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Controlling Problem within a Class of Two-Level Positive Maps

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Abstract: This paper aims to define the set of unital positive maps on $\mathbb{M}_2(\mathbb{C})$ by means of quantum Lotka–Volterra operators which are quantum analogues of the classical Lotka–Volterra operators. Furthermore, a quantum control problem within the class of quantum Lotka–Volterra operators are studied. The proposed approach will lead to the understanding of the behavior of the classical Lotka–Volterra systems within a quantum framework.

Keywords: quantum quadratic operator; control; quantum Lotka-Volterra operator

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1. Introduction

The present paper is closely related to the problem of controlling a two-level quantum system [1,2]. Let us consider a system for which the influence of the environment does not affect it [3,4]. Then, its dynamics under the action of the control $f(t)$ is governed by

$$iU_t^f = (H_0 + f(t)V)U_t^f, \quad U_{t=0}^f = I,$$

where $f \in L^1([0,T];\mathbb{R})$ and H_0, V are Hermitian matrices in $\mathbb{M}_2(\mathbb{C})$. In many physical systems, it appears several problems of maximizing of an objective functional of the form

$$J[f] = \mathrm{Tr}(\rho_T A) \tag{1}$$

which presents the quantum average of an observable A at a fixed time $T > 0$. Here $\rho_T = U_T\rho_0 U_T^*$, where ρ_0 is the initial density matrix. By defining a mapping $\Phi_T(\rho_0) = U_T\rho_0 U_T^*$, then (1) can be rewritten as follows

$$J[f] = \mathrm{Tr}(\Phi_T(\rho_0)A). \tag{2}$$

Notice that the potential of unitary control Φ_T to find extremum values of the target operator are limited, since such operators can only connect states with the same spectrum (see, for example, [5,6]). Therefore, the dynamics may be extended to non-unitary evolution by involving the set of unital positive maps. Afterwards, a more general problem can be observed: assume that a set of unital positive maps from $\mathbb{M}_2(\mathbb{C})$ to itself is given, say Σ. Consider the objective functional:

$$J[\Phi] = \mathrm{Tr}(\Phi(\rho_0)A). \tag{3}$$

The control goal is to find, for given ρ_0 and A, optimal map Φ in Σ which maximize the objective functional J. The formulated problem is a common goal in quantum control [7,8]. In [9,10], the most general physically allowed transformations of states of quantum open systems are investigated where Σ is taken as the set of all completely positive trace preserving maps. General mathematical definitions for the controlled Markov dynamics can be found in [11].

In the present paper, the set Σ is considered consisting of unital positive maps of $\mathbb{M}_2(\mathbb{C})$ associated with quantum Lotka–Volterra operators. Such types of maps have been introduced in [12] as a quantum analogue of the classical Lotka–Volterra operators [13]. Notice that set of positive maps (defined on some matrix algebra) has certain applications in quantum information theory [14–17] and entanglement witnesses [18–20].

In this paper, we define a class of quantum Lotka–Volterra operators which contains as a particular case those that were studied in [12]. We point out that construction of such types of operators are highly non-trivial, since they map $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, and checking their positivity condition is tricky. By considering conditional expectations (depending on states) from $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{C})$, and using the quantum Lotka–Volterra operators, a family of unital positive maps is introduced which depends on several parameters. If the state is taken as a trace, then the family is reduced to earlier studied maps in [12]. However, the presence of non-trivial states in the expectation makes the family very complicated for checking its positivity. For this family of positive maps, in the last section, a quantum control problem is explored. Although the investigated problem does not have physical application, the proposed approach will lead to the understanding of behavior of the classical Lotka–Volterra systems within a quantum framework.

2. Preliminaries

This section is devoted to recalling necessary definitions which will be used later on.

An algebra of 2×2 matrices over the complex field $\mathbb{C}$ is denoted as $\mathbb{M}_2(\mathbb{C})$. Furthermore, $\mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})$ denotes the tensor product of $\mathbb{M}_2(\mathbb{C})$ into itself. The symbol $\mathbf{1}$ stands for an identity matrix. In the sequel, by $\mathbb{M}_2(\mathbb{C})^{\dagger}_{+}$ we denote the set of all positive functionals defined on $\mathbb{M}_2(\mathbb{C})$. The set of all states (i.e., linear positive functionals which take value 1 at $\mathbf{1}$) defined on $\mathbb{M}_2(\mathbb{C})$ is denoted by $S(\mathbb{M}_2(\mathbb{C}))$.

It is well known that the identity $\mathbf{1}$ and Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$ form a basis for $\mathbb{M}_2(\mathbb{C})$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Therefore, any $x \in \mathbb{M}_2(\mathbb{C})$ can be written as $x = w_0\mathbf{1} + \mathbf{w}œ$ with $w_0 \in \mathbb{C}$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, where $\mathbf{w}\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3$.

