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Double Conformable Sumudu Transform

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Abstract: In this paper, we introduce a new approach to solving fractional initial and boundary value problems involving a heat equation, a wave equation, and a telegraph equation by modifying the double Sumudu transform of the fractional type. We discuss a modified double conformable Sumudu transform together with the conditions for its existence. In addition, we prove some more properties of the fractional-type Sumudu transform, including convolution and other properties, which are well known for their use in solving various symmetric and asymmetric problems in applied sciences and engineering.

Keywords: conformable fractional derivative; Sumudu transform; convolution

1. Introduction

Fractional differential equations appear widely in various applied sciences and engineering applications in order to improve the quality of modeling and better describe real-world problems, which include economic, physical, electrical, and biological applications, among many others. One can refer, for instance, to [1] and the references therein, where a good review of the applications of fractional differential equations in economics was given, and to [2] for applications in the circuit domain, in which a time-fractional RC circuit model was considered. Still, a similar fractional mathematical model can be used to better model other types of circuits, such as RLCG circuits, as in [3].

Importantly, an exciting advancement in theoretical physics and nonlinear sciences will be the development of methods for finding the exact solutions of nonlinear partial differential equations that include equations of the fractional type. Such solutions play an important role in the nonlinear sciences, which can lead to further applications.

Regarding fractional definitions, in [4], a new fractional definition that was called the *conformable fractional derivative* was introduced and was defined as follows: For a given function $\psi\colon [0,\infty) \longrightarrow \mathbb{R}$, the conformable fractional derivative of order ϑ is given by

$$D^{\vartheta}\psi(x) = \lim_{\varepsilon\to 0}\frac{\psi(x+\varepsilon x^{1-\vartheta})-\psi(x)}{\varepsilon}, \quad \vartheta \in (0,1].$$

This definition is very easy to use when calculating derivatives and solving fractional differential equations compared with other fractional definitions, such as the definitions of Liouville–Riemann and Caputo fractional derivatives. Moreover, one of its most interesting advantages is that it can be easily used to generalize many integral transforms, such as Laplace and Sumudu transforms. Various modifications of the original definition were proposed by many researchers; see, for instance, [5] and the references therein.

Recently, several powerful methods have been developed to obtain the exact solutions for conformable fractional partial differential equations, such as the reliable methods in [6,7], the single and double Laplace transform methods in [8–10], and the double Shehu transform in [11].

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Interestingly, in [9], Özkan et al. introduced a definition of the conformable fractional double Laplace transform and some of its properties, which were used to solve some conformable fractional partial differential equations and will be important in what follows here in this work. These were defined as follows: For a given function $\psi\colon [0,\infty)\times[0,\infty)\longrightarrow \mathbb{R}$, the conformable fractional double Laplace transform of order ϑ_1, $\vartheta_1 \in (0,1]$ is given as

$$\mathscr{L}_t^{\vartheta_1}\mathscr{L}_x^{\vartheta_2}[\psi(x,t)]_{(p,s)} = \Psi(p,s) = \int_0^\infty\int_0^\infty e^{-p\frac{x^{\vartheta_2}}{\vartheta_2}-s\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x.$$

Over the years, transform methods, including the Sumudu transform, have been proven to be efficient methods for solving many symmetric and asymmetric real-life problems in applied sciences and engineering. In [12], the authors presented a single Sumudu transform, and in [13], a conformable double Sumudu transform was presented in order to solve partial differential equations of the conformable fractional type. The authors of [14] used the double Sumudu transform in order to obtain solutions of a space–time telegraph equation. In [15], with the use of a double Sumudu transform, Mohamed et al. presented numerical solutions to the conformable fractional coupled Burger's equation.

In this paper, we modify the definition of the double Sumudu transform and prove some more of its properties, including its convolution properties, which have not been proven in the literature. In addition, we prove its existence under some certain conditions. Moreover, we use this modification to give exact solutions of some important conformable fractional differential equations, including the heat equation, the wave equation, and the telegraph equation. Our modification is based on the use of the conformable fractional integral defined by Khalil et al. [4], which is different from what one can find in the literature.

2. Double Conformable Sumudu Transform

In this section, we introduce the double conformable Sumudu transform and some of its properties that can be used later in order to solve some conformable fractional differential equations.

Definition 1. *A function ψ of two variables is said to be conformable and exponentially order-bounded if $|\psi(x,t)| \le Me^{k_1\frac{x^{\vartheta_2}}{\vartheta_2}+k_2\frac{t^{\vartheta_1}}{\vartheta_1}}$, where $M, k_1, k_2 > 0$ and $0 < \vartheta_1, \vartheta_2 \le 1$, for all sufficiently large x and t.*

Definition 2. *Let $\psi\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be a piecewise continuous function of conformable and exponentially order-bounded $\frac{1}{k_1}+\frac{1}{k_2}$, k_1,$k_2 > 0$. Then, the double conformable Sumudu transform of ψ is defined as:*

$$S_{\vartheta_1}^t S_{\vartheta_2}^x[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x,$$

where $\lambda_1, \lambda_2 \in \mathbb{C}$, $\vartheta_1, \vartheta_2 \in (0,1]$, and $d_{\vartheta_1}t = t^{\vartheta_1-1}dt, d_{\vartheta_2}x = x^{\vartheta_2-1}dx$.

Theorem 1. *If ψ is piecewise continuous on $[0,\infty)\times[0,\infty)$ and conformable and exponentially order-bounded for $\frac{1}{k_1}+\frac{1}{k_2}, k_1, k_2 > 0$, then the double conformable Sumudu transform $S_{\vartheta_1}^t S_{\vartheta_2}^x[\psi(x,t)]_{(\lambda_1,\lambda_2)}$ exists for $\Re e\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) > \frac{1}{k_1}+\frac{1}{k_2}$ and converges absolutely.*

Proof. Since ψ is conformable and exponentially order-bounded for $\frac{1}{k_1}+\frac{1}{k_2}, k_1, k_2 > 0$, then $\exists\ x_0, t_0, M_1, k_1, k_2 > 0$ such that

$$|\psi(x,t)| \le M_1 e^{\frac{x^{\vartheta_2}}{k_1\vartheta_2}+\frac{t^{\vartheta_1}}{k_2\vartheta_1}}, \forall x \ge x_0, \forall t \ge t_0.$$

In addition, ψ is piecewise continuous on $[0, x_0] \times [0, t_0]$, i.e.,

$$|\psi(x,t)| \leq M_2, \forall (x,t) \in [0, x_0] \times [0, t_0].$$

Since $e^{\frac{x^{\vartheta_2}}{k_1\vartheta_2}+\frac{t^{\vartheta_1}}{k_2\vartheta_1}}$ has a positive minimum on $[0,\infty) \times [0,\infty)$, we can choose a sufficiently large value of M such that

$$|\psi(x,t)| \leq M e^{\frac{x^{\vartheta_2}}{k_1\vartheta_2}+\frac{t^{\vartheta_1}}{k_2\vartheta_1}}, \forall x \geq 0, \forall t \geq 0.$$

