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On the Generalized Liouville–Caputo Type Fractional Differential Equations Supplemented with Katugampola Integral Boundary Conditions

Muath Awadalla [1,*], Muthaiah Subramanian [2], Kinda Abuasbeh [1] and Murugesan Manigandan [3]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Hafuf, Al Ahsa 31982, Saudi Arabia
[2] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641407, Tamil Nadu, India
[3] Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore 641020, Tamil Nadu, India
* Correspondence: mawadalla@kfu.edu.sa

Abstract: In this study, we examine the existence and Hyers–Ulam stability of a coupled system of generalized Liouville–Caputo fractional order differential equations with integral boundary conditions and a connection to Katugampola integrals. In the first and third theorems, the Leray–Schauder alternative and Krasnoselskii's fixed point theorem are used to demonstrate the existence of a solution. The Banach fixed point theorem's concept of contraction mapping is used in the second theorem to emphasise the analysis of uniqueness, and the results for Hyers–Ulam stability are established in the next theorem. We establish the stability of Ulam–Hyers using conventional functional analysis. Finally, examples are used to support the results. When a generalized Liouville–Caputo (ρ) parameter is modified, asymmetric results are obtained. This study presents novel results that significantly contribute to the literature on this topic.

Keywords: generalized fractional derivatives; generalized fractional integrals; coupled system; existence; fixed point

MSC: 34A08; 34B10; 34D10

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1. Introduction

We consider the nonlinear coupled fractional differential equations with generalized Liouville–Caputo derivatives

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi}_{0+}p(\tau) = f(\tau, p(\tau), q(\tau)), \tau \in \mathcal{G} := [0, \mathcal{T}], \\ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{0+}q(\tau) = g(\tau, p(\tau), q(\tau)), \tau \in \mathcal{G} := [0, \mathcal{T}], \end{cases} \tag{1}$$

enhanced with boundary conditions which are defined by:

$$\begin{cases} p(0) = 0, \quad q(0) = 0, \\ p(\mathcal{T}) = \epsilon^{\rho}\mathcal{I}^{\varsigma}_{0+}q(\varpi) = \frac{\epsilon\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^{\varpi}\frac{\theta^{\rho-1}}{(\varpi^{\rho}-\theta^{\rho})^{1-\varsigma}}q(\theta)d\theta, \\ q(\mathcal{T}) = \pi^{\rho}\mathcal{I}^{\varrho}_{0+}p(\sigma) = \frac{\pi\rho^{1-\varrho}}{\Gamma(\varrho)}\int_0^{\sigma}\frac{\theta^{\rho-1}}{(\sigma^{\rho}-\theta^{\rho})^{1-\varrho}}p(\theta)d\theta, \\ 0 < \sigma < \varpi < \mathcal{T}, \end{cases} \tag{2}$$

where ${}^{\rho}_{C}\mathcal{D}^{\xi}_{0+}$, ${}^{\rho}_{C}\mathcal{D}^{\zeta}_{0+}$ are the Liouville–Caputo-type generalized fractional derivative of order $1 < \xi, \zeta \leq 2$, ${}^{\rho}_{C}\mathcal{I}^{\varsigma}_{0+}$, ${}^{\rho}_{C}\mathcal{I}^{\varrho}_{0+}$ are the generalized fractional integral of order (Katugampola type) $\varrho, \varsigma > 0, \rho > 0, f, g : \mathcal{G} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\epsilon, \pi \in \mathbb{R}$. The strip conditions states that the value of the unknown function at the right end point $\tau = T$ of the given interval is proportional to the values of the unknown function on the strips of varying lengths. When $\rho = 1$, the generalized Liouville–Caputo equation is changed to the Caputo sense, which leads to asymmetric results. In a similar way, when $\rho = 1$, the Katugampola integrals are changed to Riemann-Liouville integrals, which leads to cases that are not symmetric. To the best of our knowledge, the stability analysis of boundary value problems (BVPs) is still in its early stages. This paper's primary contribution is to study existence and Ulam-Hyers stability analysis. In addition, we demonstrate the problem (1)–(2) employed by Leray–Schauder, Banach and Krasnoselskii's fixed point theorems to prove the existence and uniqueness of solutions. The system (1) is the well-known fractional-order coupled logistic system [1]:

$$\begin{cases} \mathcal{D}^{\alpha}u(\tau) = r_1 u(\tau) - \frac{r_1}{k_1}u(\tau)(u(\tau)+v(\tau)), \tau \in I, \\ \mathcal{D}^{\beta}v(\tau) = r_2 v(\tau) - \frac{r_2}{k_2}v(\tau)(v(\tau)+u(\tau)), \end{cases}$$

and the Lotka–Volterra prey-predator system [1]:

$$\begin{cases} \mathcal{D}^{\alpha}u(\tau) = u(\tau)(a - u(\tau)E - \gamma v(\tau)), \tau \in I, \\ \mathcal{D}^{\beta}v(\tau) = v(\tau)(-b + \gamma E v(\tau) - \beta E). \end{cases}$$

We now provide some recent results related to our problem (1)–(2). In [2], the authors discussed the existence results for coupled system of fractional differential equations Riemann–Liouville derivatives

$$\begin{cases} \mathcal{D}^{\alpha_1}_{0^+}(\mathcal{D}^{\beta_1}_{0^+}x(t)) + f(t,x(t),y(t)), t \in [0,1], \\ \mathcal{D}^{\alpha_2}_{0^+}(\mathcal{D}^{\beta_2}_{0^+}y(t)) + f(t,x(t),y(t)), t \in [0,1], \end{cases} \tag{3}$$

with the Riemann–Stieltjes integral boundary conditions:

$$\begin{cases} \mathcal{D}^{\beta_1}_{0^+}x(0) = 0,\ x(0) = 0,\ \mathcal{D}^{\beta_2}_{0^+}y(0) = 0,\ y(0) = 0, \\ x(1) = \gamma_1 \mathcal{I}^{\delta_1}_{0^+}y(\xi) + \sum_{i=1}^{p}\int_0^1 y(\tau)d\mathcal{H}_i(\tau), \\ y(1) = \gamma_2 \mathcal{I}^{\delta_2}_{0^+}x(\eta) + \sum_{j=1}^{q}\int_0^1 x(\tau)d\mathcal{K}_i(\tau), \end{cases} \tag{4}$$

where α_1 is in the interval $(0,1)$, β_1 is in the interval $(1,2)$, α_2 is in the interval $(0,1]$, β_2 is in the interval $(1,2]$, $p,q \in N$, and γ_1, γ_2, δ_1, $\delta_2 > 0$, $0 < \xi,\eta < 1$ $\mathcal{K}_j(t)$, $j = 1,\ldots,q$, $\mathcal{H}_i(t), i = 1,\ldots,p$ are bounded variation functions. Both function f and g are nonlinear. They used several theorems from fixed point index theory to prove the main results. In [3], the authors investigated existence of solutions for coupled system of fractional differential equations with Hilfer derivatives

$$\begin{cases} ({}^H\mathcal{D}^{\alpha_1,\beta_1}_{0^+}x)(t) + \lambda_1({}^H\mathcal{D}^{\alpha_1-1,\beta_1}_{0^+}x)(t) = f(t,x(t),R^{(\delta_q,\ldots,\delta_1)}x(t),y(t)), t \in [0,T], \\ ({}^H\mathcal{D}^{\alpha_2,\beta_2}_{0^+}y)(t) + \lambda_2({}^H\mathcal{D}^{\alpha_2-1,\beta_2}_{0^+}y)(t) = f(t,x(t),y(t),R^{(\zeta_q,\ldots,\zeta_1)}y(t)), t \in [0,T], \end{cases} \tag{5}$$

with Riemann–Liouville and Hadamard-type iterated integral boundary conditions:

$$\begin{cases} x(0) = 0,\ y(0) = 0, \\ x(T) = \sum_{i=1}^{m} \epsilon_i R^{(\mu_\rho,\ldots,\mu_1)}y(\eta_i)\ \eta_i \in (0,T), \\ y(T) = \sum_{j=1}^{n} \theta_j R^{(\nu_\rho,\ldots,\nu_1)}x(\xi_j)\ \xi_i \in (0,T), \end{cases} \tag{6}$$

where ${}^H\mathcal{D}^{\alpha_l,\beta_l}$ is the Hilfer fractional derivative operator of order α_l with parameters β_l, $l \in 1,2$, $1 < \alpha_l < 2$, $0 \le \beta_l \le 1$, $\lambda_1, \lambda_2, \epsilon_i, \theta_j \in \mathcal{R}\backslash\{0\}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $f,g : [0,T] \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \times\mathcal{R}$ are nonlinear continuous functions and $R^{(\phi_\tau,\ldots,\phi_1)}$, $\phi_r \in \{\delta,\zeta,\mu,\nu\}$, $r \in \{q,p,\rho | q,p,\rho \in \mathcal{N}\}$, involves the iterated Riemann–Liouville and

Hadamard fractional integral operators. They used several theorems from fixed point index theory to prove the main results. Numerous scientific and engineering phenomena are mathematically modelled using fractional order differential and integral operators. The main benefit of adopting these operators is their nonlocality, which enables the description of the materials and processes involved in the history of the phenomenon. As a result, compared to their integer-order counterparts, fractional-order models are more precise and informative. As a result of the extensive use of fractional calculus techniques in a range of real-world occurrences, such as those described in the texts cited [4–8] numerous researchers developed this significant branch of mathematical study. In recent years, a lot of research has been done on fractional differential equations with different boundary conditions. Nonlocal nonlinear fractional-order boundary value problems, in particular, have attracted a lot of attention (BVPs). The idea of nonlocal circumstances, which help to describe physical processes occurring inside the confines of a specific domain, was originally introduced in the work of Bitsadze and Samarski [9]. It is challenging to defend the assumption of a circular cross-section in computational fluid dynamics investigations of blood flow problems because to the changing shape of a blood vessel throughout the vessel. To solve that problem, integral boundary conditions have been developed. In addition, the ill-posed parabolic backward problems are solved under integral boundary conditions. Integral boundary conditions are also essential in mathematical models of bacterial self-regularization, as shown in [10]. Fractional order differential equations, as well as inclusions including Riemann–Liouville, Liouville–Caputo (Caputo), and Hadamard-type derivatives, among others, have all been included in the literature on the topic recently. For some recent works on the topic, we point the reader to several papers [11–15] and the references listed therein. The use of fractional differential systems in mathematical representations of physical and engineering processes has drawn considerable interest. See [16–22] for additional details on the theoretical evolution of such systems. The following is the remainder of the article: Section 2 introduces some fundamental definitions, lemmas, and theorems that support our main results. For the existence and uniqueness of solutions to the given system (1) and (2), we use various conditions and some standard fixed-point theorems in Section 3. Section 4 discusses the Ulam–Hyers stability of the given system (1) and (2) under certain conditions. In Section 6, examples are provided to demonstrate the main results. Finally, the consequences of existence, uniqueness, and stability for the problem (1) and (75) are provided.

2. Preliminaries

For our research, we recall some preliminary definitions of generalized Liouville–Caputo fractional derivatives and Katugampola fractional integrals.

