



Article Discrete Memristance and Nonlinear Term for Designing Memristive Maps

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Abstract: Chaotic maps have simple structures but can display complex behavior. In this paper, we apply discrete memristance and a nonlinear term in order to design new memristive maps. A general model for constructing memristive maps has been presented, in which a memristor is connected in serial with a nonlinear term. By using this general model, different memristive maps have been built. Such memristive maps process special fixed points (infinite and without fixed point). A typical memristive map has been studied as an example via fixed points, bifurcation diagram, symmetry, and coexisting iterative plots.

Keywords: discrete map; chaos; memristor; symmetry; fixed point



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1. Introduction

Discrete maps, memristor, and hidden attractor have attracted significant attention in recent years [1–4]. Discrete maps are noticeable examples of simple structures, which can exhibit complex dynamics [5,6]. Both chaos and hyperchaos were observed in discrete maps [7–9]. In the literature, numerous chaotic maps were reported range from common discrete maps to recent fractional maps. Discrete maps are suitable for developing various applications such as audio encryption, robot motion, secure systems, and so on [10–15]. Interestingly, Lu et al. develop a symmetric algorithm using trigonometric map for image encryption [16]. Application of hyperchaotic maps in generative adversarial nets is reported in [17].

Memristor is a basic circuit element presenting the relationship between charge and magnetic flux. With the development of memristor models and memristive devices, various structures of memristive systems have been established, and their dynamics have been explored [18–20]. Memristive neural networks propose novel research directions about artificial neural networks [21]. Xu et al. introduced a jerk system using a generalized memristor [22]. In the work [23], the authors investigated a memristor circuit. Asymmetric bifurcations were discovered in an asymmetric memristive jerk circuit [24]. Asymmetric coexisting bifurcations, multistability, and similar features in asymmetric circuits are the attractive features. Recently, memristors have been used to develop memristive maps [25,26].

Studies on chaotic attractors in nonlinear systems have been published in several papers [27–32]. It is noted that chaotic attractors found in a conventional system belong to a common type of self-excited attractor [33]. In order to obtain chaos, the initial states are located closely to the unstable equilibrium points. However, there are hidden attractors characterized by the existence of stable equilibrium or the absence of equilibrium [34–37]. It is challenge to find and analyze this kind of attractor [38–40].

The combination of three research topics (discrete map, memristor, and hidden attractor), as illustrated in Figure 1, opens up new research directions. For example, memristorbased systems with hidden attractors have been discovered when combining two research topics (memristor and hidden attractor). This work focuses on the overlap of three such research topics by introducing an approach to design discrete maps with a memristor. It is worth noting that the designed discrete maps belong to systems with a hidden attractor. We believe that memristor-based discrete maps have special features such as chaos, multistability, an infinite number of fixed points, etc. There are a few published works on memristor-based maps [25,26].



Figure 1. Relationships of three research topics. The overlapped area of three circles shows the memristive maps with hidden attractor.

The remainder of the work is organized as follows. A general model for designing memristive maps is proposed and analyzed in Section 2. In Section 3, a typically designed map (the MM_1 map) is investigated via fixed points, bifurcation diagram, symmetry, iterative plots, and implementation. Finally, the conclusions are presented in Section 4.

2. General Model

Motivated by the flexibility, practicability, and attention of discrete maps, memristors, and hidden attractors, we would like to introduce an approach to building new memristive maps. The general map is designed as illustrated in Figure 2. The two main components needed to construct the map are a memristor and a nonlinear term F(.), which are connected serially. The amplifiers *b* and *a* indicate the effect of the memristor and the nonlinear term on the structure, respectively. In addition, there is the presence of a controller *c* used to control the fixed points.



Figure 2. Design of map with memristor and nonlinear function F(.).

By denoting the discrete memristance as M(y(n)), the general model is characterized by

$$\begin{cases} x(n+1) = aF(bM(y(n))x(n)) + c \\ y(n+1) = y(n) + x(n) \end{cases}$$
(1)

with parameters *a*, *b*, *c*. The nonlinear term F(.) satisfying F(0) = 0. Equation (1) presents the general model of memristor-based discrete maps. It is used to design discrete maps with a memristor.

If $P(x^*, y^*)$ is the fixed point of model (1), we have

$$\begin{cases} x^* = aF(bM(y^*)x^*) + c \\ y^* = y^* + x^* \end{cases}$$
(2)

Therefore, we get

$$\begin{cases} aF(bM(y^*)x^*) + c = 0 \\ x^* = 0 \end{cases}$$
(3)

The number of fixed points in the model (1) depends on *c*. There are two attractive cases. In the first case, there is no fixed point for $c \neq 0$. In the second case, there are infinite fixed points given by $P(0, y^*)$.

Based on the proposed general model, different maps can be constructed with appropriated discrete memristance and nonlinear terms. For instance, when applying the following discrete memristance M(y(n)), and nonlinear term F(.)

$$\begin{cases} M(y(n)) = \cos(y(n)) \\ F(.) = \sin(.) \end{cases}$$
(4)

we obtain the MM_1 memristive map

$$\begin{cases} x(n+1) = a \sin(b \cos(y(n))x(n)) + c \\ y(n+1) = y(n) + x(n) \end{cases}$$
(5)

Chaotic dynamics of the MM₁ map are illustrated in Figure 3a for a = 2.6, b = 1.1, c = 0.001, and (x(0), y(0)) = (1, 2).

Three other new memristive maps are found and listed in Table 1. It is noted that Equation (1) is classified and presented in Table 1 based on the selected discrete memristance and nonlinear term. These memristive maps displayed chaos, as shown in Figure 3.



Figure 3. Cont.