Therefore, if $\frac{1}{\lambda_1} = \frac{1}{a_1} + \frac{1}{a_2} i$ and $\frac{1}{\lambda_2} = \frac{1}{b_1} + \frac{1}{b_2} i$, then

$$\left| \int_0^\gamma \int_0^\tau e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}} \psi(x,t) d_{\vartheta_1} t \, d_{\vartheta_2} x \right|$$

$$\leq \int_0^\gamma \int_0^\tau \left| e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}} M e^{\frac{x^{\vartheta_2}}{k_1\vartheta_2}+\frac{t^{\vartheta_1}}{k_2\vartheta_1}} t^{\vartheta_1-1} x^{\vartheta_2-1} \right| dt \, dx$$

$$= \int_0^\gamma \int_0^\tau \left| e^{-\left[\left(\frac{1}{a_1}+\frac{1}{a_2}i\right)\frac{x^{\vartheta_2}}{\vartheta_2}+\left(\frac{1}{b_1}+\frac{1}{b_2}i\right)\frac{t^{\vartheta_1}}{\vartheta_1}\right]} M e^{\frac{x^{\vartheta_2}}{k_1\vartheta_2}+\frac{t^{\vartheta_1}}{k_2\vartheta_1}} t^{\vartheta_1-1} x^{\vartheta_2-1} \right| dt \, dx$$

$$= \int_0^\gamma \int_0^\tau \left| e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}} \right| \left| e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)\frac{t^{\vartheta_1}}{\vartheta_1}} \right| \left| e^{\frac{-i}{a_2}\frac{x^{\vartheta_2}}{\vartheta_2}} \right| \left| e^{\frac{-i}{b_2}\frac{t^{\vartheta_1}}{\vartheta_1}} \right| M t^{\vartheta_1-1} x^{\vartheta_2-1} \, dt \, dx$$

$$= M \int_0^\gamma \left| e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}} \right| \left| e^{\frac{-i}{a_2}\frac{x^{\vartheta_2}}{\vartheta_2}} \right| x^{\vartheta_2-1} dx \int_0^\tau \left| e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)\frac{t^{\vartheta_1}}{\vartheta_1}} \right| \left| e^{\frac{-i}{b_2}\frac{t^{\vartheta_1}}{\vartheta_1}} \right| t^{\vartheta_1-1} dt$$

$$= M \int_0^{\frac{\gamma^{\vartheta_2}}{\vartheta_2}} \left| e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)h} \right| \left| e^{\frac{-i}{a_2}h} \right| dh \int_0^{\frac{\tau^{\vartheta_1}}{\vartheta_1}} \left| e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)l} \right| \left| e^{\frac{-i}{b_2}l} \right| dl \text{ (by letting } h = \frac{x^{\vartheta_2}}{\vartheta_2}, l = \frac{t^{\vartheta_1}}{\vartheta_1})$$

$$= M \int_0^{\frac{\gamma^{\vartheta_2}}{\vartheta_2}} e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)h} dh \int_0^{\frac{\tau^{\vartheta_1}}{\vartheta_1}} e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)l} dl$$

$$= M \left. \frac{e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)h}}{\frac{1}{k_1}-\frac{1}{a_1}} \right|_0^{\frac{\gamma^{\vartheta_2}}{\vartheta_2}} \left. \frac{e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)l}}{\frac{1}{k_2}-\frac{1}{b_1}} \right|_0^{\frac{\tau^{\vartheta_1}}{\vartheta_1}}$$

$$= M \left(\frac{e^{\left(\frac{1}{k_1}-\frac{1}{a_1}\right)\frac{\gamma^{\vartheta_2}}{\vartheta_2}}}{\frac{1}{k_1}-\frac{1}{a_1}} - \frac{1}{\frac{1}{k_1}-\frac{1}{a_1}} \right) \left(\frac{e^{\left(\frac{1}{k_2}-\frac{1}{b_1}\right)\frac{\tau^{\vartheta_1}}{\vartheta_1}}}{\frac{1}{k_2}-\frac{1}{b_1}} - \frac{1}{\frac{1}{k_2}-\frac{1}{b_1}} \right).$$

Now, as $\gamma \to \infty$, $\tau \to \infty$, and $\Re e\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) = \frac{1}{a_1}+\frac{1}{b_1} > \frac{1}{k_1}+\frac{1}{k_2}$, we have

$$\left| \int_0^\gamma \int_0^\tau e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}} \psi(x,t) d_{\vartheta_1} t \, d_{\vartheta_2} x \right| \leq M \left(\frac{1}{\frac{1}{k_1}-\frac{1}{a_1}} \right) \left(\frac{1}{\frac{1}{k_2}-\frac{1}{b_1}} \right).$$

□

Definition 3. *(Single conformable Sumudu transform of a function with two variables.)*

Let $\psi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a piecewise continuous function of conformable and exponentially order-bounded $\frac{1}{k_1} + \frac{1}{k_2}$, k_1,$k_2 > 0$. Then:

(1) The conformable Sumudu transform with respect to x of $\psi(x,t)$ is defined by

$$S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1)} = \frac{1}{\lambda_1}\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\psi(x,t)d_{\vartheta_2}x,$$

where $\lambda_1 \in \mathbb{C}$, $\vartheta_2 \in (0,1]$ and $d_{\vartheta_2}x = x^{\vartheta_2-1}dx$.
(2) The conformable Sumudu transform with respect to t of $\psi(x,t)$ is defined by

$$S^t_{\vartheta_1}[\psi(x,t)]_{(\lambda_2)} = \frac{1}{\lambda_2}\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t,$$

where $\lambda_2 \in \mathbb{C}$, $\vartheta_1 \in (0,1]$, and $d_{\vartheta_1}t = t^{\vartheta_1-1}dt$.

Note that if the order transformation of $\psi(x,t)$ can be changed [16], then

$$\frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x = \frac{1}{\lambda_2\lambda_1}\int_0^\infty\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\psi(x,t)d_{\vartheta_2}x\, d_{\vartheta_1}t.$$

So,

$$S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = S^x_{\vartheta_2}S^t_{\vartheta_1}[\psi(x,t)]_{(\lambda_1,\lambda_2)}.$$

Theorem 2. *Let $\psi(x,t)$, $\phi(x,t)$ be two functions that have the double conformable Sumudu transform. Then,*

(1) $S^t_{\vartheta_1}S^x_{\vartheta_2}[a\psi(x,t)+b\phi(x,t)]_{(\lambda_1,\lambda_2)} = aS^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + bS^t_{\vartheta_1}S^x_{\vartheta_2}[\phi(x,t)]_{(\lambda_1,\lambda_2)}.$

