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Fractional Jensen-Mercer Type Inequalities Involving Generalized Raina's Function and Applications

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Abstract: The aim of this paper is to derive some new generalized fractional analogues of Mercer type inequalities, essentially using the convexity property of the functions and Raina's function. We also discuss several new special cases which show that our results are, to an extent, unifying. In order to illustrate the significance of our results, we offer some interesting applications of our results to special means, error bounds, and q-digamma functions.

Keywords: Hermite–Hadamard–Mercer inequality; Raina's function; Hölder inequality; convexity; special means; error bounds; q-digamma functions

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1. Introduction

Fractional Calculus (FC) can be viewed as a generalization of classical calculus with respect to the operations of integration and differentiation of any arbitrary order. The history of fractional calculus is almost simultaneous with ordinary calculus. As compared to the ordinary counterparts in many physical problems, the concepts of fractional calculus have proven to be more useful. Consequently, in the past few decades, it has experienced a rapid development. Many classical concepts of fractional calculus have been extended and generalized according to need. One of the most significant concepts belonging to fractional calculus is Riemann-Liouville's fractional integrals.

Definition 1 ([1])**.** *Let $\Phi \in L[v_1, v_2]$. The Riemann-Liouville integrals $J^{\lambda}_{v_1+}\Phi$ and $J^{\lambda}_{v_2-}\Phi$ of order $\lambda > 0$ with $v_1 \geq 0$ are defined by*

$$J^{\lambda}_{v_1+}\Phi(u_1) = \frac{1}{\Gamma(\lambda)} \int_{v_1}^{u_1} (u_1 - \delta)^{\lambda-1}\Phi(\delta)d\delta, \quad (u_1 > v_1),$$

and

$$J^{\lambda}_{v_2-}\Phi(u_1) = \frac{1}{\Gamma(\lambda)} \int_{u_1}^{v_2} (\delta - u_1)^{\lambda-1}\Phi(\delta)d\delta, \quad (u_1 < v_2),$$

respectively, where $\Gamma(\lambda) = \int_0^{\infty} e^{-\delta}\delta^{\lambda-1}\mathrm{d}\delta$. Here is $J^{0}_{v_1+}\Phi(u_1) = J^{0}_{v_2-}\Phi(u_1) = \Phi(u_1)$.

Mubeen and Habibullah [2] introduced the class of k-Riemann-Liouville fractional integrals and discussed its basic properties. They have observed that the properties of k-Riemann-Liouville fractional integrals are quite different. Thus, we can say that k-Riemann-Liouville fractional integrals are significant generalizations of Riemann-Liouville fractional integrals.

Definition 2 ([2]). *Let $\Phi \in L[v_1, v_2]$. Then for $k > 0$, the left and right k-Riemann-Liouville fractional integrals are defined as:*

$$ {}_kI^{\lambda}_{v_1^+}\Phi(u_1) = \frac{1}{k\Gamma_k(\lambda)} \int_{v_1}^{u_1} (u_1 - \delta)^{\frac{\lambda}{k}-1}\Phi(\delta)d\delta, \quad (u_1 > v_1), $$

and

$$ {}_kI^{\lambda}_{v_2^-}\Phi(u_1) = \frac{1}{k\Gamma_k(\lambda)} \int_{u_1}^{v_2} (\delta - u_1)^{\frac{\lambda}{k}-1}\Phi(\delta)d\delta, \quad (u_1 < v_2), $$

where $\Gamma_k(\cdot)$ is the k-gamma function.

Another fractional integral defined in the literature is ψ-Riemann-Liouville fractional integrals.

Definition 3 ([1]). *Let $\psi : [v_1, v_2] \to \mathbb{R}_+$ be an increasing function possessing continuous derivative ψ'. Then, for $\lambda > 0$, the left and right ψ-Riemann-Liouville fractional integrals of Φ with respect to ψ are defined as:*

$$ I^{\lambda,\psi}_{v_1+}\Phi(u_1) = \frac{1}{\Gamma(\lambda)} \int_{v_1}^{u_1} (\psi(u_1) - \psi(\delta))^{\lambda-1}\psi'(\delta)\Phi(\delta)d\delta, \quad (u_1 > v_1), $$

and

$$ I^{\lambda,\psi}_{v_2^-}\Phi(u_1) = \frac{1}{\Gamma(\lambda)} \int_{u_1}^{v_2} (\psi(\delta) - \psi(u_1))^{\lambda-1}\psi'(\delta)\Phi(\delta)d\delta, \quad (u_1 < v_2). $$

If we take $\psi(\delta) = \delta$ that is identity function in Definition 3, then we recapture the definition of Riemann-Liouville fractional integrals.

Akkurt et al. [3] extended the notions of Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals and ψ-Riemann-Liouville fractional integrals and defined the class of ψ_k-fractional integrals as:

Definition 4. *Let $\Phi \in L[v_1, v_2]$ and $\psi : [v_1, v_2] \to \mathbb{R}_+$ be an increasing function possessing continuous derivative ψ'. Then, for $k > 0$, the left and right ψ_k-Riemann-Liouville fractional integrals of Φ with respect to ψ are defined as*

$$ {}_kI^{\lambda,\psi}_{v_1+}\Phi(u_1) = \frac{1}{k\Gamma_k(\lambda)} \int_{v_1}^{u_1} (\psi(u_1) - \psi(\delta))^{\frac{\lambda}{k}-1}\psi'(\delta)\Phi(\delta)d\delta, \quad (u_1 > v_1), $$

and

$$ {}_kI^{\lambda,\psi}_{v_2^-}\Phi(u_1) = \frac{1}{k\Gamma_k(\lambda)} \int_{u_1}^{v_2} (\psi(\delta) - \psi(u_1))^{\frac{\lambda}{k}-1}\psi'(\delta)\Phi(\delta)d\delta, \quad (u_1 < v_2). $$

Recently, Awan et al. [4] have obtained some new Hermite-Hadamard like inequalities using the Definition 4.

On the basis of k-gamma function Tunc et al. [5] introduced a new class of function (k-Raina's function), which is defined as:

$$ \mathcal{F}^{\varpi,k}_{\varrho,\lambda}(u_1) = \sum_{i=0}^{\infty} \frac{\varpi(i){u_1}^i}{k\Gamma_k(\varrho ik + \lambda)} \qquad (\varrho, \nu > 0; |u_1| < \mathbb{R}), $$

where coefficients $\varpi(i)$ with $(i \in \mathbb{N} \cup \{0\})$ are bounded sequences of positive real numbers and $\mathbb{R}$ is a set of real numbers. With the help of these functions, Tunc et al. have defined the left and right-sided k-Raina's fractional integrals, respectively, as:

$$ \left(J^{\varpi,k}_{\varrho,\lambda,v_1^+;\omega}\right)(u_1) = \int_{v_1}^{u_1} (u_1 - \delta)^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\nu}[\omega(u_1 - \delta)^{\varrho}]\Phi(\delta)\mathrm{d}\delta \qquad (u_1 > v_1 > 0), $$

$$\left(J^{\varpi,k}_{\varrho,\lambda,v_2^-;\omega}\right)(u_1)=\int_{u_1}^{v_2}(\delta-u_1)^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,v}[\omega(\delta-u_1)^{\varrho}]\Phi(\delta)\mathrm{d}\delta \qquad (0<u_1<v_2),$$

where $\varrho,\lambda>0$ and $\omega\in\mathbb{R}$.

1. For $k=1$, we obtain Raina's fractional integral, see [6].
2. If we set $k=1, i=0$ and $\varpi(0)=1$, then we obtain the classical Riemann-Liouville fractional integrals.

Set et al. [7] also introduced the following generalized fractional integrals:

Definition 5. *Let $\Phi\in L[v_1,v_2]$ and $\psi:[v_1,v_2]\to\mathbb{R}_+$ be an increasing function possessing continuous derivative ψ'. Then, for $k>0$, the left and right ψ_k-Raina fractional integrals of Φ with respect to ψ are defined as:*

$$I^{\varpi,\psi,k}_{\varrho,\lambda,v_1^+;\omega}\Phi(u_1)=\int_{v_1}^{u_1}(\psi(u_1)-\psi(\delta))^{\frac{\lambda}{k}-1}\psi'(\delta)\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(\psi(u_1)-\psi(\delta))^{\varrho}]\Phi(\delta)d\delta,$$

for $u_1>v_1$ and

$$I^{\varpi,\psi,k}_{\varrho,\lambda,v_2^-;\omega}\Phi(u_1)=\int_{u_1}^{v_2}(\psi(\delta)-\psi(u_1))^{\frac{\lambda}{k}-1}\psi'(\delta)\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(\psi(\delta)-\psi(u_1))^{\varrho}]\Phi(\delta)d\delta,$$

for $u_1<v_2$, where $\lambda,\varrho>0$ and $\omega\in\mathbb{R}$.

