac{(t-a)}{\rho}\right)^{\alpha}\right)\right) = \lambda e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha}\left(\lambda\left(\frac{(t-a)}{\rho}\right)^{\alpha}\right).$$

Proof. From Lemma 1 and the definition of Mittag–Leffler function with one parameter, we obtain

$$\begin{aligned}
&({}_a^C\mathcal{D}^{\alpha,\rho}(E_{\alpha}(\lambda(\frac{t-a}{\rho}^{\alpha})))e^{\frac{\rho-1}{\rho}(t-a)}) = \sum_{i=0}^{\infty} \frac{({}_a^C\mathcal{D}^{\alpha,\rho}(e^{\frac{\rho-1}{\rho}(t-a)})(\lambda(\frac{t-a}{\rho})^{\alpha})^i}{\Gamma(i\alpha+1)} \\
&= \sum_{i=1}^{\infty} \frac{\lambda^i \rho^{\alpha}\Gamma(\alpha i+1) e^{\frac{=\rho-1}{\rho}(t-a)})(t-a)^{\alpha i-\alpha}}{\rho^{\alpha i}\Gamma(\alpha i+1-\alpha)\Gamma(i\alpha+1)} \\
&= \lambda e^{\frac{\rho-1}{\rho}(t-a)} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}(t-a)^{\alpha(i-1)}}{\rho^{\alpha(i-1)}\Gamma(\alpha(i-1)+1)} = \lambda e^{\frac{\rho-1}{\rho}(t-a)} E_{\alpha}\left(\lambda\left(\frac{(t-a)}{\rho}\right)^{\alpha}\right).
\end{aligned}$$

□

3. Statement of the Problems

In this paper, we will consider three cases: non-instantaneous impulses, instantaneous impulses and without impulse,s and we give the relations between them.

3.1. Non-Instantaneous Impulses

Let two sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ be given such that $0 < s_{i-1} \leq t_i < s_i < t_{i+1}$, $i = 1, 2, \ldots$, and $\lim_{k\to\infty} s_k = \infty$. Let $t_0 \geq 0$ be the given fixed initial time. Without loss of generality, we will assume $0 \leq t_0 < s_0 < t_1$.

Remark 3. *The intervals $(s_k, t_{k+1}]$, $k = 0, 1, 2, \ldots$ are called intervals of non-instantaneous impulses.*

Let $J \subset \mathbb{R}$ be a given interval. Consider the following class of functions:

$$NPC^{\alpha,\rho}(J, \mathbb{R}^n) = \{u : J \to \mathbb{R}^n : \ u \in NPC(J, \mathbb{R}^n) : \text{ for any } k = 0, 1, 2, \cdots : \ t_k \in J, \\ ({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}u)(t) \quad \text{exists for} \ \ t \in (t_k, s_k] \cap J\},$$

Consider the system of *non-instantaneous impulsive delay differential equations* (NIDDE) with the generalized proportional Caputo fractional derivative

$$\begin{aligned} &({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x_t) \text{ for } \ t \in (t_k, s_k], \ k = 0, 1, 2, \ldots \\ &x(t) = \Phi_k(t, x(s_k - 0)) \quad \text{for } t \in (s_k, t_{k+1}], \ k = 0, 1, 2, \ldots, \end{aligned} \tag{6}$$

with initial condition

$$x(t + t_0) = \phi(t) \ \text{ for } t \in [-r, 0], \tag{7}$$

where $f : [t_0, s_0] \bigcup \cup_{k=1}^{\infty} [t_k, s_k] \times \mathbb{R}^n \to \mathbb{R}^n$, $\Phi_i : [s_i, t_{i+1}] \times \mathbb{R}^n \to \mathbb{R}^n$, $(i = 0, 1, 2, 3, \ldots)$, $r > 0$ is a given number, $\phi : [-r, 0] \to \mathbb{R}^n$ and $x_t = x(t + s), s \in [-r, 0]$.

Remark 4. *The functions $\Phi_k(t, x)$, $k = 1, 2, \ldots$, are called non-instantaneous impulsive functions.*

Remark 5. *For some detailed explanations about non-instantaneous impulses in generalized proportional Caputo fractional differential equations without delays, see [24].*

We will introduce the following conditions:
(A 1.1.) The function $f \in C(\cup_{k=0}^{\infty} [t_k, s_k] \times \mathbb{R}^n, \mathbb{R}^n)$.
(A 1.2.) For any natural number k the functions $\Phi_k \in C([s_k, t_k] \times \mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \ldots$.

Remark 6. *We will assume that for any initial function $\phi \in E$ the IVP for the system of NIDDE (6) and (7) has a solution $x(t; t_0, \phi) \in NPC^{\alpha,\rho}([t_0, \infty), \mathbb{R}^n)$.*

We now give a brief description of the solution of IVP for NIDDE (6) and (7). The solution $x(t; t_0, \phi)$ of (6) and (7) is given by

$$x(t; t_0, \phi) = \begin{cases} X_k(t), & \text{for } t \in (t_k, s_k], \ k = 0, 1, 2, \ldots, \\ \Phi_k(t, X_k(s_k - 0)), & \text{for } \ t \in (s_k, t_{k+1}] \ \ k = 1, 2, \ldots \end{cases} \tag{8}$$

where

- On the interval $[t_0 - r, t_0]$, the solution satisfies the initial condition (7);
- On the interval $[t_0, s_0]$, the solution coincides with $X_0(t)$ which is the solution of $({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x_t)$, $t \in (t_0, s_0]$ with initial condition (7);
- On the interval $(s_0, t_1]$, the solution $x(t; t_0, \phi)$ satisfies the equation

$$x(t; t_0, \phi) = \Phi_0(t, X_0(s_0 - 0));$$

- On the interval $(t_1, s_1]$, the solution coincides with $X_1(t)$ which is the solution of $({}^C_{t_1}\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x_t),\ t \in (t_1, s_1]$ and initial condition $x(t + t_1) = \tilde{\phi}(t),\ t \in [-r, 0]$ with
$$\tilde{\phi}(t) = \begin{cases} \Phi_0(t_1, X_0(s_0 - 0)) & t = 0 \\ x(t - t_1; t_0, \phi) & t \in [-r, 0); \end{cases}$$
- On the interval $(s_1, t_2]$, the solution $x(t; t_0, \phi)$ satisfies the equation
$$x(t; t_0, x_0) = \Phi_1(t, X_1(s_1 - 0));$$

and so on.

In connection with the study of the stability properties of zero solutions, we introduce the following assumption:

(A 1.3.) The equalities $f(t, 0) = 0$ and $\Phi_k(t, 0) \equiv 0,\ k = 0, 1, 2, \ldots,$ hold.