(2) $S^t_{\vartheta_1}S^x_{\vartheta_2}\left(e^{-\frac{x^{\vartheta_2}}{\vartheta_2 c_1}-\frac{t^{\vartheta_1}}{\vartheta_1 c_2}}\psi(x,t)\right)_{(\lambda_1,\lambda_2)} = \frac{c_1c_2}{(c_1+\lambda_1)(c_2+\lambda_2)}S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{\left(\frac{\lambda_1c_1}{\lambda_1+c_1},\frac{\lambda_2c_2}{\lambda_2+c_2}\right)}.$

(3) $S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(\gamma x,\mu t)]_{(\lambda_1,\lambda_2)} = S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{\left(\lambda_1\gamma^{\vartheta_2},\lambda_2\mu^{\vartheta_1}\right)}.$

Proof. Let $\psi(x,t)$, $\phi(x,t)$ be two functions that have the double conformable Sumudu transform. Then, we have:
For point (1):

$$\begin{aligned}
& S^t_{\vartheta_1}S^x_{\vartheta_2}[a\psi(x,t)+b\phi(x,t)]_{(\lambda_1,\lambda_2)} \\
= \;& \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}(a\psi(x,t)+b\phi(x,t))d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= \;& a\frac{1}{\lambda_2\lambda_1}\int_0^\infty\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\psi(x,t)d_{\vartheta_2}x\, d_{\vartheta_1}t + b\frac{1}{\lambda_2\lambda_1}\int_0^\infty\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\phi(x,t)d_{\vartheta_2}x\, d_{\vartheta_1}t \\
= \;& aS^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + bS^t_{\vartheta_1}S^x_{\vartheta_2}[\phi(x,t)]_{(\lambda_1,\lambda_2)}.
\end{aligned}$$

For point (2):

$$\begin{aligned}
& S^t_{\vartheta_1} S^x_{\vartheta_2}\left(e^{-\frac{x^{\vartheta_2}}{\vartheta_2 c_1}-\frac{t^{\vartheta_1}}{\vartheta_1 c_2}}\psi(x,t)\right)_{(\lambda_1,\lambda_2)} \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\left(e^{-\frac{x^{\vartheta_2}}{\vartheta_2 c_1}-\frac{t^{\vartheta_1}}{\vartheta_1 c_2}}\psi(x,t)\right)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\left(\frac{1}{\lambda_1}+\frac{1}{c_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}-\left(\frac{1}{\lambda_2}+\frac{1}{c_2}\right)\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda_1+c_1}{\lambda_1 c_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}-\left(\frac{\lambda_2+c_2}{\lambda_2 c_2}\right)\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{c_1c_2}{(c_1+\lambda_1)(c_2+\lambda_2)}\frac{1}{\frac{c_1c_2}{(c_1+\lambda_1)(c_2+\lambda_2)}}\frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda_1+c_1}{\lambda_1 c_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}-\left(\frac{\lambda_2+c_2}{\lambda_2 c_2}\right)\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{c_1c_2}{(c_1+\lambda_1)(c_2+\lambda_2)}\frac{1}{\frac{\lambda_1c_1}{(c_1+\lambda_1)}\frac{\lambda_2c_2}{(c_2+\lambda_2)}}\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda_1+c_1}{\lambda_1 c_1}\right)\frac{x^{\vartheta_2}}{\vartheta_2}-\left(\frac{\lambda_2+c_2}{\lambda_2 c_2}\right)\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{c_1c_2}{(c_1+\lambda_1)(c_2+\lambda_2)}S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{\left(\frac{\lambda_1c_1}{\lambda_1+c_1},\frac{\lambda_2c_2}{\lambda_2+c_2}\right)}.
\end{aligned}$$

Finally, we have, for point (3):

$$\begin{aligned}
& S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(\gamma x,\mu t)]_{(\lambda_1,\lambda_2)} \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(\gamma x,\mu t)d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \frac{1}{\lambda_1}\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\left[\frac{1}{\lambda_2}\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(\gamma x,\mu t)d_{\vartheta_1}t\right]d_{\vartheta_2}x \\
= & \frac{1}{\lambda_1}\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\left[\frac{1}{\lambda_2\mu^{\vartheta_1}}\int_0^\infty e^{-\frac{\zeta_2^{\vartheta_1}}{\lambda_2\vartheta_1\mu^{\vartheta_1}}}\psi(\gamma x,\zeta_2)d_{\vartheta_1}\zeta_2\right]d_{\vartheta_2}x \text{ (by letting } \zeta_2=\mu t) \\
= & \frac{1}{\lambda_1\gamma^{\vartheta_2}}\int_0^\infty e^{-\frac{\zeta_1^{\vartheta_2}}{\lambda_1\vartheta_2\gamma^{\vartheta_2}}}\left[\frac{1}{\lambda_2\mu^{\vartheta_1}}\int_0^\infty e^{-\frac{\zeta_2^{\vartheta_1}}{\lambda_2\vartheta_1\mu^{\vartheta_1}}}\psi(\zeta_1,\zeta_2)d_{\vartheta_1}\zeta_2\right]d_{\vartheta_2}\zeta_1 \text{ (by letting } \zeta_1=\gamma x) \\
= & \frac{1}{\lambda_1\gamma^{\vartheta_2}\lambda_2\mu^{\vartheta_1}}\int_0^\infty\int_0^\infty e^{-\frac{\zeta_1^{\vartheta_2}}{\lambda_1\vartheta_2\gamma^{\vartheta_2}}-\frac{\zeta_2^{\vartheta_1}}{\lambda_2\vartheta_1\mu^{\vartheta_1}}}\psi(\zeta_1,\zeta_2)d_{\vartheta_1}\zeta_2 d_{\vartheta_2}\zeta_1 \\
= & \frac{1}{\lambda_1\gamma^{\vartheta_2}\lambda_2\mu^{\vartheta_1}}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2\gamma^{\vartheta_2}}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1\mu^{\vartheta_1}}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x \text{ (since } \zeta_1,\zeta_2,x,t \text{ are dummy variables)} \\
= & S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{\left(\lambda_1\gamma^{\vartheta_2},\lambda_2\mu^{\vartheta_1}\right)},
\end{aligned}$$

and hence, the proof is complete. □

The following theorem gives some important relations, including a relation between the double Laplace transform defined in [9] and the double Sumudu transform.