If we take $k=1$ in Definition 5, then we have ψ-Raina's fractional integral which is given as:

Definition 6. *Let $\Phi\in L[v_1,v_2]$ and $\psi:[v_1,v_2]\to\mathbb{R}_+$ be an increasing function possessing continuous derivative ψ'. Then left and right ψ-Raina fractional integrals of Φ with respect to ψ are defined as*

$$I^{\varpi,\psi}_{\varrho,\lambda,v_1^+;\omega}\Phi(u_1)=\int_{v_1}^{u_1}(\psi(u_1)-\psi(\delta))^{\lambda-1}\psi'(\delta)\mathcal{F}^{\varpi}_{\varrho,\lambda}[\omega(\psi(u_1)-\psi(\delta))^{\varrho}]\Phi(\delta)d\delta,$$

for $u_1>v_1$ and

$$I^{\varpi,\psi}_{\varrho,\lambda,v_2^-;\omega}\Phi(u_1)=\int_{u_1}^{v_2}(\psi(\delta)-\psi(u_1))^{\lambda-1}\psi'(\delta)\mathcal{F}^{\varpi,}_{\varrho,\lambda}[\omega(\psi(\delta)-\psi(u_1))^{\varrho}]\Phi(\delta)d\delta,$$

for $u_1<v_2$, where $\lambda,\varrho>0$ and $\omega\in\mathbb{R}$.

Remark 1.
1. *If we take $i=0$ and $\varpi(0)=1$ in Definition 5, then we have ψ_k-Riemann-Liouville fractional integrals.*
2. *If we set $\psi(u_1)=u_1, i=0$ and $\varpi(0)=1$ in Definition 5, then we get k-Riemann-Liouville fractional integrals.*
3. *If we choose $\psi(u_1)=u_1, i=0, k=1$ and $\varpi(0)=1$ in Definition 5, then we obtain Riemann-Liouville fractional integrals.*

For further information, see [6].

The theory of convexity is one of the most significant sides of mathematical analysis in which we study the properties of convex sets and convex functions. These classical concepts have played a vital role in different branches of pure and applied sciences. Applications for integral inequalities involving convexity are numerous in a number of areas of mathematics where symmetry is crucial. The classical concept of convexity and symmetry are both connected with each other, so you may use one and apply it to the other.

A set $\mathcal{C}\subset\mathbb{R}$ is said to be convex, if

$$(1-\delta)v_1+\delta v_2\in\mathcal{C},\quad \forall v_1,v_2\in\mathcal{C},\delta\in[0,1].$$

Similarly, a function $\Phi : \mathcal{C} \to \mathbb{R}$ is said to be convex, if

$$\Phi((1-\delta)v_1 + \delta v_2) \leq (1-\delta)\Phi(v_1) + \delta\Phi(v_2), \quad \forall v_1, v_2 \in \mathcal{C}, \delta \in [0,1].$$

Another aspect of convexity theory that has made it more charming is its relation to the theory of inequalities. A huge number of inequalities known to us in the literature can be obtained by using the convexity property of the functions. One of the most studied results in this regard is Hermite-Hadamard's inequality. For some interesting details, see [8–13]. This result provides us with a necessary and sufficient condition for a function to be convex. It reads as:

Let $\Phi : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\Phi\left(\frac{v_1+v_2}{2}\right) \leq \frac{1}{v_2-v_1}\int_{v_1}^{v_2} \Phi(u_1)\mathrm{d}u_1 \leq \frac{\Phi(v_1)+\Phi(v_2)}{2}.$$

Jensen's inequality [14] is another significant pertaining to convexity property of the functions. Let Φ be a convex function on $[v_1, v_2]$, then for all $u_{1i} \in [v_1, v_2]$ and $\mu_i \in [0,1]$, $i = 1, 2, \ldots, n$, with $\sum_{i=1}^{n} \mu_i = 1$, we have

$$\Phi\left(\sum_{i=1}^{n} \mu_i u_{1i}\right) \leq \sum_{i=1}^{n} \mu_i \Phi(u_{1i}).$$

The following inequality is known in the literature as the Jensen-Mercer's inequality [15]:

$$\Phi\left(v_1 + v_2 - \sum_{i=1}^{n} \mu_i u_{1i}\right) \leq \Phi(v_1) + \Phi(v_2) - \sum_{i=1}^{n} \mu_i \Phi(u_{1i}),$$

whenever $\Phi : [v_1, v_2] \longrightarrow \mathbb{R}$ is a convex function. For more details, see [15].

Pavić [16] obtained a generalized version of the Jensen-Mercer's inequality in the following way:

Assume that $\Phi : [v_1, v_2] \to \mathbb{R}$ is a convex function, where $u_{1i} \in [v_1, v_2]$ are n-points. Let $\varpi_1, \varpi_2, \mu_i \in [0,1]$, $\varpi_3 \in [-1,1]$ be coefficients such that $\varpi_1 + \varpi_2 + \varpi_3 = \sum_{i=1}^{n} \mu_i = 1$, then

$$\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 \sum_{i=1}^{n} \mu_i u_{1i}\right) \leq \varpi_1 \Phi(v_1) + \varpi_2 \Phi(v_2) + \varpi_3 \sum_{i=1}^{n} \mu_i \Phi(u_{1i}). \tag{1}$$

Remark 2. *Note that*

1. *If we take $\varpi_1 = 1 = \varpi_2$ and $\varpi_3 = -1$ in (1), then we have Jensen-Mercer inequality.*
2. *If we set $\varpi_1 = 0 = \varpi_2$ and $\varpi_3 = 1$ in (1), then we get Jensen inequality.*
3. *If we choose $\varpi_3 = 0$ in (1), then we obtain convex inequality.*

For more details about Hermite-Hadamard-Mercer's and Jensen-Mercer's inequalities, see [17–19].

The main objective of this paper is to derive some new generalized fractional analogues of Mercer type inequalities essentially using the convexity property of the functions and with Raina's function. We discuss several new special cases which show that our results are quite unifying. In order to illustrate the significance of our results, we also present some interesting applications of our results to special means, error bounds and q-digamma functions. Finally, some conclusions and future research in this fascinating field are given. We hope that the ideas of this paper will inspire interested readers working in this field.

2. Main Results

In this section, we discuss our main results.

Auxiliary Results

We now derive two new auxiliary results. These results will play an important role in the development of our results.

Lemma 1. *Let $\Phi : [v_1, v_2] \to \mathbb{R}$ be a differentiable function on (v_1, v_2) with $v_1 < v_2$. If $\Phi' \in L[v_1, v_2]$, then*

$$\begin{aligned}
&\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{2}\\
&-\frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\Big[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\\
&+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\Big]\\
&=\frac{\varpi_3(u_2-u_1)}{2}\\
&\times\Big[\int_0^1(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\Phi'(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\\
&-\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\Phi'(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\Big],
\end{aligned}$$

for all $u_1, u_2 \in [v_1, v_2]$, $u_1 < u_2$ and $\lambda, \varrho, \omega > 0$.

Proof. Note that it is sufficient to check that $I := \frac{\varpi_3(u_2-u_1)}{2}(I_2 - I_1)$, where

$$\begin{aligned}
I_1&=\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\Phi'(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\\
&=\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))}{\varpi_3(u_1-u_2)}\Big|_0^1\\
&\quad-\int_0^1\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))}{\varpi_3(u_1-u_2)}\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\sum_{i=0}^{\infty}\frac{\omega^i(u_2-u_1)^{\varrho i}(\varpi_3)^{\varrho i}\delta^{\frac{\lambda}{k}+\varrho i}}{k\Gamma_k(\varrho ik+\lambda+k)}\right)d\delta\\
&=-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}{\varpi_3(u_2-u_1)}\\
&\quad-\int_0^1\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))}{\varpi_3(u_1-u_2)}\left(\frac{\delta^{\frac{\lambda}{k}-1}}{k}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\right)d\delta\\
&=-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}{\varpi_3(u_2-u_1)}+\frac{1}{\varpi_3 k(u_2-u_1)}\\
&\quad\times\int_0^1\delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2)d\delta\\
&=-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}{\varpi_3(u_2-u_1)}\\
&\quad+\frac{1}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}+1}}I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^+,\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)),
\end{aligned}$$

and

$$\begin{aligned}
I_2&=\int_0^1(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\Phi'(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\\
&=(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta)^{\varrho}]\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))}{\varpi_3(u_1-u_2)}\Big|_0^1\\
&\quad-\int_0^1\frac{(1-\delta)^{\frac{\lambda}{k}-1}}{\varpi_3 k(u_2-u_1)}\mathcal{F}^{\varpi}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\\
&=\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{\varpi_3(u_2-u_1)}\\
&\quad-\int_0^1\frac{(1-\delta)^{\frac{\lambda}{k}-1}}{\varpi_3 k(u_2-u_1)}\mathcal{F}^{\varpi}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))d\delta\\
&=\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{\varpi_3(u_2-u_1)}-\frac{1}{\varpi_3 k(u_2-u_1)}\\
&\quad\times\int_0^1(1-\delta)^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2)d\delta
\end{aligned}$$

$$= \frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{\varpi_3(u_2-u_1)}$$
$$- \frac{1}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}+1}} I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^-,\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)).$$

Combining I_1 and I_2 with I, we get the required result. □

We now derive our second auxiliary result.