By $D\mathbb{M}_2(\mathbb{C})$, we denote a commutative subalgebra of $\mathbb{M}_2(\mathbb{C})$ generated by $\{\mathbf{1}, \sigma_3\}$. In this setting, every element $x \in D\mathbb{M}_2(\mathbb{C})$ can be written as follows: $x = \omega_0\mathbf{1} + \omega_3\sigma_3$, where $\omega_0, \omega_3 \in \mathbb{C}$.

Lemma 1 ([21])**.** *Let $x \in \mathbb{M}_2(\mathbb{C})$. Then the following assertions hold:*

(a) *x is self-adjoint iff $w_0, \mathbf{w}$ are real;*

(b) *$x \geq 0$ iff $\|\mathbf{w}\| \leq w_0$, where $\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}$;*

(c) *A linear functional φ on $\mathbb{M}_2(\mathbb{C})$ is a state iff it can be represented by*

$$\varphi(w_0\mathbf{1} + \mathbf{w}\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f} \rangle, \tag{4}$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ such that $\|\mathbf{f}\| \leq 1$. Here as before $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{C}^3$.

Notice that a basis of $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ is formed by the system

$$\{\mathbf{1} \otimes \mathbf{1}, \sigma_i \otimes \mathbf{1}, \mathbf{1} \otimes \sigma_i, \sigma_i \otimes \sigma_j\}_{i,j=1}^{3}.$$

A linear operator $U : \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})$ such that $U(xy) = yx$ for all $x, y \in \mathbb{M}_2(\mathbb{C})$ is called a *flipped* operator.

Definition 1 ([22])**.** *A linear mapping $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})$ is said to be*

(a) *A quasi quantum quadratic operator (quasi q.q.o) if it is unital (i.e., $\Delta\mathbf{1} = \mathbf{1}\mathbf{1}$), *-preserving (i.e., $\Delta(x^*) = \Delta(x)^*$, $\forall x \in \mathbb{M}_2(\mathbb{C})$) and*

$$V_\Delta(\varphi) := \Delta^*(\varphi\varphi) \in \mathbb{M}_2(\mathbb{C})^\dagger_+ \ \text{ whenever } \varphi \in \mathbb{M}_2(\mathbb{C})^\dagger_+;$$

(b) *A quantum quadratic operator (q.q.o.) if it is unital and positive (i.e., $\Delta x \geq 0$ whenever $x \geq 0$);*

(c) *Symmetric if one has $U\Delta = \Delta$.*

It is evident that if Δ is q.q.o., then it is a quasi q.q.o. Moreover, the unitality of Δ implies any quasi q.q.o. V_Δ maps $S(\mathbb{M}_2(\mathbb{C}))$ into itself.

Remark 1. *We notice that symmetric q.q.o.s have been studied in* [23]*, which were called quantum quadratic stochastic operators. We refer the reader to* [24] *for recent reviews on quadratic operators.*

We mention that quasi quadratic quantum operators have been studied in [25]. In this regard, there is a natural question: for what sort of operators do the quasiness and the positivity coincide? This question is related to providing simpler examples of block-positive operators which have potential applications in detection of entangled witness [26].

Any unital linear map $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ can be represented as follows:

$$\Delta(x) = (w_0 + \langle \mathbf{b}, \overline{\mathbf{w}} \rangle)\mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{B}^{(1)}\mathbf{w} \cdot \sigma + \mathbf{B}^{(2)}\mathbf{w} \cdot \sigma \otimes \mathbf{1} + \sum_{m,l=1}^{3} \langle \mathbf{b}_{ml}, \overline{\mathbf{w}} \rangle \sigma_m \otimes \sigma_l, \quad (5)$$

where $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{b}_{ml} = (b_{ml,1}, b_{ml,2}, b_{ml,3})$, and $\mathbf{B}^{(k)} = (b^{(k)}_{ij})^3_{i,j=1}$, $k = 1, 2$ are real for every $i, j, k \in \{1, 2, 3\}$. Here as before $\langle \cdot, \cdot \rangle$ stands for the standard dot product in $\mathbb{C}^3$.

