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Screw Motion via Matrix Algebra in Three-Dimensional Generalized Space

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Abstract: This paper aims to investigate the screw motion in generalized space. For this purpose, firstly, the change in the screw coordinates is analyzed according to the motion of the reference frames. Moreover, the special cases of this change, such as pure rotation and translation, are discussed. Matrix multiplication and the properties of dual numbers are used to obtain dual orthogonal matrices, which are used to simplify the manipulation of screw motion in generalized space. In addition, the dual angular velocity matrix is calculated and shows that the exponential of this matrix can represent the screw displacement in the generalized space. Finally, to support our results, we give two examples of screw motion, the rotation part of which is elliptical and hyperbolic.

Keywords: generalized space; kinematics; screw motion; matrix algebra



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1. Introduction

A rigid body is a system of particles with a fixed distance from one another, so every displacement of a rigid body be an isometry. An isometry is called a rigid displacement when it has properties such as rotation and translation. The set of rigid displacements is called the Euclidean group [1,2].

The term “kinematics”, which originates from Greek, means motion, and it is a branch of mechanics that deals with the analysis of the motion of particles and rigid bodies [3].

In order to represent rigid motion in Euclidean or Lorentzian space equipped with multiple coordinate frames, it is required to determine the “rotation matrix” and “translation axis” concepts. In particular, if the axis of rotation is parallel to the axis of translation, this transformation is called screw motion and the axis is called the screw axis. These concepts are used to create homogeneous transformation matrices, which represent the position and direction of a coordinate frame relative to the other. These transformations allow us to navigate from one coordinate frame to another [4,5].

To obtain frame M from frame F , it is necessary to apply a rotation determined by R , and translation (with respect to F) given as n , where F is a fixed frame and M is a moving frame. This transformation called coordinate transformation, which is denoted as $D : F \rightarrow M$, and the coordinates of it are determined $y = Rx + n$, where y is the position of x after the displacement. In this notation, R is an $n \times n$ orthogonal matrix called a rotation matrix, and n is an n -dimensional vector called a translation. This transformation is denoted by $D = (R, n)$ [3,5].

Screw theory was introduced by Chasles in 1830 in order to explain the concept of the twist motion of a rigid body. Then, in 1865, Plücker called this a screw. Later, in 1900, Ball explained the kinematics of a rigid body by using screw theory [6–8]. The screw theory is defined as the concept of rotation and translation about a common axis; it can be thought of as any helix of rotation. A common example is the thread of a screw (hence the name screw theory). In engineering, a screw thread is defined by a pitch, i.e., the ratio of translational motion to rotational motion. To describe the direction in which the screw points, an axis is needed along which it points. Finally, to describe a point on the screw,

we need a magnitude, a measure of distance along the screw that allows us to find our position on the spiral. Thus, to describe rigid body motion using a screw, we need three components: a pitch, an axis, and an amount [9].

The derivative of motion represents the velocity of a point from frame F to frame M . Linear velocity is the instantaneous rate of change in the linear position of a point relative to some frame. The angular velocity describes the rotational motion of M with respect to F [3,5,10].

Dual numbers were initially introduced by Clifford in 1873. Their first applications to kinematics were in the study by Kotel'nikov in 1895 and in 1903. Bottema and Roth, in 1978, used dual numbers in theoretical kinematics. Then, dual quaternions were applied to spatial kinematics by Agrawal in 1987. Further information on dual quaternions can be found in the work of McCarthy from 1990 [3,5,11–15].

Vector spaces with a quadratic form generate an associative algebra with unity called Clifford algebra. The quaternion algebra, a subalgebra of Clifford algebra, is isomorphic to the rotation group. The generalized quaternions \mathbb{H} are a four-dimensional algebra that is associative but not commutative. The algebra of a generalized quaternion is a natural generalization of quaternion algebra \mathbf{H} [16–20].

According to the definition, the generalization space is a Euclidean space if $\alpha = \beta = 1$ and a semi-Euclidean space if $\alpha = 1$ and $\beta = -1$. In this study, all situations of α and β are considered, except $\alpha = \beta = 0$. Some concepts that are used in the definition of the scalar product and vector product in the generalization space are different from those in Euclidean and Lorentzian space.

Beggs gave a derivation for a screw matrix by using two different coordinate systems in 1965. In 2018, Siddika studied the screw theory in Lorentzian space \mathbb{L}^3 . In 2021, Ata and Savcı showed that unit-generalized quaternions $\mathbb{H}(\alpha, \beta)$ correspond to a rotation in generalized space $\mathbb{E}^3(\alpha, \beta)$. They obtained the screw axis of a displacement, the dual Rodrigues equation, and the Euler parameters of a spatial displacement, in generalized space. They stated that this motion gives the rotational motion in three-dimensional Euclidean space \mathbb{E}^3 and in three-dimensional semi-Euclidean space for their specific choices of $\alpha, \beta \in \mathbb{R}$. In 2022, Savcı obtained the screw axis of a displacement, the dual Rodrigues equation, and the Euler parameters of a spatial displacement, in generalized space $\mathbb{E}^3(\alpha, \beta)$ [10,20–23].