3.2. Instantaneous Impulses

Let the sequence of points $\{t_i\}_{i=1}^{\infty}$ be given such that $0 < t_i \le t_{i+1}$, $i = 1, 2, \ldots,$ and $\lim_{k\to\infty} t_k = \infty$. Let $t_0 \ge 0$ be the given fixed initial time. Without loss of generality we will assume $0 \le t_0 < t_1$.

Remark 7. *The points $t_k,\ k = 0, 1, 2, \ldots$ are called points of impulses.*

Let $J \subset \mathbb{R}$ be a given interval. Consider the following class of functions
$$PC^{\alpha,\rho}(J, \mathbb{R}^n) = \{v : J \to \mathbb{R}^n : v \in PC(J, \mathbb{R}^n) : \text{ for any } t_k \in J,\ k = 0, 1, 2, \cdots : ({}^C_{t_k}\mathcal{D}^{\alpha,\rho}v)(t) \text{ exists for } t \in (t_k, t_{k+1}] \cap J\}.$$

Consider the system of *instantaneous impulsive delay differential equations* (IDDE) with the generalized proportional Caputo fractional derivative
$$\begin{aligned} &({}^C_{t_k}\mathcal{D}^{\alpha,\rho}x)(t) = f(t, x_t) \text{ for } t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \ldots \\ &x(t_k + 0) = \Psi_k(x(t_k - 0)) \quad \text{for } k = 1, 2, \ldots, \end{aligned} \tag{9}$$

with initial condition (7), where $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\Psi_i : \mathbb{R}^n \to \mathbb{R}^n$, $(i = 1, 2, 3, \ldots)$.

Remark 8. *The functions $\Psi_k(y)$, $k = 1, 2, \ldots,$ are called impulsive functions.*

Remark 9. *In the case in Section 3.1 that both sequences coincide, i.e., $s_i = t_{i+1},\ i = 0, 1, 2, \ldots$, the system (6) is reduced to the system (9) with $\Phi_k(t, u) = \Psi_k(u),\ k = 0, 1, 2, \ldots,$ i.e., the case of non-instantaneous impulses could be considered as a generalization of the case of instantaneous impulses.*

We will introduce the following conditions:

(A 2.1.) The function $f \in C([t_0, t_1] \bigcup_{k=1}^{\infty}(t_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n)$.

(A 2.2.) The functions $\Phi_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \ldots$.

(A 2.3.) The function $f(t, 0) = 0,\ \ t \ge t_0$ and the functions $\Psi_k(0) = 0$, $k = 1, 2, \ldots$.

If condition (A 2.3) is satisfied, then for the zero initial function, the IVP for IDDE (7) and (9) has a zero solution.

Remark 10. *We will assume that for any initial function $\phi \in E$ the IVP for the system of IDDE (7) and (9) has a solution $x(t; t_0, \phi) \in PC^{\alpha,\rho}([t_0, \infty), \mathbb{R}^n)$*

3.3. No Impulses

Consider the system of *delay differential equations* (DDE) with the generalized proportional fractional derivative

$$({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}x)(t) = f(t,x_t) \text{ for } t > t_0 \tag{10}$$

with initial condition (7), where $f : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$.

Remark 11. *The system* (10) *could be considered as a partial case of* (9) *in the case when there are no impulses, i.e., in Section 3.2* $t_i = t_0,\ i = 1,2,\dots,$ *i.e., the case of instantaneous impulses could be considered as a generalization of the case of without impulses.*

Let $J \subset \mathbb{R}$ be a given interval. Consider the following classes of functions

$$\begin{aligned} C^{\alpha,\rho}(J,\mathbb{R}^n) &= \{u : J \to \mathbb{R}^n :\ u \in C(J\cap[a,\infty),\mathbb{R}^n) : \\ &\qquad ({}_{a}^{C}\mathcal{D}^{\alpha,\rho}u)(t) \ \text{ exists for } \ t \in [a,\infty)\cap J\}. \end{aligned}$$

We will introduce the following conditions:
(A 3.1.) The function $f \in C([t_0,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$.
(A 3.2.) The function $f(t,0) = 0,\ \ t \ge t_0$.

Remark 12. *We will assume that for any initial function* $\phi \in E$*, the IVP for the system of DDE* (7) *and* (10) *has a solution* $x(t;t_0,\phi) \in C^{\alpha,\rho}([t_0,\infty),\mathbb{R}^n)$

4. Mittag–Leffer-Type Stability Properties

We will study the Mittag–Leffler-type stability for NIDDE (6), IDDE (9) and DDE (10) by Lyapunov functions and an appropriate modification of the Razumikhin method.

4.1. Non-Instantaneous Impulses

Definition 1. *The zero solution of the system NIDDE* (6) *and* (7) *is said to be* generalized proportional Mittag–Leffler stable *if there exist constants* $\beta,\gamma,C,\lambda > 0$ *such that the inequality*

$$||x(t;t_0,\phi)|| \le \begin{cases} C||\phi||_0^\beta\Big(\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(s_i-t_i)}E_\alpha(-\lambda(\frac{s_i-t_i}{\rho})^\alpha)\Big)e^{\frac{\rho-1}{\rho}(t-t_k)}E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha)\Big)^\gamma, \\ \qquad t\in(t_k,s_k],\ k=0,1,\dots, \\ C||\phi||_0^\beta\Big(\prod_{i=0}^{k} e^{\frac{\rho-1}{\rho}(s_i-t_i)}E_\alpha(-\lambda(\frac{s_i-t_i}{\rho})^\alpha)\Big)^\gamma, \\ \qquad t\in(s_k,t_{k+1}],\ k=0,1,2,\dots \end{cases} \tag{11}$$

holds, where $x(t;t_0,\phi)$ *is a solution of the IVP for NIDDE* (6) *and* (7) *(with an arbitrary initial function* $\phi\in E$*).*

Remark 13. *The definition for generalized proportional Mittag–Leffler stability for NIDDE* (6) *and* (7) *depends significantly on the type of intervals—the intervals of differential equations and the intervals of non-instantaneous impulses (see, the first and the second line, respectively, in* (11)*).*

We will use the following class of Lyapunov-like functions (for more details, see the book [19]):

Definition 2. *Let* $a < b \le \infty$ *be given numbers ,* $\Omega \subset \mathbb{R}^n$*,* $0 \in \Omega$*. Then, the function* $V : [a-r,b]\times\Omega \to [0,\infty)$ *is from the class* $N\Lambda([a-r,b],\Omega)$ *if:*

- $V \in C([a,b]/\{s_k\}\times\Omega,[0,\infty))$ *and it is Lipschitz with respect to the second argument;*
- *For any* $s_k \in (a,b),\ x\in\Omega$*, there exist finite limits* $V(s_k-0,x) = \lim_{t\uparrow s_k}V(t,x)$ *and* $V(s_k+0,x) = \lim_{t\downarrow s_k}V(t,x)$*.*

We will consider the following scalar non-instantaneous impulsive differential equation (NIDE) as a comparison equation

$$\begin{aligned}
&({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}u)(t) = -\lambda u(t), \quad \text{for } t \in (t_k, s_k],\ k = 0,1,2,\dots,\\
&u(t) = \Xi_k(t, u(s_k - 0)) \text{ for } t \in (s_k, t_{k+1}],\quad k = 0,1,2,\dots,\\
&u(t_0) = u_0.
\end{aligned} \tag{12}$$

According to Lemma 4, the solution of the IVP for NIDE (12) is given by

$$u(t) = \begin{cases} u_0 e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\lambda(\frac{t-t_0}{\rho})^\alpha), & t \in [t_0, s_0] \\ \Xi_k(t, u(s_k - 0)), & t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots, \\ \Xi_{k-1}(t_k, u(s_{k-1} - 0)) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha), & t \in (t_k, s_k],\ k = 1,2,\dots. \end{cases}$$

Applying the scalar NIDE (12) as a comparison equation, we will obtain the following comparison result for NIDDE (6).