3. Quantum Lotka–Volterra Operators on $\mathbb{M}_2(\mathbb{C})$

In this section, we define a quantum analogue of Lotka–Volterra operators on $\mathbb{M}_2(\mathbb{C})$. Recall that the Lotka–Volterra operator on $\mathbb{M}_2(\mathbb{C})$ is defined as follows [12,27]:

$$\Delta_a(w_0\mathbf{1} + \mathbf{w}\sigma) = \omega_0\mathbf{1} \otimes \mathbf{1} + \frac{1}{2}\omega_3(\mathbf{1} \otimes \sigma_3 + \sigma_3 \otimes \mathbf{1}) + \frac{a}{2}\omega_3(\mathbf{1} \otimes \mathbf{1} - \sigma_3 \otimes \sigma_3). \quad (6)$$

where $|a| \leq 1$. One can see that Δ_a maps $\mathbb{M}_2(\mathbb{C})$ to $D\mathbb{M}_2(\mathbb{C}) \otimes D\mathbb{M}_2(\mathbb{C})$.

By $\tilde{\mathcal{E}} : \mathbb{M}_2(\mathbb{C}) \to D\mathbb{M}_2(\mathbb{C})$, we denote the standard projection defined by

$$\tilde{\mathcal{E}}(w_0\mathbf{1} + \mathbf{w}\sigma) = w_0\mathbf{1} + \omega_3\sigma_3.$$

Denote $\mathcal{E} = \tilde{\mathcal{E}}\tilde{\mathcal{E}}$.

Definition 2. *A symmetric q.q.o. $\Delta : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})$ is called Quantum Lotka–Volterra operator, if*

$$\mathcal{E} \circ \Delta = \Delta_a, \quad (7)$$

for some $a \in [-1, 1]$.

In what follows, we will need the following auxiliary fact.

Lemma 2. *Consider the function $f(x) = ax + b\sqrt{1 - x^2}$ where $x \in [-1, 1]$, $a, b > 0$ Then*

1. *The minimum value of f is $-a$;*

2. *The maximum value of f is $\sqrt{a^2+b^2}$.*

Lemma 3 ([12])**.** *Let $f(x) = ax^2 + bx + c$. Then the following conditions are equivalent*

(i) $f(x) \geq 0$ *for all* $x \in [0,1]$*;*

(ii) $c \geq 0$, $a+b+c \geq 0$ *and one of the following conditions is satisfied:*

 I. $a > 0$,
  (a) $b > 0$;
  (b) $-b > 2a$;
  (c) $b^2 - 4ac \leq 0$;

 II. $a < 0$.

The next theorem is the main result of this section.

Theorem 1. *Let $\Delta_{\lambda,\mu,\gamma,a} : \mathbb{M}_2(\mathbb{C}) \to \mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})$ be given as follows:*

$$\begin{aligned}\Delta_{\lambda,\mu,\gamma,a}(w_0\mathbf{1} + \mathbf{w}\sigma) =&(w_0 + \frac{a}{2}\omega_3)\mathbf{1}\otimes\mathbf{1} + \lambda\omega_1(\sigma_1\mathbf{1} + \mathbf{1}\otimes\sigma_1) + \mu\omega_2(\sigma_2\mathbf{1} + \mathbf{1}\otimes\sigma_2)\\ &+ \frac{\omega_3}{2}(\sigma_3\mathbf{1} + \mathbf{1}\otimes\sigma_3) - \frac{a}{2}\omega_3(\sigma_3\sigma_3) + \gamma\omega_1(\sigma_1\,\sigma_1),\end{aligned} \tag{8}$$

where $\lambda, \mu, \gamma \in \mathbb{R}$ and $a \in [-1,1]$. Then the following statements hold true:

(i) $\Delta_{\lambda,\mu,\gamma,a}$ *is a quantum Lotka–Volterra operator if*

$$|\gamma| \leq \sqrt{1-a^2}, \quad \max\{\lambda^2, \mu^2\} \leq \frac{(1-|a|) - \gamma^2(1+\sqrt{a^2+\gamma^2})}{4(1+|\gamma|)}. \tag{9}$$

(ii) $\Delta_{\lambda,\mu,\gamma,a}$ *is a quasi quantum quadratic operator if*

$$M \leq \frac{1-|a| - (4\gamma + \gamma^2)}{4}. \tag{10}$$

Proof.

(i) Let $x \in \mathbb{M}_2(\mathbb{C})$, $x \geq 0$, i.e., $x = w_0\mathbf{1} + \mathbf{w}\sigma$. Without loss of generality we may assume that $w_0 = 1$. The positivity of x implies $\|\mathbf{w}\| \leq 1$. From (8), one finds

$$\Delta(x) = \begin{bmatrix} 1+w_3 & y & y & \gamma w_1 \\ \overline{y} & aw_3+1 & \gamma w_1 & y \\ \overline{y} & \gamma w_1 & aw_3+1 & y \\ \gamma w_1 & \overline{y} & \overline{y} & 1-w_3 \end{bmatrix},$$

where $y = \lambda w_1 - i\mu w_2$.