Theorem 3. *Let $\psi\colon [0,\infty)\times[0,\infty)\to\mathbb{R}$ be a given function and $\vartheta_1,\vartheta_2\in(0,1]$. Then,*

(1) $S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}=\frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t\mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)}$, *where*

$$\mathscr{L}^{\vartheta_1}_t\mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{(p,s)}=\int_0^\infty\int_0^\infty e^{-p\frac{x^{\vartheta_2}}{\vartheta_2}-s\frac{t^{\vartheta_1}}{\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x.$$

(2) $S^t_{\vartheta_1}S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}=S^tS^x\left[\psi\left((\vartheta_2x)^{\frac{1}{\vartheta_2}},(\vartheta_1t)^{\frac{1}{\vartheta_1}}\right)\right]_{(\lambda_1,\lambda_2)}.$

(3) $(-1)^{m+n} S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\frac{x^{m\vartheta_2}}{\vartheta_2^m}\frac{t^{n\vartheta_1}}{\vartheta_1^n}\psi(x,t)\right]_{(\lambda_1,\lambda_2)} = \frac{1}{\lambda_1\lambda_2}\frac{\partial^{m+n}}{\partial\lambda_1^m\partial\lambda_2^n}\left[\lambda_1\lambda_2 S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}\right].$

Proof. Let ψ be a given function and let $\vartheta_1, \vartheta_2 \in (0,1]$. Then, we have:
For point (1):

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x$$
$$= \frac{1}{\lambda_1\lambda_2}\mathscr{L}t^{\vartheta_1}\mathscr{L}_x^{\vartheta_2}[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)}.$$

For point (2):

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x.$$

By letting $l = \frac{t^{\vartheta_1}}{\vartheta_1}$, $h = \frac{x^{\vartheta_2}}{\vartheta_2}$, we have

$$\frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\psi(x,t)d_{\vartheta_1}t\, d_{\vartheta_2}x = \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{h}{\lambda_1}-\frac{l}{\lambda_2}}\psi\left((\vartheta_2 h)^{\frac{1}{\vartheta_2}},(\vartheta_1 l)^{\frac{1}{\vartheta_1}}\right)dl\, dh.$$

Since t, x, l, h are dummy variables, we have

$$\frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{h}{\lambda_1}-\frac{l}{\lambda_2}}\psi\left((\vartheta_2 h)^{\frac{1}{\vartheta_2}},(\vartheta_1 l)^{\frac{1}{\vartheta_1}}\right)dl\, dh$$
$$= \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x}{\lambda_1}-\frac{t}{\lambda_2}}\psi\left((\vartheta_2 x)^{\frac{1}{\vartheta_2}},(\vartheta_1 t)^{\frac{1}{\vartheta_1}}\right)dt\, dx$$
$$= S^t S^x\left[\psi\left((\vartheta_2 x)^{\frac{1}{\vartheta_2}},(\vartheta_1 t)^{\frac{1}{\vartheta_1}}\right)\right]_{(\lambda_1,\lambda_2)}.$$

For point (3), (by Theorem 2.1 in [9]):

$$(-1)^{m+n} S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\frac{x^{m\vartheta_2}}{\vartheta_2^m}\frac{t^{n\vartheta_1}}{\vartheta_1^n}\psi(x,t)\right]_{(\lambda_1,\lambda_2)}$$
$$= \frac{1}{\lambda_1\lambda_2}(-1)^{m+n}\mathscr{L}_t^{\vartheta_1}\mathscr{L}_x^{\vartheta_2}\left[\frac{x^{m\vartheta_2}}{\vartheta_2^m}\frac{t^{n\vartheta_1}}{\vartheta_1^n}\psi(x,t)\right]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} \text{ (by 1))}$$
$$= \frac{1}{\lambda_1\lambda_2}\frac{\partial^{m+n}}{\partial\lambda_1^m\partial\lambda_2^n}\left[\mathscr{L}_t^{\vartheta_1}\mathscr{L}_x^{\vartheta_2}[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)}\right]$$
$$= \frac{1}{\lambda_1\lambda_2}\frac{\partial^{m+n}}{\partial\lambda_1^m\partial\lambda_2^n}\left[\lambda_1\lambda_2 S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}\right] \text{ (by 1)),}$$

and hence, the proof is complete. □

Theorem 4. *The Sumudu transform* $S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}$ *is as follows for some functions:*

1. $S^t_{\vartheta_1} S^x_{\vartheta_2}[c]_{(\lambda_1,\lambda_2)} = c$*, where c is constant.*
2. $S^t_{\vartheta_1} S^x_{\vartheta_2}\left[e^{a\frac{x^{\vartheta_2}}{\vartheta_2}+b\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(\lambda_1,\lambda_2)} = \frac{1}{(1-a\lambda_1)(1-b\lambda_2)}.$
3. $S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\sin a\frac{t^{\vartheta_1}}{\vartheta_1}\right]_{(\lambda_1,\lambda_2)} = \frac{a\lambda_2}{1+a^2\lambda_2^2}.$
4. $S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\cos a\frac{t^{\vartheta_1}}{\vartheta_1}\right]_{(\lambda_1,\lambda_2)} = \frac{1}{1+a^2\lambda_1^2}.$
5. $S^t_{\vartheta_1} S^x_{\vartheta_2}\left[1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(\lambda_1,\lambda_2)} = \frac{-a\lambda_2}{1-a\lambda_2}.$

6. $S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}\left[1-e^{a\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,\lambda_2)} = \frac{-a\lambda_1}{1-a\lambda_1}.$
7. $S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}\left[\left(1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right)\cos b\frac{x^{\vartheta_2}}{\vartheta_2}\right]_{(\lambda_1,\lambda_2)} = \frac{-a\lambda_2}{\left(1+b^2\lambda_1^2\right)(1-a\lambda_2)}.$

Proof. We will prove 2 and 7. The other cases are similar.
For 2., we have

$$\begin{aligned}
& S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}\left[e^{a\frac{x^{\vartheta_2}}{\vartheta_2}+b\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(\lambda_1,\lambda_2)} \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}} e^{a\frac{x^{\vartheta_2}}{\vartheta_2}+b\frac{t^{\vartheta_1}}{\vartheta_1}} d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \left(\frac{1}{\lambda_1}\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}} e^{a\frac{x^{\vartheta_2}}{\vartheta_2}} d_{\vartheta_2}x\right)\left(\frac{1}{\lambda_2}\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}} e^{b\frac{t^{\vartheta_1}}{\vartheta_1}} d_{\vartheta_1}t\right) \\
= & S_{\vartheta_2}^{x}\left(e^{a\frac{x^{\vartheta_2}}{\vartheta_2}}\right)_{\lambda_1} S_{\vartheta_1}^{t}\left(e^{b\frac{t^{\vartheta_1}}{\vartheta_1}}\right)_{(\lambda_2)} \\
= & \frac{1}{1-a\lambda_1}\frac{1}{1-b\lambda_2}=\frac{1}{(1-a\lambda_1)(1-b\lambda_2)}.
\end{aligned}$$

For 7., we have

$$\begin{aligned}
& S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}\left[\left(1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right)\cos b\frac{x^{\vartheta_2}}{\vartheta_2}\right]_{(\lambda_1,\lambda_2)} \\
= & \frac{1}{\lambda_1\lambda_2}\int_0^\infty\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\left(1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right)\cos b\frac{x^{\vartheta_2}}{\vartheta_2} d_{\vartheta_1}t\, d_{\vartheta_2}x \\
= & \left(\frac{1}{\lambda_1}\int_0^\infty e^{-\frac{x^{\vartheta_2}}{\lambda_1\vartheta_2}}\cos b\frac{x^{\vartheta_2}}{\vartheta_2} d_{\vartheta_2}x\right)\left(\frac{1}{\lambda_2}\int_0^\infty e^{-\frac{t^{\vartheta_1}}{\lambda_2\vartheta_1}}\left(1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right) d_{\vartheta_1}t\right) \\
= & S_{\vartheta_2}^{x}\left(\cos b\frac{x^{\vartheta_2}}{\vartheta_2}\right)_{\lambda_1} S_{\vartheta_1}^{t}\left(1-e^{a\frac{t^{\vartheta_1}}{\vartheta_1}}\right)_{(\lambda_2)} \\
= & \frac{1}{1+b^2\lambda_1^2}\cdot\frac{-a\lambda_2}{1-a\lambda_2}=\frac{-a\lambda_2}{\left(1+b^2\lambda_1^2\right)(1-a\lambda_2)}.
\end{aligned}$$