The space of all complex-valued Lebesgue measurable functions ϕ on (c, d) equipped with the norm is denoted by $\mathcal{Z}_b^q(c, d)$:

$$||\phi||_{\mathcal{Z}_b^q} = \left(\int_c^d |z^b \phi(z)|^q \frac{dz}{z} \right)^{\frac{1}{q}} < \infty, b \in \mathbb{R}, 1 \leq q \leq \infty.$$

Let $\mathcal{L}^1(c, d)$ represent the space of all Lebesgue measurable functions φ on (c, d) endowed with the norm:

$$\|\varphi\|_{\mathcal{L}^1} = \int_c^d |\varphi(z)| dz < \infty.$$

We further recall that $\mathcal{AC}^n(\mathcal{E}, \mathbb{R}) = \{p : \mathcal{E} \to \mathbb{R} : p, p', \ldots, p^{(n-1)} \in \mathcal{C}(\mathcal{E}, \mathbb{R})$ and $p^{(n-1)}$ is absolutely continuous. For $0 \leq \epsilon < 1$, we define $\mathcal{C}_{\epsilon,\rho}(\mathcal{E}, \mathbb{R}) = \{f : \mathcal{E} \to \mathbb{R} : (\tau^\rho - a^\rho)^\epsilon f(\tau) \in \mathcal{C}(\mathcal{E}, \mathbb{R})$ endowed with the norm $\|f\|_{\mathcal{C}_{\epsilon,\rho}} = \|(\tau^\rho - a^\rho)^\epsilon f(\tau)\|_{\mathcal{C}}$. Moreover, we define the class of functions f that have absolute continuous δ^{n-1} derivative, denoted by

$\mathcal{AC}^n_\gamma(\mathcal{E},\mathbb{R})$, as follows: $\mathcal{AC}^n_\gamma(\mathcal{E},\mathbb{R}) = \{f : \mathcal{E} \to \mathbb{R} : \gamma^{n-1}f \in \mathcal{AC}(\mathcal{E},\mathbb{R}), \gamma = \tau^{1-\rho}\frac{d}{d\tau}\}$, which is equipped with the norm $\|f\|_{\mathcal{C}^n_{\gamma,\epsilon}} = \sum_{k=0}^{n-1}\|\gamma^k f\|_{\mathcal{C}} + \|\gamma^n f\|_{\mathcal{C}_{\epsilon,\rho}}$ is defined by

$$\mathcal{C}^n_{\gamma,\epsilon}(\mathcal{E},\mathbb{R}) = \left\{f : \mathcal{E} \to \mathbb{R} : \gamma^{n-1}f \in \mathcal{C}(\mathcal{E},\mathbb{R}), \gamma^n f \in \mathcal{C}_{\epsilon,\rho}(\mathcal{E},\mathbb{R}), \gamma = \tau^{1-\rho}\frac{d}{d\tau}\right\}.$$

Notice that $\mathcal{C}^n_{\gamma,0} = \mathcal{C}^n_\gamma$. We define space $\mathcal{P} = \{p(\tau) : p(\tau) \in \mathcal{C}(\mathcal{E},\mathbb{R})\}$ equipped with the norm $||p|| = \sup\{|p(\tau)|, \tau \in \mathcal{E}\}$- this is a Banach space. Furthermore $\mathcal{Q} = \{q(\tau) : q(\tau) \in \mathcal{C}(\mathcal{E},\mathbb{R})\}$ equipped with the norm is $||q|| = \sup\{|q(\tau)|, \tau \in \mathcal{E}\}$ is a Banach space. Then the product space $(\mathcal{P} \times \mathcal{Q}, ||(p,q)||)$ is also a Banach space with norm $||(p,q)|| = ||p|| + ||q||$.

Definition 1 ([23]). *The left and right-sided generalized fractional integrals (GFIs) of $f \in \mathcal{Z}^q_b(c,d)$ of order $\xi > 0$ and $\rho > 0$ for $-\infty < c < \tau < d < \infty$, are defined as follows:*

$$({}^\rho\mathcal{I}^\xi_{c+}f)(\tau) = \frac{\rho^{1-\xi}}{\Gamma(\xi)}\int_c^\tau \frac{\theta^{\rho-1}}{(\tau^\rho - \theta^\rho)^{1-\xi}}f(\theta)d\theta, \tag{7}$$

$$({}^\rho\mathcal{I}^\xi_{d-}f)(\tau) = \frac{\rho^{1-\xi}}{\Gamma(\xi)}\int_\tau^d \frac{\theta^{\rho-1}}{(\theta^\rho - \tau^\rho)^{1-\xi}}f(\theta)d\theta. \tag{8}$$

Definition 2 ([24]). *The generalized fractional derivatives (GFDs) which are associated with GFIs (7) and (8) for $0 \le c < \tau < d < \infty$, are defined as follows:*

$$\begin{aligned}({}^\rho\mathcal{D}^\xi_{c+}f)(\tau) &= \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^n({}^\rho\mathcal{I}^{n-\xi}_{c+}f)(\tau)\\ &= \frac{\rho^{\xi-n+1}}{\Gamma(n-\xi)}\left(\tau^{1-\rho}\frac{d}{d\tau}\right)^n\int_c^\tau \frac{\theta^{\rho-1}}{(\tau^\rho - \theta^\rho)^{\xi-n+1}}f(\theta)d\theta,\end{aligned} \tag{9}$$

$$\begin{aligned}({}^\rho\mathcal{D}^\xi_{d-}f)(\tau) &= \left(-\tau^{1-\rho}\frac{d}{d\tau}\right)^n({}^\rho\mathcal{I}^{n-\xi}_{d-}f)(\tau)\\ &= \frac{\rho^{\xi-n+1}}{\Gamma(n-\xi)}\left(-\tau^{1-\rho}\frac{d}{d\tau}\right)^n\int_\tau^d \frac{\theta^{\rho-1}}{(\tau^\rho - \theta^\rho)^{\xi-n+1}}f(\theta)d\theta,\end{aligned} \tag{10}$$

if the integrals exist.

Definition 3 ([25]). *The above GFDs define the left and right-sided generalized Liouville–Caputo type fractional derivatives of $f \in \mathcal{AC}^n_\gamma[c,d]$ of order $\xi \ge 0$*

$${}^\rho_C\mathcal{D}^\xi_{c+}f(z) = {}^\rho\mathcal{D}^\xi_{c+}\left[f(\tau) - \sum_{k=0}^{n-1}\frac{\gamma^k f(c)}{k!}\left(\frac{\tau^\rho - c^\rho}{\rho}\right)^k\right](z), \gamma = z^{1-\rho}\frac{d}{dz}, \tag{11}$$

$${}^\rho_C\mathcal{D}^\xi_{d-}f(z) = {}^\rho\mathcal{D}^\xi_{d-}\left[f(\tau) - \sum_{k=0}^{n-1}\frac{(-1)^k\gamma^k f(d)}{k!}\left(\frac{d^\rho - \tau^\rho}{\rho}\right)^k\right](z), \gamma = z^{1-\rho}\frac{d}{dz}, \tag{12}$$

when $n = [\xi] + 1$.

Lemma 1 ([25]). *Let $\xi \ge 0, n = [\xi] + 1$ and $f \in \mathcal{AC}^n_\gamma[c,d]$, where $0 < c < d < \infty$. Then,*

1. *if* $\xi \notin \mathbb{N}$

$$ {}^{\rho}_{C}\mathcal{D}^{\xi}_{c^+} f(\tau) = \frac{1}{\Gamma(n-\xi)} \int_c^{\tau} \left(\frac{\tau^{\rho} - \theta^{\rho}}{\rho} \right)^{n-\xi-1} \frac{(\gamma^n f)(\theta) d\theta}{\theta^{1-\rho}} = {}^{\rho}\mathcal{I}^{n-\xi}_{c^+} (\gamma^n f)(\tau), \tag{13} $$

$$ {}^{\rho}_{C}\mathcal{D}^{\xi}_{d^-} f(\tau) = \frac{1}{\Gamma(n-\xi)} \int_{\tau}^{d} \left(\frac{\theta^{\rho} - \tau^{\rho}}{\rho} \right)^{n-\xi-1} \frac{(-1)^n (\gamma^n f)(\theta) d\theta}{\theta^{1-\rho}} = {}^{\rho}\mathcal{I}^{n-\xi}_{d^-} (\gamma^n f)(\tau). \tag{14} $$

2. *if* $\xi \in \mathbb{N}$

$$ {}^{\rho}_{C}\mathcal{D}^{\xi}_{c^+} f = \gamma^n f, \qquad {}^{\rho}_{C}\mathcal{D}^{\xi}_{d^-} f = (-1)^n \gamma^n f. \tag{15} $$

Lemma 2 ([25])**.** *Let* $f \in \mathcal{AC}^n_{\gamma}[c,d]$ *or* $\mathcal{C}^n_{\gamma}[c,d]$ *and* $\xi \in \mathbb{R}$. *Then,*

$$ {}^{\rho}\mathcal{I}^{\xi}_{c^+} {}^{\rho}_{C}\mathcal{D}^{\xi}_{c^+} f(z) = f(z) - \sum_{k=0}^{n-1} \frac{\gamma^k f(c)}{k!} \left(\frac{z^{\rho} - c^{\rho}}{\rho} \right)^k, $$

$$ {}^{\rho}\mathcal{I}^{\xi}_{d^-} {}^{\rho}_{C}\mathcal{D}^{\xi}_{d^-} f(z) = f(z) - \sum_{k=0}^{n-1} \frac{(-1)^k \gamma^k f(d)}{k!} \left(\frac{d^{\rho} - z^{\rho}}{\rho} \right)^k. $$

In particular, for $0 < \xi \leq 1$, we have

$$ {}^{\rho}\mathcal{I}^{\xi}_{c^+} {}^{\rho}_{C}\mathcal{D}^{\xi}_{c+} f(z) = f(z) - f(c), \qquad {}^{\rho}\mathcal{I}^{\xi}_{d^-} {}^{\rho}_{C}\mathcal{D}^{\xi}_{d^-} f(z) = f(z) - f(d). $$

We introduce the following notations for computational ease:

$$ \mathcal{E}_1 = \epsilon \frac{\varpi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)},\ \mathcal{E}_2 = \pi \frac{\sigma^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)},\ \widehat{\mathcal{E}} = \frac{\mathcal{T}^{\rho}}{\rho}, \tag{16} $$

$$ \mathcal{G} = \widehat{\mathcal{E}}^2 - \mathcal{E}_1 \mathcal{E}_2 \neq 0, \tag{17} $$

$$ \delta(\tau) = \left(\frac{\tau^{\rho}}{\rho \mathcal{G}} \right). \tag{18} $$

Next, we are proving a lemma, which is vital in converting the given problem to a fixed-point problem.

Lemma 3. *Given the functions* $\hat{f}, \hat{g} \in C(0,\mathcal{T}) \cup \mathcal{L}(0,\mathcal{T})$, $p, q \in \mathcal{AC}^2_{\gamma}(\mathcal{E})$ *and* $\Lambda \neq 0$. *Then the solution of the coupled BVP:*

$$ \begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi}_{0+} p(\tau) = \hat{f}(\tau), \tau \in \mathcal{E} := [0,\mathcal{T}], \\ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{0+} q(\tau) = \hat{g}(\tau), \tau \in \mathcal{E} := [0,\mathcal{T}], \\ p(0) = 0, \quad q(0) = 0, \quad p(\mathcal{T}) = \epsilon\, {}^{\rho}\mathcal{I}^{\varsigma}_{0+} q(\varpi), \quad q(\mathcal{T}) = \pi\, {}^{\rho}\mathcal{I}^{\varrho}_{0+} p(\sigma) \quad 0 < \sigma < \varpi < \mathcal{T}, \end{cases} \tag{19} $$

is given by

$$ p(\tau) = {}^{\rho}\mathcal{I}^{\xi}_{0+} \hat{f}(\tau) + \delta(\tau) \Big[\widehat{\mathcal{E}} \Big(\epsilon\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+} \hat{g}(\varpi) - {}^{\rho}\mathcal{I}^{\xi}_{0+} \hat{f}(\mathcal{T}) \Big) + \mathcal{E}_1 \Big(\pi\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+} \hat{f}(\sigma) - {}^{\rho}\mathcal{I}^{\zeta}_{0+} \hat{g}(\mathcal{T}) \Big) \Big] \tag{20} $$

and

$$ q(\tau) = {}^{\rho}\mathcal{I}^{\zeta}_{0+} \hat{g}(\tau) + \delta(\tau) \Big[\widehat{\mathcal{E}} \Big(\pi\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+} \hat{f}(\sigma) - {}^{\rho}\mathcal{I}^{\zeta}_{0+} \hat{g}(\mathcal{T}) \Big) + \mathcal{E}_2 \Big(\epsilon\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+} \hat{g}(\varpi) - {}^{\rho}\mathcal{I}^{\xi}_{0+} \hat{f}(\mathcal{T}) \Big) \Big]. \tag{21} $$