To check the positivity of the above matrix, we use the Silvester criterion, i.e., $\Delta_k \geq 0$, $k \in \{1,2,3,4\}$, where

$$\begin{aligned}\Delta_1 &= 1 + w_3\\ \Delta_2 &= (1+w_3)(1+aw_3) - |y|^2\\ \Delta_3 &= (1+aw_3 - \gamma w_1)((1+w_3)(1+aw_3+\gamma w_1) - 2|y|^2)\\ \Delta_4 &= (1+aw_3-\gamma w_1)((1+aw_3+\gamma w_1)(1-w_3^2-\gamma^2 w_1^2) - 4|y|^2 + 2\gamma w_1(y^2+\overline{y}^2)).\end{aligned}$$

Clearly, $1 + w_3 \geq 0$.

On the other hand, we can compute that $1 + aw_3 - \gamma w_1$ is an eigenvalue of $\Delta(x)$. Therefore, $1 + aw_3 - \gamma w_1$ should be non-negative, i.e.,

$$\gamma w_1 - aw_3 \leq 1. \tag{11}$$

Using Lemma 2, we infer that the maximum value of the left hand side of (11) is $\sqrt{a^2 + \gamma^2}$. So,

$$|\gamma| \leq \sqrt{1 - a^2}.$$

Now, let us consider Δ_2, then the positivity is satisfied if and only if $(1 + w_3)(1 + aw_3) \geq |y|^2$. This holds if

$$(1 + w_3)(1 + aw_3) \geq M(1 - w_3)(1 + w_3),$$

where $M = \max\{\lambda^2, \mu^2\}$, then $(a + M)w_3 \geq M - 1$. If $a \geq 0$, then the left hand side of the last inequality has its minimum value at $w_3 = -1$. Using the same argument for the case $a < 0$, we arrive at

$$M \leq \frac{1 - |a|}{2}.$$

Now, let us check the positivity of Δ_3. Keeping in mind $1 + aw_3 - \gamma w_1 \geq 0$, the positivity of Δ_3 is satisfied if $(1 + w_3)(1 + aw_3 + \gamma w_1) \geq 2|y|^2$ which is equivalent to

$$(a + 2M)w_3 + \gamma w_1 \geq 2M - 1.$$

If $a \geq 0$, by Lemma 2 the minimum value of the left hand side of the last inequality is $-(a + 2M)$. Hence, $a + 2M \leq 1 - 2M$. Therefore,

$$M \leq \frac{1 - |a|}{4}. \tag{12}$$

Finally, we have to check the positivity of Δ_4, i.e., we need to show that

$$(1 - w_3^2 - \gamma^2 w_1^2)(1 + aw_3 + \gamma w_1) \geq 4M(1 - w_3^2)(1 + |\gamma w_1|).$$

Rewriting the last inequality, one has

$$(1 - w_3^2)(1 + aw_3 + \gamma w_1 - 4M(1 + |\gamma w_1|)) \geq \gamma^2 w_1^2 (1 + aw_3 + \gamma w_1). \tag{13}$$

By Lemma 2, we infer that $\max(aw_3 + \gamma w_1) = \sqrt{a^2 + \gamma^2}$, $\min(aw_3 + \gamma w_1) = -|a|$. Hence, from (13) it follows that

$$(1 - |a| - 4M(1 + |\gamma w_1|) \geq \frac{\gamma^2 w_1^2}{1 - w_3^2}(1 + \sqrt{a^2 + \gamma^2}). \tag{14}$$

Define

$$f(w_1, w_3) := \frac{w_1^2}{1 - w_3^2}$$

over the region $w_1^2 + w_3^2 \leq 1$. It is clear that the critical point is $(0, w_3)$. Thus, the maximum value will be at the boundary, i.e., $w_1^2 = 1 - w_3^2$. Hence, the maximum value of $f(w_1, w_3)$ is 1. Therefore,

$$(1 - |a| - 4M(1 + |\gamma w_1|) \geq \gamma^2(1 + \sqrt{a^2 + \gamma^2}). \tag{15}$$

Due to $|w_1| \leq 1$, one has

$$M \leq \frac{(1 - |a|) - \gamma^2(1 + \sqrt{a^2 + \gamma^2})}{4(1 + |\gamma|)}. \tag{16}$$

By
$$\frac{(1-|a|)-\gamma^2(1+\sqrt{a^2+\gamma^2})}{4(1+|\gamma|)}\leq\frac{1-|a|}{4}$$
one obtains the positivity of Δ_4, which implies the positivity of Δ_3 as well.