Lemma 2. *Let $\Phi : [v_1, v_2] \to \mathbb{R}$ be a differentiable function on (v_1, v_2) with $v_1 < v_2$. If $\Phi' \in L[v_1, v_2]$, then*

$$\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\Bigg[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))$$
$$+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Bigg]$$
$$-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{(n+1)}$$
$$\times\left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]$$
$$=\frac{\varpi_3(u_2-u_1)}{(n+1)^2}\{\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$
$$\times\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta$$
$$-\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$
$$\times\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)d\delta\},$$

for all $u_1, u_2 \in [v_1, v_2]$, $u_1 < u_2$, $\lambda, \varrho, \omega > 0$ and $\delta \in [0,1]$.

Proof. Suppose that

$$I=\{\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$
$$\times\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta$$
$$-\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$
$$\times\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)d\delta\}=I_1-I_2.$$

Note that,

$$I_1=\int_0^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$
$$\times\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta$$
$$=\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3\delta}{n+1})^{\varrho}\right]\frac{\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2)\right)}{\varpi_3(\frac{u_1}{n+1}-\frac{u_2}{n+1})}\Big|_0^1$$
$$-\int_0^1\frac{\delta^{\frac{\lambda}{k}-1}}{k}\mathcal{F}^{\varpi}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3\delta}{n+1})^{\varrho}\right]\frac{\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2)\right)}{\varpi_3(\frac{u_1}{n+1}-\frac{u_2}{n+1})}$$
$$=-\frac{(n+1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}\right]}{\varpi_3(u_2-u_1)}\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)$$
$$+\frac{n+1}{\varpi_3(u_2-u_1)}\int_0^1\frac{\delta^{\frac{\lambda}{k}-1}}{k}\mathcal{F}^{\varpi}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$$

$$\times \Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{\delta}{n+1}u_1 + \frac{n+1-\delta}{n+1}u_2\right)\right).$$

Substituting $\Psi(u) = \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\Big((\frac{\delta}{n+1}u_1 + \frac{n+1-\delta}{n+1}u_2)\Big)$ and after some computations, we get

$$\begin{aligned}
I_1 = &-\frac{(n+1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}\right]}{\varpi_3(u_2-u_1)}\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)\\
&+\frac{(n+1)^{\frac{\lambda}{k}+1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}+1}}\\
&\times I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)),
\end{aligned}$$

and

$$\begin{aligned}
I_2 = &\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\\
&\times\Phi'\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{n+1-\delta}{n+1}u_1 + \frac{\delta}{n+1}u_2\right)\right)d\delta\\
= &\,\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\frac{\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{n+1-\delta}{n+1}u_1 + \frac{\delta}{n+1}u_2\right)\right)}{\varpi_3(\frac{u_2}{n+1}-\frac{u_1}{n+1})}\Big|_0^1\\
&-\int_0^1 \frac{\delta^{\frac{\lambda}{k}-1}}{k}\mathcal{F}^{\varpi}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\frac{\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{n+1-\delta}{n+1}u_1 + \frac{\delta}{n+1}u_2\right)\right)}{\varpi_3(\frac{u_2}{n+1}-\frac{u_1}{n+1})}\\
= &\,\frac{(n+1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}\right]}{\varpi_3(u_2-u_1)}\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\\
&-\frac{n+1}{\varpi_3(u_2-u_1)}\int_0^1 \frac{\delta^{\frac{\lambda}{k}-1}}{k}\mathcal{F}^{\varpi}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\\
&\times\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{n+1-\delta}{n+1}u_1 + \frac{\delta}{n+1}u_2\right)\right).
\end{aligned}$$

By substituting $\Psi(v) = \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\Big((\frac{n+1-\delta}{n+1}u_1 + \frac{\delta}{n+1}u_2)\Big)$ and after some computations, we get

$$\begin{aligned}
I_2 = &\frac{(n+1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}\right]}{\varpi_3(u_2-u_1)}\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\\
&-\frac{(n+1)^{\frac{\lambda}{k}+1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}+1}}\\
&\times I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)).
\end{aligned}$$

By using I_1 and I_2, it follows that

$$\begin{aligned}
I_1 - I_2 = &\frac{(n+1)^{\frac{\lambda}{k}+1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}+1}}\\
&\times\Bigg[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\\
&+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Bigg]\\
&-\frac{(n+1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}\right]}{\varpi_3(u_2-u_1)}\\
&\times\left[\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right) + \Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right].
\end{aligned}$$

By multiplying $\frac{\varpi_3(u_2-u_1)}{(n+1)^2}$ both sides of the above equality, we get the required result. □

Here, we drive some new generalized fractional Mercer type inequalities.

Theorem 1. *Suppose that $\Phi : [v_1, v_2] \to \mathbb{R}$ is a convex function, then*

$$\begin{aligned}
&\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{u_1+u_2}{2}\right)\right)\\
&\leq \frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]}\\
&\quad\times\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right.\\
&+\quad\left.I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right]\\
&\leq \frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{2}\\
&\leq \varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)+\varpi_3\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right],
\end{aligned}$$

for all $u_1, u_2 \in [v_1, v_2]$, $u_1 < u_2$ and $\lambda, \varrho, \omega > 0$.

Proof. By using the convexity property of Φ, we have

$$\begin{aligned}
&\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{u_{11}+u_{21}}{2}\right)\right)\\
&=\Phi\left(\frac{\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{11}+\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{21}}{2}\right)\\
&\leq \frac{1}{2}[\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{11})+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{21})],
\end{aligned}$$

for all $u_{11}, u_{21} \in [v_1, v_2]$. By changing the variables, we have

$$\begin{aligned}
&\Phi\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3\left(\frac{u_1+u_2}{2}\right)\right)\\
&\leq \frac{1}{2}[\Phi(\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+(1-\delta)(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\\
&\quad+\Phi((1-\delta)(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))].
\end{aligned}$$

By multiplying both sides of above inequality by $\delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]$ and then integrating the resulting inequality with respect to δ on $[0,1]$, we obtain

$$\begin{aligned}
&k\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)\\
&\leq \frac{1}{2}[\int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\\
&\quad\times\Phi(\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+(1-\delta)(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))d\delta\\
&\quad+\int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\\
&\quad\times\Phi((1-\delta)(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))d\delta]\\
&=\frac{1}{2[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\\
&\quad\times\left[\int_{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}^{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}\left[\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))-\Psi(u)\right]^{\frac{\lambda}{k}-1}\Psi'(u)\right.\\
&\times\quad\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega\left(\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))-\Psi(u)\right)^{\varrho}\right]\Phi\circ\Psi(u)du
\end{aligned}$$

$$+\int_{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}^{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}\Big[\Psi(u)-\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Big]^{\frac{\lambda}{k}-1}\Psi'(u)$$
$$\times\mathcal{F}_{\varrho,\lambda}^{\varpi,k}\Big[\omega\Big(\Psi(u)-\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Big)^{\varrho}\Big]\Phi\circ\Psi(u)du\Big]$$
$$=\frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}$$
$$\times\Big[I_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^+;\omega}^{\varpi,k}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))$$
$$+\quad I_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^-;\omega}^{\varpi,k}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Big].$$

On the other hand, again using the convexity property of Φ and using the Jensen-Mercer inequality, we have

$$\begin{aligned}&\Phi(\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+(1-\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)))\\&\quad+\Phi((1-\delta)(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\delta(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\\&\le\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)\\&\le 2[\varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)]+\varpi_3[\Phi(u_1)+\Phi(u_2)].\end{aligned}$$

Multiplying both sides of above inequalities by $\delta^{\frac{\lambda}{k}-1}\mathcal{F}_{\varrho,\lambda}^{\varpi,k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]$ and then integrating the resulting inequality with respect to δ on $[0,1]$. This completes the proof. □

Theorem 2. *Let $\Phi:[v_1,v_2]\to\mathbb{R}$ be a convex function, then*

$$\begin{aligned}&\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)\\&\le\frac{(n+1)^{\frac{\lambda}{k}}}{2k\mathcal{F}_{\varrho,\lambda+k}^{\varpi,k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}](\varpi_3(u_2-u_1))^{\frac{\lambda}{k}}}\\&\quad\times\Big[I_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{u_1+nu_2}{n+1}))]^+;\omega}^{\varpi,k}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\\&\quad+I_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{nu_1+u_2}{n+1}))]^-;\omega}^{\varpi,k}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\Big]\\&\le\varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)+\varpi_3\left(\frac{\Phi(u_1)+\Phi(u_2)}{2}\right),\end{aligned}$$

for all $u_1,u_2\in[v_1,v_2]$, $u_1<u_2$ and $\lambda,\varrho,\omega>0$.