Proof. When ${}^{\rho}\mathcal{I}_{0+}^{\xi}, {}^{\rho}\mathcal{I}_{0+}^{\zeta}$ are applied to the FDEs in (19) and Lemma 2 is used, the solution of the FDEs in (19) for $\tau \in \mathcal{E}$ is

$$p(\tau) = {}^{\rho}\mathcal{I}_{0+}^{\xi}\hat{f}(\tau) + a_1 + a_2\frac{\tau^{\rho}}{\rho} = \frac{\rho^{1-\xi}}{\Gamma(\xi)}\int_0^{\tau}\theta^{\rho-1}(\tau^{\rho}-\theta^{\rho})^{\xi-1}\hat{f}(\theta)d\theta + a_1 + a_2\frac{\tau^{\rho}}{\rho}, \tag{22}$$

$$q(\tau) = {}^{\rho}\mathcal{I}_{0+}^{\zeta}\hat{g}(\tau) + b_1 + b_2\frac{\tau^{\rho}}{\rho} = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^{\tau}\theta^{\rho-1}(\tau^{\rho}-\theta^{\rho})^{\zeta-1}\hat{g}(\theta)d\theta + b_1 + b_2\frac{\tau^{\rho}}{\rho}, \tag{23}$$

respectively, for some $a_1, a_2, b_1, b_2 \in \mathcal{R}$. Making use of the boundary conditions $p(0) = q(0) = 0$ in (22) and (23) respectively, we get $a_1 = b_1 = 0$. Next, we obtain by using the generalized integral operators ${}^{\rho}\mathcal{I}_{0+}^{\xi}, {}^{\rho}\mathcal{I}_{0+}^{\zeta}$ (22) and (23) respectively,

$$^{\rho}\mathcal{I}_{0+}^{\varrho}p(\tau) = {}^{\rho}\mathcal{I}_{0^+}^{\xi+\varrho}\hat{f}(\tau) + a_1\frac{\tau^{\rho\varrho}}{\rho^{\varrho}\Gamma(\varrho+1)} + a_2\frac{\tau^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)}, \tag{24}$$

$$^{\rho}\mathcal{I}_{0+}^{\varsigma}q(\tau) = {}^{\rho}\mathcal{I}_{0^+}^{\zeta+\varsigma}\hat{g}(\tau) + b_1\frac{\tau^{\rho\varsigma}}{\rho^{\varsigma}\Gamma(\varsigma+1)} + b_2\frac{\tau^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)}, \tag{25}$$

which, when combined with the boundary conditions $p(\mathcal{T}) = \epsilon\, {}^{\rho}\mathcal{I}_{0+}^{\varsigma}q(\varpi)$, $q(\mathcal{T}) = \pi{}^{\rho}\mathcal{I}_{0+}^{\varrho}p(\sigma)$, gives the following results:

$$^{\rho}\mathcal{I}_{0+}^{\xi}\hat{f}(\mathcal{T}) + a_1 + a_2\frac{\mathcal{T}^{\rho}}{\rho} = \epsilon{}^{\rho}\mathcal{I}_{0^+}^{\zeta+\varsigma}\hat{g}(\varpi) + b_1\frac{\epsilon\varpi^{\rho\varsigma}}{\rho^{\varsigma}\Gamma(\varsigma+1)} + b_2\frac{\epsilon\varpi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)}, \tag{26}$$

$$^{\rho}\mathcal{I}_{0+}^{\zeta}\hat{g}(\mathcal{T}) + b_1 + b_2\frac{\mathcal{T}^{\rho}}{\rho} = \pi{}^{\rho}\mathcal{I}_{0^+}^{\xi+\varrho}\hat{f}(\sigma) + a_1\frac{\pi\sigma^{\rho\varrho}}{\rho^{\varrho}\Gamma(\varrho+1)} + a_2\frac{\pi\sigma^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)}. \tag{27}$$

Next, we obtain

$$a_2\widehat{\mathcal{E}} - b_2\mathcal{E}_1 = \epsilon\, {}^{\rho}\mathcal{I}_{0^+}^{\zeta+\varsigma}\hat{g}(\varpi) - {}^{\rho}\mathcal{I}_{0+}^{\xi}\hat{f}(\mathcal{T}), \tag{28}$$

$$b_2\widehat{\mathcal{E}} - a_2\mathcal{E}_2 = \pi\, {}^{\rho}\mathcal{I}_{0^+}^{\xi+\varrho}\hat{f}(\sigma) - {}^{\rho}\mathcal{I}_{0+}^{\zeta}\hat{g}(\mathcal{T}), \tag{29}$$

by employing the notations (16) in (26) and (27) respectively. We find that when we solve the system of Equations (28) and (29) for a_2 and b_2,

$$a_2 = \frac{1}{\mathcal{G}}\Big[\widehat{\mathcal{E}}\Big(\epsilon\, {}^{\rho}\mathcal{I}_{0^+}^{\zeta+\varsigma}\hat{g}(\varpi) - {}^{\rho}\mathcal{I}_{0+}^{\xi}\hat{f}(\mathcal{T})\Big) + \mathcal{E}_1\Big(\pi\, {}^{\rho}\mathcal{I}_{0^+}^{\xi+\varrho}\hat{f}(\sigma) - {}^{\rho}\mathcal{I}_{0+}^{\zeta}\hat{g}(\mathcal{T})\Big)\Big], \tag{30}$$

$$b_2 = \frac{1}{\mathcal{G}}\Big[\mathcal{E}_2\Big(\epsilon\, {}^{\rho}\mathcal{I}_{0^+}^{\zeta+\varsigma}\hat{g}(\varpi) - {}^{\rho}\mathcal{I}_{0+}^{\xi}\hat{f}(\mathcal{T})\Big) + \widehat{\mathcal{E}}\Big(\pi\, {}^{\rho}\mathcal{I}_{0^+}^{\xi+\varrho}\hat{f}(\sigma) - {}^{\rho}\mathcal{I}_{0+}^{\zeta}\hat{g}(\mathcal{T})\Big)\Big]. \tag{31}$$

Substituting the values of a_1, a_2, b_1, b_2 in (22) and (23) respectively, we get the solution for the BVP (19). □

3. Existence Results for the Problem (1) and (2)

As a result of Lemma 3, we define an operator $\Delta : \mathcal{P}\times\mathcal{Q} \to \mathcal{P}\times\mathcal{Q}$ by

$$\Delta(p,q)(\tau) = (\Delta_1(p,q)(\tau), \Delta_2(p,q)(\tau)), \tag{32}$$

where

$$\Delta_1(p,q)(\tau) = {}^{\rho}\mathcal{I}^{\xi}_{0+} f(\tau, p(\tau), q(\tau)) + \delta(\tau)\Bigg[\widehat{\mathcal{E}}\Big(\epsilon\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+} g(\varpi, p(\varpi), q(\varpi)) - {}^{\rho}\,\mathcal{I}^{\xi}_{0+} f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)$$
$$+ \mathcal{E}_1\Big(\pi\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+} f(\sigma, p(\sigma), q(\sigma)) - {}^{\rho}\,\mathcal{I}^{\zeta}_{0+} g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)\Bigg], \tag{33}$$

$$\Delta_2(p,q)(\tau) = {}^{\rho}\mathcal{I}^{\zeta}_{0+} g(\tau, p(\tau), q(\tau)) + \delta(\tau)\Bigg[\widehat{\mathcal{E}}\Big(\pi\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+} f(\sigma, p(\sigma), q(\sigma)) - {}^{\rho}\,\mathcal{I}^{\zeta}_{0+} g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)$$
$$+ \mathcal{E}_2\Big(\epsilon\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+} g(\varpi, p(\varpi), q(\varpi)) - {}^{\rho}\,\mathcal{I}^{\xi}_{0+} f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)\Bigg]. \tag{34}$$

For brevity's sake, we'll use the following notations:

$$\mathcal{J}_1 = \frac{\Big(\mathcal{T}^{\rho\xi}(1 + |\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}, \tag{35}$$

$$\mathcal{K}_1 = |\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right), \tag{36}$$

$$\mathcal{J}_2 = |\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right), \tag{37}$$

$$\mathcal{K}_2 = \frac{\Big(\mathcal{T}^{\rho\zeta}(1 + |\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}, \tag{38}$$

$$\Phi = \min\{1 - [\psi_1(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi_1}(\mathcal{K}_1 + \mathcal{K}_2)], 1 - [\psi_2(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi_2}(\mathcal{K}_1 + \mathcal{K}_2)]\}. \tag{39}$$

Theorem 1. *Assume that $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition: $(\mathcal{A}_1)$ there exists constants $\psi_m, \hat{\psi_m} \geq 0 (m = 1, 2)$ and $\psi_0, \hat{\psi_0} > 0$ such that*

$$|f(\tau, o_1, o_2)| \leq \psi_0 + \psi_1|o_1| + \psi_2|o_2|,$$
$$|g(\tau, o_1, o_2)| \leq \hat{\psi_0} + \hat{\psi_1}|o_1| + \hat{\psi_2}|o_2|, \forall o_m \in \mathbb{R}, m = 1, 2.$$

If $\psi_1(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi_1}(\mathcal{K}_1 + \mathcal{K}_2) < 1, \psi_2(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi_2}(\mathcal{K}_1 + \mathcal{K}_2) < 1$. Then $\exists$ at least one solution for the BVP (1) and (2) on $\mathcal{E}$, where $\mathcal{J}_1, \mathcal{K}_1, \mathcal{J}_2, \mathcal{K}_2$ are given by (35)–(38) respectively.

Proof. We define operator $\Delta : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q}$ as being completely continuous in the first step. The continuity of the functions f and g implies that the operators Δ_1 and Δ_2 are continuous. As a result, the operator Δ is continuous. Let $\Psi \subset \mathcal{P} \times \mathcal{Q}$ be a bounded set to demonstrate the uniformly bounded operator Δ. Then $\hat{\mathcal{N}_1}$ and $\hat{\mathcal{N}_2}$ are positive constants such that $|f(\tau, p(\tau), q(\tau))| \leq \hat{\mathcal{N}_1}, |g(\tau, p(\tau), q(\tau))| \leq \hat{\mathcal{N}_2}, \forall (p, q) \in \Psi$. Then we have

$$
\begin{aligned}
|\Delta_1(p,q)(\tau)| \leq & {}^{\rho}\mathcal{I}^{\xi}_{0+}|f(\tau,p(\tau),q(\tau))| + |\delta(\tau)|\Bigg[|\widehat{\mathcal{E}}|\Big(|\epsilon|\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+}|g(\varpi,p(\varpi),q(\varpi))| + {}^{\rho}\,\mathcal{I}^{\xi}_{0+}|f(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big) \\
& + |\mathcal{E}_1|\Big(|\pi|\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+}|f(\sigma,p(\sigma),q(\sigma))| + {}^{\rho}\,\mathcal{I}^{\zeta}_{0+}|g(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big)\Bigg] \\
& \leq \hat{\mathcal{N}_1}\left\{\frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)} + \frac{\left(\mathcal{T}^{\rho\xi}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\xi}\Gamma(\xi+1)}\right\} \\
& + \hat{\mathcal{N}_2}\left\{\left(\frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)} + \frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)}\right)|\delta|\right\},
\end{aligned}
$$

when taking the norm and using (35) and (36), that yields for $(p,q) \in \Psi$,

$$||\Delta_1(p,q)|| \leq \mathcal{J}_1\hat{\mathcal{N}_1} + \mathcal{K}_1\hat{\mathcal{N}_2}. \tag{40}$$

Likewise, we obtain

$$
\begin{aligned}
||\Delta_2(p,q)|| \leq & \hat{\mathcal{N}_2}\left\{\frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)} + \frac{\left(\mathcal{T}^{\rho\zeta}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\zeta}\Gamma(\zeta+1)}\right\} \\
& + \hat{\mathcal{N}_1}\left\{|\delta|\left(\frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)} + \frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)}\right)\right\} \\
\leq & \mathcal{J}_2\hat{\mathcal{N}_1} + \mathcal{K}_2\hat{\mathcal{N}_2},
\end{aligned} \tag{41}
$$

using (37) and (38). Based on the inequalities (40) and (41), we can conclude that Δ_1 and Δ_2 are uniformly bounded, which indicates that the operator Δ is uniformly bounded. Next, we show that Δ is equicontinuous. Let $\tau_1, \tau_2 \in \mathcal{E}$ with $\tau_1 < \tau_2$. Then we have

$$
\begin{aligned}
& |\Delta_1(p,q)(\tau_2) - \Delta_1(p,q)(\tau_1)| \\
& \leq |{}^{\rho}\mathcal{I}^{\xi}_{0+}f(\tau_2,p(\tau_2),q(\tau_2)) - {}^{\rho}\mathcal{I}^{\xi}_{0+}f(\tau_1,p(\tau_1),q(\tau_1))| \\
& + |\delta(\tau_2)-\delta(\tau_1)|\Bigg[\widehat{\mathcal{E}}\Big(|\epsilon|\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0^+}|g(\varpi,p(\varpi),q(\varpi))| + {}^{\rho}\,\mathcal{I}^{\xi}_{0+}|f(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big) \\
& + \mathcal{E}_1\Big(|\pi|\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0^+}|f(\sigma,p(\sigma),q(\sigma))| + {}^{\rho}\,\mathcal{I}^{\zeta}_{0+}|g(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big)\Bigg] \\
\leq & \frac{\rho^{1-\xi}\hat{\mathcal{N}_1}}{\Gamma(\xi)}\left|\int_0^{\tau_1}\left[\frac{\theta^{\rho-1}}{(\tau_2^{\rho}-\theta^{\rho})^{1-\xi}} - \frac{\theta^{\rho-1}}{(\tau_1^{\rho}-\theta^{\rho})^{1-\xi}}\right]d\theta + \int_{\tau_1}^{\tau_2}\frac{\theta^{\rho-1}}{(\tau_2^{\rho}-\theta^{\rho})^{1-\xi}}d\theta\right| \\
& + |\delta(\tau_2)-\delta(\tau_1)|\left[|\widehat{\mathcal{E}}|\left(\frac{\hat{\mathcal{N}_2}|\epsilon|\varpi^{\rho\zeta+\varsigma}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)} + \frac{\hat{\mathcal{N}_1}\mathcal{T}^{\rho\xi}}{\rho^{\xi}\Gamma(\xi+1)}\right)\right] \\
& + |\delta(\tau_2)-\delta(\tau_1)|\left[|\mathcal{E}_1|\left(\frac{\hat{\mathcal{N}_1}|\pi|\sigma^{\rho\xi+\varrho}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)} + \frac{\hat{\mathcal{N}_2}\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)}\right)\right] \\
& \to 0 \text{ as } \tau_2 \to \tau_1.
\end{aligned} \tag{42}
$$

independent of (p,q) with respect to $|f(\tau,p(\tau_1),q(\tau_1))| \leq \hat{\mathcal{N}_1}$ and $|g(\tau,p(\tau_1),q(\tau_1))| \leq \hat{\mathcal{N}_2}$. Similarly, we can express $|\Delta_2(p,q)(\tau_2) - \Delta_2(p,q)(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$ independent of (p,q) in terms of the boundedness of f and g. As a result of the equicontinuity of Δ_1 and Δ_2, operator Δ is equicontinuous. As a result of the Arzela–Ascoli theorem, the operator is compact. Finally, we demonstrate that the set $\Pi(\Delta) = \{(p,q) \in \mathcal{P} \times \mathcal{Q} : \lambda\Delta(p,q);$