(ii) From (8), for every state φ (which corresponds to the vector $\mathbf{f}=(f_1,f_2,f_3)\in\mathbb{R}^3$), one finds
$$(V_{\Delta_{\lambda,\mu,\gamma,a}}(\varphi))(x)=\omega_0+(2\lambda f_1+\gamma f_1^2)\omega_1+2\mu f_2\omega_2+\big(f_3+\frac{a}{2}(1-f_3^2)\big)\omega_3.$$
Hence, the quasiness condition for $\Delta_{\lambda,\mu,\gamma,a}$ is equivalent to
$$(2\lambda f_1+\gamma f_1^2)^2+(2\mu f_2)^2+\big(f_3+\frac{a}{2}(1-f_3^2)\big)^2\leq 1,\ \text{ for all }\ \|\mathbf{f}\|\leq 1.$$
Rewriting the last inequality, we find
$$4\lambda^2f_1^2+4\mu^2f_2^2+4\lambda\gamma f_1^3+\gamma^2f_1^4+f_3^2-1+af_3(1-f_3^2)+\frac{a^2}{4}(1-f_3^2)^2\leq 0.$$
This inequality is satisfied if
$$4M(1-f_3^2)+(4|\gamma|+|\gamma^2|)(1-f_3^2)+f_3^2-1+|a||f_3|(1-f_3^2)+\frac{a^2}{4}(1-f_3^2)^2\leq 0$$
which is equivalent to
$$(1-f_3^2)(4M+4|\gamma|+\gamma^2-1+|a||f_3|+\frac{a^2}{4}(1-f_3^2))\leq 0.$$
Then
$$\frac{a^2}{4}f_3^2-|a||f_3|+1-(4|\gamma|+\gamma^2)-4M-\frac{a^2}{4}\geq 0$$
So, by Lemma 3
$$1-(4|\gamma|+\gamma^2)-4M-\frac{a^2}{4}\geq 0,\ \text{ and }\ 1-|a|-(4|\gamma|+\gamma^2)-4M\geq 0.$$
Hence,
$$M\leq\min\left\{\frac{1-\frac{a^2}{4}-(4\gamma+\gamma^2)}{4},\frac{1-|a|-(4\gamma+\gamma^2)}{4}\right\}=\frac{1-|a|-(4\gamma+\gamma^2)}{4}.$$
This completes the proof. □

Remark 2. *We stress that if $\gamma=0$, then from the proved theorem we infer that the quasiness implies the positivity of $\Delta_{\lambda,\mu,0,a}$. This type of results was established in* [12].

4. A Class of Positive Operators Corresponding to $\Delta_{\lambda,\mu,\gamma,a}$

In this section, we define a class of positive operators associated with $\Delta_{\lambda,\mu,\gamma,a}$. To do so, given a state φ on $\mathbb{M}_2(\mathbb{C})$, let us define a mapping $E_\varphi:\mathbb{M}_2(\mathbb{C})\mathbb{M}_2(\mathbb{C})\to\mathbb{M}_2(\mathbb{C})$ by
$$E_\varphi(xy)=x\varphi(y),\ \ x,y\in\mathbb{M}_2(\mathbb{C}). \tag{17}$$
It is known that E_φ is a conditional expectation.

By means of $\Delta_{\lambda,\mu,\gamma,a}$, let us define a mapping $\Phi_{\lambda,\mu,\gamma,a,\varphi}:\mathbb{M}_2(\mathbb{C})\to\mathbb{M}_2(\mathbb{C})$ by
$$\Phi_{\lambda,\mu,\gamma,a,\varphi}:=E_\varphi\circ\Delta_{\lambda,\mu,\gamma,a}. \tag{18}$$

By (8) we find

$$\Phi_{\lambda,\mu,\gamma,a,\varphi}(w_0\mathbf{1}+\mathbf{w}\sigma) = \left(w_0+\lambda f_1\omega_1+\mu f_2\omega_2+\frac{a+f_3}{2}\omega_3\right)\mathbf{1} + (\lambda+\gamma f_1)\omega_1\sigma_1+\mu\omega_2\sigma_2+\frac{1-af_3}{2}w_3\sigma_3. \quad (19)$$

We stress that if φ is taken as the normalized trace, i.e., $f_1 = f_2 = f_3 = 0$, then the mapping $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ reduces to

$$\Phi_{\lambda,\mu,a}(w_0\mathbf{1}+\mathbf{w}\sigma) = \left(w_0+\frac{a}{2}\omega_3\right)+\lambda\omega_1\sigma_1+\mu\omega_2\sigma_2+\frac{w_3}{2}\sigma_3, \quad (20)$$

which was investigated in [12]. Clearly, from (19) one sees that the structure of $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ is much complex than (20).