Since the rotational motion in Euclidean space is on the sphere, it is circular, whereas the rotational motion in Lorentz space is hyperbolic because it is on hyperboloids. With the appropriate selection of α and β in generalized space, rotational motions such as circular, hyperbolic, and elliptical can be obtained. Thus, this gives the opportunity to work in many areas, such as from the centrifugal force taken into account when calculating the slope of the road to the motion of an electric field [24] and the orbits of the planets [25,26]. Generalized space is also a scalar product space. For general information about scalar product spaces and their algebraic properties, see [16–18,20,24–29].

Screw motion is an important part of rigid body motion, and it is vital in various fields, such as multi-body dynamics, mechanical design, computational geometry, and robot mechanics. From this point of view, it is important to reveal the wider application prospects of the screw motion. With our study, the details of which are given below, screw motion has been examined for all situations in generalized space.

In this paper, we explain the change in the screw coordinates according to the motion of the reference frames and consider the specific cases of changes in the coordinates. Then, we obtain Plücker coordinates for the screw axis in generalized space. Moreover, we obtain dual orthogonal matrices to simplify the screw coordinate transformations by using matrix multiplication and the properties of dual numbers. Finally, we obtain a dual angular velocity matrix and show that screw displacement can be represented as the exponential of this matrix in generalized space $\mathbb{E}^3(\alpha, \beta)$, which includes the Euclidean and semi-Euclidean space.

2. General Information

In this section, we will provide some basic information that is used in the manuscript.

Definition 1. Let $r = (r_1, r_2, r_3)$, $s = (s_1, s_2, s_3)$ be two vectors in \mathbb{R}^3 and $\alpha, \beta \in \mathbb{R}$. The generalized scalar product is defined by

$$\langle r, s \rangle_G = \alpha r_1 s_1 + \beta r_2 s_2 + \alpha \beta r_3 s_3 \tag{1}$$

Or

$$\langle r, s \rangle_G = r^T \epsilon s$$

where $\epsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix}$.

The vector space \mathbb{R}^3 , with the generalized scalar product defined over it, is called a three-dimensional generalized space and is denoted by $\mathbb{E}^3(\alpha, \beta) = (\mathbb{R}^3, \langle, \rangle_G)$. This space includes the Euclidean and semi-Euclidean spaces, i.e.,

If $\alpha = \beta = 1$, then $\mathbb{E}^3(1, 1) = \mathbb{E}^3$ is a three-dimensional Euclidean space.

If $\alpha = 1$ and $\beta = -1$, then $\mathbb{E}^3(1, -1) = \mathbb{E}_2^3$ is a three-dimensional semi-Euclidean space.

The generalized vector product is defined as follows:

$$r \times_G s = \beta(r_2 s_3 - r_3 s_2)i - \alpha(r_1 s_3 - r_3 s_1)j + (r_1 s_2 - r_2 s_1)k,$$

where $i \times_G j = k, j \times_G k = \beta i, k \times_G i = \alpha j$.

Let $\{e_1, e_2, e_3\}$ be the orthonormal basis of this space; then, its Clifford algebra $Cl(\mathbb{E}^3(\alpha, \beta)) = Cl_{p,q}$ has the basis $\{1, e_1, e_2, e_1 e_2\}$, where $p + q = 3, e_1^2 = -\alpha, e_2^2 = -\beta$ and $e_1 e_2 = -e_2 e_1$ [18].

Definition 2. A dual number \hat{s} has the form $\hat{s} = s + \epsilon s^*$, where $s, s^* \in \mathbb{R}, \epsilon^2 = 0$ with $\epsilon \neq 0$. The set

$$\mathbb{D} = \{\hat{s} = s + \epsilon s^* : s, s^* \in \mathbb{R}, \epsilon^2 = 0, \epsilon \neq 0\}$$

is the set of dual numbers, and it forms an Abelian ring with unity and divisors of zero [30].

On the other hand, the set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \{\hat{S} : \hat{S} = S + \epsilon S^*, S, S^* \in \mathbb{R}^3\}$$

is a module over the ring \mathbb{D} .

Let us denote $\hat{S} = S + \epsilon S^* = (s_1, s_2, s_3) + \epsilon(s_1^*, s_2^*, s_3^*)$ and $\hat{R} = R + \epsilon R^* = (r_1, r_2, r_3) + \epsilon(r_1^*, r_2^*, r_3^*)$. The Lorentzian scalar product of S and R is defined as follows:

$$\langle \hat{S}, \hat{R} \rangle = \langle S, R \rangle + \epsilon(\langle S, R^* \rangle + \langle S^*, R \rangle).$$

Dual space \mathbb{D}^3 , together with the scalar product, is called the dual Lorentzian space and denoted by \mathbb{D}_1^3 . For $\hat{S} \neq 0$, the norm $\|\hat{S}\|$ of \hat{S} is defined by $\|\hat{S}\| = \sqrt{|\langle \hat{S}, \hat{S} \rangle|}$. A dual vector $\hat{S} = S + \epsilon S^*$ is called a dual-space-like vector if $\langle S, S \rangle > 0$ or $S = 0$, a dual time-like vector if $\langle S, S \rangle < 0$, and a dual null vector if $\langle S, S \rangle = 0$ for $\hat{S} \neq 0$ [30,31].

