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Some Aspects of Hyperatom Elements in Ordered Semihyperrings

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Abstract: In this paper, first, we state an operator L_R on an ordered semihyperring R. We show that if $\varphi : R \longrightarrow T$ is a monomorphism and $K \subseteq R$, then $L_T(\varphi(K)) = \varphi(L_R(K))$. Afterward, hyperatom elements in ordered semihyperrings are defined and some results in this respect are investigated. Denote by $A(R)$ the set of all hyperatoms of R. We prove that if R is a finite ordered semihyperring and $|R| \geq 2$, then for any $q \in R \setminus \{0\}$, there exists $h_q \in A^*(R) = A(R) \setminus \{0\}$ such that $h_q \leq q$. Finally, we study the L_R-graph of an ordered semihyperring and give some examples. Furthermore, we show that if $\varphi : R \longrightarrow T$ is an isomorphism, G is the L_R-graph of R and G' is the L_T-graph of T, then $G \cong G'$.

Keywords: algebraic hyperstructure; ordered semihyperring; hyperatom; L_R-graph

MSC: 16Y99

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1. Introduction and Preliminaries

The concept of the hypergroup was introduced by Marty [1] in 1934. The notion of semihyperring was introduced by Vougiouklis [2] as a generalization of semiring in 1990. The study of ordered semihypergroups is one interesting topic in hyperstructure theory. The notion of ordered semihypergroup was proposed by Heidari and Davvaz [3] in 2011. Recently, Shi et al. [4] also pioneered the notion of factorizable ordered hypergroupoids with applications. After the paper [3] was published, many authors obtained some results on ordered semihypergroups. Davvaz et al. in [5] initiated the study of pseudoorders in ordered semihypergroups. Gu and Tang in [6] and Tang et al. in [7] constructed the ordered semihypergroup from an ordered semihypergroup by using ordered regular equivalence relation.

A study on k-bi-quasi hyperideals in an ordered semihyperring is done in [8]. In [9], uni-soft interior Γ-hyperideals in ordered Γ-semihypergroups are studied. In [10], Hila et al. investigated (k, n)-absorbing hyperideals in Krasner (m, n)-hyperrings. The derivations of hyperrings [11]; the w-pseudo-orders [12]; the ordered regular equivalence relations [13]; the relative bi-(int-)Γ-hyperideals [14] and the A-**I**-Γ-hyperideals [15], have been introduced and studied. Some recent studies on ordered semihyperring theory are on derivations done by Rao et al. in [16] and Kou et al. in [17].

In 2019, Panganduyon and Canoy [18] defined zero divisor graph of a hyper BCI-algebra and discussed some of its properties. In 2021, Panganduyon et al. [19] investigated some properties of the induced topology on certain hyper BCI-algebras. Many properties of hyperatom elements were extended to hyper BCK-algebras in [20,21]. We invite the readers to [22,23] to see more about the graph theory.

A physical example of hyperstructures associated with the leptons is provided in [24]. Leptons and gauge bosons along with the interactions between their members construct

an H_v-structure [25]. Farshi et al. [26] explored some connections between graph theory and hyperstructure theory in 2014. Now, we analyze an operator L_R on an ordered semihyperring R and investigae some aspects of L_R-graph.

A mapping $\cdot : R \times R \to \mathcal{P}^*(R)$ is called a *hyperoperation* on R. If $\emptyset \neq P, Q \subseteq R$ and $r \in R$, then

$$P \cdot Q = \bigcup_{\substack{p \in P \\ q \in Q}} p \cdot q, r \cdot P = \{r\} \cdot P \text{ and } Q \cdot r = Q \cdot \{r\}.$$

$(R, \cdot)$ is called a *semihypergroup* if for every p, q, r in R,

$$p \cdot (q \cdot r) = (p \cdot q) \cdot r.$$

A triple $(R, \oplus, \odot)$ is a *semihyperring* [2] if $\oplus$ and $\odot$ are both hyperoperations such that

(1) $(R, \oplus)$ is a commutative semihypergroup;
(2) $(R, \odot)$ is a semihypergroup;
(3) for all u, v, w in R, $u \odot (v \oplus w) = u \odot v \oplus u \odot w$ and $(u \oplus v) \odot w = u \odot w \oplus v \odot w$.

If there exists 0 in R such that $u \oplus 0 = 0 \oplus u = \{u\}$ and $u \odot 0 = 0 \odot u = \{0\}$ for all u in R; then 0 is called the zero element of R.