Theorem 2. *Let $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ be given by (19). Then $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ is positive if*

$$\max(|\mu|,|\gamma f_1+\lambda|) \le \frac{2-\sqrt{4\lambda^2 f_1^2+4\mu^2 f_2^2+(a+f_3)^2}}{2}.$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$ corresponds to φ.

Proof. Let $x \in \mathbb{M}_2(\mathbb{C})$ be as in Theorem 1. Then, the matrix form of $\Phi_{\lambda,\mu,\gamma,a,\varphi}(x)$ is given by

$$\Phi_{\lambda,\mu,a,\varphi}(x) = \begin{bmatrix} W+\frac{1}{2}w_3(1+f_3)-\frac{1}{2}aw_3f_3 & \gamma w_1f_1+\bar{y} \\ \gamma w_1f_1+y & W+\frac{1}{2}w_3(-1+f_3)+\frac{1}{2}aw_3f_3 \end{bmatrix}$$

where $W = 1+\frac{1}{2}aw_3+\lambda f_1w_1+\mu f_2w_2$, $y = \lambda w_1+\mu i w_2$.

The eigenvalues of $\Phi_{\lambda,\mu,\gamma,a,\varphi}(x)$ are

$$\Lambda_1 = \lambda f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3+1-\frac{1}{2}\sqrt{(af_3-1)^2w_3^2+4(\gamma f_1+\lambda)^2w_1^2+4\mu^2w_2^2}$$
$$\Lambda_2 = \lambda f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3+1+\frac{1}{2}\sqrt{(af_3-1)^2w_3^2+4(\gamma f_1+\lambda)^2w_1^2+4\mu^2w_2^2}$$

To show the positivity of $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ it is enough to establish the positivity of Λ_1. So,

$$\lambda f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3+1-\frac{1}{2}\sqrt{(af_3-1)^2w_3^2+4(\gamma f_1+\lambda)^2w_1^2+4\mu^2w_2^2} \ge 0$$

which is equivalent to

$$\lambda f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3+1 \ge \frac{1}{2}\sqrt{(af_3-1)^2w_3^2+4(\gamma f_1+\lambda)^2w_1^2+4\mu^2w_2^2} \quad (21)$$

The inequality (21) holds if the following inequality is satisfied

$$f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3 \ge \frac{1}{2}\sqrt{(af_3-1)^2w_3^2+4\tilde{M}(1-w_3^2)}-1, \quad (22)$$

where $\tilde{M} = \max\{(\gamma f_1+\lambda)^2, \mu^2\}$. Assume that

$$F(w_1,w_2,w_3) := \lambda f_1w_1+\mu f_2w_2+\frac{1}{2}(a+f_3)w_3.$$

Therefore, we have to find the absolute minimum value of F subject to the constrain $G(w_1, w_2, w_3) := w_1^2 + w_2^2 + w_3^2 - 1$. Using Lagrange multiplier $\nabla(F) = p\nabla(G)$ one obtains

$$\lambda f_1\vec{i} + \mu f_2\vec{j} + \frac{1}{2}(a + f_3)\vec{k} = 2pw_1\vec{i} + 2pw_2\vec{j} + 2pw_3\vec{k}$$

Then

$$p = \frac{\lambda f_1}{2w_1} = \frac{\mu f_2}{2w_2} = \frac{(a + f_3)}{4w_3}$$

Thus, $w_2 = \left(\frac{\mu f_2}{\lambda f_1}\right)w_1$, $w_3 = \left(\frac{(a+f_3)}{2\lambda f_1}\right)w_1$. Plugging these values into $w_1^2 + w_2^2 + w_3^2 = 1$, one finds

$$w_1 = \pm\frac{2\lambda f_1}{\sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}},$$
$$w_2 = \pm\frac{2\mu f_1}{\sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}},$$
$$w_3 = \pm\frac{(a + f_3)}{\sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}}$$

substituting these value into $F(w_1, w_2, w_3)$, we obtain

$$\min_{w_1^2+w_2^2+w_3^2=1} F(w_1, w_2, w_3) = -\frac{1}{2}\sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}$$

Hence, by (22), one has

$$\sqrt{((af_3 - 1)^2 - 4\tilde{M})w_3^2 + 4\tilde{M}} \leq -\sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2} + 2$$

Now, if $(af_3 - 1)^2 > 4\tilde{M}$ then

$$|af_3 - 1| \leq 2 - \sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}.$$

If $(af_3 - 1)^2 \leq 4\tilde{M}$ then

$$\max(|\mu|, |\gamma f_1 + \lambda|) \leq \frac{2 - \sqrt{4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a + f_3)^2}}{2}.$$

This completes the proof. □

5. Controlling a Two-Level Quantum System

In this section, we investigate the problem of controlling a qubit, i.e., a two-level quantum system associated with $\Phi_{\lambda,\mu,\gamma,a,\varphi}$.