The dual angle between two lines is the dual number $\hat{\theta} = \theta + \epsilon d$, where θ is the angle between two lines about the common normal and d is the distance between them along their common normal. Some trigonometric and hyperbolic functions with dual variables can be written as follows:

$$\begin{aligned}
\sin \hat{\theta} &= \sin(\theta + \varepsilon d) = \sin \theta + \varepsilon d \cos \theta \\
\cos \hat{\theta} &= \cos(\theta + \varepsilon d) = \cos \theta - \varepsilon d \sin \theta \\
\sinh \hat{\theta} &= \sinh(\theta + \varepsilon d) = \sinh \theta + \varepsilon d \cosh \theta \\
\cosh \hat{\theta} &= \cosh(\theta + \varepsilon d) = \cosh \theta + \varepsilon d \sinh \theta.
\end{aligned} \tag{2}$$

These definitions arise from the Taylor series expansion of the function $f(\theta + \varepsilon d)$ about θ , which yields [5,30,31]

$$\begin{aligned}
f(\hat{\theta}) &= f(\theta + \varepsilon d) = f(\theta) + \varepsilon d f'(\theta) + (\varepsilon d)^2 f''(\theta)/2 + \dots \\
&= f(\theta) + \varepsilon d f'(\theta)
\end{aligned}$$

Definition 3. A matrix $C = \begin{bmatrix} 0 & -\beta c_3 & \beta c_2 \\ \alpha c_3 & 0 & -\alpha c_1 \\ -c_2 & c_1 & 0 \end{bmatrix}$ is called a generalized skew-symmetric

matrix (or G-skew symmetric matrix) if $C^T \epsilon = -\epsilon C$. The matrix C satisfies the equation $Cx = c \times_{\mathbb{G}} x$, where the vector $c = (c_1, c_2, c_3)$ [18].

Definition 4. The matrix R is called a generalized orthogonal matrix (or G-orthogonal matrix) if $R^T \epsilon R = |R| \epsilon$. The set of all G-orthogonal matrices under matrix multiplication operation is called a rotation group in $\mathbb{E}^3(\alpha, \beta)$ [18,20].

A rotation about the origin can be given with the equation of $Rx = y$, which is a 3×3 G-orthogonal matrix and $x \in \mathbb{E}^3(\alpha, \beta)$. The generalized Cayley formula is defined as $R = (I - C)^{-1}(I + C) = (I + C)(I - C)^{-1}$, where C is a G-skew symmetric matrix. By using the G-Cayley formula, a G-orthogonal matrix can be obtained by a G-skew symmetric matrix C , which is defined in Definition 3 [20].

Definition 5. The set of all generalized orthogonal matrices with $|R| = 1$ under the operation of matrix multiplication forms a group denoted as $SO(\alpha, \beta)(3)$, called a rotation group in $\mathbb{E}^3(\alpha, \beta)$ [18].

Definition 6. The set of displacements of the space $\mathbb{E}^3(\alpha, \beta)$ is an algebraic group. If $T_1 : F \rightarrow M_1$ and $T_2 : M_1 \rightarrow M_2$ are displacements, then $T = T_1 T_2 : F \rightarrow M_2$ is also a displacement.

The matrix representation of the combination of these two displacements is

$$\begin{bmatrix} R_1 & n_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & n_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 n_2 + n_1 \\ 0 & 1 \end{bmatrix}$$

where 0 is a row vector $(0, 0, 0)$. This representation is called a homogeneous transformation. The set of homogeneous transformations in $\mathbb{E}^3(\alpha, \beta)$ is as follows:

$$G - H(4) = \left\{ \begin{bmatrix} R & n \\ 0 & 1 \end{bmatrix}; R \in SO(\alpha, \beta)(3), n \in \mathbb{R}^3 \right\}.$$

$g = (R, n)$ is the homogeneous transformation corresponding to the displacement $D = (R, n)$ in $\mathbb{E}^3(\alpha, \beta)$. Here,

$$\begin{aligned}
g(t) : \mathbb{R} &\rightarrow G - H(4) \\
t &\rightarrow g(t) = (t_{ij})_{4 \times 4}
\end{aligned}$$

Although the displacement is not a linear transformation, the homogeneous transformation $g = (R, n)$ is a linear transformation.

