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Sandwich Theorems for a New Class of Complete Homogeneous Symmetric Functions by Using Cyclic Operator

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Abstract: In this paper, we discuss and introduce a new study on the connection between geometric function theory, especially sandwich theorems, and Viete's theorem in elementary algebra. We obtain some conclusions for differential subordination and superordination for a new formula of complete homogeneous symmetric functions class involving an ordered cyclic operator. In addition, certain sandwich theorems are found.

Keywords: symmetric function; cyclic operator; subordination; superordination; sandwich theorem

MSC: 30C45

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1. Introduction

Let $H(\mathbb{U})$ denote the analytic function class in the open unit disk $\mathbb{U} = \{|z| < 1; z \in \mathbb{C}\}$. and let $H[a,\ p]$ denotes the subclass of functions $f \in H(\mathbb{U})$ as:

$$H[a,p] = \left\{ f \in H : f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \right\}; a \in \mathbb{C},\ p \in N = \{1,\ 2, \ldots\}.$$

The resolvent $det\ (I - A)^{-1}$ of a complex matrix A is naturally an analytic function of eigenvalues $\lambda \in \mathbb{C}$, and these eigenvalues are isolated singularities. In general, any matrix has finite eigenvalues. The resolvent set of A is defined as follows:

$$\varrho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}$$

The spectrum of A is expressed by $\vartheta(A) = \mathbb{C}/\varrho(A)$.

For distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, the polynomial

$$H(z) = h_0 + h_1 z + h_2 z^2 + \cdots + h_\mathrm{n} z^\mathrm{n} + \cdots$$

are given in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ by

$$H(z) = \sum_{\mathrm{n}=0}^{\infty} h_\mathrm{n}\, z^n, h_\mathrm{n} = \sum_{1 \le i_1, \ldots, i_m \le m} \frac{m_1! m_2! \ldots\ m_m!}{\mathrm{n}!} \lambda_{i_1} \lambda_{i_2}\ \ldots \lambda_{i_m}\,; z\ \in \mathbb{C}.$$

The above formula can also be written in terms of the distinct powers of traces of the matrix as follows:

$$H(z) = \sum_{\mathrm{n}=0}^{\infty} h_\mathrm{n}\, z^\mathrm{n}; h_\mathrm{n} = \sum_{\substack{\mathrm{i}_1 + \mathrm{i}_2 + \cdots + \mathrm{i}_\mathrm{n} = k \\ \mathrm{i}_1 + 2\mathrm{i}_2 + 3\mathrm{i}_3 + \cdots + \mathrm{n}\mathrm{i}_\mathrm{n} = \mathrm{n}}} \frac{(-1)^{n-k}\ S_1^{\mathrm{i}_1} S_2^{\mathrm{i}_2} \ldots\ S_\mathrm{n}^{\mathrm{i}_\mathrm{n}}}{\mathrm{i}_1!\ \mathrm{i}_2! \ldots\ \mathrm{i}_n!\ 1^{\mathrm{i}_1}\ 2^{\mathrm{i}_2} \ldots \mathrm{n}^{\mathrm{i}_\mathrm{n}}} \tag{1}$$

This class represents the subclasses of analytical functions $H[a, n]$ and denoted by $\mathbb{H}$ such that $H[a, 1] = \mathbb{H}$ and has coefficients of the form (1), i.e., when the value of n is equal to one and can be reduced a class $\mathbb{H}$ to "the class H of normalized univalent analytical functions and composed of functions of the following form:

$$\frac{H(z) - h_0}{h_1} = z + \sum_{\mathrm{n}=2}^{\infty} a_{\mathrm{n}} z^{\mathrm{n}}; a_n = \frac{h_{\mathrm{i}}}{h_1}, \ (z \in \mathbb{U})$$

To each analytic function φ in open unit disk $\mathbb{U}$ into itself, we associated the composition operator C_φ, defined by:

$$C_\varphi f = f \circ \varphi \text{ for all } f \in H^2$$

Then, we define the ordered cyclic operator $\mathbb{T}^{\mathbb{i}}$ of f as follows:

$$(C_{\mathcal{T}^{\mathbb{i}}} f)(z) = \left(\mathcal{T}^{\mathbb{i}} \circ f\right)(z) = \mathcal{T}^{\mathbb{i}}(f(z)) = \mathcal{T}^{\mathbb{i}}\left(h_0 + \sum_{n=1}^{\infty} h_n z^n\right) = h_{\mathbb{i}} + \sum_{n=1}^{\infty} h_{n+\mathbb{i}} z^n. \tag{2}$$

Let f_1 and f_2 are *analytic* in $\mathbb{U}$, we say that the function f_1 is *subordinate* to f_2 *or*, f_2 is said to be superordinate to f_1 if there exists a Schwarz function $\mathbb{W}$ in $\mathbb{U}$ *with* $\mathbb{W}(0) = 0$, and $|\mathbb{W}(z)| < 1 \ (z \in \mathbb{U})$, where $f_1(z) = f_2(\mathbb{W}(z))$. In such a case, we write $f_1 \prec f_2$ or $f_1(z) \prec f_2(z) \ (z \in \mathbb{U})$.

Particularly, f *the function* f_2 *is univalent in* $\mathbb{U}$, then $f_1 \prec f_2$ if and only if $f_1(0) = f_2(0)$ and $f_1(\mathbb{U}) \subset f_2(\mathbb{U})$ ([1,2]).

The set of all functions q that are analytic and injective *on* $\overline{\mathbb{U}} \setminus E(\mathbb{q})$, denote *by* Q, where $\overline{\mathbb{U}} = \mathbb{U} \cup \{z \in \partial\mathbb{U}\}$, and $E(\mathbb{q}) = \{\xi \in \partial\mathbb{U} : \mathbb{q}(z) = \infty\}$, such that $\mathbb{q}'(\xi) \neq 0$ *f or* $\xi \in \partial\mathbb{U} \setminus E(\mathbb{q})$.

Definition 1 [2]. *Let* $\mathbb{h}$ *and* $\mathbb{k}$ *are two analytic functions in* $\mathbb{U}$ *and* $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. *If* $\mathbb{h}$ *and* $\phi\left(\mathbb{h}(z), z\mathbb{h}'(z), z^2\mathbb{h}''(z); z\right)$ *are univalent functions in* $\mathbb{U}$ *and if* $\mathbb{h}$ *satisfies the second-order superordination*

$$\mathbb{k}(z) \prec \phi\left(\mathbb{h}(z), z\mathbb{h}'(z), z^2\mathbb{h}''(z); z\right) \tag{3}$$

then $\mathbb{h}$ is called a solution of the differential superordination (3). A function $\mathbb{q} \in H(\mathbb{U})$ is called a subordinant of (3), if $\mathbb{h}(z) \prec \mathbb{q}(z)$ for all the functions $\mathbb{h}$ satisfying (3).

A *univalent subordinant* $\hat{\mathbb{q}}$ that satisfies $\mathbb{q}(z) \prec \hat{\mathbb{q}}(z)$ for all the *subordinants* $\mathbb{q}$ of (3), is said to be the *best* subordinant.

Definition 2 [1]. *Let* $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ *and* $\mathbb{k}(z)$ *be univalent in* $\mathbb{U}$. *If* $\mathbb{h}(z)$ *is analytic in* $\mathbb{U}$ *and satisfies the second-order differential subordination:*

$$\phi\left(\mathbb{h}(z), z\mathbb{h}'(z), z^2\mathbb{h}''(z); z\right) \prec \mathbb{k}(z) \tag{4}$$

Then, $\mathbb{h}$ is called a "solution of the differential subordination (4), and the univalent function $\mathbb{q}(z)$ is called a dominant of the solution of the differential subordination (4), or more simply dominant if $\mathbb{h}(z) \prec \mathbb{q}(z)$ for all $\mathbb{p}(z)$ satisfying (4). A univalent dominant $\hat{\mathbb{q}}(z)$ that satisfies $\hat{\mathbb{q}}(z) \prec \mathbb{q}(z)$ for all dominant $\mathbb{q}(z)$ of (4) is said to be the best dominant and is unique up to a relation of $\mathbb{U}$.