Definition 1. *[13,16] Take a semihyperring $(R, \oplus, \odot)$ and a partial order relation $\leq$. Then $(R, \oplus, \odot, \leq)$ is called an ordered semihyperring if for any $p, p', r \in R$,*

$$p \leq p' \Rightarrow \begin{cases} p \oplus r \preceq p' \oplus r, \\ p \odot r \preceq p' \odot r, \\ r \odot p \preceq r \odot p'. \end{cases}$$

For every $\emptyset \neq P, Q \subseteq R$, $P \preceq Q$ is defined by $\forall p \in P, \exists q \in Q$ such that $p \leq q$. Clearly, $P \subseteq Q$ implies $P \preceq Q$, but the converse is not valid in general. Also, R is said to be positive if $0 \leq x$ for all $x \in R$.

Let $(R, +, \cdot, \leq_R)$ and $(T, \oplus, \odot, \leq_T)$ be two ordered semihyperrings. We will say that a function $\varphi : R \to T$ is a *homomorphism* of ordered semihyperrings if for all $q, q' \in R$,

(1) $\varphi(q + q') \subseteq \varphi(q) \oplus \varphi(q')$,
(2) $\varphi(q \cdot q') \subseteq \varphi(q) \odot \varphi(q')$,
(3) $q \leq_R q'$ implies $\varphi(q) \leq_T \varphi(q')$.

We will say that $\emptyset \neq W \subseteq R$ is a *left* (resp. *right*) *hyperideal* of R if

(1) for all $a, b \in W$, $a \oplus b \subseteq W$;
(2) $R \odot W \subseteq W$ (resp. $W \odot R \subseteq W$);
(3) $(W] \subseteq W$.

The set $(W]$ is given by

$$(W] := \{r \in R \mid r \leq w \text{ for some } w \in W\}.$$

2. Main Results

In this section, for the first time, we study the operator L_R in ordered semihyperrings and present some results in this respect. Moreover, we introduce hyperatom elements in ordered semihyperrings and investigate some of their properties.

Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring and $K \subseteq R$. The set $L_R(K)$ is given by

$$L_R(K) := \{r \in R \mid r \leq k \text{ for all } k \in K\}.$$

For $K = \{x\}$, we write $L_R(x)$ instead of $L_R(\{x\})$. For $\emptyset \neq H, K \subseteq R$, we have that

(1) $L_R(\emptyset) = R$;
(2) $q \in L_R(q)$ for all $q \in R$;
(3) $L_R(K) = \bigcap_{x \in K} L_R(x)$;
(4) If $H \subseteq K$, then $L_R(K) \subseteq L_R(H)$.

Furthermore, in a positive ordered semihyperring R, $L_R(0) = 0$.

Remark 1. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring and $(u, v) \in \leq$. Then $L_R(u) \subseteq L_R(v)$. Indeed: Let $w \in L_R(u)$. Then $(w, u) \in \leq$. So, $(w, v) \in \leq$ and hence $w \in L_R(v)$. Thus, $L_R(u) \subseteq L_R(v)$.*

Proposition 1. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring and $K \subseteq R$. Then*

$$L_R(L_R(K)) \subseteq L_R(K).$$

Proof. Let $u \in L_R(L_R(K))$. Then $u \leq v$ for all $v \in L_R(K)$. By Definition of $L_R(K)$, we have $v \leq x$ for all $x \in K$. Since $\leq$ is transitive, we get $u \leq x$ for all $x \in K$. So, $u \in L_R(K)$. □

Theorem 1. *Consider $\varphi : R \longrightarrow T$ as the monomorphism. If $K \subseteq R$, then $L_T(\varphi(K)) = \varphi(L_R(K))$.*

Proof. Let $K \subseteq R$. We get

$$\begin{aligned} u \in \varphi(L_R(K)) &\Leftrightarrow \varphi^{-1}(u) \in L_R(K) \\ &\Leftrightarrow \varphi^{-1}(u) \leq_R k, \forall k \in K \\ &\Leftrightarrow u \leq_T \varphi(k), \forall k \in K \\ &\Leftrightarrow u \in L_T(\varphi(K)). \end{aligned}$$

Hence, $L_T(\varphi(K)) = \varphi(L_R(K))$. □

Definition 2. *We say that an element x of an ordered semihyperring $(R, \oplus, \odot, \leq)$ is a hyperatom element if*

(1) for any $z \in R$, $z \leq x$ implies $z = 0$ or $z = x$;
(2) $x \odot z \neq 0$ implies $(x \odot z) \odot m = z$ for some $m \in R$.