$0 < \lambda < 1\}$ is bounded. Let $(p,q) \in \Pi(\Delta)$.Then $(p,q) = \lambda\Delta(p,q)$. For any $\tau \in \mathcal{E}$, we have $p(\tau) = \lambda\Delta_1(p,q)(\tau), q(\tau) = \lambda\Delta_2(p,q)(\tau)$. By utilizing $(\mathcal{A}_1)$ in (33), we obtain

$$\begin{aligned}|p(\tau)| \leq &{}^{\rho}\mathcal{I}^{\xi}_{0+}(\psi_0, \psi_1|p(\tau)|, \psi_2|q(\tau)|)\\ &+ |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big(|\epsilon|^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}(\hat{\psi}_0 + \hat{\psi}_1|p(\varpi)| + \hat{\psi}_2|q(\varpi)|) +^{\rho}\mathcal{I}^{\xi}_{0+}(\psi_0 + \psi_0|p(\mathcal{T})| + \psi_2|q(\mathcal{T})|)\Big)\\ &+ |\mathcal{E}_1|\Big(|\pi|^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}(\psi_0 + \psi_1|p(\sigma)| + \psi_2|q(\sigma)|) +^{\rho}\mathcal{I}^{\zeta}_{0+}(\hat{\psi}_0 + \hat{\psi}_1|p(\mathcal{T})| + \hat{\psi}_2|q(\mathcal{T})|)\Big)\Bigg),\end{aligned}$$

which results when taking the norm for $\tau \in \mathcal{E}$,

$$||p|| \leq (\psi_0 + \psi_1||p|| + \psi_2||q||)\mathcal{J}_1 + (\hat{\psi}_0 + \hat{\psi}_1||p|| + \hat{\psi}_2||q||)\mathcal{K}_1. \tag{43}$$

Similarly, we are capable of obtaining that

$$||q|| \leq (\hat{\psi}_0 + \hat{\psi}_1||p|| + \hat{\psi}_2||q||)\mathcal{K}_2 + (\psi_0 + \psi_1||p|| + \psi_2||q||)\mathcal{J}_2. \tag{44}$$

From (43) and (44), we get

$$\begin{aligned}||p|| + ||q|| =&\psi_0(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_0(\mathcal{K}_1 + \mathcal{K}_2) + ||p||[\psi_1(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_1(\mathcal{K}_1 + \mathcal{K}_2)]\\ &+ ||q||[\psi_1(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_1(\mathcal{K}_1 + \mathcal{K}_2)],\end{aligned}$$

which results, with $||(p,q)|| = ||p|| + ||q||$,

$$||(p,q)|| \leq \frac{\psi_0(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_0(\mathcal{K}_1 + \mathcal{K}_2)}{\Phi}.$$

As a result, $\Pi(\Delta)$ is bounded. Thus, the nonlinear alternative of Leray–Schauder [26] is valid and the operator Δ has at least one fixed point. It implies that the BVP (1) and (2) contain at least one solution on $\mathcal{E}$. □

Theorem 2. *Assume that $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition: $(\mathcal{A}_2)$ there exists constants $\phi_m, \hat{\phi_m} \geq 0 (m = 1, 2)$ such that*

$$\begin{aligned}|f(\tau, o_1, o_2) - f(\tau, \hat{o}_1, \hat{o}_2)| \leq&\phi_1|o_1 - \hat{o}_1| + \phi_2|o_2 - \hat{o}_2|,\\ |g(\tau, o_1, o_2) - g(\tau, \hat{o}_1, \hat{o}_2)| \leq&\hat{\phi}_1|o_1 - \hat{o}_1| + \hat{\phi}_2|o_2 - \hat{o}_2|, \forall o_m, \hat{o}_m \in \mathbb{R}, m = 1, 2.\end{aligned}$$

Furthermore, there exist $\mathcal{S}_1, \mathcal{S}_2 > 0$ such that $|f(\tau, 0, 0)| \leq \mathcal{S}_1, |g(\tau, 0, 0)| \leq \mathcal{S}_2$, Then, given that

$$(\mathcal{J}_1 + \mathcal{J}_2)(\phi_1 + \phi_2) + (\mathcal{K}_1 + \mathcal{K}_2)(\hat{\phi}_1 + \hat{\phi}_2) < 1, \tag{45}$$

the BVP (1) and (2) has a unique solution on $\mathcal{E}$, where $\mathcal{J}_1, \mathcal{K}_1, \mathcal{J}_2, \mathcal{K}_2$ are given by (35)–(38) respectively.

Proof. Let us fix $\varphi \leq \frac{(\mathcal{J}_1+\mathcal{J}_2)\mathcal{S}_1+(\mathcal{K}_1+\mathcal{K}_2)\mathcal{S}_2}{1-((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi}_1+\hat{\phi}_2))}$ and demonstrate that $\Delta\mathcal{B}_\varphi \subset \mathcal{B}_\varphi$ when operator Δ is given by (32) and $\mathcal{B}_\varphi = \{(p,q) \in \mathcal{P} \times \mathcal{Q} : ||(p,q)|| \leq \varphi\}$. For $(p,q) \in \mathcal{B}_\varphi, \tau \in \mathcal{E}$

$$\begin{aligned}|f(\tau, p(\tau), q(\tau))| &\leq \phi_1|p(\tau)| + \phi_2|q(\tau)| + \mathcal{S}_1\\ &\leq \phi_1||p|| + \phi_2||q|| + \mathcal{S}_1,\end{aligned}$$

and

$$|g(\tau,p(\tau),q(\tau))| \leq \hat{\phi}_1|p(\tau)| + \hat{\phi}_2|q(\tau)| + \mathcal{S}_2$$
$$\leq \hat{\phi}_1||p|| + \hat{\phi}_2||q|| + \mathcal{S}_2. \tag{46}$$

This guides to

$$\begin{aligned}
|\Delta_1(p,q)(\tau)| \leq & {}^{\rho}\mathcal{I}^{\xi}_{0+}\Big[|f(\tau,p(\tau),q(\tau)) - f(\tau,0,0)| + |f(\tau,0,0)|\Big] \\
& + |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big(|\epsilon|^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}g[(\varpi,p(\varpi),q(\varpi)) - g(\varpi,0,0)| + |g(\varpi,0,0)|] \\
& + {}^{\rho}\mathcal{I}^{\xi}_{0+}f[(\mathcal{T},p(\mathcal{T}),q(\mathcal{T})) - f(\mathcal{T},0,0)| + |f(\mathcal{T},0,0)]|\Big) \\
& + |\mathcal{E}_1|\Big(|\pi|^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}f[f(\sigma,p(\sigma),q(\sigma)) - f(\sigma,0,0)| + |f(\sigma,0,0)|] \\
& + {}^{\rho}\mathcal{I}^{\zeta}_{0+}[|g(\mathcal{T},p(\mathcal{T}),q(\mathcal{T})) - g(\mathcal{T},0,0)| + |g(\mathcal{T},0,0)|]\Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
\leq & (\phi_1||p|| + \phi_2||q|| + \mathcal{S}_1)\left\{\frac{\left(\mathcal{T}^{\rho\xi}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right\} \\
& + (\hat{\phi}_1||p|| + \hat{\phi}_2||q|| + \mathcal{S}_2)\left\{|\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right)\right\}
\end{aligned}$$

$$||\Delta_1(p,q)|| \leq (\phi_1||p|| + \phi_2||q|| + \mathcal{S}_1)\mathcal{J}_1 + (\hat{\phi}_1||p|| + \hat{\phi}_2||q|| + \mathcal{S}_2)\mathcal{K}_1. \tag{47}$$

Similarly, we obtain

$$\begin{aligned}
|\Delta_2(p,q)(\tau)| \leq & (\hat{\phi}_1||p|| + \hat{\phi}_2||q|| + \mathcal{S}_2)\left\{\frac{\left(\mathcal{T}^{\rho\zeta}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right\} \\
& + (\phi_1||p|| + \phi_2||q|| + \mathcal{S}_1)\left\{|\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right)\right\} \\
||\Delta_2(p,q)|| \leq & (\hat{\phi}_1||p|| + \hat{\phi}_2||q|| + \mathcal{S}_2)\mathcal{K}_2 + (\phi_1||p|| + \phi_2||q|| + \mathcal{S}_1)\mathcal{J}_2.
\end{aligned} \tag{48}$$

As a result, (47) and (48) follow $||\Delta(p,q)|| \leq \varphi$, and thus $\Delta\mathcal{B}_{\varphi} \subset \mathcal{B}_{\varphi}$. Now, for $(p_1,q_1),(p_2,q_2) \in \mathcal{P} \times \mathcal{Q}$ and any $\tau \in \mathcal{E}$, we get

$$\begin{aligned}
& |\Delta_1(p_1,q_1)(\tau) - \Delta_1(p_2,q_2)(\tau)| \\
\leq & {}^{\rho}\mathcal{I}^{\xi}_{0+}|f(\tau,p_1(\tau),q_1(\tau)) - f(\tau,p_2(\tau),q_2(\tau))| \\
& + |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big(|\epsilon|\ {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}|g(\varpi,p_1(\varpi),q_1(\varpi)) - g(\varpi,p_2(\varpi),q_2(\varpi))| \\
& + {}^{\rho}\mathcal{I}^{\xi}_{0+}|f(\mathcal{T},p_1(\mathcal{T}),q_1(\mathcal{T})) - f(\mathcal{T},p_2(\mathcal{T}),q_2(\mathcal{T}))|\Big) \\
& + |\mathcal{E}_1|\Big(|\pi|\ {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}|f(\sigma,p_1(\sigma),q_1(\sigma)) - f(\sigma,p_2(\sigma),q_2(\sigma))| \\
& + {}^{\rho}\mathcal{I}^{\zeta}_{0+}|g(\mathcal{T},p_1(\mathcal{T}),q_1(\mathcal{T})) - g(\mathcal{T},p_2(\mathcal{T}),q_2(\mathcal{T}))|\Big)\Bigg)
\end{aligned}$$

$$\leq(\phi_1||p_1-p_2||+\phi_2||q_1-q_2||)\left\{\frac{\left(\mathcal{T}^{\rho\xi}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\xi}\Gamma(\xi+1)}+\frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right\}$$
$$+(\hat{\phi}_1||p_1-p_2||+\hat{\phi}_2||q_1-q_2||)\left\{|\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)}+\frac{|\widehat{\mathcal{E}}||\epsilon|\omega^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right)\right\}$$

$$\leq(\mathcal{J}_1(\phi_1+\phi_2)+\mathcal{K}_1(\hat{\phi}_1+\hat{\phi}_2))(||p_1-p_2||+||q_1-q_2||).$$