□

Theorem 5. *Let $\psi(x,t)$and $\phi(x,t)$ have a double Sumudu transform. Then,*

$$S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}[(\psi*\phi)(x,t)]_{(\lambda_1,\lambda_2)} = \lambda_1\lambda_2 S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}[\psi(x,t)]_{(\lambda_1,\lambda_2)} S_{\vartheta_1}^{t} S_{\vartheta_2}^{x}[\phi(x,t)]_{(\lambda_1,\lambda_2)},$$

where

$$(\psi*\phi)(x,t) = \int_0^x\int_0^t \psi(\zeta,\eta)\phi(x-\zeta,t-\eta)d\zeta\, d\eta.$$

Proof. Using Theorem 3 (part (2)) and Theorem 3.1 in [17], we have

$$\begin{aligned}
& S^t_{\vartheta_1} S^x_{\vartheta_2}[(\psi * \phi)(x,t)]_{(\lambda_1,\lambda_2)} \\
= \; & S^t S^x\left[(\psi*\phi)\left((\vartheta_2 x)^{\frac{1}{\vartheta_2}}, (\vartheta_1 t)^{\frac{1}{\vartheta_1}}\right)\right]_{(\lambda_1,\lambda_2)} \\
= \; & \lambda_1\lambda_2 S^t S^x\left[\psi\left((\vartheta_2 x)^{\frac{1}{\vartheta_2}}, (\vartheta_1 t)^{\frac{1}{\vartheta_1}}\right)\right]_{(\lambda_1,\lambda_2)} S^t S^x\left[\phi\left((\vartheta_2 x)^{\frac{1}{\vartheta_2}}, (\vartheta_1 t)^{\frac{1}{\vartheta_1}}\right)\right]_{(\lambda_1,\lambda_2)} \\
= \; & \lambda_1\lambda_2 S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} S^t_{\vartheta_1} S^x_{\vartheta_2}[\phi(x,t)]_{(\lambda_1,\lambda_2)}.
\end{aligned}$$

□

Theorem 6. *The double conformable Sumudu transforms of the ϑ_1-th- and ϑ_2-th-order fractional partial derivatives are given by:*

$$\begin{aligned}
S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_2}_x \psi(x,t)\right]_{(\lambda_1,\lambda_2)} &= \frac{1}{\lambda_1} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}[\psi(0,t)]_{(0,\lambda_2)}, \\
S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)} &= \frac{1}{\lambda_2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)},
\end{aligned}$$

where $D^{\vartheta_2}_x \psi(x,t)$and $D^{\vartheta_1}_t \psi(x,t)$ are ϑ_2-th- and ϑ_1-th-order fractional partial derivatives, respectively.

Proof. Using Theorem 2.5 in [12], Lemma 2.1 in [9], and Theorem 3, we have

$$\begin{aligned}
& S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_2}_x \psi(x,t)\right]_{(\lambda_1,\lambda_2)} \\
= \; & \frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x\left[D^{\vartheta_2}_x \psi(x,t)\right]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} \\
= \; & \frac{1}{\lambda_1\lambda_2}\left[\frac{1}{\lambda_1}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \mathscr{L}^{\vartheta_1}_t[\psi(0,t)]_{\left(0,\frac{1}{\lambda_2}\right)}\right] \\
= \; & \frac{1}{\lambda_1}\frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t[\psi(0,t)]_{\left(0,\frac{1}{\lambda_2}\right)} \\
= \; & \frac{1}{\lambda_1}\frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \frac{1}{\lambda_1}\frac{1}{\lambda_2}\mathscr{L}^{\vartheta_1}_t[\psi(0,t)]_{\left(0,\frac{1}{\lambda_2}\right)} \\
= \; & \frac{1}{\lambda_1} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}[\psi(0,t)]_{(0,\lambda_2)}.
\end{aligned}$$

In addition,

$$\begin{aligned}
S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)} &= \frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x\left[D^{\vartheta_1}_t \psi(x,t)\right]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} \\
&= \frac{1}{\lambda_1\lambda_2}\left[\frac{1}{\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \mathscr{L}^{\vartheta_2}_x[\psi(x,0)]_{\left(\frac{1}{\lambda_1},0\right)}\right] \\
&= \frac{1}{\lambda_2}\frac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x[\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \frac{1}{\lambda_2}\frac{1}{\lambda_1}\mathscr{L}^{\vartheta_2}_x[\psi(x,0)]_{\left(\frac{1}{\lambda_1},0\right)} \\
&= \frac{1}{\lambda_2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)}.
\end{aligned}$$

□

Theorem 7. *Let $\vartheta_1, \vartheta_2 \in (0,1]$ and $m, n \in \mathbb{N}$ such that $\psi \in C^k(\mathbb{R}^+ \times \mathbb{R}^+)$, $k = \max(m,n)$. Let the conformable Sumudu transforms of the functions $\psi(x,t)$, $D_x^{i\vartheta_2}\psi(x,t)$, $D_t^{j\vartheta_1}\psi(x,t)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ exist. Then,*

$$
\begin{aligned}
S_{\vartheta_1}^t S_{\vartheta_2}^x & \left[D_x^{m\vartheta_2}\psi(x,t)\right]_{(\lambda_1,\lambda_2)} \\
&= \tfrac{1}{\lambda_1^m} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} - \tfrac{1}{\lambda_1^m} S_{\vartheta_1}^t [\psi(0,t)]_{(0,\lambda_2)} \\
&\quad - \sum_{i=1}^{m-1} \left(\tfrac{1}{\lambda_1}\right)^{m-i} S_{\vartheta_1}^t \left[D_x^{i\vartheta_2}\psi(0,t)\right]_{(0,\lambda_2)},
\end{aligned}
$$

$$
\begin{aligned}
S_{\vartheta_1}^t S_{\vartheta_2}^x & \left[D_t^{n\vartheta_1}\psi(x,t)\right]_{(\lambda_1,\lambda_2)} \\
&= \tfrac{1}{\lambda_2^n} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} - \tfrac{1}{\lambda_2^n} S_{\vartheta_2}^x [\psi(x,0)]_{(\lambda_1,0)} \\
&\quad - \sum_{j=1}^{n-1} \left(\tfrac{1}{\lambda_2}\right)^{n-j} S_{\vartheta_2}^x \left[D_t^{j\vartheta_1}\psi(x,0)\right]_{(\lambda_1,0)}.
\end{aligned}
$$