Lemma 6. *Suppose:*

1. *The function $x^*(t) = x(t; t_0, \phi) \in NPC^{\alpha,\rho}([t_0, \infty), \Delta)$ is a solution of the NIDDE (6) and (7), where $\Delta \subset \mathbb{R}^n$.*
2. *The functions $\Xi_k \in C([s_k, t_{k+1}] \times \mathbb{R}, \mathbb{R})$ and $\Xi_k(t, u) \leq u$ for $t \in [s_k, t_{k+1}]$, $u \geq 0$, $k = 0,1,2,\dots$.*
3. *The function $V \in N\Lambda([t_0 - r, \infty), \Delta)$ and*
   (i) for any $t \in (t_k, s_k]$ with $k = 0,1,\dots$ such that

$$\begin{aligned} &V(t, x^*(t)) \frac{e^{\frac{1-\rho}{\rho}(t-t_k)}}{E_\alpha\Big(-\lambda\Big(\frac{(t-t_k)}{\rho}\Big)^\alpha\Big)} \\ &\geq \sup_{s \in [t-r,t] \cap [t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\Big(-\lambda\Big(\frac{(s-t_k)}{\rho}\Big)^\alpha\Big)} V(s, x^*(s)) \end{aligned} \tag{13}$$

   the inequality

$${}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} V(t, x^*(t)) \leq -\lambda V(t, x^*(t))$$

   holds where $\lambda > 0$ is a given number.

   (ii) For any $k = 0,1,\dots$ the inequalities

$$V(t, \Phi_k(t, x^*(s_k - 0))) \leq \Xi_k(t, V(s_k - 0, x^*(s_k - 0))) \ \text{for } t \in (s_k, t_{k+1}].$$

   hold.

Then, the inequality

$$\begin{aligned} &V(t, x^*(t)) \\ &\leq \begin{cases} M\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-\lambda(\frac{s_i-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha), \\ \qquad t \in (t_k, s_k],\ k = 0,1,\dots, \\ M\Big(\prod_{i=0}^{k} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-\lambda(\frac{s_i-t_i}{\rho})^\alpha)\Big), \quad t \in (s_k, t_{k+1}], k = 0,1,2,\dots, \end{cases} \end{aligned} \tag{14}$$

holds where $M = \max_{s \in [-r,0]} V(t_0 + s, \phi(s))$.

Proof. *Case 1.* Let $t \in [t_0, s_0]$. Define the function $m(t) = V(t, x^*(t))$ for $t \in [t_0 - r, s_0]$. Then, the function $m(t) \in C^{\alpha,\rho}([t_0, s_0], \mathbb{R}_+)$ and the inequality $m(t_0) = V(t_0, \phi(0)) \leq \sup_{s\in[-r,0]} V(t_0 + s, \phi(s)) = M$ hold. We will prove that

$$m(t) < Me^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big) + \varepsilon e^{\frac{\rho-1}{\rho}(t-t_0)}, \quad t \in [t_0, s_0], \tag{15}$$

where $\varepsilon > 0$ is a small enough number. Note for $t = t_0$ inequality (15) holds. Assume (15) is not true on $(t_0, s_0]$. Therefore, there exists $t^* \in (t_0, s_0]$ such that

$$\begin{aligned} m(t) &< Me^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big) + \varepsilon e^{\frac{\rho-1}{\rho}(t-t_0)}, \ t \in [t_0, t^*), \\ m(t^*) &= e^{\frac{\rho-1}{\rho}(t^*-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big) + \varepsilon e^{\frac{\rho-1}{\rho}(t^*-t_0)}. \end{aligned} \tag{16}$$

Consider the function $\xi(t) = m(t) - Me^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big) - \varepsilon e^{\frac{\rho-1}{\rho}(t-t_0)}$ for $t \in [t_0, s_0]$. According to Lemma 3 with $u(t) \equiv \xi(t)$ the inequality $({}^{\ c}_{t_0}\mathcal{D}^{\alpha,\rho}\xi)(t)|_{t=t^*} > 0$ holds. Therefore, according to Lemma 5 and Remark 2, we obtain

$$({}^{\ c}_{t_0}\mathcal{D}^{\alpha,\rho} m)(t)|_{t=t^*} > -\lambda Me^{\frac{\rho-1}{\rho}(t^*-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big). \tag{17}$$

Case 1.1. Let $r < t^* - t_0$. Then, $t^* - r > t_0$ and $[t^* - r, t^*] \subset (t_0, t^*]$, i.e., $[t^* - r, t^*] \cap [t_0, t^*] = [t^* - r, t^*]$. Therefore, since the function $E_\alpha(-\lambda t)$ is decreasing for $t \in (t_0, t^*]$, i.e., $\frac{1}{E_\alpha\Big(-\lambda\Big(\frac{(t-t_0)}{\rho}\Big)^\alpha\Big)} \leq \frac{1}{E_\alpha\Big(-\lambda\Big(\frac{(t^*-t_0)}{\rho}\Big)^\alpha\Big)}$ for $t \in [t^* - r, t^*]$ by (16), we obtain

$$\begin{aligned} m(t)\frac{e^{\frac{1-\rho}{\rho}(t-t_0)}}{E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big)} &< M + \varepsilon\frac{1}{E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big)} \\ &\leq M + \varepsilon\frac{1}{E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big)} \\ &= m(t^*)\frac{e^{\frac{1-\rho}{\rho}(t^*-t_0)}}{E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big)}, \ t \in [t^* - r, t^*], \end{aligned} \tag{18}$$

i.e., inequality (13) is satisfied for $t = t^*$.