Similarly, we obtain

$$|\Delta_2(p_1,q_1)(\tau)-\Delta_2(p_2,q_2)(\tau)|$$
$$\leq(\hat{\phi}_1||p_1-p_2||+\hat{\phi}_2||q_1-q_2||)\left\{\frac{\left(\mathcal{T}^{\rho\zeta}(1+|\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\zeta}\Gamma(\zeta+1)}+\frac{|\delta||\epsilon||\mathcal{E}_2|\omega^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right\}$$
$$+(\phi_1||p_1-p_2||+\phi_2||q_1-q_2||)\left\{|\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)}+\frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right)\right\}$$
$$\leq(\mathcal{J}_2(\phi_1+\phi_2)+\mathcal{K}_2(\hat{\phi}_1+\hat{\phi}_2))(||p_1-p_2||+||q_1-q_2||)).$$

Thus we obtain

$$||\Delta_1(p_1,q_1)(\tau)-\Delta_1(p_2,q_2)(\tau)||\leq(\mathcal{J}_1(\phi_1+\phi_2)+\mathcal{K}_1(\hat{\phi}_1+\hat{\phi}_2))(||p_1-p_2||+||q_1-q_2||). \tag{49}$$

In a similar manner,

$$||\Delta_2(p_1,q_1)(\tau)-\Delta_2(p_2,q_2)(\tau)||\leq(\mathcal{J}_2(\phi_1+\phi_2)+\mathcal{K}_2(\hat{\phi}_1+\hat{\phi}_2))(||p_1-p_2||+||q_1-q_2||). \tag{50}$$

Hence, using (49) and (50) we can get

$$||\Delta(p_1,q_1)(\tau)-\Delta(p_2,q_2)(\tau)||\leq((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi}_1+\hat{\phi}_2))$$
$$(||p_1-p_2||+||q_1-q_2||).$$

As a consequence of condition $((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi}_1+\hat{\phi}_2))<1$, Δ is a contraction operator. As an outcome of the Banach fixed point theorem, we can conclude that operator has a unique fixed point, which is the unique solution of the problem (1), and (2). □

For brevity's sake, we'll use the following notations:

$$\hat{\Omega}_1=\mathcal{J}_1-\frac{\mathcal{T}^{\rho\xi}}{\rho^{\xi}\Gamma(\xi+1)}+\mathcal{K}_1, \tag{51}$$
$$\hat{\Omega}_2=\mathcal{J}_2-\frac{\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)}+\mathcal{K}_2. \tag{52}$$

Theorem 3. *Assume that $f,g:\mathcal{E}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions satisfying the assumption $(\mathcal{A}_2)$ in Theorem 2. Furthermore, there exist positive constants $\mathcal{U}_1,\mathcal{U}_2$ such that $\forall\tau\in\mathcal{E}$ and $r_i\in\mathbb{R}, i=1,2$.*

$$|f(\tau,r_1,r_2)|\leq\mathcal{U}_1,\qquad |g(\tau,r_1,r_2)|\leq\mathcal{U}_2. \tag{53}$$

If

$$\frac{\mathcal{T}^{\rho\xi}(\phi_1+\phi_2)}{\rho^{\xi}\Gamma(\xi+1)}+\frac{\mathcal{T}^{\rho\zeta}(\hat{\phi}_1+\hat{\phi}_2)}{\rho^{\zeta}\Gamma(\zeta+1)}<1, \tag{54}$$

then the BVP (1), and (2) has at least one solution on $\mathcal{E}$.

Proof. Let us define a closed ball $\mathcal{B}_\varphi = \{(p,q) \in \mathcal{P} \times \mathcal{Q} : ||(p,q)|| \leq \varphi\}$ as follows and split Δ_1, Δ_2 as:

$$\begin{aligned}\Delta_{1,1}(p,q)(\tau) =&\delta(\tau)\Bigg(\widehat{\mathcal{E}}\Big(\epsilon\, {}^{\rho}\mathcal{I}_{0+}^{\zeta+\varsigma} g(\varpi, p(\varpi), q(\varpi)) - {}^{\rho}\,\mathcal{I}_{0+}^{\xi} f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big) \\ &+ \mathcal{E}_1\Big(\pi\, {}^{\rho}\mathcal{I}_{0+}^{\xi+\varrho} f(\sigma, p(\sigma), q(\sigma)) - {}^{\rho}\,\mathcal{I}_{0+}^{\zeta} g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)\Bigg), \qquad (55)\\ \Delta_{1,1}(p,q)(\tau) =&{}^{\rho}\mathcal{I}_{0+}^{\xi} f(\tau, p(\tau), q(\tau)), \qquad (56)\end{aligned}$$

$$\begin{aligned}\Delta_{2,1}(p,q)(\tau) =&\delta(\tau)\Bigg(\widehat{\mathcal{E}}\Big(\pi\, {}^{\rho}\mathcal{I}_{0+}^{\xi+\varrho} f(\sigma, p(\sigma), q(\sigma)) - {}^{\rho}\,\mathcal{I}_{0+}^{\zeta} g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big) \\ &+ \mathcal{E}_2\Big(\epsilon\, {}^{\rho}\mathcal{I}_{0+}^{\zeta+\varsigma} g(\varpi, p(\varpi), q(\varpi)) - {}^{\rho}\,\mathcal{I}_{0+}^{\xi} f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))\Big)\Bigg), \qquad (57)\\ \Delta_{2,2}(p,q)(\tau) =&{}^{\rho}\mathcal{I}_{0+}^{\xi} g(\tau, p(\tau), q(\tau)). \qquad (58)\end{aligned}$$

In the Banach space $\mathcal{P} \times \mathcal{Q}, \Delta_1(p,q)(\tau) = \Delta_{1,1}(p,q)(\tau) + \Delta_{1,2}(p,q)(\tau)$, and $\Delta_2(p,q)(\tau) = \Delta_{2,1}(p,q)(\tau) + \Delta_{2,2}(p,q)(\tau)$ on $\mathcal{B}_\varphi$ are closed, bounded and convex subsets of $\mathcal{P} \times \mathcal{Q}$. Let us fix $\varphi \leq \max\{\mathcal{J}_1\mathcal{U}_1 + \mathcal{K}_1\mathcal{U}_2, \mathcal{J}_2\mathcal{U}_1 + \mathcal{K}_2\mathcal{U}_2\}$ and show that $\Delta\mathcal{B}_\varphi \subset \mathcal{B}_\varphi$ to verify Krasnoselskii's theorem [27] condition (i), If we choose $p = (p_1, p_2), q = (q_1, q_2) \in \mathcal{B}_\varphi$, and utilizing condition (53), we obtain

$$\begin{aligned}&|\Delta_{1,1}(p,q)(\tau) + \Delta_{1,2}(p,q)(\tau)| \\ \leq&{}^{\rho}\mathcal{I}_{0+}^{\xi}|f(\tau, p(\tau), q(\tau))| \\ &+ |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big(|\epsilon|\, {}^{\rho}\mathcal{I}_{0+}^{\zeta+\varsigma}|g(\varpi, p(\varpi), q(\varpi))| + {}^{\rho}\,\mathcal{I}_{0+}^{\xi}|f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))|\Big) \\ &+ |\mathcal{E}_1|\Big(|\pi|{}^{\rho}\mathcal{I}_{0+}^{\xi+\varrho}|f(\sigma, p(\sigma), q(\sigma))| + {}^{\rho}\,\mathcal{I}_{0+}^{\zeta}|g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))|\Big)\Bigg) \\ \leq&\,\mathcal{U}_1\left\{\frac{\left(\mathcal{T}^{\rho\xi}(1 + |\delta||\widehat{\mathcal{E}}|)\right)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right\} \\ &+ \mathcal{U}_2\left\{|\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right)\right\} \\ \leq&\,\mathcal{U}_1\mathcal{J}_1 + \mathcal{U}_2\mathcal{K}_1 \leq \varphi.\end{aligned}$$

In a similar manner, we can find that

$$|\Delta_{2,1}(p,q)(\tau) + \Delta_{2,2}(p,q)(\tau)| \leq \mathcal{U}_1\mathcal{J}_2 + \mathcal{U}_2\mathcal{K}_2 \leq \varphi.$$

Clearly the above two inequalities lead to the fact that $\Delta_1(p,q) + \Delta_2(p,q) \in \mathcal{B}_\varphi$. Thus, we define operator $(\Delta_{1,2}, \Delta_{2,2})$ as a contraction-satisfying condition (iii) of Krasnoselskii's theorem [27]. For $(p_1, q_1), (p_2, q_2) \in \mathcal{B}_\varphi$, we have

$$\begin{aligned}|\Delta_{1,2}(p_1,q_1)(\tau)-\Delta_{1,2}(p_2,q_2)(\tau)| \leq & \frac{\rho^{1-\xi}}{\Gamma(\xi)}\int_0^\tau \frac{\theta^{\rho-1}}{(\tau^\rho-\theta^\rho)^{1-\xi}} \\ & \times |f(\theta,p_1(\theta),q_1(\theta))-f(\theta,p_2(\theta),q_2(\theta))|d\theta \\ \leq & \frac{\mathcal{T}^{\rho\xi}}{\rho^\xi\Gamma(\xi+1)}(\phi_1||p_1-p_2||+\phi_2||q_1-q_2||) \end{aligned} \tag{59}$$

and

$$\begin{aligned}|\Delta_{2,1}(p_1,q_1)(\tau)-\Delta_{2,1}(p_2,q_2)(\tau)| \leq & \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^\tau \frac{\theta^{\rho-1}}{(\tau^\rho-\theta^\rho)^{1-\zeta}} \\ & \times |g(\theta,p_1(\theta),q_1(\theta))-g(\theta,p_2(\theta),q_2(\theta))|d\theta \\ \leq & \frac{\mathcal{T}^{\rho\zeta}}{\rho^\zeta\Gamma(\zeta+1)}(\hat{\phi}_1||p_1-p_2||+\hat{\phi}_2||q_1-q_2||). \end{aligned} \tag{60}$$

As a result (59) and (60),

$$\begin{aligned}&|(\Delta_{1,2},\Delta_{2,2})(p_1,q_1)(\tau)-(\Delta_{1,2},\Delta_{2,2})(p_2,q_2)(\tau)| \\ \leq & \frac{\mathcal{T}^{\rho\xi}(\phi_1+\phi_2)}{\rho^\xi\Gamma(\xi+1)}+\frac{\mathcal{T}^{\rho\zeta}(\hat{\phi}_1+\hat{\phi}_2)}{\rho^\zeta\Gamma(\zeta+1)}(||p_1-p_2||+||q_1-q_2||),\end{aligned}$$

is a contraction by (54). Therefore, condition (iii) of the Theorem is satisfied. Following that, we can establish that the operator $(\Delta_{1,1},\Delta_{2,1})$ satisfies the Krasnoselskii theorem's [27] condition (ii). We can infer the continuous existence of the $(\Delta_{1,1},\Delta_{2,1})$ operator by examining the continuity of the f,g functions. For each $(p,q)\in\mathcal{B}_\varphi$ we have

$$\begin{aligned}&|\Delta_{1,1}(p,q)(\tau)| \\ \leq & |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big(|\epsilon|\,{}^\rho\mathcal{I}_{0+}^{\zeta+\varsigma}|g(\varpi,p(\varpi),q(\varpi))|+{}^\rho\,\mathcal{I}_{0+}^{\xi}|f(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big) \\ & +|\mathcal{E}_1|\Big(|\pi|\,{}^\rho\mathcal{I}_{0+}^{\xi+\varrho}|f(\sigma,p(\sigma),q(\sigma))|+{}^\rho\,\mathcal{I}_{0+}^{\zeta}|g(\mathcal{T},p(\mathcal{T}),q(\mathcal{T}))|\Big)\Bigg) \\ & \leq \mathcal{U}_1\left\{\frac{\Big(\mathcal{T}^{\rho\xi}(|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^\xi\Gamma(\xi+1)}+\frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right\} \\ & \quad +\mathcal{U}_2\left\{|\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^\zeta\Gamma(\zeta+1)}+\frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right)\right\} \\ = & \hat{\Omega}_1,\end{aligned}$$

$$\begin{aligned}|\Delta_{2,1}(p,q)(\tau)| \leq & \mathcal{U}_2\left\{\frac{\Big(\mathcal{T}^{\rho\zeta}(|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^\zeta\Gamma(\zeta+1)}+\frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right\} \\ & +\mathcal{U}_1\left\{|\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^\xi\Gamma(\xi+1)}+\frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right)\right\} \\ = & \hat{\Omega}_2,\end{aligned}$$

which leads to

$$||(\Delta_{1,1},\Delta_{2,1})(p,q)||\leq\hat{\Omega}_1+\hat{\Omega}_2.$$

From the above inequalities, the set $(\Delta_{1,1}, \Delta_{2,1})\mathcal{B}_\varphi$ is uniformly bounded. The following step will demonstrate that the set $(\Delta_{1,1}, \Delta_{2,1})\mathcal{B}_\varphi$ is equicontinuous. For $\tau_1, \tau_2 \in \mathcal{E}$ with $\tau_1 < \tau_2$ and for any $(p, q) \in \mathcal{B}_\varphi$ we get