Proof. Using Theorem 2.5 in [12], Theorem 2.2 in [9], and Theorem 3, we have

$$
\begin{aligned}
S_{\vartheta_1}^t S_{\vartheta_2}^x & \left[D_x^{m\vartheta_2}\psi(x,t)\right]_{(\lambda_1,\lambda_2)} \\
&= \tfrac{1}{\lambda_1\lambda_2} \mathscr{L}_t^{\vartheta_1} \mathscr{L}_x^{\vartheta_2} \left[D_x^{m\vartheta_2}\psi(x,t)\right]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} \\
&= \tfrac{1}{\lambda_1\lambda_2} \Bigg[\left(\tfrac{1}{\lambda_1}\right)^m \mathscr{L}_t^{\vartheta_1} \mathscr{L}_x^{\vartheta_2} [\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \left(\tfrac{1}{\lambda_1}\right)^{m-1} \mathscr{L}_t^{\vartheta_1} [\psi(0,t)]_{\left(0,\frac{1}{\lambda_2}\right)} \\
&\quad - \sum_{i=1}^{m-1} \left(\tfrac{1}{\lambda_1}\right)^{m-i-1} \mathscr{L}_t^{\vartheta_1} \left[D_x^{i\vartheta_2}\psi(0,t)\right]_{\left(0,\frac{1}{\lambda_2}\right)} \Bigg] \\
&= \left(\tfrac{1}{\lambda_1}\right)^m \tfrac{1}{\lambda_1\lambda_2} \mathscr{L}_t^{\vartheta_1} \mathscr{L}_x^{\vartheta_2} [\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \left(\tfrac{1}{\lambda_1}\right)^m \tfrac{1}{\lambda_2} \mathscr{L}_t^{\vartheta_1} [\psi(0,t)]_{\left(0,\frac{1}{\lambda_2}\right)} \\
&\quad - \sum_{i=1}^{m-1} \left(\tfrac{1}{\lambda_1}\right)^{m-i} \tfrac{1}{\lambda_2} \mathscr{L}_t^{\vartheta_1} \left[D_x^{i\vartheta_2}\psi(0,t)\right]_{\left(0,\frac{1}{\lambda_2}\right)} \\
&= \tfrac{1}{\lambda_1^m} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} - \tfrac{1}{\lambda_1^m} S_{\vartheta_1}^t [\psi(0,t)]_{(0,\lambda_2)} \\
&\quad - \sum_{i=1}^{m-1} \left(\tfrac{1}{\lambda_1}\right)^{m-i} S_{\vartheta_1}^t \left[D_x^{i\vartheta_2}\psi(0,t)\right]_{(0,\lambda_2)}.
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{n\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)} \\
&= \tfrac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x \left[D^{n\vartheta_1}_t \psi(x,t)\right]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} \\
&= \tfrac{1}{\lambda_1\lambda_2}\Bigg[\left(\tfrac{1}{\lambda_2}\right)^n \mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x [\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \left(\tfrac{1}{\lambda_2}\right)^{n-1}\mathscr{L}^{\vartheta_2}_x[\psi(x,0)]_{\left(\frac{1}{\lambda_1},0\right)} \\
&\qquad - \sum_{j=1}^{n-1}\left(\tfrac{1}{\lambda_2}\right)^{n-1-j}\mathscr{L}^{\vartheta_2}_x\left[D^{j\vartheta_1}_t \psi(x,0)\right]_{\left(\frac{1}{\lambda_1},0\right)}\Bigg] \\
&= \left(\tfrac{1}{\lambda_2}\right)^n \tfrac{1}{\lambda_1\lambda_2}\mathscr{L}^{\vartheta_1}_t \mathscr{L}^{\vartheta_2}_x [\psi(x,t)]_{\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right)} - \left(\tfrac{1}{\lambda_2}\right)^n \tfrac{1}{\lambda_1}\mathscr{L}^{\vartheta_2}_x[\psi(x,0)]_{\left(\frac{1}{\lambda_1},0\right)} \\
&\qquad - \sum_{j=1}^{n-1}\left(\tfrac{1}{\lambda_2}\right)^{n-j}\tfrac{1}{\lambda_1}\mathscr{L}^{\vartheta_2}_x\left[D^{j\vartheta_1}_t \psi(x,0)\right]_{\left(\frac{1}{\lambda_1},0\right)} \\
&= \tfrac{1}{\lambda_2^n} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \tfrac{1}{\lambda_2^n} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)} \\
&\qquad - \sum_{j=1}^{n-1}\left(\tfrac{1}{\lambda_2}\right)^{n-j} S^x_{\vartheta_2}\left[D^{j\vartheta_1}_t \psi(x,0)\right]_{(\lambda_1,0)}.
\end{aligned}
$$

□

3. Solution of Some Conformable Partial Differential Equations

In this section, we apply the double conformable fractional Sumudu transform to solve the following homogeneous and non-homogeneous fractional heat equations, the homogenous fractional wave equation, and the non-homogenous fractional telegraph equation.

Problem 1. *Consider the following homogeneous fractional heat equation:*

$$D^{\vartheta_1}_t \psi(x,t) = D^{2\vartheta_2}_x \psi(x,t),$$

where

$$
\begin{aligned}
\psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}}, \\
\psi(x,0) &= e^{\frac{x^{\vartheta_2}}{\vartheta_2}}, \\
D^{\vartheta_2}_x \psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}},
\end{aligned}
$$

$\vartheta_1, \vartheta_2 \in (0,1]$, $x, t > 0$, *and* $D^{\vartheta_2}_x$ *and* $D^{\vartheta_1}_t$ *denote the* ϑ_2*-th- and* ϑ_1*-th-order fractional partial conformable fractional derivative of* $\psi(x,t)$.

Solution 1. *By applying the conformable Sumudu transform, we have*

$$S_{\vartheta_1}^t S_{\vartheta_2}^x \left[D_t^{\vartheta_1}\psi(x,t)\right]_{(\lambda_1,\lambda_2)} = S_{\vartheta_1}^t S_{\vartheta_2}^x \left[D_x^{2\vartheta_2}\psi(x,t)\right]_{(\lambda_1,\lambda_2)},$$

$$\frac{1}{\lambda_2} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2} S_{\vartheta_2}^x [\psi(x,0)]_{(\lambda_1,0)}$$

$$= \frac{1}{\lambda_1^2} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1^2} S_{\vartheta_1}^t [\psi(0,t)]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S_{\vartheta_1}^t \left[D_x^{\vartheta_2}\psi(0,t)\right]_{(0,\lambda_2)},$$

$$\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1^2}\right) S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)}$$