According to condition 3(i) the inequality

$$\begin{aligned} ({}^{\ c}_{t_0}\mathcal{D}^{\alpha,\rho} m)(t)|_{t=t^*} &= ({}^{\ c}_{t_0}\mathcal{D}^{\alpha,\rho} V(t, x^*(t)))|_{t=t^*} \leq -\lambda V(t^*, x^*(t^*)) \\ &= -\lambda m(t^*) = -\lambda Me^{\frac{\rho-1}{\rho}(t^*-t_0)} E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big) - \lambda\varepsilon e^{\frac{\rho-1}{\rho}(t^*-t_0)} \end{aligned} \tag{19}$$

holds.

From inequalities (17) and (19), it follows that $-\lambda\varepsilon e^{\frac{\rho-1}{\rho}(t^*-t_0)} > 0$. The obtained contradiction proves the inequality (15) on $[t_0, s_0]$.

Case 1.2. Let $r \geq t^* - t_0$. Then, $t^* - r \leq t_0$ and $[t^* - r, t^*] \cap [t_0, t^*] = [t_0, t^*] = \{t_0\} \cup (t_0, t^*]$. Similar to the proof in Case 1.1, we obtain the inequality

$$m(t)\frac{e^{\frac{1-\rho}{\rho}(t-t_0)}}{E_\alpha\Big(-\lambda\Big(\frac{t-t_0}{\rho}\Big)^\alpha\Big)} \leq m(t^*)\frac{e^{\frac{1-\rho}{\rho}(t^*-t_0)}}{E_\alpha\Big(-\lambda\Big(\frac{t^*-t_0}{\rho}\Big)^\alpha\Big)}, \ t \in (t_0, t^*].$$

For $t = t_0$, apply (16), $E_\alpha(0) = 1$ and obtain $m(t^*)\frac{e^{\frac{1-\rho}{\rho}(t^*-t_0)}}{E_\alpha\left(-\lambda\left(\frac{t^*-t_0}{\rho}\right)^\alpha\right)} > M \geq m(t_0)$.

Therefore, inequality (13) holds for $t = t^*$.

Thus, condition 3(i) is applicable and as in Case 1.1 we obtain a contradiction.

The contradiction proves inequality (15). From inequality (15) as $\varepsilon \to 0$ follows the validity of (14) on $[t_0, s_0]$.

Case 2. Let $t \in (s_0, t_1]$. Then, $x^*(t) = \Phi_1(t, x^*(s_0 - 0))$. From conditions 2, 3(ii) for $k = 0$ and Case 1, we obtain

$$\begin{aligned} V(t, x^*(t)) &= V(t, \Phi_0(t, x^*(s_0 - 0))) \leq \Xi_0(t, V(s_0 - 0, x^*(s_0 - 0))) \\ &\leq V(s_0 - 0, x^*(s_0 - 0)) \\ &\leq M e^{\frac{\rho-1}{\rho}(s_0 - t_0)} E_\alpha\left(-\lambda\left(\frac{s_0 - t_0}{\rho}\right)^\alpha\right), \quad t \in (s_0, t_1]. \end{aligned}$$

Therefore, inequality (14) holds on $(s_0, t_1]$.

Case 3. Let $t \in (t_1, s_2]$. Define the function

$$m_1(t) = \begin{cases} V(t_1, x^*(t_1)) & \text{for } t \in [t_1 - r, t_1], \\ V(t, x^*(t)) & \text{for } t \in (t_1, s_1]. \end{cases}$$

Then, the function $m_1(t) \in C^{\alpha,\rho}([t_1, s_1], \mathbb{R}_+)$. Denote $M_1 = V(t_1, x^*(t_1))$. Then,

$$\left(\max_{s\in[-r,0]} m_1(t_1 + s)\right) = M_1$$

and according to Case 2, the inequality

$$M_1 < M e^{\frac{\rho-1}{\rho}(s_0 - t_0)} E_\alpha\left(-\lambda\left(\frac{s_0 - t_0}{\rho}\right)^\alpha\right)$$

holds.

Similar to the proof of inequality (15) in Case 1, we have the validity of the inequality

$$m_1(t) < M_1 e^{\frac{\rho-1}{\rho}(t - t_1)} E_\alpha\left(-\lambda\left(\frac{t - t_1}{\rho}\right)^\alpha\right) + \varepsilon e^{\frac{\rho-1}{\rho}(t - t_1)}, \ t \in [t_1, s_1].$$

Thus,

$$\begin{aligned} m_1(t) &< M e^{\frac{\rho-1}{\rho}(s_0 - t_0)} E_\alpha\left(-\lambda\left(\frac{s_0 - t_0}{\rho}\right)^\alpha\right) e^{\frac{\rho-1}{\rho}(t - t_1)} E_\alpha\left(-\lambda\left(\frac{t - t_1}{\rho}\right)^\alpha\right) \\ &+ \varepsilon e^{\frac{\rho-1}{\rho}(t - t_1)}, \quad t \in (t_1, s_1]. \end{aligned} \tag{20}$$

Taking the limit in (20) as $\varepsilon \to 0$ we obtain the claim of Lemma 6 on $(t_1, s_1]$.

Continue this process and an induction argument proves the claim in Lemma 6. □

Remark 14. *The condition (13) is a modified Razumikhin condition applied in connection with generalized proportional fractional derivatives.*

Remark 15. *The inequality (13) in condition 3(i) of Lemma 6 could be replaced by*

$$V(t, x^*(t)) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s - t_k)}}{E_\alpha\left(-\lambda\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)} V(s, x^*(s)) \tag{21}$$

Note that if (21) holds, then inequality (13) is also satisfied.

Remark 16. *If the condition (21) is satisfied, then the classical Razumikhin condition* $V(t, x^*(t)) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} V(s, x^*(s))$ *holds.*

Remark 17. *The condition 3(i) is satisfied only at some particular points of t from the studied interval.*

We study the generalized Mittag–Leffler stability properties of the zero solution of NIDDE (6).

Theorem 1. *Suppose :*

1. *Conditions (A 1.1)–(A 1.3) are satisfied.*
2. *There exists a function* $V \in N\Lambda([t_0 - r, \infty), \mathbb{R}^n)$ *such that*
   - *(i)* *There exist positive constants* A, B, a, b *such that the inequalities* $A||x||^a \leq V(t, x) \leq B||x||^{ab}$, $t \geq t_0$, $x \in \mathbb{R}^n$ *hold.*
   - *(ii)* *For any point* $t \in (t_k, s_k]$ *with* $k = 0, 1, 2, \dots$ *and any function* $\psi \in C^{\alpha,\rho}(t_k, [t - r, t], \mathbb{R}^n)$ *such that* $\left({}^C_{t_k}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = f(t, \psi_t)$ *and*

$$\begin{aligned} &V(t,\psi(t))\frac{e^{\frac{1-\rho}{\rho}(t-t_k)}}{E_\alpha\left(-D\left(\frac{(t-t_k)}{\rho}\right)^\alpha\right)} \\ &\geq \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-D\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)} V(s,\psi(s)) \end{aligned} \tag{22}$$

   the inequality

$$\left({}^C_{t_k}\mathcal{D}^{\alpha,\rho}V(t,\psi(t))\right) \leq -DV(t,\psi(t)) \tag{23}$$

   holds where $D > 0$ *is a given number.*
   - *(iii)* *For any* $k = 0, 1, \dots$ *and* $u \in \mathbb{R}^n$, *the inequalities*

$$V(t, \Phi_k(t, u)) \leq C||u||^a \text{ for } t \in (s_k, t_{k+1}].$$

   hold where $C \in (0, A]$.