$$|\Delta_{1,1}(p,q)(\tau_2) - \Delta_{1,1}(p,q)(\tau_1)|$$

$$\begin{aligned}
\leq & |\delta(\tau_2) - \delta(\tau_1)| \Bigg(|\widehat{\mathcal{E}}| \Big(|\epsilon| \, {}^{\rho}\mathcal{I}_{0+}^{\zeta+\varsigma} |g(\omega, p(\omega), q(\omega))| + {}^{\rho}\mathcal{I}_{0+}^{\xi} |f(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))| \Big) \\
& + |\mathcal{E}_1| \Big(|\pi| \, {}^{\rho}\mathcal{I}_{0+}^{\xi+\varrho} |f(\sigma, p(\sigma), q(\sigma))| + {}^{\rho}\mathcal{I}_{0+}^{\zeta} |g(\mathcal{T}, p(\mathcal{T}), q(\mathcal{T}))| \Big) \Bigg) \\
\leq & |\delta(\tau_2) - \delta(\tau_1) \Bigg(\mathcal{U}_1 \Big(\frac{\Big(\mathcal{T}^{\rho\xi}(|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)} \Big) \\
& + \mathcal{U}_2 |\delta| \Big(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\omega^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)} \Big) \Bigg).
\end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
& |\Delta_{2,1}(p,q)(\tau_2) - \Delta_{2,1}(p,q)(\tau_1)| \\
\leq & |\delta(\tau_2) - \delta(\tau_1) \Bigg(\mathcal{U}_2 \Big(\frac{\Big(\mathcal{T}^{\rho\zeta}(|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\delta||\epsilon||\mathcal{E}_2|\omega^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)} \Big) \\
& + \mathcal{U}_1 \Bigg(|\delta| \Bigg(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)} \Bigg) \Bigg) \Bigg).
\end{aligned}$$

Therefore $|(\Delta_{1,1}, \Delta_{2,1}(\tau_2)) - (\Delta_{1,1}, \Delta_{2,1}(\tau_1))| \to 0$ as $\tau_2 \to \tau_1$ independent of $(p, q) \in \mathcal{B}_\varphi$ Thus the set $(\Delta_{1,1}, \Delta_{2,1})\mathcal{B}_\varphi$ is equicontinuous. As an outcome, the Arzela–Ascoli theorem implies that the operator $(\Delta_{1,1}, \Delta_{2,1})$ is compact on $\mathcal{B}\varphi$. Krasnoselskii's theorem [27] statement leads us to the conclusion that the problem (1) and (2) has at least one solution on $\mathcal{E}$. □

4. Example

Consider the following Liouville–Caputo type generalized FDEs coupled system:

$$\begin{cases} {}^{\frac{3}{4}}_{C}\mathcal{D}_{0+}^{\frac{5}{4}} p(\tau) = f(\tau, p(\tau), q(\tau)), \tau \in \mathcal{E} := [0,1], \\ {}^{\frac{3}{4}}_{C}\mathcal{D}_{0+}^{\frac{31}{20}} q(\tau) = g(\tau, p(\tau), q(\tau)), \tau \in \mathcal{E} := [0,1], \end{cases} \tag{61}$$

supplemented with boundary conditions:

$$\Big\{ p(0) = 0, \; q(0) = 0, \; p(1) = \tfrac{1}{6} {}^{\frac{3}{4}}\mathcal{I}^{\frac{13}{20}} q(\tfrac{7}{10}), \; q(1) = \tfrac{1}{7} {}^{\frac{3}{4}}\mathcal{I}^{\frac{17}{20}} p(\tfrac{1}{2}), \tag{62}$$

where $\xi = \frac{5}{4}, \zeta = \frac{31}{20}, \rho = \frac{3}{4}, \mathcal{T} = 1, \epsilon = \frac{1}{6}, \omega = \frac{7}{10}, \pi = \frac{1}{7}, \sigma = \frac{1}{2}, \varsigma = \frac{13}{20}, \varrho = \frac{17}{20}$ and

$$f(\tau, p(\tau), q(\tau)) = \frac{(1+\tau)}{30} \left(\frac{|p(\tau)|}{1+|p(\tau)|} + \frac{1}{3}\cos(q(\tau)) + 3\tau \right), \tag{63}$$

$$g(\tau, p(\tau), q(\tau)) = \frac{e^{-\tau}}{25} \left(\frac{\sqrt{\tau}+1}{5} + \frac{1}{6}\cos(p(\tau)) + \frac{|q(\tau)|}{1+|q(\tau)|} \right). \tag{64}$$

With $\psi_0 = \frac{1}{10}, \psi_1 = \frac{1}{30}, \psi_2 = \frac{1}{90}, \hat{\psi}_0 = \frac{1}{125}, \hat{\psi}_1 = \frac{1}{25}$, and $\hat{\psi}_2 = \frac{1}{150}$, the functions f and g clearly satisfy the $(\mathcal{A}_1)$ condition. Next, we find that $(\mathcal{J}_1) = 2.5370237266984113$,

$(\mathcal{K}_1) = 0.17111607453629377, \mathcal{J}_2 = 0.0906406939922634, \mathcal{K}_2 = 2.274156747108814, \mathcal{J}_i, \mathcal{K}_i$ $(i = 1,2)$ are respectively given by (35),(36),(37) and (38), based on the data available. Thus $\psi_1(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_1(\mathcal{K}_1 + \mathcal{K}_2) \approxeq 0.18539972688882678 < 1, \psi_2(\mathcal{J}_1 + \mathcal{J}_2) + \hat{\psi}_2(\mathcal{K}_1 + \mathcal{K}_2) \approxeq$ $0.04549809015197488 < 1$, all the conditions of Theorem 1 are satisfied, and there is at least one solution for problem (61) and (62) on $[0,1]$ with f and g given by (63) and (64) respectively.

In addition, we'll use

$$f(\tau, p(\tau), q(\tau)) = \frac{\tau}{3} + \frac{3}{4(\tau + 16)} + \frac{|p(\tau)|}{1 + |p(\tau)|} + \frac{2}{75}\cos(q(\tau)), \tag{65}$$

$$g(\tau, p(\tau), q(\tau)) = \frac{(1 + e^{-\tau})}{4} + \frac{19}{400}\cos(p(\tau)) + \frac{1}{60}\frac{|q(\tau)|}{1 + |q(\tau)|}, \tag{66}$$

to demonstrate Theorem 2. It is simple to demonstrate that f and g are continuous and satisfy the assumption $(\mathcal{A}_2)$ with $\phi_1 = \frac{3}{64}, \phi_2 = \frac{2}{75}, \hat{\phi}_1 = \frac{19}{400}$ and $\hat{\phi}_2 = \frac{1}{60}$. All the assumptions of Theorem 2 are also satisfied with $(\mathcal{J}_1 + \mathcal{J}_2)(\phi_1 + \phi_2) + (\mathcal{K}_1 + \mathcal{K}_2)(\hat{\phi}_1 + \hat{\phi}_2) \approxeq 0.35014782699385444 < 1$. As a result, Theorem 2 holds true, and the problem (61) and (62) with f and g given by (65) and (66) respectively, has a unique solution on [0,1].

5. Ulam–Hyers Stability Results for the Problem (1) and (2)

The U–H stability of the solutions to the BVP (1) and (2) will be discussed in this section using the integral representation of their solutions defined by

$$p(\tau) = \Delta_1(p,q)(\tau),\ q(\tau) = \Delta_2(p,q)(\tau), \tag{67}$$

where Δ_1 and Δ_2 are given by (33) and (34). Consider the following definitions of nonlinear operators

$$\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}(\mathcal{E}, \mathbb{R}) \times \mathcal{C}(\mathcal{E}, \mathbb{R}) \to \mathcal{C}(\mathcal{E}, \mathbb{R}),$$

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi}_{0+}p(\tau) - f(\tau, p(\tau), q(\tau)) = \mathcal{H}_1(p,q)(\tau), \tau \in \mathcal{E}, \\ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{0+}q(\tau) - g(\tau, p(\tau), q(\tau)) = \mathcal{H}_1(p,q)(\tau), \tau \in \mathcal{E}. \end{cases}$$

It considered the following inequalities for some $\hat{\lambda}_1, \hat{\lambda}_2 > 0:$

$$||\mathcal{H}_1(p,q)|| \le \hat{\lambda}_1, ||\mathcal{H}_2(p,q)|| \le \hat{\lambda}_2. \tag{68}$$

Definition 4. *The coupled system (1) and (2) is said to be U–H stable if $\mathcal{V}_1, \mathcal{V}_2 > 0$ and there exists a unique solution $(p,q) \in \mathcal{C}(\mathcal{E}, \mathbb{R})$ of a problem (1) and (2) with*

$$||(p,q) - (p^*, q^*)|| \le \mathcal{V}_1\hat{\lambda}_1 + \mathcal{V}_2\hat{\lambda}_2,$$

$\forall (p,q) \in \mathcal{C}(\mathcal{E}, \mathbb{R})$ of inequality (68).

Theorem 4. *Assume that $(\mathcal{A}_2)$ holds. Then the problem (1) and (2) is U–H stable.*

Proof. Let $(p,q) \in \mathcal{C}(\mathcal{E}, \mathbb{R}) \times \mathcal{C}(\mathcal{E}, \mathbb{R})$ be the (1)–(2) solution of the problem that satisfies (33) and (34). Let (p,q) be any solution that meets the condition (68):

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi}_{0+}p(\tau) = f(\tau, p(\tau), q(\tau)) + \mathcal{H}_1(p,q)(\tau), \tau \in \mathcal{E}, \\ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{0+}q(\tau) = g(\tau, p(\tau), q(\tau)) + \mathcal{H}_1(p,q)(\tau), \tau \in \mathcal{E}, \end{cases}$$

so,

$$\begin{aligned} p^*(\tau) = {} & \Delta_1(p^*,q^*)(\tau) +^{\rho} \mathcal{I}^{\xi}_{0+}\mathcal{H}_1(p,q)(\tau) \\ & + \delta(\tau)\Bigg(\widehat{\mathcal{E}}\Big[\epsilon\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}\mathcal{H}_2(p,q)(\varpi) -^{\rho} \mathcal{I}^{\xi}_{0+}\mathcal{H}_1(p,q)(\mathcal{T})\Big] \\ & + \mathcal{E}_1\Big[\pi\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}\mathcal{H}_1(p,q)(\sigma) -^{\rho} \mathcal{I}^{\zeta}_{0+}\mathcal{H}_2(p,q)(\mathcal{T})\Big]\Bigg). \end{aligned}$$

It follows that

$$\begin{aligned} |\Delta_1(p^*,q^*)(\tau) - p^*(\tau)| \leq & {}^{\rho}\mathcal{I}^{\xi}_{0+}|\mathcal{H}_1(p,q)(\tau)| \\ & + |\delta(\tau)|\Bigg(|\widehat{\mathcal{E}}|\Big[|\epsilon|\, {}^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}|\mathcal{H}_2(p,q)(\varpi)| +^{\rho} \mathcal{I}^{\xi}_{0+}|\mathcal{H}_1(p,q)(\mathcal{T})|\Big] \\ & + |\mathcal{E}_1|\Big[|\pi|\, {}^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}|\mathcal{H}_1(p,q)(\sigma)| +^{\rho} \mathcal{I}^{\zeta}_{0+}|\mathcal{H}_2(p,q)(\mathcal{T})|\Big]\Bigg) \\ \leq & \ \hat{\lambda}_1\left\{\frac{\Big(\mathcal{T}^{\rho\xi}(1+|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right\} \\ & + \hat{\lambda}_2\left\{|\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right)\right\} \\ \leq & \mathcal{J}_1\hat{\lambda}_1 + \mathcal{K}_1\hat{\lambda}_2. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} |\Delta_2(p^*,q^*)(\tau) - q^*(\tau)| \leq & \hat{\lambda}_2\left\{\frac{\Big(\mathcal{T}^{\rho\zeta}(1+|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right\} \\ & + \hat{\lambda}_1\left\{|\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\pi||\widehat{\mathcal{E}}|\sigma^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right)\right\} \\ \leq & \mathcal{J}_2\hat{\lambda}_1 + \mathcal{K}_2\hat{\lambda}_2, \end{aligned}$$

where $\mathcal{J}_1, \mathcal{K}_1, \mathcal{J}_2$, and $\mathcal{K}_2$ are defined in (35)–(38), respectively. As an outcome, we deduce from operator Δ's fixed-point property, which is defined by (33) and (34),

$$\begin{aligned} |p(\tau) - p^*(\tau)| = & |p(\tau) - \Delta_1(p^*,q^*)(\tau) + \Delta_1(p^*,q^*)(\tau) - p^*(\tau)| \\ \leq & |\Delta_1(p,q)(\tau) - \Delta_1(p^*,q^*)(\tau)| + |\Delta_1(p^*,q^*)(\tau) - p^*(\tau)| \\ \leq & ((\mathcal{J}_1\phi_1 + \mathcal{K}_1\hat{\phi}_1) + (\mathcal{J}_1\phi_2 + \mathcal{K}_1\hat{\phi}_2))||(p,q) - (p^*,q^*)|| \\ & + \mathcal{J}_1\hat{\lambda}_1 + \mathcal{K}_1\hat{\lambda}_2. \end{aligned} \tag{69}$$

$$\begin{aligned} |q(\tau) - q^*(\tau)| = & |q(\tau) - \Delta_2(p^*,q^*)(\tau) + \Delta_2(p^*,q^*)(\tau) - q^*(\tau)| \\ \leq & |\Delta_2(p,q)(\tau) - \Delta_2(p^*,q^*)(\tau)| + |\Delta_2(p^*,q^*)(\tau) - q^*(\tau)| \\ \leq & ((\mathcal{J}_2\phi_1 + \mathcal{K}_2\hat{\phi}_1) + (\mathcal{J}_2\phi_2 + \mathcal{K}_2\hat{\phi}_2))||(p,q) - (p^*,q^*)|| \\ & + \mathcal{J}_2\hat{\lambda}_1 + \mathcal{K}_2\hat{\lambda}_2. \end{aligned} \tag{70}$$