The tangent operators of $G - H(4)$ must satisfy the relation

$$\begin{aligned}\dot{g}g^{-1} &= \begin{bmatrix} \dot{R} & \dot{n} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}n \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} A & s \\ 0 & 0 \end{bmatrix} \\ &= (A, s)\end{aligned}$$

where $A = \dot{R}R^{-1}$ is the 3×3 angular velocity matrix and $s = -An + \dot{n}$ is its three-dimensional linear velocity vector. The tangent direction of the motion $g(t)$ at $t = t_0$ is defined by $\dot{g}(t_0)$.

Similarly, the condition for $SO(\alpha, \beta)(3)$ to be tangent operators is obtained from the relation $R^T \epsilon R = \epsilon$ in $\mathbb{E}^3(\alpha, \beta)$. Differentiating both sides, we obtain

$$\dot{R}^T \epsilon R + R^T \epsilon \dot{R} = 0$$

which can be written as

$$R^T \epsilon \dot{R} = (-R^T \epsilon \dot{R})^T.$$

The last equation shows that $R^T \epsilon \dot{R} = \Phi$ is a skew-symmetric matrix that is called the angular velocity matrix of the rotation $R(t)$ in $\mathbb{E}^3(\alpha, \beta)$. If we can calculate that $\dot{R} = R\epsilon^{-1}\Phi$, let $\epsilon^{-1}\Phi = \Omega$, then $\dot{R} = R\Omega$. Note that Ω is a G-skew-symmetric matrix.

$$\dot{R}(t) = R(t) \Omega,$$

The solution of this matrix-differential equation gives a one-parameter group of rotations as follows [23]:

$$R(t) = e^{t\Omega}.$$

In the following section, the translation matrix between the coordinate transformations will be obtained and the changes in the screw motion depending on the coordinate transformations will be examined.

3. Coordinate Transformations of a Screw in $\mathbb{E}^3(\alpha, \beta)$

The tangent operator of g is $L = (A, s)$, where A is 3×3 G-skew symmetric matrix and s is a three-dimensional vector. Let the vector a correspond to matrix A . Then, the six-dimensional vector $l = (a, s)$ is called a screw. In this section, we will analyze the change in the coordinates of the screw depending on the change in reference frame for the motion.

Given fixed F and moving frames M , let a motion be described by displacement $g(t) : F \rightarrow M$, with parameter t . The displacements $T : F \rightarrow F'$ and $T : M \rightarrow M'$ define $g'(t) = Tg(t)T^{-1}$. Then, the motion can be given as $g'(t) : F' \rightarrow M'$. The tangent operator L' of $g'(t)$ is obtained from the tangent operator L of $g(t)$ as follows:

$$\begin{aligned}L' &= (\dot{g}')(g')^{-1} \\ &= (T\dot{g}T^{-1})(Tg^{-1}T^{-1}) \\ &= T\dot{g}g^{-1}T^{-1} \\ &= TLT^{-1}.\end{aligned}$$

If we rewrite this equation more clearly for displacement $T = (R, n)$, we obtain

$$\begin{aligned} L^i &= \begin{bmatrix} R & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^{-1} & -R^{-1}n \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} RAR^{-1} & -RAR^{-1}n + Rs \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (3)$$

From the G-skew symmetric matrix $A^i = RAR^{-1}$, we can obtain the vector a^i . From the relationship between the generalized cross-product and G-skew symmetric matrices, we have

$$\begin{aligned} -RAR^{-1}n &= -A^i n \\ &= -a^i \times_G n \\ &= n \times_G a^i \end{aligned}$$

Thus, the screw $l = (a, s)$ becomes $l^i = (a^i, s^i)$, where

$$\begin{aligned} a^i &= R a \\ s^i &= n \times_G R a + R s \end{aligned}$$

This transformation can be written as

$$\begin{bmatrix} a^i \\ s^i \end{bmatrix} = \begin{bmatrix} R & 0 \\ TR & R \end{bmatrix} \begin{bmatrix} a \\ s \end{bmatrix}, \quad (4)$$

Here, T is a G-skew symmetric matrix corresponding to vector n , and 0 is the 3×3 square zero matrix. Equation (4) describes how the screw changes with the change in reference frames.

In the following section, it will be examined how the screw motion, which changes depending on the coordinate transformations, changes the components that make up the screw, and their specific cases will be discussed.

4. The Forms of a Screw in $\mathbb{E}^3(\alpha, \beta)$

Let us consider a specific case of the change in coordinates in Equation (4). It is the screw motion obtained as a pure translation $g = (I, n)$ of the fixed and moving frames. Thus, $l = (a, s)$ transforms to

$$l^i = (a, n \times_G a + s).$$

This implies that the linear velocity s of the screw changes depending on the changes in the origin of the reference frames.