Then, the zero solution of NIDDE (6) with the zero initial function is generalized proportional Mittag–Leffler stable with $C = \sqrt[a]{\frac{B}{A}}, \beta = b,\ \lambda = D,\ \gamma = \frac{1}{a}$.

Proof. Let $\phi \in E$ be an arbitrary initial function and now let $x(t) = x(t; t_0, \phi) \in NPC^{\alpha,\rho}([t_0, \infty), \mathbb{R}^n)$ be the solution of the IVP for NIDDE (6) and (7). Let $t^* \in (t_k, s_k]$ with k a non-negative integer, be such that the inequality (22) holds with $\psi(t) = x(t)$. Note that $x \in C^{\alpha,\rho}(t_k, [t^* - r, t^*], \mathbb{R}^n)$ and $\left({}^C_{t_k}\mathcal{D}^{\alpha,\rho}x\right)(t)|_{t=t^*} = f(t^*, x_{t^*})$. Then, according to condition 2(ii) of Theorem 1, the inequality (23) holds, i.e., we have

$$\left({}^C_{t_k}\mathcal{D}^{\alpha,\rho}V(t, x(t))\right)|_{t=t^*} \leq -DV(t^*, x(t^*)),$$

i.e., the condition 3(i) of Lemma 6 is satisfied with $\lambda = D$.

Let $k = 0, 1, \dots$ be an arbitrary number. Then, from conditions 2(i) and 2(iii) of Theorem 1, we obtain $V(t, \Phi_k(t, x(s_k - 0))) \leq C||x(s_k - 0)||^a \leq \frac{C}{A}V(s_k - 0, x(s_k - 0))$, i.e., condition 3(ii) of Lemma 6 is satisfied with $\Xi_k(t, u) = \frac{C}{A}u \leq u$ according to the choice of the constants A, C.

According to Lemma 6, the inequality

$$V(t,x(t)) \leq \begin{cases} M\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-D(\frac{s_i-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-D(\frac{t-t_k}{\rho})^\alpha), \\ \quad t\in(t_k,s_k],\ k=0,1,\dots, \\ M\Big(\prod_{i=0}^{k} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-D(\frac{s_i-t_i}{\rho})^\alpha)\Big),\ \ t\in(s_k,t_{k+1}],\ k=0,1,2,\dots. \end{cases} \tag{24}$$

holds where $M \leq B||\phi||_0^{ab}$.

Thus, from condition 2(i) of Theorem 1, we obtain

$$||x(t)|| \leq \begin{cases} \sqrt[a]{\frac{B}{A}}||\phi||_0^b \Big(\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-D(\frac{s_i-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-D(\frac{t-t_k}{\rho})^\alpha)\Big)^{\frac{1}{a}}, \\ \quad t\in(t_k,s_k],\ k=0,1,\dots, \\ \sqrt[a]{\frac{B}{A}}||\phi||_0^b \Big(\prod_{i=0}^{k} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-D(\frac{s_i-t_i}{\rho})^\alpha)\Big)^{\frac{1}{a}}, \\ \quad t\in(s_k,t_{k+1}],\ k=0,1,2,\dots. \end{cases} \tag{25}$$

Thus, the zero solution of (6) is generalized Mittag-Leffler stable with $C = \sqrt[a]{\frac{B}{A}}, \beta = b,\ \lambda = D,\ \gamma = \frac{1}{a}$. □

Corollary 1. *Let the conditions of Theorem 1 be satisfied where the inequality (22) is replaced by*

$$V(t,\psi(t)) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\Big(-D\Big(\frac{(s-t_k)}{\rho}\Big)^\alpha\Big)} V(s,\psi(s)) \tag{26}$$

Then, the zero solution of NIDDE (6) with the zero initial function is generalized proportional Mittag–Leffler stable.

Proof. If the inequality (26) is satisfied for the point t, then we obtain

$$V(t,\psi(t)) \frac{e^{\frac{1-\rho}{\rho}(t-t_k)}}{E_\alpha\Big(-D\Big(\frac{(t-t_k)}{\rho}\Big)^\alpha\Big)} \geq V(t,\psi(t)),$$

i.e., inequality (22) is satisfied. □

Corollary 2. *Let the conditions of Theorem1 be satisfied where the condition 2(ii) is replaced by 2(ii)* for any point $t \in (t_k,s_k]$ with $k = 0,1,2,\dots$ and any function $\psi \in C^{\alpha,\rho}(t_k,[t-r,t],\mathbb{R}^n)$ such that $\Big({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi\Big)(t) = f(t,\psi_t)$ and*

$$V(t,\psi(t)) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\Big(-D\Big(\frac{(s-t_k)}{\rho}\Big)^\alpha\Big)} V(s,\psi(s)) \tag{27}$$

the inequality

$$\Big({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} V(t,\psi(t))\Big) \leq -D \sup_{s\in[t-r,t]\cap[t_k,t]} ||\psi(s)||^{ab} \tag{28}$$

holds where $D > 0$ is a given number.

Then, the zero solution of NIDDE (6) with the zero initial function is generalized proportional Mittag–Leffler stable.

Proof. From condition 2(iii) of Theorem 1 and inequality (27), we have that $||\psi(s)||^{ab} \geq V(s, \psi(s)),\ s \in [t-r, t] \cap [t_k, t]$, i.e.,

$$-D\left(\sup_{s\in[t-r,t]\cap[t_k,t]} ||\psi(s)||^{ab}\right) \leq -D\left(\sup_{s\in[t-r,t]\cap[t_k,t]} V(s, \psi(s))\right) = -DV(t, \psi(t)).$$

Thus, from inequality (28) we have inequality (23). □

Corollary 3. *Let the conditions of Theorem1 be satisfied where the inequality* (23) *is replaced by*

$$ {}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} V(t, \psi(t)) \leq 0, \tag{29}$$

and condition 2(i) is changed by

2(i) There exist positive constants A, B such that the inequalities $A||x|| \leq V(t, x) \leq B||x||,\ t \geq t_0,\ \ x \in \mathbb{R}^n$ hold.*

Then, the zero solution of NIDDE (6) with the zero initial function is stable.