From the above Equations (69) and (70) it follows that

$$\begin{aligned}||(p,q)-(p^*,q^*)|| \leq &(\mathcal{J}_1+\mathcal{J}_2)\hat{\lambda_1}+(\mathcal{K}_1+\mathcal{K}_2)\hat{\lambda_2} \\ &+((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi_1}+\hat{\phi_2}))||(p,q)-(p^*,q^*)||.\end{aligned}$$

$$\begin{aligned}||(p,q)-(p^*,q^*)|| \leq &\frac{(\mathcal{J}_1+\mathcal{J}_2)\hat{\lambda_1}+(\mathcal{K}_1+\mathcal{K}_2)\hat{\lambda_2}}{1-((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi_1}+\hat{\phi_2}))} \\ \leq &\mathcal{V}_1\hat{\lambda_1}+\mathcal{V}_2\hat{\lambda_2},\end{aligned}$$

with

$$\mathcal{V}_1=\frac{\mathcal{J}_1+\mathcal{J}_2}{1-((\mathcal{J}_1+|\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+|\mathcal{K}_2)(\hat{\phi_1}+\hat{\phi_2}))},$$

$$\mathcal{V}_2=\frac{\mathcal{K}_1+\mathcal{K}_2}{1-((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi_1}+\hat{\phi_2}))}.$$

Hence, the problem (1)–(2) is U–H stable. □

6. Example

Consider the following Liouville–Caputo type generalized FDEs coupled system:

$$\begin{cases} {}^{\frac{19}{20}}_{C}\mathcal{D}^{\frac{5}{4}}_{0+}p(\tau)=\frac{\sqrt{\tau}}{2}+\frac{1}{5(\tau+25)}\frac{|p(\tau)|}{1+|p(\tau)|}+\frac{3}{80}\cos(q(\tau)), \tau\in[0,1], \\ {}^{\frac{19}{20}}_{C}\mathcal{D}^{\frac{31}{20}}_{0+}q(\tau)=\frac{\tau}{5}+\frac{17}{300}\cos(p(\tau))+\frac{1}{70}\frac{|q(\tau)|}{1+|q(\tau)|}, \tau\in[0,1], \end{cases} \tag{71}$$

supplemented with boundary conditions:

$$\left\{p(0)=0,\ q(0)=0,\ p(1)=\tfrac{5}{6}{}^{\frac{19}{20}}\mathcal{I}^{\frac{13}{20}}q(\tfrac{9}{20}),\ q(1)=\tfrac{6}{7}{}^{\frac{19}{20}}\mathcal{I}^{\frac{17}{20}}p(\tfrac{13}{20}),\right. \tag{72}$$

where $\xi=\frac{5}{4}, \zeta=\frac{31}{20}, \rho=\frac{19}{20}, \mathcal{T}=1, \epsilon=\frac{5}{6}, \varpi=\frac{9}{20}, \pi=\frac{6}{7}, \sigma=\frac{13}{20}, \varsigma=\frac{13}{20}, \varrho=\frac{17}{20}$ and

$$|f(\tau,p_1(\tau),q_1(\tau))-f(\tau,p_2(\tau),q_2(\tau))|=\frac{1}{125}|p_1(\tau)-p_2(\tau)|+\frac{3}{80}|q_1(\tau)-q_2(\tau)|, \tag{73}$$

$$|g(\tau,p_1(\tau),q_1(\tau))-g(\tau,p_2(\tau),q_2(\tau))|=\frac{17}{300}|p_1(\tau)-p_2(\tau)|+\frac{1}{70}|q_1(\tau)-q_2(\tau)|. \tag{74}$$

With $\phi_1=\frac{1}{125}, \phi_2=\frac{3}{80}, \hat{\phi_1}=\frac{17}{300}$, and $\hat{\phi_2}=\frac{1}{70}$, the functions f and g clearly satisfy the $(\mathcal{A}_2)$ condition. Next, we find that $(\mathcal{J}_1)=1.9529307397739033, (\mathcal{K}_1)=0.21135021378560123, \mathcal{J}_2=0.42682560046779994, \mathcal{K}_2=1.6225052940838325, \mathcal{J}_i, \mathcal{K}_i(i=1,2)$ are respectively given by (35),(36),(37) and (38), based on the data available. Thus $((\mathcal{J}_1+\mathcal{J}_2)(\phi_1+\phi_2)+(\mathcal{K}_1+\mathcal{K}_2)(\hat{\phi_1}+\hat{\phi_2}))\approxeq 0.2383953280869716<1$, all the conditions of Theorem 5.2 are satisfied, and there is a unique solution for problem (71) and (72) on $[0,1]$, which is stable for Ulam–Hyers, with f and g given by (73) and (74) respectively.

7. Existence Results for the Problem (1) and (75)

Furthermore, we are investigating the system (1) under the following conditions:

$$\begin{cases} p(0)=0, \quad q(0)=0, \\ p(\mathcal{T})=\epsilon{}^{\rho}\mathcal{I}^{\varsigma}_{0+}q(\varpi)=\frac{\epsilon\rho^{1-\varsigma}}{\Gamma(\varsigma)}\int_0^{\varpi}\frac{\theta^{\rho-1}}{(\varpi^{\rho}-\theta^{\rho})^{1-\varsigma}}q(\theta)d\theta, \\ q(\mathcal{T})=\pi{}^{\rho}\mathcal{I}^{\varrho}_{0+}p(\varpi)=\frac{\pi\rho^{1-\varrho}}{\Gamma(\varrho)}\int_0^{\varpi}\frac{\theta^{\rho-1}}{(\varpi^{\rho}-\theta^{\rho})^{1-\varrho}}p(\theta)d\theta, \\ 0<\varpi<\mathcal{T}. \end{cases} \tag{75}$$

Bear in mind that the conditions (2) contain strips of varying lengths, whereas the one in (75) contains only one strip of the same length $(0, \varpi)$. We introduce the following notations for computational ease:

$$\mathcal{E}_1 = \epsilon \frac{\varpi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)},\ \mathcal{E}_2 = \pi \frac{\varpi^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)},\ \widehat{\mathcal{E}} = \frac{\mathcal{T}^\rho}{\rho}, \tag{76}$$

$$\mathcal{G} = \widehat{\mathcal{E}}^2 - \mathcal{E}_1\mathcal{E}_2 \neq 0, \tag{77}$$

$$\delta(\tau) = \left(\frac{\tau^\rho}{\rho\mathcal{G}}\right). \tag{78}$$

Lemma 4. *Given the functions* $\hat{f}, \hat{g} \in C(0,\mathcal{T}) \cap \mathcal{L}(0,\mathcal{T}), p, q \in \mathcal{AC}^2_\gamma(\mathcal{E})$ *and* $\Lambda \neq 0$. *Then the solution of the coupled BVP:*

$$\begin{cases} {}^\rho_C\mathcal{D}^\xi_{0+}p(\tau) = \hat{f}(\tau), \tau \in \mathcal{E} := [0,\mathcal{T}], \\ {}^\rho_C\mathcal{D}^\zeta_{0+}q(\tau) = \hat{g}(\tau), \tau \in \mathcal{E} := [0,\mathcal{T}], \\ p(0) = 0,\ q(0) = 0,\ p(\mathcal{T}) = \epsilon^\rho\mathcal{I}^\varsigma_{0+}q(\varpi),\ q(\mathcal{T}) = \pi^\rho\mathcal{I}^\varrho_{0+}p(\varpi),\ 0 < \varpi < \mathcal{T}, \end{cases} \tag{79}$$

is given by

$$p(\tau) = {}^\rho\mathcal{I}^\xi_{0+}\hat{f}(\tau) + \delta(\tau)\left(\left[\epsilon\, {}^\rho\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) - {}^\rho\mathcal{I}^\xi_{0+}\hat{f}(\mathcal{T})\right] + \left[\pi\, {}^\rho\mathcal{I}^{\xi+\varsigma}_{0+}\hat{f}(\varpi) - {}^\rho\mathcal{I}^\zeta_{0+}\hat{g}(\mathcal{T})\right]\right) \tag{80}$$

and

$$q(\tau) = {}^\rho\mathcal{I}^\zeta_{0+}\hat{g}(\tau) + \delta(\tau)\left(\left[\pi\, {}^\rho\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{f}(\varpi) - {}^\rho\mathcal{I}^\zeta_{0+}\hat{g}(\mathcal{T})\right] + \left[\epsilon\, {}^\rho\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) - {}^\rho\mathcal{I}^\xi_{0+}\hat{f}(\mathcal{T})\right]\right). \tag{81}$$

Proof. When ${}^\rho\mathcal{I}^\xi_{0+}, {}^\rho\mathcal{I}^\zeta_{0+}$ are applied to the FDEs in (79) and Lemma 4 is used the solution of the FDEs in (79) for $\tau \in \mathcal{E}$ is

$$p(\tau) = {}^\rho\mathcal{I}^\xi_{0+}\hat{f}(\tau) + a_1 + a_2\frac{\tau^\rho}{\rho} = \frac{\rho^{1-\xi}}{\Gamma(\xi)}\int_0^\tau \theta^{\rho-1}(\tau^\rho - \theta^\rho)^{\xi-1}\hat{f}(\theta)d\theta + a_1 + a_2\frac{\tau^\rho}{\rho}, \tag{82}$$

$$q(\tau) = {}^\rho\mathcal{I}^\zeta_{0+}\hat{g}(\tau) + b_1 + b_2\frac{\tau^\rho}{\rho} = \frac{\rho^{1-\zeta}}{\Gamma(\zeta)}\int_0^\tau \theta^{\rho-1}(\tau^\rho - \theta^\rho)^{\zeta-1}\hat{g}(\theta)d\theta + b_1 + b_2\frac{\tau^\rho}{\rho}, \tag{83}$$

respectively, for some $a_1, a_2, b_1, b_2 \in \mathcal{R}$. Making use of the boundary conditions $p(0) = q(0) = 0$ in (82) and (83) respectively, we get $a_1 = b_1 = 0$. We obtain by using the generalized integral operators ${}^\rho\mathcal{I}^\varrho_{0+}, {}^\rho\mathcal{I}^\zeta_{0+}$ (82) and (83) respectively,

$${}^\rho\mathcal{I}^\varrho_{0+}p(\tau) = {}^\rho\mathcal{I}^{\xi+\varrho}_{0+}\hat{f}(\tau) + a_1\frac{\tau^{\rho\varrho}}{\rho^\varrho\Gamma(\varrho+1)} + a_2\frac{\tau^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)}, \tag{84}$$

$${}^\rho\mathcal{I}^\varsigma_{0+}q(\tau) = {}^\rho\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\tau) + b_1\frac{\tau^{\rho\varsigma}}{\rho^\varsigma\Gamma(\varsigma+1)} + b_2\frac{\tau^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)}, \tag{85}$$

which, when combined with the boundary conditions $p(\mathcal{T})$=$\epsilon^\rho\mathcal{I}^\varsigma_{0+}q(\varpi), q(\mathcal{T}) = \pi^\rho\mathcal{I}^\varrho_{0+}p(\varpi)$, gives the following results:

$${}^\rho\mathcal{I}^\xi_{0+}\hat{f}(\mathcal{T}) + a_1 + a_2\frac{\mathcal{T}^\rho}{\rho} = \epsilon^\rho\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) + b_1\frac{\epsilon\varpi^{\rho\varsigma}}{\rho^\varsigma\Gamma(\varsigma+1)} + b_2\frac{\epsilon\varpi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)}, \tag{86}$$

$$ {}^{\rho}\mathcal{I}^{\zeta}_{0+}\hat{g}(\mathcal{T}) + b_1 + b_2\frac{\mathcal{T}^{\rho}}{\rho} = \pi^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}\hat{f}(\varpi) + a_1\frac{\pi\varpi^{\rho\varrho}}{\rho^{\varrho}\Gamma(\varrho+1)} + a_2\frac{\pi\varpi^{\rho(\varrho+1)}}{\rho^{\varrho+1}\Gamma(\varrho+2)}. \tag{87} $$