Recently, Miller and Mocanu [1] obtained sufficient conditions on the functions $\mathbb{k}$, $\mathbb{q}$, and ϕ for which the following implication holds:

$$\mathbb{k}(z) \prec \phi\left(\mathbb{h}(z), z\mathbb{h}'(z), z^2\mathbb{h}''(z); z\right) \to \mathbb{q}(z) \prec \mathbb{h}(z)$$

Using these results, Bulboaca [3] considered some classes of first-order differential superordinations, as well as integral operators preserving superordination [4]. Ali et al. [5,6], Atshan and Hadi [7], Atshan and Ali [8,9], and (see [10–17]) obtained results of subordination and superordination to analytic functions in $\mathbb{U}$. Lately, Al-Ameedee et al. [18,19], Atshan et al. [20–22], Bulboaca [23], Selvaraj and Karthikeyan [24] and (see [25–34]) got sandwich results to some classes of analytic functions. Further differential subordination results can be found in [35,36] for different orders.

Lemma 1. *Let $H \in \mathbb{H}$ and , $h_n \in \mathbb{C}$, we define the ordered cyclic operator of degree 1*

$$\mathcal{T}^1 : \mathbb{H} \to \mathbb{H}, \quad by$$

$$\mathcal{T}^1(H(z)) = \sum_{k=0}^{n} h_{n+1} z^n,\ z \in \mathbb{U}$$

and

$$\mathcal{T}^{\mathbb{i}}(H(z)) = \sum_{k=0}^{n} h_{n+\mathbb{i}} z^n,\ z \in \mathbb{U}$$

where

$$\mathcal{T}^0(H(z)) = H(z).$$

Proof 1. Let $H(z) \in \mathbb{H}$, then

$$\mathcal{T}^1(H(z)) = (C_{\mathcal{T}^1} H)(z) = \left(H \circ \mathcal{T}^1\right)(z) = \mathcal{T}^1\left(\sum_{k=1}^{\infty} h_k z^k\right) = \sum_{n=1}^{\infty} h_{n+1} z^n,$$

and

$$\begin{aligned}\mathcal{T}^2(H(z)) &= (C_{\mathcal{T}^2} f)(z) = (H \circ \mathcal{T}^2)(z) = (H \circ \mathcal{T} \circ \mathcal{T})(z) = \mathcal{T}\left(\mathcal{T}\left(\sum_{k=1}^{\infty} h_k z^k\right)\right) \\ &= \mathcal{T}\left(\sum_{n=1}^{\infty} h_{n+1} z^n\right) = \sum_{n=1}^{\infty} h_{n+2} z^n,\end{aligned}$$

and so on

$$\mathcal{T}^{\mathbb{i}}(f(z)) = (C_{\mathcal{T}^{\mathbb{i}}} H)(z) = \left(f \circ \mathcal{T}^{\mathbb{i}}\right)(z) = \left(H \circ \underbrace{\mathcal{T} \ldots \circ \mathcal{T}}_{\mathbb{i}-times}\right)(z) = \mathcal{T}\left(\mathcal{T}\left(\ldots \mathcal{T}\left(\sum_{n=0}^{\infty} h_n z^n\right) \ldots\right)\right) = \sum_{n=0}^{\infty} h_{n+\mathbb{i}} z^n. \quad (5)$$

□

This completes the proof.

By simple calculation and using Newton's identities, we obtain

$$z\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)' = -\mathcal{T}^{\mathbb{i}}(H(z))\mathbb{S}(z).$$

Also

$$\frac{z\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}}(H(z))} = -\mathbb{S}(z) = \frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}. \quad (6)$$

2. Preliminaries

In order to demonstrate our results of differential subordination and superordination, the following definitions and known results are used.

Definition 3. [37]. *A polynomial* $p(x_1, x_2, \ldots, x_{\mathfrak{n}})$ *is called a symmetric polynomial if it satisfies:*

$$p(x_{\varphi(1)}, x_{\varphi(2)}, \ldots, x_{\varphi(\mathfrak{n})}) = p(x_1, x_2, \ldots, x_{\mathfrak{n}})$$

for all permutations φ *of {1, ... , n} such that* $\Lambda_{\mathfrak{n}}$ *denoted to the space of all symmetric polynomials in* $x_1, x_2, \ldots, x_{\mathfrak{n}}$.

Definition 4. [37]. *Suppose* $x_1, x_2, \ldots, x_{\mathfrak{n}}$ *are the n roots of a polynomial*

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

then

$$e_0 = 1, e_1(x_1, x_2, \ldots, x_{\mathfrak{n}}) = \sum_{i=1}^{\mathfrak{n}} x_i = -a_1, e_2(x_1, x_2, \ldots, x_{\mathfrak{n}}) = \sum_{1 \le i_1 < i_2 \le n}^{n} x_{i_1} x_{i_2} = a_2,$$
$$\ldots, e_m(x_1, x_2, \ldots, x_{\mathfrak{n}}) = \sum_{1 \le i_1 < \ldots < i_m \le \mathfrak{n}}^{n} x_{i_1} \ldots x_{i_m} = (-1)^m a_m, \ldots, e_n(x_1, x_2, \ldots, x_{\mathfrak{n}}) = x_1 x_2 \ldots x_{\mathfrak{n}}$$

The polynomial $e_m(x_1, x_2, \ldots, x_{\mathfrak{n}})$ *is called the m-th symmetric polynomial in* $x_1, x_2, \ldots, x_{\mathfrak{n}}$.

Definition 5. [37]. *For each* $k \ge 0$*, the complete symmetric polynomial is the sum of all monomials of k-degree as follows:*

$$h_k(x_1, x_2, \ldots, x_{\mathfrak{n}}) = \sum_{d_1 + \ldots + d_{\mathfrak{n}} = k} {x_1}^{d_1} \ldots {x_{\mathfrak{n}}}^{d_{\mathfrak{n}}}$$

Particularly $h_0(x_1, x_2, \ldots, x_{\mathfrak{n}}) = 1$. It is not hard to see that

$$h_k(x_1, x_2, \ldots, x_{\mathfrak{n}}) = \sum_{\lambda \in \mathrm{P(k,n)}} m_\lambda(x_1, x_2, \ldots, x_n)$$

such that m_λ is the partition of k.

Thus, for each $k \in \mathrm{Z}^+$, there exists exactly one complete homogeneous symmetric polynomial of k-degree in $\mathfrak{n}$ variables.

Lemma 2. [37]. *The symmetry between h and e suggests the introduction of the following map* $\omega : \Lambda_{\mathfrak{n}} \to \Lambda_{\mathfrak{n}}$ *such that*

$$\omega\left(\sum_{m_1, \ldots, m_{\mathfrak{n}}} a_{m_1, \ldots, m_{\mathfrak{n}}} e_1^{m_1} + \cdots + e_{\mathfrak{n}}^{m_{\mathfrak{n}}}\right) = \sum_{m_1, \ldots, m_{\mathfrak{n}}} a_{m_1, \ldots, m_{\mathfrak{n}}} h_1^{m_1} + \cdots + h_{\mathfrak{n}}^{m_{\mathfrak{n}}},$$

and

$$\omega\left(\sum_{m_1, \ldots, m_{\mathfrak{n}}} a_{m_1, \ldots, m_{\mathfrak{n}}} e_1^{m_1} + \cdots + e_{\mathfrak{n}}^{m_{\mathfrak{n}}}\right) = \sum_{m_1, \ldots, m_{\mathfrak{n}}} a_{m_1, \ldots, m_{\mathfrak{n}}} e_1^{m_1} + \cdots + e_{\mathfrak{n}}^{m_{\mathfrak{n}}}.$$

It has the following properties:

(1) ω is a ring isomorphism, i.e.,

$$\omega(\rho + q) = \omega(\rho) + \omega(q),\ \omega(\rho \cdot q) = \omega(\rho) \cdot \omega(q),$$

for $\rho,\ q \in \mathfrak{n}$.

(2) $\omega(e_i) = h_i$ *and* $\omega(h_i) = e_i$.

(3) $\omega^2 = id$.

We see that $\Lambda_{\mathfrak{n}} = \mathbb{C}[h_1, \ldots, h_{\mathfrak{n}}]$. In other words, if we define for $\lambda' = \left(\lambda'_1, \ldots, \lambda'_k\right) \in \rho'(k, \mathfrak{n})$.

$$h_{\lambda'}(x_1, x_2, \ldots, x_{\mathfrak{n}}) = h_{\lambda'_1}(x_1, x_2, \ldots, x_{\mathfrak{n}}) \ldots h_{\lambda'_k}(x_1, x_2, \ldots, x_{\mathfrak{n}}),$$

$\{h_{\lambda'}\}_{\lambda' \in \rho'(k, \mathfrak{n})}$ is a basis of $\Lambda_k^{\mathfrak{n}}$.