Remark 2. *Consider the ordered semihyperring $(R, \oplus, \odot, \leq)$. Then R is hyperatomic if and only if $L_R(q) = \{q\}$ or $L_R(q) = \{0, q\}$ for any $q \in R$.*

Theorem 2. *Let $(R, \oplus, \odot, \leq)$ be an ordered semihyperring such that $|R| > 2$. We denote by $A(R)$ the set if all hyperatom elements of R. If $m, m' \in A(R)$, then $L_R(\{m, m'\}) = \{0\}$ or $\emptyset$.*

Proof. Let $m, m' \in A(R)$. Consider the following situations.

Case 1. $0 \notin L_R(m)$ for all $m \in R$.
Since $m \in A(R)$, we get $L_R(m) = \{m\}$ for all $m \in R$. Thus, $L_R(\{m, m'\}) = \emptyset$.

Case 2. $0 \in L_R(m)$ for all $m \in R$.
Then, $L_R(m) = \{0, m\}$. By Remark 2, we have $L_R(m') = \{m'\}$ or $L_R(m') = \{0, m'\}$. Hence, $L_R(\{m, m'\}) = \{0\}$ or $\emptyset$. □

Theorem 3. *Let $(R, \oplus, \odot, \leq)$ be a finite ordered semihyperring. Then, for any $q \in R \setminus \{0\}$, there exists $h_q \in A^*(R) = A(R) \setminus \{0\}$ such that $h_q \leq q$, where $|R| \geq 2$.*

Proof. Let $q \in R \setminus \{0\}$. Consider the following situations.

Case 1. q is a hyperatom.
Then take $h_q = q$.

Case 2. q is not a hyperatom.
Then, there exists $q_1 \in R \setminus \{0, q\}$ such that $q_1 \leq q$.

Subcase 1. q_1 is a hyperatom.
Then take $h_q = q_1$.

Subcase 2. q_1 is not a hyperatom.
Then, there exists $q_2 \in R \setminus \{0, q, q_1\}$ such that $q_2 \leq q_1 \leq q$. Since R is finite, we have

$$q_r \leq q_{r-1} \leq \cdots \leq q_2 \leq q_1 \leq q,$$

where $q_r \in R \setminus \{0, q, q_1, \cdots, q_{r-1}\}$. So, $q_r = 0$ if $0 \in L_R(q)$ or $h_q = q_r \in A^*(R)$ and $h_q \leq q$. □

Corollary 1. *Let $(R, \oplus, \odot, \leq)$ be a finite ordered semihyperring. Then there exists $h \in R \setminus \{0\}$ such that $h \leq q$ for any $q \in R \setminus \{0\}$, where $|R| \geq 2$ if and only if $A^*(R) = \{h\}$.*

Proof. Necessity. Take any $m \in A^*(R)$. By hypothesis, there exists $h \in R \setminus \{0\}$ such that $h \leq m$. Since m is a hyperatom element and $h \neq 0$, we get $h = m$. So, $A^*(R) = \{h\}$.
Sufficiency. Let $A^*(R) = \{h\}$ and $q \in R \setminus \{0\}$. By Theorem 3, we have $h \leq q$. □

Definition 3. *The L_R-graph G of a finite ordered semihyperring R is the graph whose vertex set $V(G) = R$ and edge set $E(G)$ satisfying the following condition:*

$$\forall q, q' \in R \text{ and } q \neq q', qq' \in E(G) \Longleftrightarrow L_R(\{q, q'\}) = \{0\}.$$

Example 1. *Consider the ordered semihyperring $(R, \oplus, \odot, \leq)$, where $\oplus$ is a symmetrical hyperoperation (as shown in Tables 1 and 2):*

Table 1. Table of $\oplus$ for Example 1.

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{a, b\}$	b	c
b	b	b	$\{0, b\}$	c
c	c	c	c	$\{0, c\}$

Table 2. Table of $\odot$ for Example 1.

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

$$0 \leq a \leq b \leq c.$$

Then

$$L_R(\{0, a\}) = \{0\},$$
$$L_R(\{0, b\}) = \{0\},$$
$$L_R(\{0, c\}) = \{0\},$$
$$L_R(\{a, b\}) = \{0, a\},$$
$$L_R(\{a, c\}) = \{0, a\},$$

and

$$L_R(\{b,c\}) = \{0,a,b\}.$$

The L_R-graph G of R is shown in Figure 1:

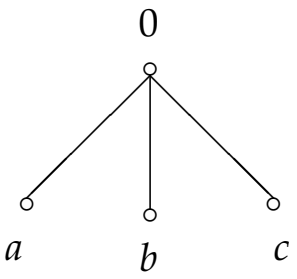


Figure 1. L_R-graph for Example 1.