By Σ_a we denote the set of all $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ which is positive. One can check that Σ_a is a convex set. Denote $\vartheta = \bigcup_{-1\leq a\leq 1} \Sigma_a$. Define

$$\mathcal{J}(\Phi) = Tr(\Phi(\rho_0)O), \quad \Phi \in \vartheta, \tag{23}$$

where $\rho_0 \geq 0$, $O \in \mathbb{M}_2(\mathbb{C}), O^* = O$.

The main aim of this section is to maximize $\underset{\Phi\in\vartheta}{\mathcal{J}}(\Phi)$. Let us first observe that any Hermitian operator $O \in \mathbb{M}_2(\mathbb{C})$ can be diagonalized as

$$\bar{O} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \tilde{\lambda_1}\mathbf{1} + \tilde{\lambda_2}\sigma_3. \tag{24}$$

where $\tilde{\lambda_1} = \frac{\lambda_1+\lambda_2}{2}$, $\tilde{\lambda_1} = \frac{\lambda_1-\lambda_2}{2}$. For the sake of simplicity, we choose $\lambda_1 = 1,\ \lambda_2 = 0$. Now, substituting $\Phi_{\lambda,\mu,\gamma,a,\varphi}$ and $\bar{O}$ into (23) one finds

$$\mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi}) = Tr(\Phi_{\lambda,\mu,\gamma,a,\phi}(\rho_0)\bar{O}) = \lambda f_1 w_1 + \mu f_2 w_2 + \frac{1}{2}(a+1+(1-a)f_3)w_3, \tag{25}$$

here $\rho_0 = \mathbf{1} + \mathbf{w}\cdot\boldsymbol{\sigma}$ is the initial density matrix with $||\mathbf{w}|| \leq 1$. The next theorem is the main result of this section.

Theorem 3. *Let $\mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi})$ be given by (25), then the following statements hold true:*

(i) *If $w_1w_3 \neq 0$, then*

$$\max \mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi}) = K\left(\frac{1}{2}\sqrt{1+(3-\sqrt{6})\sqrt{4-\sqrt{6}}} + \frac{1}{8}(3+\sqrt{6})\sqrt{4-\sqrt{6}} - \frac{\sqrt{6}}{4} + \frac{7}{8}\right),$$

where $K = \max\{|w_1|, |w_3|\}$.

(ii) *If $w_1 = 0,\ w_3 \neq 0$, then* $\max \mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi}) = \left(\frac{2\sqrt{2}+1}{4}\right)|w_3|$.

(iii) *If $w_1 \neq 0,\ w_3 = 0$, then* $\max \mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi}) = \frac{1}{\sqrt{2}}|w_1|$.

(iv) *If $w_1 = 0,\ w_3 = 0$, then* $\max \mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi}) = 0$.

Proof. Let us first denote $\mathcal{J}(\Phi_{\lambda,\mu,\gamma,a,\phi})$ by $F(\lambda,\mu,\gamma,a,\phi)$. Therefore, we have to find the maximum value of

$$F(\lambda,\mu,\gamma,a,\phi) := \lambda f_1 A + \mu f_2 B + \frac{1}{2}(a+1+(1-a)f_3)C,$$

where $w_1 = A,\ w_2 = B$, and $w_3 = C$ subject to the constrain

$$G(\lambda,\mu,\gamma,a,\phi) := 4\lambda^2 f_1^2 + 4\mu^2 f_2^2 + (a+f_3)^2 - (2-2\mu)^2 \leq 0. \tag{26}$$

Using Lagrange multiplier, one has

$$\nabla F(\lambda,\mu,\gamma,a,\phi) = p\nabla G(\lambda,\mu,\gamma,a,\phi).$$

Then we obtain the following system of equations:

$$\begin{cases} 2(a+f_3)p & = & \frac{1}{2}(1-f_3)C \\ 8\lambda f_1^2 p & = & f_1 A \\ (8\mu f_2^2 - 8\mu + 8)p & = & f_2 B \\ 8\lambda^2 f_1 p & = & \lambda A \\ 8\mu^2 f_2 p & = & \mu B \\ 2(a+f_3)p & = & \frac{1}{2}(1-a)C \end{cases} \tag{27}$$

Now, we analyze the system (27). Suppose that $\mu f_2 \neq 0$ then, from the fifth equation of (27), one has $p = \frac{B}{8\mu f_2}$, and, inserting this value into the third equation of (27), we arrive at $\mu = 1$, but it contradicts the constrain (26). Therefore, $\mu f_2 = 0$. Consequently, our problem is reduced to

$$F(\lambda,\gamma,a,\phi) = \lambda f_1 A + \frac{1}{2}(a+1+(1-a)f_3)C$$

subject to the constrain

$$G(\lambda,\gamma,a,\phi) = 4\lambda^2 f_1^2 + (a+f_3)^2 - 2 \leq 0.$$

Now, we consider the following cases with respect to values of A and C.