Theorem 1. [37]. *For $r \geq 1$, the $r-th$ Newton polynomial (power sum) in $x_1, x_2, \ldots, x_\mathfrak{n}$ is $\rho_r(x_1, x_2, \ldots, x_\mathfrak{n}) = x_1^r + \cdots + x_\mathfrak{n}^r$.*

The generating function for them is

$$P_\mathfrak{n} = \sum_{r\geq 1} \rho_r(x_1, x_2, \ldots, x_\mathfrak{n})t^{r-1} = \sum_{i=1}^{\mathfrak{n}} \sum_{r\geq 1} x_i^r t^{r-1} = \sum_{i=1}^{\mathfrak{n}} \frac{x_i}{1-x_i t} = \frac{d}{dt} log \frac{1}{\prod_{i=1}^{\mathfrak{n}} 1-x_i t}.$$

By comparing this formula, we get:

$$P_\mathfrak{n}(t) = \frac{\mathrm{H}_\mathfrak{n}'(t)}{\mathrm{H}_\mathfrak{n}(t)} = \frac{\mathrm{E}_\mathfrak{n}'(-t)}{\mathrm{E}_\mathfrak{n}(-t)},$$

and by applying ω , yields to:
$\omega(\rho_\mathfrak{n})(t) = h_\mathfrak{n}(-t)$ or equivalently,
$\omega(\rho_r) = (-1)^{r-1} h_r$, one also has $H_\mathfrak{n}'(t) = \rho_\mathfrak{n}(t) H_\mathfrak{n}(t)$, $\mathrm{E}_\mathfrak{n}'(t) = \rho_\mathfrak{n}(t)\mathrm{E}_\mathfrak{n}(-t)$.

Equivalently

$$k\, h_k = \sum_{r=1}^{k} \rho_r h_{k-r}\ , also\ k\, e_k = \sum_{r=1}^{k} (-1)^{r-1} \rho_r e_{k-r}$$

These are called the Newton formulas.

Definition 6. [38]. *A permutation matrix is a square matrix that has inputs* $0,\ 1$ *derived from the identity matrix of the same size by a permutation of rows. There are* $n!$ *permutation matrices of size* n.

Definition 7. [39]. *A bounded linear operator T on a Hilbert space H is called cyclic operator if there exists a vector $x \in H$ and the set span $\{T^\mathfrak{n} x :\ \mathfrak{n} =\ 0,\ 1,\ \ldots\}$ is dense in H. The vector x is called a cyclic vector for the operator T.*

Theorem 2. [39]. *Let $S,\ T,\ X$ b bounded operators on a Hilbert space H satisfying a conjugate relation $SX = XT$, if T is cyclic operator and X has a dense range, then S is cyclic operator too.*

Proposition 1. [40]. *Let T be "an operator on a Hilbert space H that has diagonal matrix" $A = diag(\lambda_1, \lambda_2, , \ldots)$ with respect to some orthonormal basis $\{e_n\}$. Then T is cyclic if and only if the diagonal entries $\{\lambda_j\}$ are distinct.*

Definition 8. [35,41]. *The set of all functions $f(z)$ that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, denote by Q, where $\overline{\mathbb{U}} = \mathbb{U} \cup \{z \in \partial\mathbb{U}\}$ and*

$$E(f) = \{\xi \in \partial\mathbb{U} : f(z) = \infty\}, \tag{7}$$

such that $f'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(f)$.

Lemma 3. [1]. *Let the function $\mathfrak{q}(z)$ be univalent in the open unit disc $\mathbb{U}$ and let θ and φ be analytic in a domain D containing $\mathfrak{q}(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in \mathfrak{q}(\mathbb{U})$. Put $Q(z) = z\mathfrak{q}'(z)\varphi(\mathfrak{q}(z))$ and $\mathbb{h}(z) = \theta(\mathfrak{q}(z) + Q(z))$.*

Suppose that

(1) $Q(z)$ is starlike univalent in $\mathbb{U}$,
(2) $Re\left(\frac{z\mathbb{h}'(z)}{Q(z)}\right) > 0,\ z \in \mathbb{U}$.

If $\mathbb{h}$ is analytic in $\mathbb{U}$ *with* $\mathbb{h}(0) = \mathbb{q}(0),\ \mathbb{h}(\mathbb{U}) \subseteq D$ and

$$\theta(\mathbb{h}(z)) + z\mathbb{h}^{'}(z)\varphi(\mathbb{h}(z)) \prec \theta(\mathbb{q}(z)) + z\mathbb{q}^{'}(z)\varphi(\mathbb{q}(z)), \tag{8}$$

then $\mathbb{h}(z) \prec \mathbb{q}(z)$, *and* $\mathbb{q}(z)$ is the best dominant.

Lemma 4. **[35].** *Let* $\mathbb{q}(z)$ *be convex univalent function in open unit disk* $\mathbb{U}$, *let* $\psi \in \mathbb{C},\ \ \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ *with*

$$Re\left(1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}^{'}(z)}\right) > max\left\{0, -Re\left(\frac{\psi}{\gamma}\right)\right\}. \tag{9}$$

If $\mathbb{h}(z)$ is analytic in $\mathbb{U}$ with $\mathbb{h}(0) = \mathbb{q}(0)$ and

$$\psi\mathbb{h}(z) + \gamma z\mathbb{h}^{'}(z) \prec \psi\mathbb{q}(z) + \gamma z\mathbb{q}^{'}(z), \tag{10}$$

then $\mathbb{h}(z) \prec \mathbb{q}(z); z \in \mathbb{U}$ and $\mathbb{q}$ is the *best* dominant.

Lemma 5. **[23].** *Let* $\mathbb{q}(z)$ *be convex univalent in the unit disk* $\mathbb{U}$ *and let* θ *and* φ *be analytic in a domain* D *containing* $\mathbb{q}(\mathbb{U})$. *Suppose that*

(1) $Re\left\{\frac{\theta^{'}(\mathbb{q}(z))}{\varphi(\mathbb{q}(z))}\right\} > 0\ \ for\ \ z \in \mathbb{U},$

(2) $z\mathbb{q}^{'}(z)\varphi(\mathbb{q}(z))$*is starlike univalent in* $z \in \mathbb{U}$.

If $\mathbb{h}(z) \in\ H[\mathbb{q}(0), 1] \cap Q$, *with* $\mathbb{h}(\mathbb{U}) \subseteq D$, *and* $\theta(\mathbb{h}(z) + z\mathbb{h}^{'}(z)\varphi(\mathbb{h}(z))$ is univalent in $\mathbb{U}$, and

$$\theta(\mathbb{q}(z)) + z\mathbb{q}^{'}(z)\varphi(\mathbb{q}(z)) \prec \theta(\mathbb{h}(z)) + z\mathbb{h}^{'}(z)\varphi(\mathbb{h}(z)), \tag{11}$$

then $\mathbb{q}(z) \prec \mathbb{h}(z)$, *and* $\mathbb{q}(z); z \in \mathbb{U}$ is the *best* subordinant.

Lemma 6. **[2].** *Let* $\mathbb{q}(z)$ *be convex univalent in* $\mathbb{U}$ *and* $\mathbb{q}(0) = 1$. *Let* $\gamma \in \mathbb{C}$, *that* $Re\{\gamma\} > 0$. *If* $\mathbb{h}(z) \in H[\mathbb{q}(0), 1] \cap Q$ *and* $\mathbb{h}(z) + \gamma z\mathbb{h}\prime(z)$ *is univalent in* $\mathbb{U}$, *then*

$$\mathbb{q}(z) + \gamma z\mathbb{q}^{'}(z) \prec \mathbb{h}(z) + \gamma z\mathbb{h}^{'}(z), \tag{12}$$

which implies that $\mathbb{q}(z) \prec \mathbb{h}(z)$ *and* $\mathbb{q}(z)$ *is the best subordinant.*

3. Derivation of the Formula for h_ks in Terms of s_ks

If $A = (aij)_{m\times m},\ a_{ij}\ \in\ \mathbb{C}$, be a diagonal complex matrix, the rational polynomial $\frac{1}{det\ (I\ -\ A)}$ which factors into $\frac{1}{\prod_{l=1}^{m}(1-z\lambda_l)}$ where λ_l is the eigenvalues of the matrix. The coefficients $h_1,\ h_2,\ \cdots,\ h_n\ , \ldots$ of this polynomial

$$H(z) = h_0 + h_1 z + h_2 z^2 + \ldots + h_k z^k + \ldots; h_0 = 1, z \in \mathbb{U}$$

are given in terms of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ by

$$h_k(\lambda_1, \lambda_2\ , \ldots, \lambda_m) = \sum_{i_1+\cdots+i_m=k}^{n} {\lambda_1}^{i_1} \ldots\ {\lambda_m}^{i_m}\ ,$$

$$h_k(\lambda_1, \lambda_2\ , \ldots, \lambda_m) = \sum_{1\le \mathrm{i}_1,\ldots,\mathrm{i}_k \le m} \frac{m_1!m_2! \ldots\ m_m!}{k!} \lambda_{\mathrm{i}_1}\lambda_{\mathrm{i}_2}\ \ldots \lambda_{\mathrm{i}_m}$$

The symmetrical powers of eigenvalue.λ_i are defined by $S_\mathfrak{n} = trace\ (A^\mathfrak{n}) = \sum_{l=1}^{m} \lambda_l^\mathfrak{n}$. The summation here is over all the eigenvalues of A.