Linear velocity vector a of $l = (a, s)$ can be split into two components: one that is parallel to a and another that is orthogonal to a . Let pa be a parallel component; then, p is obtained as follows:

$$p = \frac{\langle s, a \rangle_G}{\langle a, a \rangle_G}. \quad (5)$$

The orthogonal part is $s - pa$. Here, we can find a point k on the screw axis such that

$$k \times_G a = s - pa.$$

The solution of the above equation is

$$k = \frac{a \times_G s}{\langle a, a \rangle_G}. \quad (6)$$

Let us change the reference frames by the translation $g = (I, -k)$; then, $l = (a, s)$ is transformed to the screw $l' = (a, -k \times_G a + s)$. Since $k \times_G a$ orthogonal to a , hence

$$-k \times_G a + s = pa.$$

Therefore, in this reference frame, the screw has the form $l' = (a, pa)$. This implies that in the new frame, the angular and linear velocity vectors are in the same direction of a . It is clear that the body is moving in a screw motion at time t . The axis of this screw motion is line $d = k + ta$, called the instantaneous screw axis in $\mathbb{E}^3(\alpha, \beta)$. The general screw $l = (a, s)$ can be written in the standard form using equation $s = k \times_G a + pa$

$$l = (a, k \times_G a + pa) \quad (7)$$

The line $d = k + ta$ and constant p are the axis and the pitch of screw l , respectively.

When $p = 0$ in Equation (7), we encounter a critical case of the screw where there is no velocity vector. Let the screw be rewritten:

$$l = (a, k \times_G a).$$

There is an angular velocity and no linear velocity relative to $d = k + ta$. Thus, the line l can state the screw, and the components of the l are known as the Plücker coordinates of the line.

When $p = 0$, and the screw $l = (0, s)$, we obtain another special case where the screw degenerates into a line screw. This means that the motion with no angular velocity is pure translation.

In the generalized scalar product in Equation (1), we have four cases, depending on the selection of the signs of α and β , which are $\alpha, \beta > 0$, $\alpha > 0$ and $\beta < 0$, $\alpha < 0$ and $\beta > 0$, $\alpha, \beta < 0$. However, from the definitions of the scalar product, the last three cases are the same. Since all possible selections of α and β can be covered by two conditions $\alpha, \beta > 0$ and $\alpha > 0, \beta < 0$, in the following section, we will consider only these two cases.

In the following section, a dual orthogonal matrix of an orthogonal matrix will be obtained and this matrix will be used to describe screw motion.

5. Generalized Dual Orthogonal Matrices and Screw Motion in $\mathbb{E}^3(\alpha, \beta)$

Now, we wish to simplify the screw coordinate transformations via matrix multiplication and the properties of dual numbers. Let $l = (w, s)$ and $l' = (a', s')$ be screws associated with the dual vectors $\hat{a} = a + \epsilon s$ and $\hat{a}' = a' + \epsilon s'$, respectively. Thus, the screw transformation in Equation (4) can be rewritten as

$$\hat{a}' = \hat{R} \hat{a}$$

where

$$\hat{R} = R + \epsilon T R \quad (8)$$

and T is a G-skew symmetric matrix.

Theorem 1. Dual matrix \hat{R} is a G-dual orthogonal matrix.

Proof. Let dual matrix \hat{R} be given as in (9), with R as a G-orthogonal matrix and T as a G-skew symmetric matrix. Then, since

$$\begin{aligned} \hat{R}^T \epsilon \hat{R} &= (R + \epsilon TR)^T \epsilon (R + \epsilon TR) \\ &= R^T \epsilon R + \epsilon (R^T \epsilon TR + R^T T^T \epsilon R) + \epsilon^2 (\dots) \\ &= R^T \epsilon R + \epsilon (R^T \epsilon TR + R^T (-\epsilon T) R) \\ &= R^T \epsilon R + \epsilon (R^T \epsilon TR - R^T \epsilon TR) \\ &= R^T \epsilon R \\ &= \epsilon \end{aligned}$$

where $\varepsilon^2 = 0$ and $R^T \varepsilon = -\varepsilon R$. \square

Case I ($\alpha > 0$ and $\beta > 0$): By using Equation (8), the G-orthogonal matrix related to the screw displacement has its angle θ about the axis y , and a distance n along axis y , and we can obtain

$$\hat{Y} = R + \varepsilon T R = \begin{bmatrix} \cos \hat{\theta} & 0 & \sqrt{\beta} \sin \hat{\theta} \\ 0 & 1 & 0 \\ -\frac{\sin \hat{\theta}}{\sqrt{\beta}} & 0 & \cos \hat{\theta} \end{bmatrix} \quad (9)$$

where

$$R = \begin{bmatrix} \cos \theta & 0 & \sqrt{\beta} \sin \theta \\ 0 & 1 & 0 \\ -\frac{\sin \theta}{\sqrt{\beta}} & 0 & \cos \theta \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & \sqrt{\beta} t \\ 0 & 0 & 0 \\ -\frac{t}{\sqrt{\beta}} & 0 & 0 \end{bmatrix}.$$