Proof. By using the convexity property of Φ, we conclude that

$$\begin{aligned}&\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_{11}+u_{21}}{2}\right)\right)\\&\le\frac{1}{2}[\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{11})+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_{21})]\end{aligned}$$

By setting $u_{11}=\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2$ and $u_{21}=\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2$ for $u_1,u_2\in[v_1,v_2]$ and $\delta\in[0,1]$ in the above inequality, we find

$$\begin{aligned}&2\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)\\&\le\left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)\right.\\&\quad\left.+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)\right].\end{aligned}$$

By multiplying both sides of above inequality by $\delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3\delta}{n+1})^{\varrho}\right]$ and then integrating the resulting inequality with respect to δ on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 2\delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)d\delta \\
&\le \int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta \\
&\quad+\int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)d\delta.
\end{aligned}$$

This implies

$$\begin{aligned}
&2k\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right) \\
&\le \int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta \\
&\quad+\int_0^1 \delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right] \\
&\quad\times\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)d\delta=\frac{(n+1)^{\frac{\lambda}{k}}}{(\varpi_3(u_2-u_1))^{\frac{\lambda}{k}}} \\
&\quad\times\int_{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{u_1+nu_2}{n+1}))}^{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}\left[\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))-\Psi(u)\right]^{\frac{\lambda}{k}-1}\Psi'(u) \\
&\quad\times\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega\left(\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))-\Psi(u)\right)^{\varrho}\right]\Phi\circ\Psi(u)du \\
&\quad+\frac{(n+1)^{\frac{\lambda}{k}}}{(\varpi_3(u_2-u_1))^{\frac{\lambda}{k}}} \\
&\quad\times\int_{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)}^{\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{nu_1+u_2}{n+1}))}\left[\Psi(u)-\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right]^{\frac{\lambda}{k}-1}\Psi'(u) \\
&\quad\times\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega\left(\Psi(u)-\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right)^{\varrho}\right]\Phi\circ\Psi(u)du \\
&=\frac{(n+1)^{\frac{\lambda}{k}}}{(\varpi_3(u_2-u_1))^{\frac{\lambda}{k}}} \\
&\quad\times\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{u_1+nu_2}{n+1}))]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right. \\
&\quad\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{nu_1+u_2}{n+1}))]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right].
\end{aligned}$$

This implies

$$
\begin{aligned}
&\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right) \\
&\le \frac{(n+1)^{\frac{\lambda}{k}}}{2k\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3}{n+1}\right)^{\varrho}\right](\varpi_3(u_2-u_1))^{\frac{\lambda}{k}}} \\
&\quad\times\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{u_1+nu_2}{n+1}))]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right. \\
&\quad\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{nu_1+u_2}{n+1}))]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right].
\end{aligned}
$$

Again using the convexity property of the functions and the Jensen-Mercer inequality, we obtain

$$
\begin{aligned}
&\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta}{n+1}u_1+\frac{n+1-\delta}{n+1}u_2\right)\right) \\
&+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right) \\
&\le 2[\varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)]+\varpi_3(\Phi(u_1)+\Phi(u_2)).
\end{aligned}
$$

Multiplying both sides of the above inequality by $\delta^{\frac{\lambda}{k}-1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]$ and then integrating the resulting inequality with respect to δ on $[0,1]$ completes the proof. □

Using Lemmas 1 and 2, we can derive the following next results.

Theorem 3. *Under the assumptions of Lemma 1, if $|\Phi'|$ is convex on $[v_1,v_2]$, then*

$$
\begin{aligned}
&\left|\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{2}\right. \\
&-\frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}} \\
&\times\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right. \\
&\left.\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right]\right| \\
&\le|\varpi_3|(u_2-u_1)\mathcal{F}^{\varpi_0,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}] \\
&\times\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right],
\end{aligned}
$$

where $\varpi_0(i):=\varpi(i)\left(1-\frac{1}{2^{\frac{\lambda}{k}+\varrho i}}\right)$ and for all $u_1,u_2\in[v_1,v_2]$ with $u_1<u_2$ and $\lambda,\varrho,\omega>0$.

Proof. By using Lemma 1 and the Jensen-Mercer inequality, we have

$$
\begin{aligned}
&\left|\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\frac{\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)+\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)}{2}\right. \\
&-\frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}} \\
&\times\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)]^{-};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right.
\end{aligned}
$$

$$
\begin{aligned}
&+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)]^{+};\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\Big]\Bigg| \\
\leq\ & \frac{|\varpi_3|(u_2-u_1)}{2} \\
&\times\int_0^1\left|(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]-\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\right| \\
&\times\left|\Phi'(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\delta u_1+(1-\delta)u_2))\right|d\delta \\
\leq\ & \frac{|\varpi_3|(u_2-u_1)}{2} \\
&\times\int_0^1\left|(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]-\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]\right| \\
&\times[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\delta|\Phi'(u_1)|+\varpi_3(1-\delta)|\Phi'(u_2)|]d\delta \\
=\ & \frac{|\varpi_3|(u_2-u_1)}{2} \\
&\times\int_0^{\frac{1}{2}}(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]-\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}] \\
&\times[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\delta|\Phi'(u_1)|+\varpi_3(1-\delta)|\Phi'(u_2)|]d\delta \\
&+\frac{\varpi_3(u_2-u_1)}{2} \\
&\times\int_{\frac{1}{2}}^1\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]-(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}] \\
&\times[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\delta|\Phi'(u_1)|+\varpi_3(1-\delta)|\Phi'(u_2)|]d\delta \\
=\ & \frac{\varpi_3(u_2-u_1)}{2}(L_1+L_2).
\end{aligned}
$$

By calculating the constants L_1 and L_2, we obtain

$$
\begin{aligned}
L_1 =\ & (\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|) \\
&\times\left(\int_0^{\frac{1}{2}}[(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]-\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]]d\delta\right) \\
&+\varpi_3|\Phi'(u_1)|\left(\int_0^{\frac{1}{2}}\delta(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]d\delta\right. \\
&\left.-\int_0^{\frac{1}{2}}\delta^{\frac{\lambda}{k}+1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]d\delta\right) \\
&+\varpi_3|\Phi'(u_2)|\left(\int_0^{\frac{1}{2}}(1-\delta)^{\frac{\lambda}{k}+1}\mathcal{F}^{\varpi,k}_{\varrho,\varpi_1+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]d\delta\right. \\
&\left.-\int_0^{\frac{1}{2}}(1-\delta)\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]d\delta\right) \\
=\ & (\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|)\left(\mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_2,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right) \\
&+\varpi_3|\Phi'(u_1)|\left(\mathcal{F}^{\varpi_3,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_4,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right) \\
&+\varpi_3|\Phi'(u_2)|\left(\mathcal{F}^{\varpi_5,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_6,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
L_1 =\ & (\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|)\left(\frac{1}{\frac{\lambda}{k}+\varrho i+1}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i}(\frac{\lambda}{k}+\varrho i+1)}\right) \\
&+\varpi_3|\Phi'(u_1)|\left(\frac{1}{(\frac{\lambda}{k}+\varrho i+1)(\frac{\lambda}{k}+\varrho i+2)}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right)
\end{aligned}
$$

$$+\varpi_3|\Phi'(u_2)|\left(\frac{1}{\frac{\lambda}{k}+\varrho i+2}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right),$$

and

$$\begin{aligned}
L_2 &= (\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|)\\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left[\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]-(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]\right]d\delta\right)\\
&\quad+\varpi_3|\Phi'(u_1)|\left(\int_{\frac{1}{2}}^{1}\delta^{\frac{\lambda}{k}+1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]d\delta\right.\\
&\quad\left.-\int_{\frac{1}{2}}^{1}\delta(1-\delta)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]d\delta\right)\\
&+\varpi_3|\Phi'(u_2)|\left(\int_{\frac{1}{2}}^{1}(1-\delta)\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3\delta)^{\varrho}]d\delta\right.\\
&\quad\left.-\int_{\frac{1}{2}}^{1}(1-\delta)^{\frac{\lambda}{k}+1}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3(1-\delta))^{\varrho}]d\delta\right)\\
&=(\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|)\left(\mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_2,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right)\\
&\quad+\varpi_3|\Phi'(u_1)|\left(\mathcal{F}^{\varpi_5,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_6,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right)\\
&+\varpi_3|\Phi'(u_2)|\left(\mathcal{F}^{\varpi_3,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]-\mathcal{F}^{\varpi_4,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}\varpi_3{}^{\varrho}]\right)\\
&=(\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|)\left(\frac{1}{\frac{\lambda}{k}+\varrho i+1}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i}(\frac{\lambda}{k}+\varrho i+1)}\right)\\
&\quad+\varpi_3|\Phi'(u_1)|\left(\frac{1}{\frac{\lambda}{k}+\varrho i+2}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right)\\
&+\varpi_3|\Phi'(u_2)|\left(\frac{1}{(\frac{\lambda}{k}+\varrho i+1)(\frac{\lambda}{k}+\varrho i+2)}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\varpi_1(k)&:=\varpi(k)\left(\frac{1}{\frac{\lambda}{k}+\varrho i+1}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right),\\
\varpi_2(k)&:=\varpi(k)\left(\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}\right),\\
\varpi_3(k)&:=\varpi(k)\left(\frac{1}{\frac{\lambda}{k}+\varrho i+1}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}+\frac{1}{2^{\frac{\lambda}{k}+\varrho i+2}(\frac{\lambda}{k}+\varrho i+2)}-\frac{1}{\frac{\lambda}{k}+\varrho i+2}\right),\\
\varpi_4(k)&:=\varpi(k)\left(\frac{1}{2^{\frac{\lambda}{k}+\varrho i+2}(\frac{\lambda}{k}+\varrho i+2)}\right),\\
\varpi_5(k)&:=\varpi(k)\left(\frac{1}{\frac{\lambda}{k}+\varrho i+2}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+2}(\frac{\lambda}{k}+\varrho i+2)}\right),\\
\varpi_6(k)&:=\varpi(k)\left(\frac{1}{2^{\frac{\lambda}{k}+\varrho i+1}(\frac{\lambda}{k}+\varrho i+1)}-\frac{1}{2^{\frac{\lambda}{k}+\varrho i+2}(\frac{\lambda}{k}+\varrho i+2)}\right).
\end{aligned}$$