Consider the formal power series $\mathbb{S}(z) = \sum_{n=0}^{\infty} S_{\mathfrak{n}}\, z^n$ and $H(z) = \sum_{\mathfrak{n}=0}^{\infty} (-1)^{n-1} h_{\mathfrak{n}}\, z^{\mathfrak{n}}$. It is a convenience to take $S_0 = 0$ and $h_0 = 1$.

By using identities similar to Newton's identities and applying ω, one gets:

$\omega(h_{\mathfrak{n}})(t) = a_{\mathfrak{n}}(-t)$ or equivalently, $\omega(h_{\mathfrak{n}}) = (-1)^{n-1} a_{\mathfrak{n}}$, we obtain identities equivalent to the formal differential equation:

$$\mathbb{S}(z)H(z) + zH^{'}(z) = 0.$$

This may be resolved by separating the variables:

$$\mathbb{S}(z) = -\frac{z\mathrm{H}^{'}(z)}{\mathrm{H}(z)},$$

and

$$\int \mathbb{S}(z)dz = -\int \frac{z\mathrm{H}^{'}(z)}{\mathrm{H}(z)}dz = -\ln \mathrm{H}(z) + c.$$

We can include the term on the left side for each term to be obtained

$$\int \sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}}\, z^{n-1} dz = \sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}} = -\ln \mathrm{H}(z) + c.$$

When $z = 0$, the left side is 0, and the right side is c. Therefore, $c = 0$, and we have two sets of power whose coefficients imply $h_{\mathfrak{n}}$ and $S_{\mathfrak{n}}$.

Since $\ln H(z) = -\sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}$, and that yields $H(z) = e^{-\sum_{n=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}}$.

Expansion using the power series to the exponential function,

$$H(z) = 1 - \frac{1}{1!}\left(\sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right) + \frac{1}{2!}\left(\sum_{n=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right)^2 - \frac{1}{3!}\left(\sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right)^3 + \cdots.$$

Hence, collect coefficients of z^n in this series, as above,

$$\begin{aligned}(-1)^{n-1} h_{\mathfrak{n}} &= \tfrac{1}{1!}\tfrac{S_{\mathfrak{n}}}{\mathfrak{n}} + \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2=2\\ \mathfrak{i}_1,\mathfrak{i}_2\geq 1}} \tfrac{1}{2!}\tfrac{S_{\mathfrak{i}_1}}{\mathfrak{i}_1}\tfrac{S_{\mathfrak{i}_2}}{\mathfrak{i}_2} - \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\mathfrak{i}_3=3\\ \mathfrak{i}_1,\mathfrak{i}_2,\mathfrak{i}_3\geq 1}} \tfrac{1}{3!}\tfrac{S_{\mathfrak{i}_1}}{\mathfrak{i}_1}\tfrac{S_{\mathfrak{i}_2}}{\mathfrak{i}_2}\tfrac{S_{\mathfrak{i}_3}}{\mathfrak{i}_3} + \cdots \\ &= \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{n}}=k\\ \mathfrak{i}_1+2\mathfrak{i}_2+3\mathfrak{i}_3+\cdots+\mathfrak{n}\mathfrak{i}_{\mathfrak{n}}=\mathfrak{n}}} \frac{(-1)^{n-k}\, S_1^{\mathfrak{i}_1} S_2^{\mathfrak{i}_2} \ldots\, S_{\mathfrak{n}}^{\mathfrak{i}_{\mathfrak{n}}}}{\mathfrak{i}_1!\ \mathfrak{i}_2! \ldots\ \mathfrak{i}_n!\ 1^{\mathfrak{i}_1}\ 2^{\mathfrak{i}_2} \ldots \mathfrak{n}^{\mathfrak{i}_{\mathfrak{n}}}}.\end{aligned}$$

One also has

$$h_{\mathfrak{n}} = \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{n}}=k\\ \mathfrak{i}_1+2\mathfrak{i}_2+3\mathfrak{i}_3+\cdots+\mathfrak{n}\mathfrak{i}_{\mathfrak{n}}=\mathfrak{n}}} \frac{(-1)^{n-k}\, S_1^{\mathfrak{i}_1} S_2^{\mathfrak{i}_2} \ldots\ S_{\mathfrak{n}}^{\mathfrak{i}_{\mathfrak{n}}}}{\mathfrak{i}_1!\ \mathfrak{i}_2! \ldots\ \mathfrak{i}_n!\, 1^{\mathfrak{i}_1}\ 2^{\mathfrak{i}_2} \ldots \mathfrak{n}^{\mathfrak{i}_{\mathfrak{n}}}}$$

and, indeed, the extent of the coefficient.

$$\frac{n!}{\mathfrak{i}_1! 1^{\mathfrak{i}_1}\ \mathfrak{i}_2! 2^{\mathfrak{i}_2} \ldots\ \mathfrak{i}_n!\ n^{\mathfrak{i}_n}}.$$

This corresponds to the number of permutations of n symbols consisting of $\mathfrak{i}_j$ - cycles of length $j = 1, 2, \ldots\ \mathfrak{n}$,

$$h_{\mathfrak{n}} = \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_{\mathfrak{n}}=k\\ \mathfrak{i}_1+2\mathfrak{i}_2+3\mathfrak{i}_3+\cdots+\mathfrak{n}\mathfrak{i}_{\mathfrak{n}}=\mathfrak{n}}} \frac{\mathfrak{n}!}{\mathfrak{n}!} . \frac{(-1)^{n-k}\, S_1^{\mathfrak{i}_1} S_2^{\mathfrak{i}_2} \ldots\ S_{\mathfrak{n}}^{\mathfrak{i}_{\mathfrak{n}}}}{\mathfrak{i}_1!\ \mathfrak{i}_2! \ldots\ \mathfrak{i}_n!\ 1^{\mathfrak{i}_1}\ 2^{\mathfrak{i}_2} \ldots \mathfrak{n}^{\mathfrak{i}_{\mathfrak{n}}}}.$$

It also provides the computations, viz

$$\sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_\mathfrak{n}=k \\ \mathfrak{i}_1+2\mathfrak{i}_2+3\mathfrak{i}_3+\cdots+\mathfrak{n}\mathfrak{i}_\mathfrak{n}=\mathfrak{n}}} \frac{\mathfrak{n}!}{i_1!1^{\mathfrak{i}_1}\ \mathfrak{i}_2!2^{\mathfrak{i}_2}\ldots\ \mathfrak{i}_\mathfrak{n}!\ \mathfrak{n}^{\mathfrak{i}_n}},$$

it equals to $|s(\mathfrak{n},k)| = (-1)^{n-k}s(\mathfrak{n},k)$, where $s(\mathfrak{n},k)$ is the well-known Sterling numbers of the first kind.

$$h_n = \sum_{\substack{\mathfrak{i}_1+\mathfrak{i}_2+\cdots+\mathfrak{i}_\mathfrak{n}=k \\ \mathfrak{i}_1+2\mathfrak{i}_2+3\mathfrak{i}_3+\cdots+\mathfrak{n}\mathfrak{i}_\mathfrak{n}=\mathfrak{n}}} \frac{s(\mathfrak{n},k)\ S_1^{\mathfrak{i}_1}S_2^{\mathfrak{i}_2}\ldots\ S_\mathfrak{n}^{\mathfrak{n}}}{\mathfrak{n}!}.$$