Clearly, G is a connected graph.

The L_R-graph of an ordered semihyperring is not connected in general. The following example validates it.

Example 2. *Consider the ordered semihyperring $(R, \oplus, \odot, \leq)$, where $\oplus$ is a symmetrical hyperoperation (see Tables 3 and 4):*

Table 3. Table of $\oplus$ for Example 2.

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	$\{0, b\}$	$\{0, b, c\}$
c	c	a	$\{0, b, c\}$	$\{0, c\}$

Table 4. Table of $\odot$ for Example 2.

$\odot$	0	a	b	c
0	0	0	0	0
a	0	a	$\{0, b\}$	0
b	0	0	0	0
c	0	$\{0, c\}$	0	0

$$\leq := \{(0,0),(a,a),(b,b),(c,c),(0,b),(0,c)\}.$$

Then

$$L_R(\{0,a\}) = \emptyset,$$
$$L_R(\{0,b\}) = \{0\},$$
$$L_R(\{0,c\}) = \{0\},$$
$$L_R(\{a,b\}) = \emptyset,$$
$$L_R(\{a,c\}) = \emptyset,$$

and

$$L_R(\{b,c\}) = \{0\}.$$

The L_R-graph G of R is shown in Figure 2:

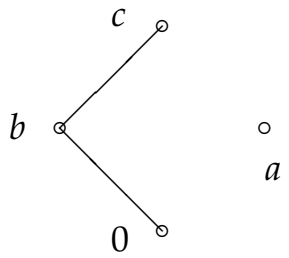


Figure 2. L_R-graph for Example 2.

Clearly, G is a disconnected graph.

Theorem 4. *Let $(R, \oplus, \odot, \leq)$ be a positive ordered semihyperring. If $L_R(\{0, q\}) \neq \emptyset$ for all $q \in R$, then $L_R(\{0, q\}) = \{0\}$.*

Proof. Let $L_R(\{0, q\}) \neq \emptyset$ for all $q \in R$ and $q' \in L_R(\{0, q\})$. Since R is positive, we get

$$L_R(0) = \{r \in R \mid r \leq 0\} = \{0\}.$$

So,

$$q' \in L_R(\{0, q\}) = L_R(0) \cap L_R(q) = \{0\} \cap L_R(q).$$

It means that $q' = 0$. Hence, $L_R(\{0, q\}) = \{0\}$ for all $q \in R$. □

Corollary 2. *Let G be the L_R-graph of a positive ordered semihyperring R. If $L_R(\{0, q\}) \neq \emptyset$ for all $q \in R$, then $0q \in E(G)$ for all $q \in R \setminus \{0\}$.*

Theorem 5. *Let φ be an isomorphism from an ordered semihyperring R into an ordered semihyperring T. If G is the L_R-graph of R and G' is the L_T-graph of T, then $G \cong G'$.*

Proof. Consider $\varphi : R \longrightarrow T$ as the isomorphism. We have $V(G) = R$ and $V(G') = T$. So, By hypothesis, there exists a one-to-one correspondence between $V(G)$ and $V(G')$. On the other hand,

$$\forall q, q' \in R \text{ and } q \neq q', qq' \in E(G) \Longleftrightarrow L_R(\{q, q'\}) = \{0\}.$$

By Theorem 1, we get

$$\begin{aligned} L_T(\varphi(\{q, q'\})) &= L_T(\{\varphi(q), \varphi(q')\}) \\ &= \varphi(L_R(\{q, q'\}) \\ &= \varphi(0) \\ &= 0. \end{aligned}$$

Thus,

$$qq' \in E(G) \Longleftrightarrow L_T(\{\varphi(q), \varphi(q')\}) = \{0\}.$$

Hence,

$$qq' \in E(G) \Longleftrightarrow \varphi(q)\varphi(q') \in E(G').$$

So, $G \cong G'$. □

3. Conclusions

In this study, we have looked into the idea of hyperatom elements on an ordered semihyperring. Moreover, we introduced an operator L_R on an ordered semihyperring R and investigated some aspects of L_R-graph. Furthermore, we have proved that if φ is an isomorphism from an ordered semihyperring R into an ordered semihyperring T, G is the L_R-graph of R and G' is the L_T-graph of T, then $G \cong G'$. In the future, we will intend to study hyperatom elements of ordered semihypergroups.

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