(i) Assume that $AC \neq 0$, then from the first and the last equation of (27) we obtain that $f_3 = a$. Next, assume that $\lambda f_1 = 0$, then we have critical points at $f_3 = \pm\frac{1}{\sqrt{2}}$. Plugging it into F, we obtain $\max F = \left(\frac{2\sqrt{2}+1}{4}\right)|C|$. Now, let us suppose $\lambda f_1 \neq 0$, then from the second equation of (27) one finds $p = \frac{A}{8\lambda f_1}$, then inserting this value into the first equation, we obtain $\lambda f_1 = \frac{A}{C}\frac{f_3}{1-f_3}$. Then plugging the obtained ones into G yields

$$4\frac{A^2}{C^2}\left(\frac{f_3}{1-f_3}\right)^2 + 4f_3^2 - 2.$$

Rewriting the last expression gives

$$\left(\frac{f_3}{1-f_3}\right) = \pm\frac{C\sqrt{2-4f_3^2}}{2A},\ |f_3| \leq \frac{1}{\sqrt{2}}.$$

Now, plugging the value $\left(\frac{f_3}{1-f_3}\right)$ into F yields

$$F = A\frac{\sqrt{2-4f_3^2}}{2} - \frac{1}{2}C(1-f_3)^2 + C.$$

Let $K = \max\{|A|,|C|\}$, then

$$F \leq K\left(\frac{\sqrt{2-4f_3^2}}{2} - \frac{1}{2}(1-f_3)^2 + 1\right).$$

Now, let us find the maximum value of $\frac{\sqrt{2-4f_3^2}}{2} - \frac{1}{2}(1-f_3)^2 + 1$ over the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. The last function's critical points are:

$$X = \pm\frac{1}{\sqrt{2}},\quad Y = \frac{1}{2} - \frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{-6+4\sqrt{6}}.$$

One can easily check that the maximum value reaches at Y, and its value is

$$K\left(\frac{1}{2}\sqrt{-4\left(\frac{1}{2} - \frac{\sqrt{6}}{4} + \frac{\sqrt{-6+4\sqrt{6}}}{4}\right)^2 + 2} + \frac{1}{16}(\sqrt{6}+2)\sqrt{-6+4\sqrt{6}} - \frac{\sqrt{6}}{4} + \frac{7}{8}\right).$$

Simplifying the last expression, we arrive at the required assertion.

(ii) Let $A = 0, C \neq 0$. Then, $F = \frac{1}{2}(1 + 2f_3 - f_3^2)$ and $G = 4f_3^2 - 2$. So, one has that the critical points at $f_3 = \pm\frac{1}{\sqrt{2}}$. Hence, the maximum value is $\left(\frac{2\sqrt{2}+1}{4}\right)|C|$.

(iii) Assume that $A \neq 0, C = 0$. Then $F = \lambda f_1 A$ and $G = 4\lambda^2 f_1^2 - 2$. Similarly, the maximum value of F is $\frac{1}{\sqrt{2}}|A|$.

(iv) Let $A = 0, C = 0$. In this case $F = G = 0$. So, the maximum value is 0. This completes the proof.

□

6. Conclusions

In the present paper, we have introduced a class of quantum Lotka–Volterra operators which contains as a particular case those that were studied in [12]. The provided construction of such types of maps is highly non-trivial, since they map $\mathbb{M}_2(\mathbb{C})$ into $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, and checking their positivity condition is tricky. Moreover, considering conditional expectations (depending on states) from $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ to $\mathbb{M}_2(\mathbb{C})$, and using the introduced class of maps, a family of unital positive maps is defined which depends on several parameters. We stress that if the state is taken as a trace, then the family is reduced to earlier studied maps in [12]. However, the presence of non-trivial states makes the family very complicated for checking its positivity, which has been done in Section 4. Within such a family of positive maps, in the last section, a quantum control problem is explored. The proposed approach will lead to the understanding of the behavior of the classical Lotka–Volterra systems within a quantum framework. The constructed unital positive maps will also serve to be entangled witnesses. Moreover, it would be interesting to find conditions on the parameters for which the considered family satisfies the Kadison–Schwarz property, which has certain applications in the approximation of positive maps [28].

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