Proof. Inequality (29) is a partial case of (23) with $D = 0$, then use $E_\alpha(0) = 1$ and inequality (25) and we obtain $||x(t)|| \leq \frac{B}{A}||\phi||_0$ for $t \geq t_0$, which proves the stability of the solution. □

Example 1. *. Consider the scalar IVP for NIDDE*

$$\begin{aligned}
&\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} x\right)(t) = -\frac{2+t}{t+1}(x(t) - 0.5x_t^{(k)}), \quad \text{for } t \in (t_k, s_k],\ k = 0, 1, 2, \ldots, \\
&x(t) = 0.5(\sin t)x(s_k - 0) \text{ for } t \in (s_k, t_{k+1}],\ k = 0, 1, 2, \ldots, \\
&x(t_0 + s) = \phi(s),\ s \in [-r, 0],
\end{aligned} \tag{30}$$

where for any $t \in (t_k, s_k]$ we denote $x_t^{(k)}(s) = x(t+s), s \in [max\{-r, t_k - t\}, 0]$.

The scalar IVP for NIDDE (30) with $\phi(s) \equiv 0$ has a zero solution.

Consider the Lyapunov function $V(t, x) = x^2$. Then, condition 2(i) of Theorem 1 is satisfied with $A = 0.25,\ B = 1, a = 2, b = 1$. Let k be a whole number and the point $t \in (t_k, s_k]$ and the function $\psi \in C^{\alpha,\rho}(t_k, [t-r, t], \mathbb{R})$ be such that $\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = -\frac{2+t}{t+1}(\psi(t) - 0.5\psi_t^{(k)})$ and

$$\psi^2(t) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)}\psi^2(s). \tag{31}$$

Then applying $\sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)}\psi^2(s) \geq \sup_{s\in[t-r,t]\cap[t_k,t]} \psi^2(s)$ we obtain

$$\begin{aligned}
&\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi^2\right)(t) \leq 2\psi(t)\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) \\
&= -2\frac{2+t}{t+1}(\psi^2(t) - 0.5\psi(t)\psi_t^{(k)}) \\
&\leq \frac{2+t}{t+1}(-2\psi^2(t) + 0.5\psi^2(t) + 0.5(\psi_t^{(k)})^2) \\
&\leq \frac{2+t}{t+1}(-2\psi^2(t) + 0.5\psi^2(t) + 0.5\sup_{s\in[t-r,t]\cap[t_k,t]} \psi^2(s)) \\
&\leq \frac{2+t}{t+1}(-1.5\psi^2(t) + 0.5\psi^2(t)) = -\frac{2+t}{t+1}\psi^2(t) \\
&< -V(t, \psi(t)).
\end{aligned} \tag{32}$$

Let $t \in (s_k, t_{k+1}]$ where $k = 0, 1, 2, \ldots$. Then, $(0.5\sin t\ u)^2 \leq 0.25u^2 = 0.25|u|^2$.

Therefore, the conditions of Corollary 1 are satisfied with $D = 1, C = A = 0.25, B = 1, a = 2, b = 1$. According to Corollary 1 the zero solution of the scalar NIDDE (30) is generalized proportional Mittag–Leffler stable with $C = \sqrt{4} = 2, \beta = 1,\ \lambda = 1,\ \gamma = 0.5$, i.e., the inequality

$$||x(t)|| \le \begin{cases} 2||\phi||_0 \sqrt{\Big(\prod_{i=0}^{k-1} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-(\frac{s_i-t_i}{\rho})^\alpha)\Big) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-(\frac{t-t_k}{\rho})^\alpha)}, \\ \qquad t \in (t_k, s_k],\ k = 0,1,\dots, \\ 2||\phi||_0 \sqrt{\prod_{i=0}^{k} e^{\frac{\rho-1}{\rho}(s_i-t_i)} E_\alpha(-(\frac{s_i-t_i}{\rho})^\alpha)}, \\ \qquad t \in (s_k, t_{k+1}],\ k = 0,1,2,\dots. \end{cases}$$

holds.

Remark 18. *The Mittag–Leffler type stability for the Caputo fractional differential equations (with $\rho = 1$) is studied in [25].*

4.2. Instantaneous Impulses

As mentioned in Remark 9, the case of non-instantaneous impulses could be considered as a generalization of the case of instantaneous impulses. That is why we can translate the results from the previous section to instantaneous impulses.

Definition 3. *The zero solution of the system IDDE (7) and (9) (with $\phi \equiv 0$) is said to be* generalized proportional Mittag–Leffler stable *if there exist constants $\beta, \gamma, C, \lambda > 0$ such that the inequality*

$$||x(t;t_0,\phi)|| \le C||\phi||_0^\beta \Big(e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha)\Big)^\gamma, \quad t \in (t_k, t_{k+1}],\ k = 0,1,\dots, \tag{33}$$

holds, where $x(t;t_0,\phi)$ is a solution on the IVP for IDDE (7) and (9) with an arbitrary initial function $\phi \in E$.

We will use some comparison results for IDDE (9) by applying piecewise continuous Lyapunov functions and we introduce a class of Lyapunov-like functions:

Definition 4. *Let $a < b \le \infty$ be given numbers , $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$. Then, the function $V : [a-r, b] \times \Omega \to [0, \infty)$ is from the class $P\Lambda([a-r, b], \Omega)$ if:*

- *$V \in C([a, b]/\{t_k\} \times \Omega, [0, \infty))$ and it is Lipschitz with respect to the second argument;*
- *For any $t_k \in (a, b)$, $x \in \Omega$, there exist finite limits $V(t_k - 0, x) = \lim_{t\uparrow t_k} V(t, x)$ and $V(t_k + 0, x) = \lim_{t\downarrow t_k} V(t, x)$.*

The comparison scalar equation (IDE) is

$$\begin{aligned} &({}^C_{t_k}\mathcal{D}^{\alpha,\rho}u)(t) = -\lambda u(t), \quad \text{for } t \in (t_k, t_{k+1}],\ k = 0,1,2,\dots, \\ &u(t) = \Xi_k(u(t_k - 0)) \text{ for } k = 1,2,\dots, \\ &u(t_0) = u_0. \end{aligned} \tag{34}$$

According to Lemma 4, the solution of the IVP for IDE (34) is given by

$$u(t) = \begin{cases} u_0 e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\lambda(\frac{t-t_0}{\rho})^\alpha) & t \in [t_0, t_1] \\ \Xi_k(u(t_k-0)) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha) & t \in (t_k, t_{k+1}],\ k = 1,2,\dots. \end{cases}$$

The auxiliary Lemma, corresponding to Lemma 6, reduces to

Lemma 7. *Suppose:*

1. *The function $x^*(t) = x(t; t_0, \phi) \in PC^{\alpha,\rho}([t_0, \infty), \Delta)$ is a solution of the IDDE (7) and (9) where $\Delta \subset \mathbb{R}^n$.*
2. *The functions $\Xi_k \in C(\mathbb{R}, \mathbb{R})$ and $\Xi_k(u) \le u$ for $u \ge 0$, $k = 1, 2, \ldots$.*
3. *The function $V \in P\Lambda([t_0 - r, \infty), \Delta)$ and*
   (i) *For any $t \in (t_k, t_{k+1}]$ with $k = 0, 1, \ldots,$ such that*

$$\begin{aligned} V(t, x^*(t)) & \frac{e^{\frac{1-\rho}{\rho}(t-t_k)}}{E_\alpha\left(-\lambda\left(\frac{(t-t_k)}{\rho}\right)^\alpha\right)} \\ \ge & \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-\lambda\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)} V(s, x^*(s)) \end{aligned} \tag{35}$$

   the inequality

$${}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} V(t, x^*(t)) \le -\lambda V(t, x^*(t))$$

   holds where $\lambda > 0$ is a given number.