In addition, the screw displacement has its angle ϕ about the axis x , and a distance n along axis x

$$\hat{X} = R + \varepsilon T R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \hat{\phi} & -\sqrt{\alpha} \sin \hat{\phi} \\ 0 & \frac{\sin \hat{\phi}}{\sqrt{\alpha}} & \cos \hat{\phi} \end{bmatrix} \quad (10)$$

where

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sqrt{\alpha} \sin \phi \\ 0 & \frac{\sin \phi}{\sqrt{\alpha}} & \cos \phi \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\alpha} t \\ 0 & \frac{t}{\sqrt{\alpha}} & 0 \end{bmatrix}$$

The dual orthogonal transformation

$$\hat{D} = \hat{Y} \hat{X}$$

and

$$\hat{D} = \begin{bmatrix} \cos \hat{\theta} & \frac{\sqrt{\beta} \sin \hat{\theta} \sin \hat{\phi}}{\sqrt{\alpha}} & \sqrt{\beta} \sin \hat{\theta} \cos \hat{\phi} \\ 0 & \cos \hat{\phi} & -\sqrt{\alpha} \sin \hat{\phi} \\ -\frac{\sin \hat{\theta}}{\sqrt{\beta}} & -\frac{\cos \hat{\theta} \sin \hat{\phi}}{\sqrt{\alpha}} & -\cos \hat{\theta} \cos \hat{\phi} \end{bmatrix}.$$

Case II ($\alpha > 0$ and $\beta < 0$): In this case, we have two subcases where the x -axis and y -axis are timelike or spacelike.

Case IIa Let x -axis and y -axis be timelike; then, we have Case 1.

Case IIb Let the x -axis and y -axis be spacelike; then,

$$\hat{Y} = R + \varepsilon T R = \begin{bmatrix} \cosh \hat{\theta} & 0 & \sqrt{-\beta} \sinh \hat{\theta} \\ 0 & 1 & 0 \\ \frac{\sinh \hat{\theta}}{\sqrt{-\beta}} & 0 & \cosh \hat{\theta} \end{bmatrix} \quad (11)$$

where

$$R = \begin{bmatrix} \cosh \theta & 0 & \sqrt{-\beta} \sinh \theta \\ 0 & 1 & 0 \\ \frac{\sinh \theta}{\sqrt{-\beta}} & 0 & \cosh \theta \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & \sqrt{-\beta} t \\ 0 & 0 & 0 \\ \frac{t}{\sqrt{-\beta}} & 0 & 0 \end{bmatrix}.$$

In the same way, the screw displacement has its angle ϕ about the axis y , and a distance n along axis y

$$\hat{X} = R + \varepsilon T R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \hat{\phi} & \sqrt{\alpha} \sinh \hat{\phi} \\ 0 & \frac{\sinh \hat{\phi}}{\sqrt{\alpha}} & \cosh \hat{\phi} \end{bmatrix} \quad (12)$$

Here,

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \phi & \sqrt{\alpha} \sinh \phi \\ 0 & \frac{\sinh \phi}{\sqrt{\alpha}} & \cosh \phi \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{\alpha} t \\ 0 & \frac{t}{\sqrt{\alpha}} & 0 \end{bmatrix}.$$

Additionally, the dual orthogonal transformation $\widehat{D} = \widehat{Y} \widehat{X}$

$$\widehat{D} = \begin{bmatrix} \cosh \widehat{\theta} & \frac{\sqrt{\beta} \sinh \widehat{\theta} \sinh \widehat{\phi}}{\sqrt{\alpha}} & \sqrt{\beta} \sinh \widehat{\theta} \cosh \widehat{\phi} \\ 0 & \cosh \widehat{\phi} & \sqrt{\alpha} \sinh \widehat{\phi} \\ \frac{\sinh \widehat{\theta}}{\sqrt{\beta}} & \frac{\cosh \widehat{\theta} \sinh \widehat{\phi}}{\sqrt{\alpha}} & \cosh \widehat{\theta} \cosh \widehat{\phi} \end{bmatrix}.$$

In Case I and Case II, \widehat{D} is used to refer to the coordinate frames in successive members of a manipulator.

Example 1. Case 1. Let $\alpha = 1$ and $\beta = 2$. Let the screw displacement have its elliptical angle $\theta = 0.6435011$ about the axis y , and a distance n along axis y . Substituting Equation (9) into Equation (8), we obtain the G-dual orthogonal matrix as

$$\begin{aligned} \widehat{R} &= R + \varepsilon T R \\ &= \begin{bmatrix} \frac{4}{5} & 0 & \frac{3\sqrt{2}}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5\sqrt{2}} & 0 & \frac{4}{5} \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{4}{5} & 0 & \frac{3\sqrt{2}}{5} \\ 0 & 1 & 0 \\ -\frac{3}{5\sqrt{2}} & 0 & \frac{4}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{5} - \varepsilon \frac{3}{5} & 0 & \sqrt{2}(\frac{3}{5} + \varepsilon \frac{4}{5}) \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2}(\frac{3}{5} + \varepsilon \frac{4}{5}) & 0 & \frac{4}{5} - \varepsilon \frac{3}{5} \end{bmatrix}, \end{aligned}$$

where G-skew symmetric matrix T corresponding to translation vector $n = (0, 1, 0)$ is

$$T = \begin{bmatrix} 0 & 0 & \sqrt{\beta} t \\ 0 & 0 & 0 \\ -\frac{t}{\sqrt{\beta}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$$

and for $\alpha = 1$ and $\beta = 2$, the matrix ε is

$$\varepsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Note that $R^T \varepsilon R = \varepsilon$ and $\widehat{R}^T \varepsilon \widehat{R} = \varepsilon$ are the G-orthogonal matrix and G-dual orthogonal matrix, respectively.