Next, we obtain

$$ a_2\widehat{\mathcal{E}} - b_2\mathcal{E}_1 = \epsilon^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) -^{\rho}\mathcal{I}^{\xi}_{0+}\hat{f}(\mathcal{T}), \tag{88} $$

$$ b_2\widehat{\mathcal{E}} - a_2\mathcal{E}_2 = \pi^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}\hat{f}(\varpi) -^{\rho}\mathcal{I}^{\zeta}_{0+}\hat{g}(\mathcal{T}), \tag{89} $$

by employing the notations (76)–(78) in (86) and (87) respectively. We find that when we solve the system of Equations (88) and (89) for a_2 and b_2,

$$ a_2 = \frac{1}{\mathcal{G}}\Big[\widehat{\mathcal{E}}\Big(\epsilon^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) -^{\rho}\mathcal{I}^{\xi}_{0+}\hat{f}(\mathcal{T})\Big) + \mathcal{E}_1\Big(\pi^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}\hat{f}(\varpi) -^{\rho}\mathcal{I}^{\zeta}_{0+}\hat{g}(\mathcal{T})\Big)\Big], \tag{90} $$

$$ b_2 = \frac{1}{\mathcal{G}}\Big[\mathcal{E}_2\Big(\epsilon^{\rho}\mathcal{I}^{\zeta+\varsigma}_{0+}\hat{g}(\varpi) -^{\rho}\mathcal{I}^{\xi}_{0+}\hat{f}(\mathcal{T})\Big) + \widehat{\mathcal{E}}\Big(\pi^{\rho}\mathcal{I}^{\xi+\varrho}_{0+}\hat{f}(\varpi) -^{\rho}\mathcal{I}^{\zeta}_{0+}\hat{g}(\mathcal{T})\Big)\Big]. \tag{91} $$

Substituting the values of a_1, a_2, b_1, b_2 in (82) and (83) respectively, we get the solution for (79). $\square$

For brevity's sake, we'll use the following notations:

$$ \mathcal{J}_1 = \frac{\Big(\mathcal{T}^{\rho\xi}(1+|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\delta||\pi||\mathcal{E}_1|\varpi^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}, \tag{92} $$

$$ \mathcal{K}_1 = |\delta|\left(\frac{|\mathcal{E}_1|\mathcal{T}^{\rho\zeta}}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\widehat{\mathcal{E}}||\epsilon|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}\right), \tag{93} $$

$$ \mathcal{J}_2 = |\delta|\left(\frac{\mathcal{T}^{\rho\xi}|\mathcal{E}_2|}{\rho^{\xi}\Gamma(\xi+1)} + \frac{|\pi||\widehat{\mathcal{E}}|\varpi^{\rho(\xi+\varrho)}}{\rho^{\xi+\varrho}\Gamma(\xi+\varrho+1)}\right), \tag{94} $$

$$ \mathcal{K}_2 = \frac{\Big(\mathcal{T}^{\rho\zeta}(1+|\delta||\widehat{\mathcal{E}}|)\Big)}{\rho^{\zeta}\Gamma(\zeta+1)} + \frac{|\delta||\epsilon||\mathcal{E}_2|\varpi^{\rho(\zeta+\varsigma)}}{\rho^{\zeta+\varsigma}\Gamma(\zeta+\varsigma+1)}. \tag{95} $$

To finish up, we will go over the results of existence, uniqueness, and Ulam–Hyers stability for problems (1) and (75), respectively. For reasons that are similar to those in Sections 3–6, we are not providing the proof.

Corollary 1. *Assume that $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition: $(\mathcal{A}_1)$ there exists constants $\psi_m, \hat{\psi_m} \leq 0 (m = 1,2)$ and $\psi_0, \hat{\psi_0} > 0$ such that*

$$ \begin{aligned} |f(\tau, o_1, o_2)| &\leq \psi_0 + \psi_1|o_1| + \psi_2|o_2|, \\ |g(\tau, o_1, o_2)| &\leq \hat{\psi_0} + \hat{\psi_1}|o_1| + \hat{\psi_2}|o_2|, \forall o_m \in \mathbb{R}, m = 1,2. \end{aligned} $$

If $\psi_1(\hat{\mathcal{J}_1} + \hat{\mathcal{J}_2}) + \hat{\psi_1}(\hat{\mathcal{K}_1} + \hat{\mathcal{K}_2}) < 1, \psi_2(\hat{\mathcal{J}_1} + \hat{\mathcal{J}_2}) + \hat{\psi_2}(\hat{\mathcal{K}_1} + \hat{\mathcal{K}_2}) < 1$. Then at least one solution for the BVP (1) and (75) on $\mathcal{E}$, where $\hat{\mathcal{J}_1}, \hat{\mathcal{K}_1}, \hat{\mathcal{J}_2}, \hat{\mathcal{K}_2}$ are given by (92)–(95) respectively.

Corollary 2. *Assume that $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the condition: ($\mathcal{A}_2$) there exists constants $\phi_m, \hat{\phi_m} \leq 0 (m = 1, 2)$ such that*

$$|f(\tau, o_1, o_2) - f(\tau, \hat{o}_1, \hat{o}_2)| \leq \phi_1 |o_1 - \hat{o}_1| + \phi_2 |o_2 - \hat{o}_2|,$$
$$|g(\tau, o_1, o_2) - g(\tau, \hat{o}_1, \hat{o}_2)| \leq \hat{\phi}_1 |o_1 - \hat{o}_1| + \hat{\phi}_2 |o_2 - \hat{o}_2|, \forall o_m, \hat{o}_m \in \mathbb{R}, m = 1, 2.$$

Moreover, there exist $\mathcal{S}_1, \mathcal{S}_2 > 0$ such that $|f(\tau, 0, 0)| \leq \mathcal{S}_1, |f(\tau, 0, 0)| \leq \mathcal{S}_2$, Then, given that

$$(\mathcal{J}_1 + \mathcal{J}_2)(\phi_1 + \phi_2) + (\mathcal{K}_1 + \mathcal{K}_2)(\hat{\phi}_1 + \hat{\phi}_2) < 1, \tag{96}$$

the BVP (1) and (75) has a unique solution on $\mathcal{E}$, where $\hat{\mathcal{J}}_1, \hat{\mathcal{K}}_1, \hat{\mathcal{J}}_2, \hat{\mathcal{K}}_2$ are given by (92)–(95) respectively.

Corollary 3. *Assume that $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying the assumption ($\mathcal{A}_2$) in Theorem 2. Further more, there exist positive constants $\mathcal{U}_1, \mathcal{U}_2$ such that $\forall \tau \in \mathcal{E}$ and $r_i \in \mathbb{R}, i = 1, 2$.*

$$|f(\tau, r_1, r_2)| \leq \mathcal{U}_1, \qquad |g(\tau, r_1, r_2)| \leq \mathcal{U}_2. \tag{97}$$

If

$$\frac{\mathcal{T}^{\rho\xi}(\phi_1 + \phi_2)}{\rho^{\xi}\Gamma(\xi + 1)} + \frac{\mathcal{T}^{\rho\zeta}(\hat{\phi}_1 + \hat{\phi}_2)}{\rho^{\zeta}\Gamma(\zeta + 1)} < 1, \tag{98}$$

then the BVP (1), and (75) has at least one solution on $\mathcal{E}$.

Corollary 4. *Assume that (A2) holds. Then the problem (1) and (75) is Ulam–Hyers stable.*

8. Asymmetric Cases

Remark 1. *If $\rho = 1$, the problem (1) generalized Liouville–Caputo type reduces to the classical Caputo form.*

$$\begin{cases} {}^{C}\mathcal{D}^{\xi}_{0^+} p(\tau) = f(\tau, p(\tau), q(\tau)), \tau \in \mathcal{G} := [0, \mathcal{T}], \\ {}^{C}\mathcal{D}^{\zeta}_{0^+} q(\tau) = g(\tau, p(\tau), q(\tau)), \tau \in \mathcal{G} := [0, \mathcal{T}]. \end{cases} \tag{99}$$

Remark 2. *If $\rho = 1$ in the boundary conditions (2) and (75) generalized Riemann–Liouville integral boundary conditions reduces to the Riemann–Liouville integral conditions respectively.*

$$\begin{cases} p(0) = 0, \quad q(0) = 0, \\ p(\mathcal{T}) = \epsilon \mathcal{I}^{\varsigma}_{0+} q(\varpi) = \frac{\epsilon}{\Gamma(\varsigma)} \int_0^{\varpi} (\varpi - \theta)^{\varsigma - 1} q(\theta) d\theta, \\ q(\mathcal{T}) = \pi \mathcal{I}^{\varrho}_{0+} p(\sigma) = \frac{\pi}{\Gamma(\varrho)} \int_0^{\sigma} (\sigma - \theta)^{\varrho - 1} p(\theta) d\theta, \\ 0 < \sigma < \varpi < \mathcal{T}, \end{cases} \tag{100}$$

and

$$\begin{cases} p(0) = 0, \quad q(0) = 0, \\ p(\mathcal{T}) = \epsilon \mathcal{I}^{\varsigma}_{0+} q(\varpi) = \frac{\epsilon}{\Gamma(\varsigma)} \int_0^{\varpi} (\varpi - \theta)^{\varsigma - 1} q(\theta) d\theta, \\ q(\mathcal{T}) = \pi \mathcal{I}^{\varrho}_{0+} p(\varpi) = \frac{\pi}{\Gamma(\varrho)} \int_0^{\varpi} (\varpi - \theta)^{\varpi - 1} p(\theta) d\theta, \\ 0 < \varpi < \mathcal{T}. \end{cases} \tag{101}$$

Remark 3. *If $\rho = 1$ and $\varsigma = \varrho = 1$ in the boundary conditions (2) and (75) generalized Riemann–Liouville integral boundary conditions reduces to the classical integral conditions respectively.*

$$\left\{ p(0) = 0,\ q(0) = 0,\ p(\mathcal{T}) = \epsilon \int_0^{\varpi} q(\theta)d\theta,\ q(\mathcal{T}) = \pi \int_0^{\sigma} p(\theta)d\theta\ 0 < \sigma < \varpi < \mathcal{T} \right. \tag{102}$$

and

$$\left\{ p(0) = 0,\ q(0) = 0,\ p(\mathcal{T}) = \epsilon \int_0^{\varpi} q(\theta)d\theta,\ q(\mathcal{T}) = \pi \int_0^{\varpi} p(\theta)d\theta\ 0 < \varpi < \mathcal{T}. \right. \tag{103}$$

9. Conclusions

This paper employs coupled nonlinear generalized Liouville–Caputo fractional differential equations and Katugampola fractional integral operators to solve a novel class of boundary value problems. Applying the techniques of fixed-point theory to discover the existence criterion for solutions is efficient. While the second outcome provides a sufficient criterion to establish the problem's unique solution, the first and third results define various criteria for the presence of solutions to the given problem. In the fourth section, the Hyers–Ulam stability of the solution was determined. In the remarks, we have shown the asymmetric cases of the assigned problem. Moreover, the form of the solution in these kinds of remarks can be used to study the positive solution and its asymmetry in more depth. We conclude that our results are novel and can be viewed as an expansion of the qualitative analysis of fractional differential equations. Our results are novel in this configuration and add to the literature on nonlinear coupled generalized Liouville–Caputo fractional differential equations with nonlocal boundary conditions utilizing Katugampola-type integral operators. Future research could focus on various conceptions of stability and existence in relation to a Lotka–Volterra prey-predator system/coupled logistic system.

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