   (ii) *For any $k = 1, \ldots,$ the inequalities*

$$V(t_k - 0, \Psi_k(x^*(t_k - 0))) \le \Xi_k(V(t_k - 0, x^*(t_k - 0))),$$

   hold.

   Then, the inequality

$$V(t, x^*(t)) \le \left(\max_{s\in[-r,0]} V(t_0 + s, \phi(s))\right) e^{\frac{\rho-1}{\rho}(t-t_k)} E_\alpha(-\lambda(\frac{t-t_k}{\rho})^\alpha), \tag{36}$$

$t \in (t_k, t_{k+1}]$, $k = 0, 1, \ldots,$

holds.

Remark 19. *The comparison scalar Equation (34) is chosen such that its explicit solution is known and condition 3(i) will be satisfied for the Lyapunov function.*

Theorem 2. *Suppose:*

1. *Conditions (A 2.1)–(A 2.3) are satisfied.*
2. *There exists a function $V \in P\Lambda([t_0 - r, \infty), \mathbb{R}^n)$ such that*
   (i) *There exist positive constants A, B, a, b such that the inequalities $A||x||^a \le V(t, x) \le B||x||^{ab}$, $t \ge t_0$, $x \in \mathbb{R}^n$ hold.*
   (ii) *For any point $t \in (t_k, t_{k+1}]$ with $k = 0, 1, 2, \ldots$ and any function $\psi \in C^{\alpha,\rho}(t_k, [t - r, t], \mathbb{R}^n)$ such that $\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = f(t, \psi_t)$ and*

$$\begin{aligned} V(t, \psi(t)) & \frac{e^{\frac{1-\rho}{\rho}(t-t_k)}}{E_\alpha\left(-D\left(\frac{(t-t_k)}{\rho}\right)^\alpha\right)} \\ \ge & \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-D\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)} V(s, \psi(s)) \end{aligned} \tag{37}$$

the inequality

$$ {}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} V(t,\psi(t)) \le -DV(t,\psi(t))r \tag{38} $$

holds where $D > 0$ is a given number.

(iii) *For any $k = 1, 2, \dots$ and $u \in \mathbb{R}^n$ the inequalities*

$$ V(t, \Psi_k(u)) \le C||u||^a \text{ for } t \in (t_k, t_{k+1}]. $$

hold where $C \in (0, A]$.

Then, the zero solution of IDDE (9) with the zero initial function is generalized proportional Mittag–Leffler stable with $C = \sqrt[a]{\frac{B}{A}}, \beta = b,\ \lambda = D,\ \gamma = \frac{1}{a}$.

Now we will provide an example illustrating the application of the given above sufficient conditions. To be able to compare both cases about non-instantaneous impulses and instantaneous impulses we will consider the scalar IVP for NIDDE (30) with appropriate changes.

Example 2. . *Consider the scalar IVP for IDDE*

$$ \begin{aligned} &\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho} x\right)(t) = -\frac{2+t}{t+1}(x(t) - 0.5x_t^{(k)}) \quad \text{for } t \in (t_k, t_{k+1}],\ k = 0,1,2,\dots, \\ &x(t_k + 0) = 0.5(\sin t_k)x(t_k - 0) \quad \text{for } k = 1,2,\dots, \\ &x(t_0 + s) = \phi(s),\ s \in [-r, 0]. \end{aligned} \tag{39} $$

The scalar IVP for IDDE (39) with $\phi(s) \equiv 0$ has a zero solution.

Let $V(t,x) = x^2$. Thus, the condition 2(i) of Theorem 2 is satisfied with $A = 0.25, B = 1, a = 2, b = 1$.

Let k be a given natural number and $t \in (t_k, t_k + 1)$, and the function $\psi \in C^{\alpha,\rho}(t_k, [t - r, t], \mathbb{R})$ be such that

$$ \left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = -\frac{2+t}{t+1}(\psi(t) - 0.5\psi_t^{(k)}) $$

and

$$ \psi^2(t) \ge \sup_{s\in[t-r,t]\cap[t_k,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-\left(\frac{(s-t_k)}{\rho}\right)^\alpha\right)}\psi^2(s). $$

Then, we obtain $\left({}_{t_k}^{C}\mathcal{D}^{\alpha,\rho}\psi^2\right)(t) < -V(t,\psi(t))$ (see (32)), i.e., condition 2(ii) of Theorem 2 is satisfied with $D = 1$.

For any $k = 1, 2, \dots$ we obtain $(0.5 \sin t_k\, u)^2 \le 0.25u^2 = 0.25|u|^2$, i.e., the condition 2(iii) of Theorem 2 is satisfied with $C = 0.25$.

According to Theorem 2, the zero solution of the scalar IDDE (39) is a generalized proportional Mittag–Leffler stable with $C = 2, \beta = 1,\ \lambda = 1,\ \gamma = 0.5$, i.e., the inequality

$$ ||x(t;t_0,\phi)|| \le 2||\phi||_0\sqrt{e^{\frac{\rho-1}{\rho}(t-t_i)}E_\alpha(-(\frac{t-t_k}{\rho})^\alpha)},\ t \in (t_k, t_{k+1}],\ k = 0,1,\dots $$

holds (compare with the special case $t_{k+1} = s_k,\ k = 0,1,2,\dots$ of Example 1) .

4.3. No Impulses

As mentioned in Remark 11 the case of instantaneous impulses could be considered as a generalization of the case of no impulses, i.e., the system (10) could be considered as a partial case of (9) with $t_i = t_0,\ i = 1,2,\dots$. That is why we can translate the results from the previous section to the case without impulses.

Definition 5. *The zero solution of the system DDE (10) (with $\phi \equiv 0$) is said to be* generalized proportional Mittag-Leffler stable *if there exist constants $\beta, \gamma, C, \lambda > 0$ such that the inequality*

$$||x(t; t_0, \phi)|| \leq C||\phi||_0^\beta \left(e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\lambda(\frac{t-t_0}{\rho})^\alpha) \right)^\gamma, \ t \geq t_0, \tag{40}$$

holds, where $x(t; t_0, \phi)$ is a solution on the IVP for DDE (7) and (10).