$$= \frac{1}{\lambda_2} S_{\vartheta_2}^x [\psi(x,0)]_{(\lambda_1,0)} - \frac{1}{\lambda_1^2} S_{\vartheta_1}^t [\psi(0,t)]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S_{\vartheta_1}^t \left[D_x^{\vartheta_2}\psi(0,t)\right]_{(0,\lambda_2)},$$

$$\frac{\lambda_1^2-\lambda_2}{\lambda_2\lambda_1^2} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)}$$

$$= \frac{1}{\lambda_2} S_{\vartheta_2}^x \left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)} - \frac{1}{\lambda_1^2} S_{\vartheta_1}^t \left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S_{\vartheta_1}^t \left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right],$$

$$\frac{\lambda_1^2-\lambda_2}{\lambda_2\lambda_1^2} S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{\lambda_2}\frac{1}{1-\lambda_1} - \frac{1}{\lambda_1^2}\frac{1}{1-\lambda_2} - \frac{1}{\lambda_1}\frac{1}{1-\lambda_2},$$

$$S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_1^2-\lambda_1^2\lambda_2-(1+\lambda_1)(\lambda_2-\lambda_1\lambda_2)}{(\lambda_2-\lambda_1\lambda_2)\left(\lambda_1^2-\lambda_1^2\lambda_2\right)} \cdot \frac{\lambda_2\lambda_1^2}{\lambda_1^2-\lambda_2},$$

$$S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_1^2-\lambda_1^2\lambda_2-\lambda_2+\lambda_1\lambda_2+\lambda_1^2\lambda_2}{\lambda_2\lambda_1^2(1-\lambda_1)(1-\lambda_2)} \cdot \frac{\lambda_2\lambda_1^2}{\lambda_1^2-\lambda_2},$$

$$S_{\vartheta_1}^t S_{\vartheta_2}^x [\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{(1-\lambda_1)(1-\lambda_2)},$$

$$\psi(x,t) = e^{\frac{x^{\vartheta_2}}{\vartheta_2}+\frac{t^{\vartheta_1}}{\vartheta_1}},$$

which is the exact solution of our homogeneous fractional heat equation.

Problem 2. *Consider the following non-homogeneous fractional heat equation:*

$$D_t^{\vartheta_1}\psi(x,t) = D_x^{2\vartheta_2}\psi(x,t) + \cos\frac{x^{\vartheta_2}}{\vartheta_2},$$

where

$$\begin{aligned} \psi(0,t) &= 1 - e^{-\frac{t^{\vartheta_1}}{\vartheta_1}}, \\ \psi(x,0) &= 0, \\ D_x^{\vartheta_2}\psi(0,t) &= 0, \end{aligned}$$

$\vartheta_1, \vartheta_2 \in (0,1]$, $x, t > 0$, *and* $D_x^{\vartheta_2}$ *and* $D_t^{\vartheta_1}$ *denote the* ϑ_2*-th- and* ϑ_1*-th-order fractional partial conformable fractional derivative of* $\psi(x,t)$.

Solution 2. *By applying the conformable Sumudu transform, we have*

$$S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)}$$
$$= S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{2\vartheta_2}_x \psi(x,t)\right]_{(\lambda_1,\lambda_2)} + S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\cos\frac{x^{\vartheta_2}}{\vartheta_2}\right]_{(\lambda_1,\lambda_2)},$$

$$\frac{1}{\lambda_2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)}$$
$$= \frac{1}{\lambda_1^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1^2} S^t_{\vartheta_1}[\psi(0,t)]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}\left[D^{\vartheta_2}_x \psi(0,t)\right]_{(0,\lambda_2)} + S^t_{\vartheta_1} S^x_{\vartheta_2}\left[\cos\frac{x^{\vartheta_2}}{\vartheta_2}\right]_{(\lambda_1,\lambda_2)},$$

$$\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1^2}\right) S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}$$
$$= \frac{1}{\lambda_2} S^x_{\vartheta_2}[0]_{(\lambda_1,0)} - \frac{1}{\lambda_1^2} S^t_{\vartheta_1}\left[1 - e^{-\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}[0]_{(0,\lambda_2)} + \frac{1}{1+\lambda_1^2},$$

$$\frac{\lambda_1^2 - \lambda_2}{\lambda_2 \lambda_1^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = -\frac{1}{\lambda_1^2}\frac{\lambda_2}{1+\lambda_2} + \frac{1}{1+\lambda_1^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \left[\frac{-\lambda_2}{\lambda_1^2(1+\lambda_2)} + \frac{1}{1+\lambda_1^2}\right] \cdot \frac{\lambda_2 \lambda_1^2}{\lambda_1^2 - \lambda_2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_2 \lambda_1^2}{\left(\lambda_1^2 + \lambda_1^2 \lambda_2\right)\left(1+\lambda_1^2\right)},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_2}{(1+\lambda_2)\left(1+\lambda_1^2\right)},$$

$$\psi(x,t) = \left(1 - e^{-\frac{t^{\vartheta_1}}{\vartheta_1}}\right)\cos\frac{x^{\vartheta_2}}{\vartheta_2},$$

which is the exact solution of our non-homogeneous fractional heat equation.

Problem 3. *Consider the following homogeneous fractional wave equation:*

$$D^{2\vartheta_1}_t \psi(x,t) = D^{2\vartheta_2}_x \psi(x,t),$$

where

$$\begin{aligned} \psi(x,0) &= e^{\frac{x^{\vartheta_2}}{\vartheta_2}}, \\ \psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}}, \\ D^{\vartheta_2}_x \psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}}, \\ D^{\vartheta_1}_t \psi(x,0) &= e^{\frac{x^{\vartheta_2}}{\vartheta_2}}, \end{aligned}$$

$\vartheta_1, \vartheta_2 \in (0,1]$, $x, t > 0$, *and* $D^{\vartheta_2}_x$ *and* $D^{\vartheta_1}_t$ *denote the* ϑ_2*-th- and* ϑ_1*-th-order fractional partial conformable fractional derivative of* $\psi(x,t)$.

Solution 3. *By applying the conformable Sumudu transform, we have*

$$S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{2\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)} = S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{2\vartheta_2}_x \psi(x,t)\right]_{(\lambda_1,\lambda_2)},$$

$$\frac{1}{\lambda_2^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2^2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,0)\right]_{(\lambda_1,0)}$$

$$= \frac{1}{\lambda_1^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1^2} S^t_{\vartheta_1}[\psi(0,t)]_{(0,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}\left[D^{\vartheta_2}_x \psi(0,t)\right]_{(0,\lambda_2)},$$

$$\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}\right) S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}$$

$$= \frac{1}{\lambda_2^2} S^x_{\vartheta_2}\left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)} + \frac{1}{\lambda_2} S^x_{\vartheta_2}\left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)} - \frac{1}{\lambda_1^2} S^t_{\vartheta_1} S^x_{\vartheta_2}\left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_1} S^t_{\vartheta_1}\left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(0,\lambda_2)},$$