Example 2. Case 2. Let $\alpha = 1$ and $\beta = -2$. Let the screw displacement have its hyperbolic angle $\theta = 0.51082562376$ about the y spacelike axis, and a distance n along y . Substituting Equation (11) into Equation (8), we have G-dual orthogonal matrix

$$\begin{aligned} \widehat{R} &= R + \epsilon T R \\ &= \begin{bmatrix} \frac{17}{15} & 0 & \frac{8\sqrt{2}}{15} \\ 0 & 1 & 0 \\ \frac{8}{15\sqrt{2}} & 0 & \frac{17}{15} \end{bmatrix} + \epsilon \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{17}{15} & 0 & \frac{8\sqrt{2}}{15} \\ 0 & 1 & 0 \\ \frac{8}{15\sqrt{2}} & 0 & \frac{17}{15} \end{bmatrix} \\ &= \begin{bmatrix} \frac{17}{15} + \epsilon \frac{8}{15} & 0 & \sqrt{2}(\frac{8}{15} + \epsilon \frac{17}{15}) \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2}(\frac{8}{15} + \epsilon \frac{17}{15}) & 0 & \frac{8}{15} + \epsilon \frac{17}{15} \end{bmatrix}, \end{aligned}$$

where G-skew symmetric matrix T corresponding to the translation vector $n = (0, 1, 0)$ is

$$T = \begin{bmatrix} 0 & 0 & \sqrt{-\beta}t \\ 0 & 0 & 0 \\ \frac{t}{\sqrt{-\beta}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}$$

and for $\alpha = 1$ and $\beta = -2$, the matrix ϵ is

$$\epsilon = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Again, note that $R^T \epsilon R = \epsilon$ and $\widehat{R}^T \epsilon \widehat{R} = \epsilon$ are the G-orthogonal matrix and G-dual orthogonal matrix, respectively.

In the following section, the exponential equation of the obtained dual orthogonal matrix for rotation given in (2) will be obtained for the screw motion, and the screw components will be defined by this exponential equation.

6. Generalized Dual Exponential Matrix in $\mathbb{E}^3(\alpha, \beta)$

Let $\widehat{R}(t)$ be a G-dual orthogonal matrix parametrized with t ; then, we have the equation $\widehat{R}^T \epsilon \widehat{R} = \epsilon$. Differentiating both sides, we obtain

$$(\widehat{R})^T \epsilon \dot{\widehat{R}} + \widehat{R}^T \epsilon \dot{\widehat{R}} = 0$$

which can be written as

$$\widehat{R}^T \epsilon \dot{\widehat{R}} = -(\widehat{R}^T \epsilon \dot{\widehat{R}})^T.$$

The last equation shows that $\widehat{R}^T \epsilon \dot{\widehat{R}} = \widehat{\Phi}$ is a skew-symmetric matrix in $\mathbb{E}^3(\alpha, \beta)$. Here, $\dot{\widehat{R}} = \widehat{R} \epsilon^{-1} \widehat{\Phi}$. Let $\epsilon^{-1} \widehat{\Phi} = \widehat{\Omega}$, and $\dot{\widehat{R}} = \widehat{R} \widehat{\Omega}$. We obtain G-dual skew-symmetric matrix $\widehat{\Omega}$, called the dual angular velocity matrix in $\mathbb{E}^3(\alpha, \beta)$. Let \widehat{w} be a dual vector corresponding to G-dual skew symmetric matrix $\widehat{\Omega}$ in $\mathbb{E}^3(\alpha, \beta)$. Then, it can be written as $\widehat{w} = w + \epsilon w^*$. Let us split w^* into components parallel and perpendicular to w , as in Section 3, written as $w^* = k \times_G w + pw$. Here,

$$k = \frac{w \times_G w^*}{\langle w, w \rangle_G} \quad \text{and} \quad p = \frac{\langle w, w^* \rangle_G}{\langle w, w \rangle_G}.$$

Thus, dual vector \widehat{w} is written as

$$\begin{aligned} \widehat{w} &= w + \epsilon(k \times_G w + pw) \\ &= (1 + \epsilon p)(w + \epsilon k \times_G w) \end{aligned} \tag{13}$$

where $1 + \epsilon p$ is a dual scalar. The dual vector $w + \epsilon k \times_G w$ describes an instantaneous screw axis and p is the pitch of the screw motion defined by $\widehat{R}(t)$.