Remark 20. *In the case $\rho = 1$, Definition 5 is the same as in [26].*

We will use some comparison results for DDE (10) by applying Lyapunov functions:

Definition 6. *Let $a < b \leq \infty$ be given numbers , $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$. Then, the function $V : [a-r, b] \times \Omega \to [0, \infty)$ is from the class $\Lambda([a-r, b], \Omega)$ if $V \in C([a, b]/\{t_k\} \times \Omega, [0, \infty))$ and it is Lipschitz with respect to the second argument.*

The comparison scalar equation (DE) is

$$\begin{aligned} &({}_{t_0}^C\mathcal{D}^{\alpha,\rho}u)(t) = -\lambda u(t), \quad \text{for } t > t_0, \\ &u(t_0) = u_0. \end{aligned} \tag{41}$$

According to Lemma 4, the solution of the IVP for DE (41) is given by $u(t) = u_0 e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\lambda(\frac{t-t_0}{\rho})^\alpha)$. $t \geq t_0$.

The auxiliary Lemma, corresponding to Lemma 6 reduces to

Lemma 8. *Suppose:*

1. *The function $x^*(t) = x(t; t_0, \phi) \in C^{\alpha,\rho}([t_0, \infty), \Delta)$ is a solution of the DDE (7) and (10), where $\Delta \subset \mathbb{R}^n$.*
2. *The function $V \in C\Lambda([t_0 - r, \infty), \Delta)$ and for any point $t > t_0$ such that*

$$\begin{aligned} &V(t, x^*(t)) \frac{e^{\frac{1-\rho}{\rho}(t-t_0)}}{E_\alpha\left(-\lambda\left(\frac{(t-t_0)}{\rho}\right)^\alpha\right)} \\ &\geq \sup_{s \in [t-r,t] \cap [t_0,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_0)}}{E_\alpha\left(-\lambda\left(\frac{(s-t_0)}{\rho}\right)^\alpha\right)} V(s, x^*(s)) \end{aligned} \tag{42}$$

the inequality

$$\left({}_{t_0}^C\mathcal{D}^{\alpha,\rho} V(t, x^*(t))\right) \leq -\lambda V(t, x^*(t))$$

holds where $\lambda > 0$ is a given number.

Then, the inequality

$$V(t, x^*(t)) \leq \max_{s \in [-r,0]} V(t_0 + s, \phi(s)) e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-\lambda(\frac{t-t_0}{\rho})^\alpha), \ \ t > t_0$$

holds.

Theorem 3. *Suppose:*

1. *Conditions (A 3.1), (A 3.2) are satisfied.*
2. *There exists a function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ such that*
   (i) *There exist positive constants A, B, a, b such that $C \leq A$ and the inequalities $A||x||^a \leq V(t, x) \leq B||x||^{ab}$, $t \geq t_0$, $x \in \mathbb{R}^n$ hold.*

(ii) *For any point $t > t_0$ and any function $\psi \in C^{\alpha,\rho}(t_0, [t-r, t], \mathbb{R}^n)$ such that $\left({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = f(t, \psi_t)$ and*

$$V(t, \psi(t)) \frac{e^{\frac{1-\rho}{\rho}(t-t_0)}}{E_\alpha\left(-\lambda\left(\frac{(t-t_0)}{\rho}\right)^\alpha\right)} \geq \sup_{s\in[t-r,t]\cap[t_0,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_k)}}{E_\alpha\left(-\lambda\left(\frac{(s-t_0)}{\rho}\right)^\alpha\right)} V(s, \psi(s)) \quad (43)$$

the inequality

$${}_{t_0}^{C}\mathcal{D}^{\alpha,\rho} V(t, \psi(t)) \leq -DV(t,\ \psi(t)) \quad (44)$$

holds where $D > 0$ is a given number.

Then, the zero solution of DDE (10) with the zero initial function is generalized proportional Mittag–Leffler stable with constants $C = \sqrt[a]{\frac{B}{A}}, \beta = b,\ \lambda = D,\ \gamma = \frac{1}{a}$.

Example 3. *Consider the scalar IVP for DDE*

$$\begin{aligned} &\left({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho} x\right)(t) = -\frac{2+t}{t+1}(x(t) - 0.5 \sup_{s\in[-r,0]} x(t+s)),\ \ t > t_0, \\ &x(t_0 + s) = \phi(s),\ s \in [-r, 0]. \end{aligned} \quad (45)$$

The scalar IVP for DDE (45) with $\phi(s) \equiv 0$ has a zero solution.

Let $V(t, x) = x^2$. Thus, the condition 2(i) of Theorem 3 is satisfied with $A = 0.25, B = 1, a = 2, b = 1$.

Let $t > t_0$ and the function $\psi \in C^{\alpha,\rho}(t_0, [t-r, t], \mathbb{R})$ be such that $\left({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}\psi\right)(t) = -\frac{2+t}{t+1}(\psi(t) - 0.5 \sup_{s\in[-r,0]} \psi(t+s)$ and $\psi^2(t) \geq \sup_{s\in[t-r,t]\cap[t_0,t]} \frac{e^{\frac{1-\rho}{\rho}(s-t_0)}}{E_\alpha\left(-\left(\frac{(s-t_0)}{\rho}\right)^\alpha\right)} \psi^2(s)$. Then, we obtain

$$\left({}_{t_0}^{C}\mathcal{D}^{\alpha,\rho}\psi^2\right)(t) < -V(t, \psi(t))$$

(see (32)), i.e., condition 2(ii) of Theorem 3 is satisfied with $D = 1$.

According to Theorem 3, the zero solution of the scalar DDE (45) is generalized proportional Mittag–Leffler stable with $C = 2, \beta = 1,\ \lambda = 1,\ \gamma = 0$, i.e., the inequality

$$||x(t; t_0, \phi)|| \leq 2||\phi||_0 \sqrt{e^{\frac{\rho-1}{\rho}(t-t_0)} E_\alpha(-(\frac{t-t_0}{\rho})^\alpha)},\ \ t \geq t_0.$$

holds (compare with the special case of $t_0 = t_k,\ k = 1, 2, \ldots$ of Example 2).

5. Conclusions

In this paper, a system of nonlinear differential equations with finite delay and with a generalized proportional Caputo fractional derivative is studied. The basic cases are presented: the case when there are non-instantaneous impulses in the equations, the case when there are instantaneous impulses in the equations, and the case without any impulses in all equations. The appropriate initial value problem is set up in all these cases, and the relation between them is discussed. It is shown that the case of non-instantaneous impulses is a generalization of the case of instantaneous impulses, and the case of instantaneous impulses could be considered as a generalization of the case without any impulses. These statements could be applied to study various qualitative properties of the solutions. In this paper, based on the application of Lyapunov functions and an appropriate modification of the Razumikhin method, the Mittag–Leffler type stability is investigated.

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