By substituting L_1 and L_2, we obtain the desired inequality. □

Theorem 4. *Under the all assumptions of Lemma 2, if $|\Phi'|$ is convex on $[v_1, v_2]$, then*

$$\left|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\right.$$

$$
\begin{aligned}
&\times \left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^+;\omega} \Phi\circ\Psi\left(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)\right)\right.\\
&\left. + I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^-;\omega} \Phi\circ\Psi\left(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)\right)\right]\\
&- \frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^\varrho(\frac{\varpi_3}{n+1})^\varrho]}{n+1}\\
&\times \left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)\right.\\
&\left.\left. + \Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\right|\\
&\leq \frac{2k|\varpi 3|(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^\varrho(\frac{\varpi_3}{n+1})^\varrho]}{(n+1)^2}\\
&\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right],
\end{aligned}
$$

for all $u_1, u_2 \in [v_1, v_2],\ u_1 < u_2,\ \lambda, \varrho, \omega > 0,$ *and* $n \in \mathbb{N}$.

Proof. By using the Lemma 2 and Jensen-Mercer inequality, we have

$$
\begin{aligned}
&\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\\
&\times \left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^+;\omega} \Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right.\\
&\left.+ I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^-;\omega} \Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right]\\
&- \frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^\varrho(\frac{\varpi_3}{n+1})^\varrho]}{n+1}\\
&\times \left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]|\\
&\leq \frac{|\varpi_3(|u_2-u_1)}{(n+1)^2}\{\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^\varrho\left(\frac{\varpi_3\delta}{n+1}\right)^\varrho\right]\\
&\times \left|\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta u_1}{n+1}+\frac{n+1-\delta}{n+1}u_2\right)\right)d\delta\right|\\
&+\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^\varrho\left(\frac{\varpi_3\delta}{n+1}\right)^\varrho\right]\\
&\times \left|\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta}{n+1}u_2\right)\right)\right|d\delta\}\\
&\leq \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\left\{\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^\varrho\left(\frac{\varpi_3\delta}{n+1}\right)^\varrho\right]\right.\\
&\times [\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{\delta}{n+1}\right)|\Phi'(u_1)|+\varpi_3\left(\frac{n+1-\delta}{n+1}\right)|\Phi'(u_2)|]d\delta\\
&+\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^\varrho\left(\frac{\varpi_3\delta}{n+1}\right)^\varrho\right]\\
&\left.\times [\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{n+1-\delta}{n+1}\right)|\Phi'(u_1)|+\varpi_3\left(\frac{\delta}{n+1}\right)|\Phi'(u_2)|]d\delta\right\}\\
&\leq \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\left\{\int_0^1 \delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^\varrho\left(\frac{\varpi_3\delta}{n+1}\right)^\varrho\right]\right.\\
&\times [2\varpi_1|\Phi'(v_1)|+2\varpi_2|\Phi'(v_2)|+\varpi_3(|\Phi'(u_1)|+|\Phi'(u_2)|)]d\delta\}
\end{aligned}
$$

$$\leq \frac{|\varpi_3|(u_2-u_1)}{2(n+1)^2}\left\{\int_0^1 \delta^{\frac{\lambda}{k}} \mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\right.$$
$$\left.\times\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right]d\delta\right\}$$
$$\leq \frac{2k|\varpi_3|(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{(n+1)^2}$$
$$\times\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right].$$

This completes the proof. □

Theorem 5. *Under the assumptions of Lemma 2, if $|\Phi'|^q$ is convex on $[v_1, v_2]$ with $q > 1$, then*

$$\left|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2))\right.\right.$$
$$\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1))\right]$$
$$-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{n+1}$$
$$\left.\times\left[\Phi\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\right|$$
$$\leq \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\quad \mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3}{n+1}\right)^{\varrho}\right]$$
$$\times\left[\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right],$$

where $\varpi_1(i) := \varpi(i)\left(\frac{k}{\varrho pki+p\lambda+k}\right)^{\frac{1}{p}}$ and for all $u_1, u_2 \in [v_1, v_2]$ with $u_1 < u_2$, $\lambda, \varrho, \omega > 0$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By using Lemma 2 and the Hölder's inequality, we get

$$\left|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2))\right.\right.$$
$$\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1))\right]$$
$$-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{n+1}$$
$$\left.\times\left[\Phi\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\right|$$
$$\leq \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\left(\int_0^1\left(\delta^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3\delta}{n+1}\right)^{\varrho}\right]\right)^p d\delta\right)^{\frac{1}{p}}$$
$$\times\left[\left(\int_0^1\left|\Phi'\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta u_2}{n+1}\right)\right)\right|^q d\delta\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\left|\Phi'\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{\delta u_1}{n+1}+\frac{n+1-\delta}{n+1}u_2\right)\right)\right|^q d\delta\right)^{\frac{1}{q}}\right]$$

$$
\begin{aligned}
&= \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\sum_{i=1}^{\infty}\frac{\varpi_k(i)[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]^i}{k\Gamma_k(\varrho ki+\lambda+k)}\left(\int_0^1 \delta^{(\varrho i+\frac{\lambda}{k})p}\mathrm{d}\delta\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\int_0^1\left|\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{n+1-\delta}{n+1}u_1+\frac{\delta u_2}{n+1}\right)\right)\right|^q d\delta\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\left|\Phi'\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{\delta u_1}{n+1}+\frac{n+1-\delta}{n+1}u_2\right)\right)\right|^q d\delta\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

By using the Jensen-Mercer inequality and taking into account the convexity of $|\Phi'|^q$, we find

$$
\begin{aligned}
&\left|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{u_1+nu_2}{n+1}))]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2))\right.\right. \\
&\left.+I^{\varpi,k}_{[\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3(\frac{nu_1+u_2}{n+1}))]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1))\right] \\
&-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{n+1} \\
&\left.\times\left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\right| \\
&\le \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\sum_{i=1}^{\infty}\frac{\varpi_k(i)[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]^i}{k\Gamma_k(\varrho ki+\lambda+k)}\left(\frac{k}{\varrho pki+p\lambda+k}\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\int_0^1\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{n+1-\delta}{n+1}|\Phi'(u_1)|^q+\frac{\delta}{n+1}|\Phi'(u_2)|^q\right)d\delta\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\int_0^1\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{\delta}{n+1}|\Phi'(u_1)|^q+\frac{n+1-\delta}{n+1}|\Phi'(u_2)|^q\right)d\delta\right)^{\frac{1}{q}}\right] \\
&= \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\sum_{i=1}^{\infty}\frac{\varpi_k(i)[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]^i}{k\Gamma_k(\varrho ki+\lambda+k)}\left(\frac{k}{\varrho pki+p\lambda+k}\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right] \\
&= \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2}\quad\mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3}{n+1}\right)^{\varrho}\right] \\
&\quad\times\left[\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof. □

3. Special Cases

In this section, we discuss some special cases of results obtained in the previous section.