Dual angular velocity $\widehat{\Omega}$ is a constant because $\widehat{R}^T \varepsilon \widehat{R} = \varepsilon$. The solution of the matrix differential equation,

$$\dot{\widehat{R}} = \widehat{R} \widehat{\Omega}$$

with the initial condition $\widehat{R}(0) = I$

$$\widehat{R}(t) = e^{t\widehat{\Omega}}. \quad (14)$$

$\widehat{R}(t)$ is a screw motion; its axis $w + \varepsilon k \times_G w$ is obtained via $\widehat{\Omega}$. We have two cases where $k \times_G w$ is either equal to zero or not.

Let $k \times_G w = 0$. This means that the origin of the fixed frame F is on the screw axis. Since Equation (13) can be written as $\widehat{w} = (1 + \varepsilon p)w$, \widehat{R} has the special G-skew symmetric matrix such that $\widehat{\Omega} = (1 + \varepsilon p)\Omega$. Using the identity $(1 + \varepsilon p)^n = (1 + n\varepsilon p)$, Equation (15) can be rewritten as

$$\begin{aligned} e^{(1+\varepsilon p)\Omega t} &= I + (1 + \varepsilon p)\Omega t + \frac{1}{2}(1 + \varepsilon p)^2 \Omega^2 t^2 + \dots \\ &= (I + \Omega t + \frac{1}{2}\Omega^2 t^2 + \dots) \\ &\quad + \varepsilon p \Omega t (I + \Omega t + \frac{1}{2}\Omega^2 t^2 + \dots) \end{aligned} \quad (15)$$

Now, assume that $R(t)$ is the exponential Ω . Thus, Equation (16) defines the G-dual orthogonal matrix $\widehat{R}(t) = R(t) + \varepsilon p t \Omega R(t)$. Consequently, this transformation describes a screw motion with rotation about the angle $\phi(t) = \omega t$, with axis w , ($\omega = \|w\|$) and the amount $n(t) = p t w$. Moreover, ω is the angular velocity, $s = p \omega$ is its linear velocity, and p is the pitch of the motion.

Let $k \times_G w \neq 0$, which means that the origin is not on the screw axis. Then, $\widehat{R}(t) = \widehat{T} \exp((1 + \varepsilon p)\Omega t) \widehat{T}^{-1}$, where \widehat{T} is the translation $\widehat{T} = I + \varepsilon K$ and K is the G-skew symmetric matrix obtained via k . Hence,

$$\begin{aligned} \widehat{T} \exp((1 + \varepsilon p)t\Omega) \widehat{T}^{-1} &= \exp(\widehat{T}((1 + \varepsilon p)t\Omega) \widehat{T}^{-1}) \\ &= \exp((1 + \varepsilon p)(\Omega + \varepsilon(K\Omega - \Omega K))t) \end{aligned} \quad (16)$$

Since $(K\Omega - \Omega K)$ is the G-skew symmetric matrix, we obtain it from vector $k \times_G w$, and the matrix $(\Omega + \varepsilon(K\Omega - \Omega K))$ describes a screw axis $w + \varepsilon k \times_G w$. Thus, we can write

$$\begin{aligned} \widehat{R}(t) &= R(t) + \varepsilon(p t \Omega R(t) + K R(t) - R(t) K) \\ &= R(t) + \varepsilon(p t \Omega + K - R K R^{-1}) R(t) \end{aligned}$$

G-skew symmetric matrix $p t \Omega + K - R K R^{-1}$ defines the translation component of the motion and its vector form

$$n(t) = p t w + (I - R(t))k.$$

The above equation defines a displacement affiliated with a screw motion with pitch p and axis $w + \varepsilon k \times_G w$ in $\mathbb{E}^3(\alpha, \beta)$.

7. Conclusions

Screw motion has been studied by many authors in Euclidean and Lorentzian spaces. In this paper, screw motion is studied in generalized space. Thus, the theory of screw motion in the three-dimensional Euclidean and Lorentzian spaces is generalized into generalized space. The results obtained in this study reveal that the theory of screw motion is independent of the choice of the metric tensor in the Euclidean and Lorentzian spaces.

As is known, screw motion is a combination of a rotation around an axis and translational along the same axis. The rotation part of the screw motion is circular rotation in Euclidean space and hyperbolic rotation in Lorentz space. However, the rotation part of the screw motion in generalized space can be circular rotation, hyperbolic rotation, elliptical rotation, etc., depending on the choice of alpha and beta. Thus, this gives the opportunity to study a space where rotation motion is richer. The first case of the example in this study is

the screw motion whose rotation part is elliptical rotation, and the second case is the screw motion whose rotation part is hyperbolic rotation. As mentioned briefly in the Introduction, generalized space has many application areas, such as mechanics, electromagnetism, and astrophysics.

The study of screw motion in the more general space is more difficult and complex than in the Euclidean and Lorentzian. Nevertheless, this study will make an important contribution to future studies to better understand and interpret screw motion. Screw motion in other scalar product spaces is also an open problem for further study.

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