4. Derivation of an Ordered Cyclic Operator

Let C_m be an ordered cyclic subgroup of symmetric group S_m and $\mathcal{M}_m$ be an ordered cyclic matrix for symmetric matrices of size m as follows:

$$m_m^0 = I_m = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, m_m^1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \ldots,$$

$$m_m^{m-1} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

We define the isomorphism $\mathcal{T} : (C_m, \circ) \to (\mathcal{M}_m, .)$ such that $\mathcal{T}\left(c_m^{\mathbb{i}}\right) = m_m^{\mathbb{i}}$ for all $\mathbb{i}$. Also these matrices act on the sequence $\{a_n\}$ as follows:

$$m_m^{\mathbb{i}} \begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1} \end{bmatrix} = \begin{bmatrix} a_{k+\mathbb{i}} \\ a_{k+\mathbb{i}+1} \\ \vdots \\ a_{k+\mathbb{i}+n-1} \end{bmatrix}, \text{ it follows from that :}$$

$$m_m^{\mathbb{i}}\left(C^T(p)\begin{bmatrix} a_k \\ a_{k+1} \\ \vdots \\ a_{k+n-1} \end{bmatrix}\right) = m_m^{\mathbb{i}}\left(\begin{bmatrix} a_{k+1} \\ a_{k+2} \\ \vdots \\ a_{k+n} \end{bmatrix}\right) = \begin{bmatrix} a_{k+\mathbb{i}+1} \\ a_{k+\mathbb{i}+2} \\ \vdots \\ a_{k+\mathbb{i}+n} \end{bmatrix}.$$

It's satisfies for each powers of companion matrix, thus the action of ordered cyclic operator of elementary symmetric polynomials and so on by taking $tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^k\right); k = 1, 2, \ldots, n$, that yields to:

$$\begin{gathered} tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^1\right) = -a_{1+\mathbb{i}} = S_1, \\ tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^2\right) = a_{1+\mathbb{i}}^2 - 2a_{2+\mathbb{i}} = S_2; a_{2+\mathbb{i}} = \tfrac{1}{2}\left((S_1)^2 - S_2\right), \\ tr\left(m_m^{\mathbb{i}}\left(C^T(p)\right)^3\right) = -a_{1+\mathbb{i}}^3 - a_{3+\mathbb{i}} + 3a_{1+\mathbb{i}}a_{2+i}a_{3+\mathbb{i}} = S_3; a_{3+\mathbb{i}} \\ = \tfrac{1}{3}\left(-(S_1)^3 + 3S_1\ S_2 - S_3\right)\ \ldots. \end{gathered}$$

It follows from acting of ordered cyclic operator on characteristic polynomial $\mathcal{T}^{\mathbb{i}}(p(z)) = \mathcal{T}^{\mathbb{i}}\left(a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n\right) = a_{\mathbb{i}} z^n + a_{1+\mathbb{i}} z^{n-1} + \cdots + a_{n-1+\mathbb{i}} z + a_{n+\mathbb{i}}; n\ mod\ \mathbb{i}$.

By applying ω, one gets:

$\omega(a_n)(t) = h_n(-t)$ or equivalently,

$\omega(a_n) = (-1)^{n-1} h_n$. One also has

$$\mathcal{T}^{\mathbb{i}}(H(z)) = \mathcal{T}^{\mathbb{i}}\left(h_0 + h_1 z + h_2 z^2 + \ldots + h_k z^k + \ldots\right) \\ = h_{\mathbb{i}} + h_{1+\mathbb{i}} z + h_{2+\mathbb{i}} z^2 + \ldots + h_{k+\mathbb{i}} z^k + \cdots \ .$$

Consider the formal power series $\mathbb{S}(z) = \sum_{n=0}^{\infty} S_n\, z^n$ and $\mathcal{T}^{\mathbb{i}}(H(z)) = \sum_{n=0}^{\infty} (-1)^{n-1} h_{n+\mathbb{i}}\, z^n$. It is convenient to take $S_0 = 0\ \ and\ \ a_0 = 1$.

Then, by using identities similar to Newton's identities and applying ω, one gets:

$\omega(h_n)(t) = a_n(-t)$ or equivalently, $\omega(h_n) = (-1)^{n-1} a_n$, we obtain identities equivalent to the formal differential equation:

$$\mathbb{S}(\mathrm{z})\mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) + \mathrm{z}\mathcal{T}^{\mathbb{i}}\left(\mathrm{H}'(\mathrm{z})\right) = 0,$$

$$\mathbb{S}(\mathrm{z})\mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) + \mathrm{z}\mathcal{T}^{\mathbb{i}}(\acute{\mathrm{H}}(\mathrm{z})) = \left(\sum_{\mathrm{n}=0}^{\infty} \mathrm{S_n}\, \mathrm{z^n}\right)\left(\sum_{\mathrm{n}=0}^{\infty} (-1)^{\mathrm{n}-1} \mathrm{h}_{\mathrm{n}+\mathbb{i}}\, \mathrm{z^n}\right) + \\ \sum_{\mathrm{n}=0}^{\infty} (-1)^{\mathrm{n}-1} \mathrm{h}_{\mathrm{n}+\mathbb{i}}\, \mathrm{z^n}.$$

This can be solved by separating the variables:

$$\mathbb{S}(\mathrm{z}) = -\frac{\mathrm{z}\mathcal{T}^{\mathbb{i}}\left(\mathrm{H}'(\mathrm{z})\right)}{\mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z}))}; \mathbb{i} = 0, 1, 2, \ldots,$$

and

$$\int \mathbb{S}(\mathrm{z})\mathrm{dz} = -\int \frac{\mathrm{z}\mathcal{T}^{\mathbb{i}}\left(\mathrm{H}'(\mathrm{z})\right)}{\mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z}))}\mathrm{dz} = -\ln \mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) + c.$$

We can include the term on the left side for each term to be obtained

$$\int \sum_{\mathfrak{n}=1}^{\infty} \mathrm{S}_{\mathfrak{n}}\, \mathrm{z}^{\mathrm{n}-1}\mathrm{dz} = \sum_{\mathfrak{n}=1}^{\infty} \mathrm{S}_{\mathfrak{n}} \frac{\mathrm{z^n}}{\mathfrak{n}} = -\ln \mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) + c.$$

When $\mathrm{z} = 0$, the left side is 0 and the right side is c. Therefore, $c = 0$ and we have two sets of power whose coefficients imply $\mathrm{h}_{\mathrm{n}+\mathbb{i}}$ and $\mathrm{S_n}$.

Since $\ln \mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) = -\sum_{\mathrm{n}=1}^{\infty} \mathrm{S_n} \frac{\mathrm{z^n}}{\mathrm{n}}$, and that yields $\mathcal{T}^{\mathbb{i}}(\mathrm{H}(\mathrm{z})) = \mathrm{e}^{-\sum_{\mathrm{n}=1}^{\infty} \mathrm{S_n} \frac{\mathrm{z^n}}{\mathrm{n}}}$.

Expansion using the power series to the exponential function,

$$\mathcal{T}^{\mathbb{i}}(H(z)) = 1 - \frac{1}{1!}\left(\sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right) + \frac{1}{2!}\left(\sum_{\mathfrak{n}=1}^{\infty} S_n \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right)^2 - \frac{1}{3!}\left(\sum_{\mathfrak{n}=1}^{\infty} S_{\mathfrak{n}} \frac{z^{\mathfrak{n}}}{\mathfrak{n}}\right)^3 + \cdots .$$

Hence, collect coefficients of z^n in this series, as above,

$$h_{n+\mathbb{i}} = \sum_{\substack{\mathrm{i}_1 + \mathrm{i}_2 + \cdots + \mathrm{i}_{(n+\mathbb{i})} = k \\ \mathrm{i}_1 + 2\mathrm{i}_2 + 3\mathrm{i}_3 + \cdots + (n+\mathbb{i}) i\mathrm{i}_{(n+\mathbb{i})} = (n+\mathbb{i})}} \frac{(-1)^{n-k}\, S_1^{\mathrm{i}_1} S_2^{\mathrm{i}_2} \ldots\ S_{(n+\mathbb{i})}^{\mathrm{i}_{(n+\mathbb{i})}}}{\mathrm{i}_1!\ \ \mathrm{i}_2! \ldots\ \mathrm{i}_n!\ 1^{\mathrm{i}_1}\ 2^{\mathrm{i}_2} \ldots (n+\mathbb{i})^{i_{(n+\mathbb{i})}}}.$$