Corollary 1. *Under the assumptions of Theorem 1, if we choose $\Psi(u_1)=u_1$, then*

$$
\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)
$$

$$\leq \frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]}$$
$$\times\Big[I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1)^+;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2))$$
$$+I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2)^-;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1))\Big]$$
$$\leq \frac{\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1)+\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2)}{2}$$
$$\leq \varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)+\varpi_3\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right].$$

Corollary 2. *Under the assumptions of Theorem 1, if we take $\varpi_1=1=\varpi_2$ and $\varpi_3=-1$, then*

$$\Phi\left(v_1+v_2-\left(\frac{u_1+u_2}{2}\right)\right)$$
$$\leq \frac{1}{2k(u_2-u_1)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(-1)^{\varrho}]}\Big[I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-u_1)]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_2))$$
$$+\quad I^{\varpi,k}_{[\Psi^{-1}(\ v_1+v_2-u_2)]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_1))\Big]$$
$$\leq \frac{\Phi(v_1+v_2-u_1)+\Phi(v_1+v_2-u_2)}{2}$$
$$\leq \Phi(v_1)+\Phi(v_2)-\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right].$$

Corollary 3. *Under the assumptions of Theorem 2, if we choose $\Psi(u_1)=u_1$, then*

$$\Phi\left(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+u_2}{2}\right)\right)$$
$$\leq \frac{(n+1)^{\frac{\lambda}{k}}}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}$$
$$\times\left[I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))^+;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2))\right.$$
$$\left.+\quad I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))^-;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1))\right]$$
$$\leq \varpi_1\Phi(v_1)+\varpi_2\Phi(v_2)+\varpi_3\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right].$$

Corollary 4. *Under the assumptions of Theorem 2, if we take $\varpi_1=1=\varpi_2$ and $\varpi_3=-1$, then*

$$\Phi\left(v_1+v_2-\frac{u_1+u_2}{2}\right)\leq \frac{(n+1)^{\frac{\lambda}{k}}}{2k(u_2-u_1)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{-1}{n+1})^{\varrho}\right]}$$
$$\times\left[I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-\frac{u_1+nu_2}{n+1})]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_2))\right.$$
$$\left.+\ I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-\frac{nu_1+u_2}{n+1})]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_1))\right]$$
$$\leq \frac{\Phi(v_1+v_2-u_1)+\Phi(v_1+v_2-u_2)}{2}$$
$$\leq \Phi(v_1)+\Phi(v_2)-\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right].$$

Corollary 5. *Under the assumptions of Theorem 2, if we choose $\varpi_1 = 0 = \varpi_2$ and $\varpi_3 = 1$, then*

$$\begin{aligned}
\Phi\left(\frac{u_1+u_2}{2}\right) &\le \frac{(n+1)^{\frac{\lambda}{k}}}{2k(u_2-u_1)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{1}{n+1})^{\varrho}\right]} \\
&\times\left[I^{\varpi,k}_{[\Psi^{-1}(\frac{u_1+nu_2}{n+1})]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(u_2)) + I^{\varpi,k}_{[\Psi^{-1}(\frac{nu_1+u_2}{n+1})]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(u_1))\right] \\
&\le \frac{\Phi(u_1)+\Phi(u_2)}{2}.
\end{aligned}$$

Corollary 6. *If we choose $\varpi_1 = 0 = \varpi_2$ and $\varpi_3 = 1$ in Corollary 3, then we obtain the Hermite-Hadamard's inequality for generalized k-Raina's integral*

$$\begin{aligned}
\Phi\left(\frac{u_1+u_2}{2}\right) &\le \frac{(n+1)^{\frac{\lambda}{k}}}{2(u_2-u_1)^{\frac{\lambda}{k}}\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}(\frac{1}{n+1})^{\varrho}\right]} \\
&\times\left[I^{\varpi,k}_{(\frac{u_1+nu_2}{n+1})^+;\omega}\Phi(u_2) + I^{\varpi,k}_{(\frac{nu_1+u_2}{n+1})^-;\omega}\Phi(u_1)\right] \le \frac{\Phi(u_1)+\Phi(u_2)}{2}.
\end{aligned}$$

Corollary 7. *Under the all assumptions of Theorem 3, if we take $\Psi(u_1) = u_1$, then*

$$\begin{aligned}
&\Bigg|\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}]\frac{\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1)+\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2)}{2} \\
&\quad - \frac{1}{2k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\times[I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2)^-;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1) \\
&\quad + I^{\varpi,k}_{(\varpi_1v_1+\varpi_2v_2+\varpi_3u_1)^+;\omega}\Phi(\varpi_1v_1+\varpi_2v_2+\varpi_3u_2)]\Bigg| \\
&\le \varpi_3(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\varpi_3)^{\varrho}] \\
&\quad\times\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right].
\end{aligned}$$

Corollary 8. *Under the all assumptions of Theorem 3, if we choose $\varpi_1 = 0 = \varpi_2$ and $\varpi_3 = 1$, then*

$$\begin{aligned}
&\Bigg|\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}]\left[\frac{\Phi(u_1)+\Phi(u_2)}{2}\right] - \frac{1}{2k[(u_2-u_1)]^{\frac{\lambda}{k}}} \\
&\quad\times\left[I^{\varpi,k}_{[\Psi^{-1}(u_2)]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(u_1)) + I^{\varpi,k}_{[\Psi^{-1}(u_1)]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(u_2))\right]\Bigg| \\
&\le (u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}]\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right).
\end{aligned}$$

Corollary 9. *Under the all assumptions of Theorem 3, if we take $\varpi_1 = 1 = \varpi_2$ and $\varpi_3 = -1$, then*

$$\begin{aligned}
&\Bigg|\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}]\frac{\Phi(v_1+v_2-u_1)+\Phi(v_1+v_2-u_2)}{2} - \frac{1}{2k[(u_2-u_1)]^{\frac{\lambda}{k}}} \\
&\times\Big[I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-u_1)]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_2)) \\
&+ I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-u_2)]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_1))\Big]\Bigg|
\end{aligned}$$

$$\leq (u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(-1)^{\varrho}]\left[|\Phi'(v_1)|+|\Phi'(v_2)|-\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right].$$

Corollary 10. *Under the all assumptions of Theorem 4, if we choose $\Psi(u_1)=u_1$, then*

$$\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\Bigg[I^{\varpi,k}_{(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right))^+;\omega}\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)\\
&\quad+I^{\varpi,k}_{(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right))^-;\omega}\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)\Bigg]\\
&\quad-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{n+1}\\
&\quad\times\left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\Bigg|\\
&\leq\frac{2k|\varpi_3|(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{(n+1)^2}\\
&\quad\times\left[\varpi_1|\Phi'(v_1)|+\varpi_2|\Phi'(v_2)|+\varpi_3\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right].
\end{aligned}$$

Corollary 11. *Under the all assumptions of Theorem 4, if we take $\varpi_1=0=\varpi_2$ and $\varpi_3=1$, then*

$$\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k(u_2-u_1)^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(\frac{u_1+nu_2}{n+1})]^+;\omega}\Phi\circ\Psi\left(\Psi^{-1}(u_2)\right)+I^{\varpi,k}_{[\Psi^{-1}(\frac{nu_1+u_2}{n+1})]^-;\omega}\Phi\circ\Psi\left(\Psi^{-1}(u_1)\right)\right]\\
&\quad-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{1}{n+1})^{\varrho}]}{n+1}\left[\Phi\left(\frac{u_1+nu_2}{n+1}\right)+\Phi\left(\frac{nu_1+u_2}{n+1}\right)\right]\Bigg|\\
&\leq\frac{2k(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\frac{1}{n+1})^{\varrho}]}{(n+1)^2}\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right).
\end{aligned}$$

Corollary 12. *Under the all assumptions of Theorem 4, if we choose $\varpi_1=1=\varpi_2$ and $\varpi_3=-1$, then*

$$\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k(u_2-u_1)^{\frac{\lambda}{k}}}\Bigg[I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-(\frac{u_1+nu_2}{n+1}))]^-;\omega}\Phi\circ\Psi\left(\Psi^{-1}(v_1+v_2-u_2)\right)\\
&\quad+I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-(\frac{nu_1+u_2}{n+1}))]^+;\omega}\Phi\circ\Psi\left(\Psi^{-1}(v_1+v_2-u_1)\right)\Bigg]\\
&\quad-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{-1}{n+1})^{\varrho}]}{n+1}\\
&\quad\times\left[\Phi\left(v_1+v_2-\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(v_1+v_2-\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\Bigg|\\
&\leq\frac{2k(u_2-u_1)\mathcal{F}^{\varpi,k}_{\varrho,\lambda+2k}[\omega(u_2-u_1)^{\varrho}(\frac{-1}{n+1})^{\varrho}]}{(n+1)^2}\\
&\quad\times\left[|\Phi'(v_1)|+|\Phi'(v_2)|-\left(\frac{|\Phi'(u_1)|+|\Phi'(u_2)|}{2}\right)\right].
\end{aligned}$$