$$\frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)}$$

$$= \frac{1}{\lambda_2^2}\frac{1}{1-\lambda_1} + \frac{1}{\lambda_2}\frac{1}{1-\lambda_1} - \frac{1}{\lambda_1^2}\frac{1}{1-\lambda_2} - \frac{1}{\lambda_1}\frac{1}{1-\lambda_2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \left[\frac{1+\lambda_2}{\lambda_2^2(1-\lambda_1)} - \frac{1+\lambda_1}{\lambda_1^2(1-\lambda_2)}\right]\cdot \frac{\lambda_1^2\lambda_2^2}{\lambda_1^2-\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_1^2-\lambda_2^2\lambda_1^2-\lambda_2^2+\lambda_2^2\lambda_1^2}{\lambda_1^2\lambda_2^2(1-\lambda_1)(1-\lambda_2)}\cdot \frac{\lambda_1^2\lambda_2^2}{\lambda_1^2-\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2(1-\lambda_1)(1-\lambda_2)}\cdot \frac{\lambda_1^2\lambda_2^2}{\lambda_1^2-\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{(1-\lambda_1)(1-\lambda_2)},$$

$$\psi(x,t) = e^{\frac{x^{\vartheta_2}}{\vartheta_2} + \frac{t^{\vartheta_1}}{\vartheta_1}},$$

which is the exact solution of our homogeneous fractional wave equation.

Problem 4. *Consider the following non-homogeneous fractional telegraph equation:*

$$-D^{2\vartheta_1}_t \psi(x,t) + D^{2\vartheta_2}_x \psi(x,t) - D^{\vartheta_1}_t \psi(x,t) - \psi(x,t) = -2e^{\frac{x^{\vartheta_2}}{\vartheta_2} + \frac{t^{\vartheta_1}}{\vartheta_1}},$$

where

$$\begin{aligned} \psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}}, \\ \psi(x,0) &= e^{\frac{x^{\vartheta_2}}{\vartheta_2}}, \\ D^{\vartheta_2}_x \psi(0,t) &= e^{\frac{t^{\vartheta_1}}{\vartheta_1}}, \\ D^{\vartheta_1}_t \psi(x,0) &= e^{\frac{x^{\vartheta_2}}{\vartheta_2}}, \end{aligned}$$

$\vartheta_1, \vartheta_2 \in (0,1]$, $x,t > 0$, *and* $D^{\vartheta_2}_x$ *and* $D^{\vartheta_1}_t$ *denote the* ϑ_2*-th- and* ϑ_1*-th-order fractional partial conformable fractional derivative of* $\psi(x,t)$.

Solution 4. *By applying the conformable Sumudu transform, we have*

$$S^t_{\vartheta_1} S^x_{\vartheta_2}\left[-2e^{\frac{x^{\vartheta_2}}{\vartheta_2}+\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(\lambda_1,\lambda_2)}$$

$$= -S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{2\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)} + S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{2\vartheta_2}_x \psi(x,t)\right]_{(\lambda_1,\lambda_2)} - S^t_{\vartheta_1} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,t)\right]_{(\lambda_1,\lambda_2)}$$

$$-S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)},$$

$$\frac{-2}{(1-\lambda_1)(1-\lambda_2)}$$

$$= -\frac{1}{\lambda_1^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + \frac{1}{\lambda_1^2} S^t_{\vartheta_1}[\psi(0,t)]_{(0,\lambda_2)} + \frac{1}{\lambda_1} S^t_{\vartheta_1}\left[D^{\vartheta_2}_x \psi(0,t)\right]_{(0,\lambda_2)}$$

$$+\frac{1}{\lambda_2^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} - \frac{1}{\lambda_2^2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}\left[D^{\vartheta_1}_t \psi(x,0)\right]_{(\lambda_1,0)}$$

$$-\frac{1}{\lambda_2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + \frac{1}{\lambda_2} S^x_{\vartheta_2}[\psi(x,0)]_{(\lambda_1,0)} - S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)},$$

$$\frac{-2}{(1-\lambda_1)(1-\lambda_2)}$$

$$= \left(-\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} - \frac{1}{\lambda_2} - 1\right) S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + \frac{1}{\lambda_1^2} S^t_{\vartheta_1}\left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(0,\lambda_2)} + \frac{1}{\lambda_1} S^t_{\vartheta_1}\left[e^{\frac{t^{\vartheta_1}}{\vartheta_1}}\right]_{(0,\lambda_2)}$$

$$-\frac{1}{\lambda_2^2} S^x_{\vartheta_2}\left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)} - \frac{1}{\lambda_2} S^x_{\vartheta_2}\left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)} + \frac{1}{\lambda_2} S^x_{\vartheta_2}\left[e^{\frac{x^{\vartheta_2}}{\vartheta_2}}\right]_{(\lambda_1,0)},$$

$$\frac{-2}{(1-\lambda_1)(1-\lambda_2)}$$

$$= \frac{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2}{\lambda_1^2\lambda_2^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + \frac{1}{\lambda_1^2(1-\lambda_2)} + \frac{1}{\lambda_1(1-\lambda_2)} - \frac{1}{\lambda_2^2(1-\lambda_1)},$$

$$\frac{-2}{(1-\lambda_1)(1-\lambda_2)} = \frac{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2}{\lambda_1^2\lambda_2^2} S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} + \frac{-\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_2-\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2(1-\lambda_1)(1-\lambda_2)},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \left[\frac{-2}{(1-\lambda_1)(1-\lambda_2)} - \frac{-\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_2-\lambda_1^2+\lambda_2^2}{\lambda_1^2\lambda_2^2(1-\lambda_1)(1-\lambda_2)}\right] \cdot \frac{\lambda_1^2\lambda_2^2}{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{-2\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_2^2-\lambda_1^2\lambda_2+\lambda_1^2-\lambda_2^2}{\lambda_1^2\lambda_2^2(1-\lambda_1)(1-\lambda_2)} \cdot \frac{\lambda_1^2\lambda_2^2}{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2}{(1-\lambda_1)(1-\lambda_2)} \cdot \frac{1}{-\lambda_2^2+\lambda_1^2-\lambda_1^2\lambda_2-\lambda_1^2\lambda_2^2},$$

$$S^t_{\vartheta_1} S^x_{\vartheta_2}[\psi(x,t)]_{(\lambda_1,\lambda_2)} = \frac{1}{(1-\lambda_1)(1-\lambda_2)},$$

$$\psi(x,t) = e^{\frac{x^{\vartheta_2}}{\vartheta_2}+\frac{t^{\vartheta_1}}{\vartheta_1}},$$

which is the exact solution of our non-homogeneous fractional telegraph equation.

4. Conclusions

The conformable fractional integral defined by Khalil et al. [4] was used to modify the double conformable Sumudu transform. Moreover, with the use of the conformable double Laplace transform defined in [9], some more properties of the transform, including its convolution properties, in addition to the existence of the transform for functions satisfying certain exponential conditions, were proved. Finally, exact solutions of some important partial differential equations of the conformable fractional type—namely, the heat equation, the wave equation, and the telegraph equation—were given.

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