5. Differential Subordination Results

Theorem 3. *Let* $\mathbb{q}(z)$*be convex univalent in with* $\mathbb{q}(0) = 1$, $\lambda > 0$, *and* $\eta \in \mathbb{C} \setminus \{0\}$. *Assume that*

$$Re\left\{1 + \frac{z\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\}, \tag{13}$$

if

$$e_1(z) = (1+\eta)\left(\mathcal{T}^{\mathbb{i}}(H(z))\right) - \eta\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}\right), \tag{14}$$

$$e_1(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z), \tag{15}$$

then

$$\mathcal{T}^{\mathbb{i}}H(z) \prec \mathbb{q}(z), \tag{16}$$

and $\mathbb{q}(z)$ *is the best dominant.*

Proof 2. If we consider the analytic function

$$\mathbb{p}(z) = \mathcal{T}^{\mathbb{i}}H(z); \;\; \mathbb{i} = 0, 1, 2, \ldots, \; z \in \mathbb{U}, \tag{17}$$

differentiating (17) with respect to z , we have

$$\mathbb{p}'(z) = \left(\mathcal{T}^{\mathbb{i}}H(z)\right)', \mathrm{z}\mathbb{p}'(z) = -\mathbb{p}(z)\mathbb{S}(\mathrm{z}), \frac{\mathrm{z}\mathbb{p}'(z)}{\mathbb{p}(z)} = -\mathbb{S}(z).$$

Now, using the identity (6), we obtain

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))},$$

Therefore,

$$z\mathbb{p}'(z) = \frac{z\,\mathcal{T}^{\mathbb{i}}(H(z))\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}\;. \tag{18}$$

Since $e_1(z) = (1+\eta)\left(\mathcal{T}^{\mathbb{i}}(H(z))\right) - \eta\left(\mathcal{T}^{\mathbb{i}}(H(z))\right)\left(\frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}\right)$,

$$e_1(z) \prec \mathbb{q}(z) + \eta z\mathbb{q}'(z).$$

The subordination (15) is equivalent to

$$\mathbb{p}(z) + \eta z\mathbb{p}'(z) \prec \mathbb{q}(z) + \eta\lambda z\mathbb{q}'(z). \tag{19}$$

Application of Lemma 3 with $\beta = \eta$, $\alpha = 1$, we obtain (16). □
Taking $\mathbb{q}(z) = \frac{1+Az}{1+\mathrm{B}z}$ $(-1 \leq \mathrm{B} < \mathrm{A} \leq 1)$, in Theorem 3, we get the following result.

Corollary 1. *Let* $\eta \in \mathbb{C} \setminus \{0\}$, *and suppose that*

$$Re\left(\frac{1+z}{1-z}\right) > max\left\{0, -Re\left(\frac{1}{\eta}\right)\right\}.$$

If $H \in \mathbb{H}$ satisfies the following subordination condition:

$$\left(\mathcal{T}^{\mathrm{i}}(H(z))\right) + \left(\mathcal{T}^{\mathrm{i}}(H(z))\right)\left(\frac{z\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathrm{i}+1}(H(z))}\right) \prec \frac{1+z}{1-z} + \frac{2z}{(1-z)^2},$$

then

$$z\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right)' \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ *is the best dominant.*

Theorem 4. *Let* $\mathbb{q}(z)$ *be convex univalent in unit disk* $\mathbb{U}$ *with* $\mathbb{q}(0) = 1$, $\mathbb{q}(z) \neq 0$ *and* $\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$ *is starlike univalent* $\mathbb{U}$, $\eta \in \mathbb{C} \setminus \{0\}$, *and* $a, \lambda, \mu, \sigma, \varrho, \alpha, \xi \in \mathbb{C}$, $H \in \mathbb{H}$, *and suppose that* H *and* $\mathbb{q}$ *satisfy the next two conditions*

$$tz\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right) + (1-t)z\left(\mathcal{T}^{\mathrm{i}}(H(z))\right) \neq 0, \ (z \in \mathbb{U}, \ 0 \le t \le 1), \tag{20}$$

and

$$Re\left\{1 + \frac{\alpha}{\eta}\mathbb{q}(z) + \frac{2\xi}{\eta}(\mathbb{q}(z))^2 - z\frac{\mathbb{q}'(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}''(z)}{\mathbb{q}'(z)}\right\} > 0. \tag{21}$$

If

$$e_2(z) = a + \lambda\mathbb{q}\mathbb{q}(z) + \mu\xi\mathbb{q}^2(z) + \varrho\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \tag{22}$$

such that

$$\begin{aligned} e_2(z) = \ & \sigma + \alpha\left(tz\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right) + (1-t)z\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)\right) \\ & + \xi\left(tz\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right) + (1-t)z\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)\right) \\ & + \eta\left[\frac{tz\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right)' + (1-t)z\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)'}{t\mathcal{T}^{\mathrm{i}+1}(H(z)) + (1-t)\mathcal{T}^{\mathrm{i}}(H(z))}\right], \end{aligned} \tag{23}$$

and

$$e_2(z) \prec \sigma + \alpha\mathbb{q}(z) + \xi(\mathbb{q}(z))^2 + \eta\,\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \tag{24}$$

then

$$tz\mathcal{T}^{\mathrm{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathrm{i}}(H(z)) \prec \mathbb{q}(z), \tag{25}$$

and $\mathbb{q}(z)$ is the best dominant.

Proof 3. Suppose that $\mathbb{p}(z)$ is an analytic function and is defined as:

$$\mathbb{p}(z) = t\mathcal{T}^{\mathrm{i}+1}(H(z)) + (1-t)\mathcal{T}^{\mathrm{i}}(H(z)). \tag{26}$$

Then, the function $\mathbb{p}(z)$ is analytic in $\mathbb{U}$ and $\mathbb{p}(0) = 1$, differentiating (26) with respect to z, we get

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{tz\left(\mathcal{T}^{\mathrm{i}+1}(H(z))\right)' + (1-t)z\left(\mathcal{T}^{\mathrm{i}}(H(z))\right)'}{t\mathcal{T}^{\mathrm{i}+1}(H(z))f(z) + (1-t)\mathcal{T}^{\mathrm{i}}(H(z))}. \tag{27}$$

By setting $\theta(\omega) = \sigma + \alpha\omega + \xi\omega^2$ *and* $\phi(\omega) = \frac{\eta}{\omega}$, it can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C} \setminus \{0\}$ *and* $\phi(\omega) \neq 0$, $\omega \in \mathbb{C} \setminus \{0\}$.In addition, we get

$$\begin{aligned} Q(z) = z\mathbb{q}'(z)\phi(z) \ & = \eta\,\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}, \quad (z \in \mathbb{U}) \text{ and } \ \mathbb{h}(z) = \theta(\mathbb{q}(z)) + Q(z) \\ & = \sigma + \alpha\mathbb{q}'(z) + \xi(\mathbb{q}(z))^2 + \eta\,\frac{z\mathbb{q}\prime(z)}{\mathbb{q}(z)}, \end{aligned}$$

It is clear that $Q(z)$ is starlike univalent in $\mathbb{U}$, and that

$$Re\left(\frac{z\mathbb{h}'(z)}{Q(z)}\right) = Re\left(1 + \frac{\alpha}{\eta}\mathbb{q}(z) + \frac{2\xi}{\eta}(\mathbb{q}(z))^2 - z\frac{\mathbb{q}'(z)}{\mathbb{q}(z)} + z\frac{\mathbb{q}''(z)}{\mathbb{q}'(z)}\right) > 0; (z \in \mathbb{U}).$$

By using (27), hypothesis (24) can be equivalently written as

$$\theta(\mathbb{p}(z)) + z\mathbb{p}'(z)\phi(\mathbb{p}(z)) \prec \phi(\mathbb{q}(z)) + z\mathbb{q}'(z)\phi(\mathbb{q}(z)),$$

Thus, by applying Lemma 4, the function $\mathbb{q}(z)$ is the best dominant. □

6. Differential Superordination Results

Theorem 5. *Let $\mathbb{q}(z)$ be a convex univalent function in $\mathbb{U}$ with $\mathbb{q}(0) = 1$, $Re\{\eta\} > 0$. Let $H \in \mathbb{H}$, satisfies*

$$\mathcal{T}^{\mathbb{i}}(H(z)) \in H[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ defined by (14) is *u*nivalent in $\mathbb{U}$ and

$$\mathbb{q}(z) + \eta z\mathbb{q}'(z) \prec e_1(z), \tag{28}$$

then

$$\mathbb{q}(z) \prec \mathcal{T}^{\mathbb{i}}(H(z)), \tag{29}$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof . Suppose that $\mathbb{p}(z)$ is an analytic function and is defined as :

$$\mathbb{p}(z) = \mathcal{T}^{\mathbb{i}}(H(z)). \tag{30}$$

Differentiating (30) with respect to z, we have

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{z\left(\mathcal{T}^{\mathbb{i}+1}(H(z))\right)'}{\mathcal{T}^{\mathbb{i}+1}(H(z))}. \tag{31}$$

After some computation and using (6), from (31), we get

$$e_1(z) = \mathbb{p}(z) + \eta z\mathbb{p}'(z),$$

and by applying Lemma 5, we get the following result.