Corollary 13. *Under the all assumptions of Theorem 5, if we take $\Psi(u_1)=u_1$, then*

$$\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[\varpi_3(u_2-u_1)]^{\frac{\lambda}{k}}}\Bigg[I^{\varpi,k}_{[\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)]^+;\omega}\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_2)\\
&+I^{\varpi,k}_{[\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)]^-;\omega}\Phi(\varpi_1 v_1+\varpi_2 v_2+\varpi_3 u_1)\Bigg]-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{\varpi_3}{n+1})^{\varrho}]}{n+1}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(\varpi_1 v_1+\varpi_2 v_2+\varpi_3\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\Bigg| \\
\le\ & \frac{|\varpi_3|(u_2-u_1)}{(n+1)^2} \quad \mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{\varpi_3}{n+1}\right)^{\varrho}\right] \\
&\times \left[\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right. \\
&+\left.\left(\varpi_1|\Phi'(v_1)|^q+\varpi_2|\Phi'(v_2)|^q+\varpi_3\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 14. *Under the all assumptions of Theorem 5, if we choose $\varpi_1=0=\varpi_2$ and $\varpi_3=1$, then*

$$
\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k[(u_2-u_1)]^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(\frac{u_1+nu_2}{n+1})]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(u_2))+I^{\varpi,k}_{[\Psi^{-1}(\frac{nu_1+u_2}{n+1})]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(u_1))\right] \\
&-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{1}{n+1})^{\varrho}]}{n+1}\left[\Phi\left(\frac{u_1+nu_2}{n+1}\right)+\Phi\left(\frac{nu_1+u_2}{n+1}\right)\right]\Bigg| \\
\le\ & \frac{(u_2-u_1)}{(n+1)^2} \quad \mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{1}{n+1}\right)^{\varrho}\right] \\
&\times\left[\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)^{\frac{1}{q}}\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Corollary 15. *Under the all assumptions of Theorem 5, if we take $\varpi_1=1=\varpi_2$ and $\varpi_3=-1$, then*

$$
\begin{aligned}
&\Bigg|\frac{(n+1)^{\frac{\lambda}{k}-1}}{k(u_2-u_1)^{\frac{\lambda}{k}}}\left[I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-(\frac{u_1+nu_2}{n+1}))]^-;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_2))\right. \\
&+\left.I^{\varpi,k}_{[\Psi^{-1}(v_1+v_2-(\frac{nu_1+u_2}{n+1}))]^+;\omega}\Phi\circ\Psi(\Psi^{-1}(v_1+v_2-u_1))\right] \\
&-\frac{\mathcal{F}^{\varpi,k}_{\varrho,\lambda+k}[\omega(u_2-u_1)^{\varrho}(\frac{-1}{n+1})^{\varrho}]}{n+1} \\
&\times\left[\Phi\left(v_1+v_2-\left(\frac{u_1+nu_2}{n+1}\right)\right)+\Phi\left(v_1+v_2-\left(\frac{nu_1+u_2}{n+1}\right)\right)\right]\Bigg| \\
\le\ & \frac{(u_2-u_1)}{(n+1)^2} \quad \mathcal{F}^{\varpi_1,k}_{\varrho,\lambda+k}\left[\omega(u_2-u_1)^{\varrho}\left(\frac{-1}{n+1}\right)^{\varrho}\right] \\
&\times\left[\left(|\Phi'(v_1)|^q+|\Phi'(v_2)|^q-\left(\frac{(2n+1)|\Phi'(u_1)|^q+|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right. \\
&+\left.\left(|\Phi'(v_1)|^q+|\Phi'(v_2)|^q-\left(\frac{|\Phi'(u_1)|^q+(2n+1)|\Phi'(u_2)|^q}{2(n+1)}\right)\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

4. Applications

In this section, we discuss some applications of our main results.

Applications to Special Means

First of all, we recall some previously known concepts regarding special means. For $v_1\neq v_2$, we have

1. The arithmetic mean: $A(v_1, v_2) = \frac{v_1+v_2}{2}$,
2. The generalized logarithmic mean: $L_n(v_1, v_2) = \left[\frac{v_2^{n+1}-v_1^{n+1}}{(v_2-v_1)(n+1)}\right]^{\frac{1}{n}}$, where n belongs to $\mathbb{Z} \setminus \{-1, 0\}$.

Proposition 1. *Under all the assumptions of Theorem 1, we have*

$$\begin{aligned}&A^{\frac{r}{l}+2}(2(\varpi_1 v_1 + \varpi_2 v_2), \varpi_3(u_1 + u_2))\\&\leq \frac{l}{r+2l} L_{\frac{r}{l}+2}^{\frac{r}{l}+2}[\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1, \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2]\\&\leq \frac{2l}{r+2l} A\left(v_1^{\frac{r}{l}+2}, v_2^{\frac{r}{l}+2}\right) + \frac{\varpi_3 l}{r+2l} A\left(u_1^{\frac{r}{l}+2}, u_2^{\frac{r}{l}+2}\right).\end{aligned}$$

Proof. The proof is directly obtained from Theorem 1, taking $\lambda = k = 1$, $i = 0$, convex function $\Phi(x) = \frac{l}{r+2l} x^{\frac{r}{l}+2}$, and using the fact $\varpi(0) = 1$. □

Proposition 2. *Under all the assumptions of Theorem 3, we have*

$$\begin{aligned}&\Bigg|\frac{l}{r+2l} A\left((\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1)^{\frac{r}{l}+2}, (\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2)^{\frac{r}{l}+2}\right)\\&\quad - \frac{l}{r+2l} L_{\frac{r}{l}+2}^{\frac{r}{l}+2}[\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1, \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2]\Bigg|\\&\leq \frac{|\varpi_3|(u_2 - u_1)}{2}\left[2A\left(\varpi_1 v_1^{\frac{r}{l}+1}, \varpi_2 v_2^{\frac{r}{l}+1}\right) + \varpi_3 A\left(u_1^{\frac{r}{l}+1}, u_2^{\frac{r}{l}+1}\right)\right].\end{aligned}$$

Proof. The proof is directly obtained from Theorem 3, choosing $\lambda = k = 1$, $i = 0$, convex function $\Phi(x) = \frac{l}{r+2l} x^{\frac{r}{l}+2}$, and using the fact $\varpi(0) = 1$. □

Proposition 3. *Under the assumptions of Theorem 4, we have*

$$\begin{aligned}&\Bigg|A^{\frac{r}{l}+2}(2(\varpi_1 v_1 + \varpi_2 v_2) + \varpi_3(u_1 + u_2))\\&\quad - L_{\frac{r}{l}+2}^{\frac{r}{l}+2}[\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1, \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2]\Bigg|\\&\leq \frac{|\varpi_3|(u_2 - u_1)}{8}\left[2A\left(\varpi_1 v_1^{\frac{r}{l}+1}, \varpi_2 v_2^{\frac{r}{l}+1}\right) + \varpi_3 A\left(u_1^{\frac{r}{l}+1}, u_2^{\frac{r}{l}+1}\right)\right].\end{aligned}$$

Proof. The proof is directly obtained from Theorem 4, taking $\lambda = k = 1$, $i = 0$, convex function $\Phi(x) = \frac{l}{r+2l} x^{\frac{r}{l}+2}$, and using the fact $\varpi(0) = 1$. □

Proposition 4. *Under the assumptions of Theorem 5, we have*

$$\begin{aligned}&\Bigg|A^{\frac{r}{q}+2}(2(\varpi_1 v_1 + \varpi_2 v_2) + \varpi_3(u_1 + u_2))\\&\quad - \frac{q}{r+2q} L_{\frac{r}{q}+2}^{\frac{r}{q}+2}[\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1, \varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2]\Bigg|\\&\leq \frac{|\varpi_3|(u_2 - u_1)}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(2A\left(\varpi_1 v_1^{(\frac{r}{l}+1)q}, \varpi_2 v_2^{(\frac{r}{l}+1)q}\right) + \frac{\varpi_3}{2} A\left(3u_1^{(\frac{r}{l}+1)q}, u_2^{(\frac{r}{l}+1)q}\right)\right)^{\frac{1}{q}}\right.\\&\quad \left.+\left(2A\left(\varpi_1 v_1^{(\frac{r}{l}+1)q}, \varpi_2 v_2^{(\frac{r}{l}+1)q}\right) + \frac{\varpi_3}{2} A\left(u_1^{(\frac{r}{l}+1)q}, 3u_2^{(\frac{r}{l}+1)q}\right)\right)^{\frac{1}{q}}\right].\end{aligned}$$

Proof. The proof is directly obtained from Theorem 5, choosing $\lambda = k = 1$, $i = 0$, convex function $\Phi(x) = \frac{l}{r+2l} x^{\frac{r}{l}+2}$, and using the fact $\varpi(0) = 1$. $\square$

Here, we present an error bound in connection with Theorem 3. For this, we fix $\varpi_1, \varpi_2 \in [0,1]$ and $\varpi_3 \in (0,1]$ with $\varpi_1 + \varpi_2 + \varpi_3 = 1$. For $v_2 > v_1 > 0$, let $\hat{S} : v_1 = u_{10} < u_{11} < u_{12} < \ldots < u_{1i} < u_{1i+1} < \ldots < u_{1n} = v_2$ be a partition of $[v_1, v_2]$ and $u_{1i1}, u_{1i2} \in [u_{1i}, u_{1i+1}]$.

$$T(\hat{S}, \Phi) := \varpi_3 \sum_{i=0}^{n} \left[\frac{\Phi(\varpi_1 u_{1i} + \varpi_2 u_{1i+1} + \varpi_3 u_{1i1}) + \Phi(\varpi_1 u_{1i} + \varpi_2 u_{1i+1} + \varpi_3 u_{1i2})}{2} \right] h_i,$$
$$\int_{\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1}^{\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2} \Phi(u_1) \mathrm{d}u_1 := T(\hat{S}, \Phi) + \bar{R}(\hat{S}, \Phi),$$

where $T(\hat{S}, \Phi)$ is the trapezium quadrature formula and $\bar{R}(\hat{S}, \Phi)$ is the remainder term (error estimation).