Taking $\mathbb{q}(z) = \frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1)$, in Theorem 5, we get the following corollary. □

Corollary 2. *Let* $-1 \le B < A \le 1$, $\eta \in \mathbb{C} \setminus \{0\}$ *with* $Re\{\eta\} > 0$, *also let*

$$\mathcal{T}^{\mathbb{i}}(H(z)) \in H[\mathbb{q}(0), 1] \cap Q.$$

If the function $e_1(z)$ given by (14) is univalent in $\mathbb{U}$ and $H \in \mathbb{H}$ satisfies the following superordination condition

$$\frac{1 + Az}{1 + BBz} + \eta\frac{(A - B)z}{(1 + Bz)^2} \prec e_1(z),$$

then

$$\frac{1 + Az}{1 + Bz} \prec \mathcal{T}^{\mathbb{i}}(H(z)),$$

and the function $\frac{1+Az}{1+\mathrm{B}z}$ is the best subordinant.

Theorem 6. *Let $\mathbb{q}(z)$ be convex univalent in unit disk $\mathbb{U}$, with $\mathbb{q}(0) = 1$, $\mathbb{q}(z) \neq 0$, and $\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}$, is starlike in $\mathbb{U}$, let $\eta \in \mathbb{C} \setminus \{0\}$ and $\sigma, \alpha \in \mathbb{C}$. Further, assume that $\mathbb{q}$ satisfies*

$$Re\left\{(\alpha + 2\xi\mathbb{q}(z))\frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\} > 0, \qquad (z \in \mathbb{U}). \tag{32}$$

Let $H(z) \in \mathbb{H}$, and suppose that $H(z)$ satisfies the next condition

$$tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z)) \neq 0; (z \in \mathbb{U}),\ (0 \leq t \leq 1) \tag{33}$$

and

$$tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z)) \in H[\mathbb{q}(0), 1] \cap Q. \tag{34}$$

If the function $e_2(z)$, given by (22) is univalent in $\mathbb{U}$, and

$$\sigma + \alpha\mathbb{q}(z) + \xi(\mathbb{q}(z))^2 + \eta\,\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)} \prec e_2(z), \tag{35}$$

then

$$\mathbb{q}(z) \prec tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z)), \tag{36}$$

and $\mathbb{q}(z)$ is the best subordinant.

Proof 4. Let the function $\mathbb{p}(z)$ defined on $\mathbb{U}$ by (24).

Then, a computation shows that

$$\frac{z\mathbb{p}'(z)}{\mathbb{p}(z)} = \frac{tz\mathcal{T}^{\mathbb{i}+1}(H(z))' + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z))'}{t\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)\mathbb{T}\mathcal{T}^{\mathbb{i}}(H(z))}, \tag{37}$$

by setting $\Theta(w) = \sigma + \alpha w + \xi w^2$, $and \phi(w) = \frac{\eta}{w}, (w \in \mathbb{C} \setminus \{0\})$.

We see that $\Theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$, and that $\phi(w) \neq 0$ $(w \in \mathbb{C} \setminus \{0\})$. In addition, we get

$$Q(z) = z\mathbb{q}'(z)\phi(\mathbb{q}(z)) = \eta\frac{z\mathbb{q}'(z)}{\mathbb{q}(z)}\ ; z \in \mathbb{U}.$$

It observed *that* $Q(z)$ is starlike univalent *in* $\mathbb{U}$, and that

$$Re\left(\frac{z\Theta'(\mathbb{q}(z))}{\phi(\mathbb{q}(z))}\right) = Re\left\{\alpha + 2\xi\mathbb{q}(z)\frac{\mathbb{q}(z)\mathbb{q}'(z)}{\eta}\right\} > 0\,.$$

By making use of (37), hypothesis (35) can be written as:

$$\Theta(\mathbb{q}(z)) + z\mathbb{q}'(z)\phi(\mathbb{q}(z)) \prec \Theta(\mathbb{p}(z)) + z\mathbb{p}'(z)\phi(\mathbb{p}(z)),$$

Thus, the proof is complete by applying Lemma 6. □

7. Sandwich Results

Combination Theorem 3 with Theorem 5 to obtain the following theorem.

Theorem 7. *Let $\mathbb{q}_1$ and $\mathbb{q}_2$ be convex univalent functions in $\mathbb{U}$ with $\mathbb{q}_1(0) = \mathbb{q}_2(0) = 1$ and $\mathbb{q}_2$ satisfies (13). Suppose that $Re\{y\} > 0$, $\lambda \in \mathbb{C} \setminus \{0\}$. If $H \in \mathbb{H}$, such that*

$$\mathcal{T}^{\mathbb{i}}(H(z)) \in H[\mathbb{q}(0), 1] \cap Q,$$

and the function $e_1(z)$ is univalent in $\mathbb{U}$ and satisfies

$$\mathbb{q}_1(z) + \eta z \mathbb{q}_1{}'(z) \prec e_1(z) \prec \mathbb{q}_2(z) + \eta z \mathbb{q}_2{}'(z), \tag{38}$$

where $e_1(z)$ is given by (14), then

$$\mathbb{q}_1(z) \prec \mathcal{T}^{\mathbb{i}}(H(z)) \prec \mathbb{q}_2(z),$$

where $\mathbb{q}_1$ and $\mathbb{q}_2$, are respectively the best subordinant and best dominant of (38).

Combining Theorem 4 with Theorem 6, we obtain the following sandwich theorem :

Theorem 8. *Let $\mathbb{q}_j$, be two convex univalent functions in $\mathbb{U}$, such that $\mathbb{q}_j(0) = 1$, $\mathbb{q}_j(z) \neq 0$, and $\frac{z\mathbb{q}_j'(z)}{\mathbb{q}_j(z)}$ $(j = 1, 2)$ is starlike univalent in $\mathbb{U}$, let $\lambda, \eta \in \mathbb{C} \setminus \{0\}$ and $\sigma, \alpha, \xi \in \mathbb{C}$. Suppose that $\mathbb{q}_1$ and $\mathbb{q}_2$ satisfies (22) and (32), respectively.*

If $H \in \mathbb{H}$, and suppose that H satisfies the next condition

$$tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z)) \neq 0, \ (z \in U, \ 0 \leq t \leq 1),$$

and

$$tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(f(z)) \in H[q(0), 1] \cap Q.$$

If the function $e_2(z)$ given by (23) is univalent in $\mathbb{U}$, and

$$\begin{aligned} \sigma + \alpha\mathbb{q}_1(z) + \xi(\mathbb{q}_1(z))^2 + \eta \frac{z\mathbb{q}_1{}'(z)}{\mathbb{q}_1(z)} \prec e_2(z) \\ \prec \sigma + \alpha\mathbb{q}_2(z) + \xi(\mathbb{q}_2(z))^2 + \eta \frac{z\mathbb{q}_2{}'(z)}{\mathbb{q}_2(z)}, \end{aligned} \tag{39}$$

then

$$\mathbb{q}_1(z) \prec tz\mathcal{T}^{\mathbb{i}+1}(H(z)) + (1-t)z\mathcal{T}^{\mathbb{i}}(H(z)) \prec \mathbb{q}_2(z),$$

where $\boldsymbol{q_1}$ ***and*** $\boldsymbol{q_2}$ are the best subodinant and best dominant respectively of (7.2).

8. Conclusions

We introduce a new study on the connection between geometric function theory, especially sandwich theorems, and Viete's theorem in elementary algebra. We obtain some conclusions for differential subordination and superordination for a new formula of complete homogeneous symmetric functions class involving an ordered cyclic operator. In addition, certain sandwich theorems are found. These properties and results are symmetrical with differential superordination properties to form sandwich theorems. We have different results than the other authors. We have opened some windows to allow authors to generalize our new subclasses in order to obtain new results in the theory of univalent and multivalent functions using the results of the paper.