Proposition 5. *Under all the assumptions of Theorem 3, then*

$$|\bar{R}(\hat{S}, \Phi)| \le \frac{{\varpi_3}^2}{2} \sum_{i=0}^{n} h_i^2 \left[\varpi_1 |\Phi'(u_{1i})| + \varpi_2 |\Phi'(u_{1i+1})| + \varpi_3 \frac{|\Phi'(u_{1i1})| + |\Phi'(u_{1i2})|}{2} \right],$$

where $h_i := u_{1i+1} - u_{1i}$.

Proof. Using Theorem 3 over subinterval $[u_{1i}, u_{1i+1}]$ of closed interval $[v_1, v_2]$, taking sum with respect to index i from 0 to $n-1$, and choosing $\lambda = k = 1$, we get our required result. $\square$

Remark 3. *Using the same technique as in above results, we can derive many new error estimations using our main results. We omit here their proofs and the details are left to the interested reader.*

Now we provide some applications to q-digamma function [20]. Suppose $0 < q < 1$, then q-digamma function $\chi_q(u)$ is given as:

$$\begin{aligned} \chi_q(u) &= -\ln(1-q) + \ln(q) \sum_{i=0}^{\infty} \frac{q^{i+u}}{1-q^{i+u}} \\ &= -\ln(1-q) + \ln(q) \sum_{i=0}^{\infty} \frac{q^{iu}}{1-q^{iu}}. \end{aligned}$$

For $q > 1$ and $u > 0$, then q-digamma function $\chi_q(u)$ can be given as:

$$\begin{aligned} \chi_q(u) &= -\ln(q-1) + \ln(q) \left[u - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-(i+u)}}{1-q^{-(i+u)}} \right] \\ &= -\ln(q-1) + \ln(q) \left[u - \frac{1}{2} - \sum_{i=0}^{\infty} \frac{q^{-iu}}{1-q^{-iu}} \right]. \end{aligned}$$

From the above definition, it is clear that $\chi'_q(x)$ is completely monotone function on $(0, \infty)$ for $q > 0$. This implies that $\chi'_q(x)$ is convex function. Using this fact, we can establish the following new important results regarding q-digamma function.

Proposition 6. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned} &\chi'_q\left(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 \frac{u_1 + u_2}{2} \right) \\ &\le \frac{\chi_q(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2) - \chi_q(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1)}{u_2 - u_1} \end{aligned}$$

$$\leq \varpi_1 \chi_q'(v_1) + \varpi_2 \chi_q'(v_2) + \varpi_3 \frac{\chi_q'(u_1) + \chi_q'(u_2)}{2}.$$

Proof. The proof directly follows from Theorem 1, taking $\lambda = k = 1, i = 0, \Phi(x) = \chi_q'(x)$ and using the fact $\varpi(0) = 1$. □

Proposition 7. *Under the assumptions of Theorem 3, we have*

$$\begin{aligned} &\left|\frac{\chi_q'(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2) + \chi_q'(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1)}{2}\right. \\ &\quad \left. - \frac{\chi_q(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_2) - \chi_q(\varpi_1 v_1 + \varpi_2 v_2 + \varpi_3 u_1)}{u_2 - u_1}\right| \\ &\leq \frac{|\varpi_3|(u_2 - u_1)}{2}\left[\varpi_1 \chi''_q(v_1) + \varpi_2 \chi''_q(v_2) + \varpi_3 \frac{\chi''_q(u_1) + \chi''_q(u_2)}{2}\right]. \end{aligned}$$

Proof. The proof directly follows from Theorem 3, choosing $\lambda = k = 1, i = 0, \Phi(x) = \chi_q'(x)$ and using the fact $\varpi(0) = 1$. □

Remark 4. *Using the same technique as in above proved relations, we can obtain several interesting inequalities pertaining to q-digamma function applying our main results. We omit here their proofs.*

5. Graphical Analysis

In this section, we discuss the pictorial illustrations of our main outcome, which are helpful to understand theoretical results.
If we choose $\Phi(x) = x^2$ with $v_1 = 0, v_2 = 3, u_1 = 1$ and $u_2 = 2$ in Theorem 3 (Figure 1), then we have

$$\left|\frac{5}{2\lambda\Gamma(\lambda)} - \frac{1}{2\Gamma(\lambda)}\left[\frac{5}{\lambda} - \frac{2}{1+\lambda} + \frac{2}{2+\lambda}\right]\right| \leq \frac{3(2^\lambda - 1)}{2^\lambda \Gamma(\lambda+2)}.$$

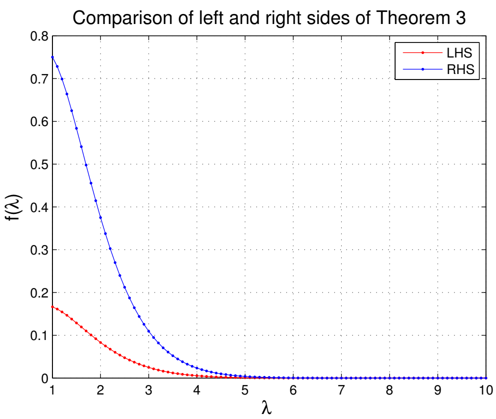

Figure 1. This is an image showing the comparison between left and right sides of Theorem 3.

If we choose $\Phi(x) = x^2$ with $v_1 = 0, v_2 = 3, u_1 = 1$ and $u_2 = 2$ in Theorem 4 (Figure 2), then we have

$$\left|\frac{18\lambda^2 + 54\lambda + 40}{8\lambda\gamma(\lambda)(\lambda^2 + 3\lambda + 2)} - \frac{9}{4\Gamma(\lambda+1)}\right| \leq \frac{3}{2\Gamma(\lambda+2)}.$$

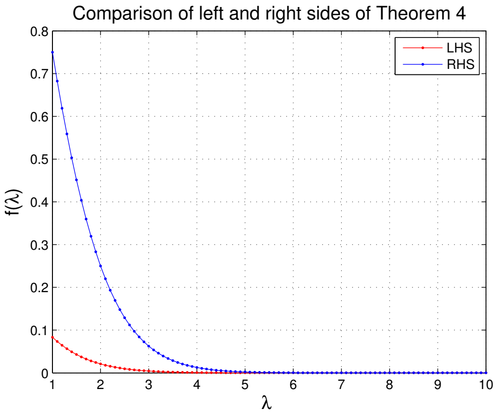

Figure 2. This is an image showing the comparison between left and right sides of Theorem 4.

If we choose $\Phi(x) = x^2$ with $v_1 = 0, v_2 = 3, u_1 = 1$ and $u_2 = 2$ in Theorem 5 (Figure3), then we have

$$\left| \frac{18\lambda^2 + 54\lambda + 40}{8\lambda\gamma(\lambda)(\lambda^2 + 3\lambda + 2)} - \frac{9}{4\Gamma(\lambda+1)} \right| \leq \frac{1}{4\Gamma(\lambda+1)} \left(\frac{1}{2\Gamma(\lambda) + 3} \right)^{\frac{1}{2}} \left(\sqrt{29} + \sqrt{26} \right).$$

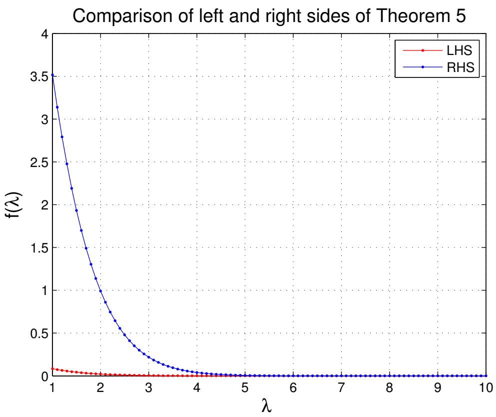

Figure 3. This is an image showing the comparison between left and right sides of Theorem 5.

6. Conclusions

In this paper, we have derived some new generalized fractional analogues of Mercer type inequalities applying the convexity property of the functions and Raina's function. We also discussed several new special cases which showed that our results are quite unifying. The efficiency of our results was demonstrated with several interesting applications from our results pertaining to special means, error bounds and q-digamma functions. To the best of our knowledge, these results are new in the literature. Since the class of convex functions have large applications in many mathematical areas, they can be applied to obtain several results in convex analysis, special functions, quantum mechanics, related optimization theory, and mathematical inequalities. They may stimulate further research in different areas of pure and applied sciences. Using the techniques and ideas of this paper may inspire and motivate further research in this energetic field.

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