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References

1. Miller, S.S.; Mocanu, P.T. Differential Subordinations: Theory and Applications. In *Series on Monographs and Texbooks in Pure and Applied Mathematics*; Marcel Dekker, Inc.: New York, NY, USA, 2000.
2. Miller, S.S.; Mocanu, P.T. Subordinant of differential superordinations. *Complex Var.* **2003**, *48*, 815–826. [CrossRef]
3. Bulboaca, T. Classes of first order differential superordinations. *Demonstr. Math.* **2002**, *35*, 287–292. [CrossRef]
4. Bulboaca, T. A class of superordination-preserving integral operators. *Indag. Math.* **2002**, *13*, 301–311. [CrossRef]
5. Ali, R.M.; Ravichandran, V.; Khan, M.H.; Subramanian, K.G. Differential sandwich theorems for certain analytic functions. *Far East J. Math. Sci.* **2004**, *15*, 87–94.
6. Ali, R.M.; Ravichandran, V.; Seenivasagan, N. Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions. *Bull. Malays. Math. Sci. Soc.* **2008**, *31*, 193–207.
7. Atshan, W.G.; Hadi, R.A. Some differential subordination and superordination results of p-valent functions defined by differential operator. *J. Phys. Conf. Ser.* **2020**, *1664*, 012043. [CrossRef]
8. Atshan, W.G.; Ali, A.A.R. On Some sandwich theorems of analytic functions involving Noor-Sălăgean operator. *Adv. Math. Sci. J.* **2020**, *9*, 8455–8467. [CrossRef]
9. Atshan, W.G.; Ali, A.A.R. On sandwich theorems results for certain univalent functions defined by generalized operators. *Iraqi J. Sci.* **2021**, *62*, 2376–2383. [CrossRef]
10. Farzana, H.A.; Stephen, B.A.; Jeyaramam, M.P. Third–order differential subordination of analytic function defined by functional derivative operator. *Ann. Stiint. Univ. Al. I. Cuzal Iasi Mat. New Ser.* **2016**, *62*, 105–120.
11. Jeyaraman, M.P.; Suresh, T.K. Third-order differential subordination of analytic functions. *Acta Univ. Apulensis Math. Inform.* **2013**, *35*, 187–202.
12. Murugusundaramoorthy, G.; Magesh, N. An application of second order differential inequalities based on linear and integral operators. *Int. J. Math. Sci. Eng. Appl.* **2008**, *2*, 105–114.
13. Ponnusamy, S.; Juneja, O.P. Third-order differential inequalities in the complex plane. In *Current Topics in Analytic Function Theory*; World Scientific Publishing Company: Singapore; Hackensack, NJ, USA; London, UK; Hongkong, China, 1992.
14. Raducanu, D. Third-order differential subordinations for analytic functions associated with generalized Mittag-Leffler functions. *Mediterr. J. Math.* **2017**, *14*, 1–18. [CrossRef]
15. Shanmugam, T.N.; Sivasubramanian, S.; Srivastava, H.M. Differential Sandwich theorems for certain subclasses of analytic functions involving multiplier transformations. *Integral Transform. Spec. Funct.* **2006**, *17*, 889–899. [CrossRef]
16. Gochhayat, P. Sandwich-type results for a class of functions defined by a generalized differential operator. *Mat. Vesn.* **2013**, *65*, 178–186.
17. Kavitha, S.; Sivasubramanian, S.; Jayasankar, R. Differential subordination and superordination results for Cho-Kwon-Srivastava operator. *Comput. Math. Appl.* **2012**, *64*, 1789–1803. [CrossRef]
18. Al-Ameedee, S.A.; Atshan, W.G.; Al-Maamori, F.A. On sandwich results of univalent functions defined by a linear operator. *J. Interdiscip. Math.* **2020**, *23*, 803–809. [CrossRef]
19. Al-Ameedee, S.A.; Atshan, W.G.; Al-Maamori, F.A. Some new results of differential subordinations for Higher-order derivatives of multivalent functions. *J. Phys. Conf. Ser.* **2021**, *1804*, 012111. [CrossRef]
20. Atshan, W.G.; Battor, A.H.; Abaas, A.F. Some sandwich theorems for meromorphic univalent functions defined by new integral operator. *J. Interdiscip. Math.* **2021**, *24*, 579–591. [CrossRef]
21. Atshan, W.G.; Hiress, R.A.; Altinkaya, S. On third-order differential subordination and superordination properties of analytic functions defined by a generalized operator. *Symmetry* **2022**, *14*, 418. [CrossRef]
22. Atshan, W.G.; Kulkarni, S.R. On application of differential subordination for certain subclass of meromorphically p-valent functions with positive coefficients defined by linear operator. *J. Inequalities Pure Appl. Math.* **2009**, *102009*, 53.
23. Bulboaca, T. *Differential Subordinations and Superordinations, Recent Results*; House of Scientific Book Publishing: Cluj-Napoca, Romania, 2005.
24. Selvaraj, C.; Karthikeyan, K.R. Differential subordinations and superordinations for certain subclasses of analytic functions. *Far East J. Math. Sci.* **2008**, *29*, 419–430.
25. Barlcz, A.; Deniz, E.; Çaglar, M.; Orhan, H. Differential subordinations involving generalized Bessel functions. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 1255–1280. [CrossRef]
26. Atshan, W.G.; Hassan, H.Z.; Yalcin, S. On third-order differential subordination results for univalent functions defined by differential operator. *Uzb. Math. J.* **2021**, *65*, 26–42.
27. Darweesh, A.M.; Atshan, W.G.; Battor, A.H.; Lupas, A.A. Third-order differential subordination results for analytic functions associated with a certain differential operator. *Symmetry* **2022**, *14*, 99. [CrossRef]
28. Mihsin, B.K.; Atshan, W.G.; Alhily, S.S.; Lupas, A.A. New results on fourth- order differential subordination and superordination for univalent analytic functions involving a linear operator. *Symmetry* **2022**, *14*, 324. [CrossRef]

29. Sabri, M.A.; Atshan, W.G.; El-Seidy, E. On sandwich-type results for a subclass of certain univalent functions using a new Hadamard product operator. *Symmetry* **2022**, *14*, 931. [CrossRef]
30. Theyab, S.D.; Atshan, W.G.; Lupaș, A.A.; Abdullah, H.K. New results on higher-order differential subordination and superordination for univalent analytic functions using a new operator. *Symmetry* **2022**, *14*, 1576. [CrossRef]
31. Tang, H.; Srivastava, H.M.; Li, S.; Ma, L. Third-order differential subordination and superordination results for meromorphically multivalent functions associated with the Liu-Srivastava operator. *Abstr. Appl. Anal.* **2014**, *2014*, 1–11. [CrossRef]
32. Gochhayat, P.; Prajapati, A. Applications of third order differential subordination and superordination involving generalized Struve function. *Filomat* **2019**, *33*, 3047–3059. [CrossRef]
33. El-Ashwah, R.M.; Aouf, M.K. Differential subordination and superordination for certain subclasses of p-valent functions. *Math. Comput. Model.* **2010**, *51*, 349–360. [CrossRef]
34. Cho, N.E.; Bulboacă, T.; Srivastava, H.M. A general family of integral and associated subordination and superordination properties of analytic function classes. *Appl. Math. Comput.* **2012**, *219*, 2278–2288.
35. Antonino, J.A.; Miller, S.S. Third-order differential inequalities and subordinations in the complex plane. *Complex Var. Elliptic Equ.* **2011**, *56*, 439–445. [CrossRef]
36. Atshan, W.G.; Battor, A.H.; Abaas, A.F. On third-order differential subordination results for univalent analytic functions involving an operator. *J. Phys. Conf. Ser.* **2020**, *1664*, 012044. [CrossRef]
37. MacDonald, I.G. *Symmetric Functions and Hall Polynomials*, 2nd ed.; Claredon Press: Oxford, UK, 1995.
38. Golub, G.H.; van Loan, C.F. *Matrix Computations*, 3rd ed.; Johns Hopkins: Baltimore, MD, USA, 1996.
39. Herrero, D.A. Hypercyclic operators and chaos. *J. Oper. Theory* **1992**, *28*, 93–103.
40. Herrero, D.A.; Larson, D.R.; Wogen, W.R. Semitriangular operatos. *Houst. J. Math.* **1991**, *17*, 477–499.
41. Dziok, J.; Srivastava, H.M. Classes of analytic functions associated with the generalized hypergeometric function. *Appl. Math. Comput.* **1999**, *103*, 1–13. [CrossRef]