Figure 3. Iterative plots obtained from : (a) MM_1 map, (b) MM_2 map, (c) MM_3 map, and (d) MM_4 map.

Table 1. Constructed maps using memristor and nonlinear term.

Map	Equations	Parameters	(x(0), y(0))
MM ₁	$x(n+1) = a\sin(b\cos(y(n))x(n)) + c$	a = 2.6, b = 1.1	x(0) = 1
	y(n+1) = y(n) + x(n)	c = 0.001	y(0) = 2
MM_2	$x(n+1) = a\sin(b\sin(y(n))x(n)) + c$	a = 2.6, b = 1	x(0) = 1
	y(n+1) = y(n) + x(n)	c = 0.001	y(0) = 2
MM_3	$x(n+1) = a \tanh(b \cos(y(n))x(n)) + c$	a = 2.6, b = 1.3	x(0) = 1
	y(n+1) = y(n) + x(n)	c = 0.001	y(0) = 2
MM_4	$x(n+1) = a \tanh(b \sin(y(n))x(n)) + c$	a = 2.7, b = 1.1	x(0) = 1
	y(n+1) = y(n) + x(n)	c = 0.001	y(0) = 2

3. MM₁ **Map**

We focus on the MM₁ map described by Equation (5) to illustrate noticeable features of such newly constructed maps. The fixed point $P(x^*, y^*)$ of the MM₁ map must satisfy the condition

$$\begin{cases} x^* = a \sin(b \cos(y^*) x^*) + c \\ y^* = y^* + x^* \end{cases}$$
(6)

In the first case, there is no fixed point for $c \neq 0$. In the second case, there are infinite fixed points given by $P(0, y^*)$. The Jacobian matrix of the map is

$$\mathbf{J} = \begin{bmatrix} ab\cos(b\cos(y^*)x^*)\cos(y^*) & -ab\cos(b\cos(y^*)x^*)x^*\sin(y^*) \\ 1 & 1 \end{bmatrix}$$
(7)

Therefore, at $P(0, y^*)$, we obtain

$$\mathbf{J}|_{P(0,y^*)} = \begin{bmatrix} ab\cos(y^*) & 0\\ 1 & 1 \end{bmatrix}$$
(8)

As a result, we get its characteristic equation

$$\lambda^2 + ab\cos(y^*)\lambda = 0 \tag{9}$$

From the characteristic equation, the fixed point is stable for

$$\cos^2(y^*) < \frac{1}{a^2b^2} \tag{10}$$

otherwise $P(0, y^*)$ is unstable.

For $c \neq 0$, there is no fixed point in the map. We study the map with b = 1.1, c = 0.001 and varying *a*. Both the bifurcation plot in Figure 4a and Lyapunov exponents in Figure 4b exhibit dynamical behavior in the range $a \in [2.3, 2.7]$. Positive Lyapunov exponents appear when increasing *a* and indicate chaos in the map.



Figure 4. (a) Bifurcation diagram, (b) Lyapunov exponents of MM_1 map when changing *a* from 2.3 to 2.7. The red and blue colors present the first and second Lyapunov exponents, respectively.

It is noted that the map is symmetrical via the transformation

$$(x,y) \leftrightarrow (x,y \pm 2k\pi)$$
 (11)

Figure 5 show seven iterative plots, which coexist with a = 2.6, b = 1.1, c = 0.001 for different values of (x(0), y(0)).



Figure 5. Coexisting iterative plots are observed when changing (x(0), y(0)): $(0.1, 0.2 + 6\pi)$ (yellow), $(0.1, 0.2 + 4\pi)$ (magenta), $(0.1, 0.2 + 2\pi)$ (red), (0.1, 0.2) (black), $(0.1, 0.2 - 2\pi)$ (blue), $(0.1, 0.2 - 4\pi)$ (green), $(0.1, 0.2 - 6\pi)$ (cyan).

3.2. *Case* 2 *When* c = 0

When c = 0, the map is given by

$$\begin{cases} x(n+1) = a \sin(b \cos(y(n))x(n)) \\ y(n+1) = y(n) + x(n) \end{cases}$$
(12)

The symmetry of the MM₁ map is confirmed by

$$(x,y) \leftrightarrow (-x,-y) \tag{13}$$

The coexistence of two iterative plots is shown in Figure 6 for a = 2.6, b = 1.1 with two values (x(0), y(0)) = (1, 2) (black color) and (x(0), y(0)) = (-1, -2) (red color).



Figure 6. The presence of two iterative plots for (x(0), y(0)) = (1, 2) (black) and (x(0), y(0)) = (-1, -2) (red).

Similar to Case 1, the MM₁ map can generate various iterative plots when keeping a = 2.6, b = 1.1 and changing (x(0), y(0)) (see Figure 7).



Figure 7. Coexistence of seven iterative plots for different values (x(0), y(0)): $(0.1, 0.2 + 6\pi)$ (yellow), $(0.1, 0.2 + 4\pi)$ (magenta), $(0.1, 0.2 + 2\pi)$ (red), (0.1, 0.2) (black), $(0.1, 0.2 - 2\pi)$ (blue), $(0.1, 0.2 - 4\pi)$ (green), $(0.1, 0.2 - 6\pi)$ (cyan).

3.3. Discussion

The purposed memristive maps display attractive behavior. In addition, such maps can be implemented using a hardware platform, for example, microcontroller or FPGA. We have used a microcontroller to realize the MM₁ map by applying the general approach [13]. The microcontroller-based approach is a simple one and can be implemented conveniently [41]. A display is connected to the microcontroller directly to show the iterative plot, which is reported in Figure 8. The feasibility and chaos of the map are suitable for practical applications.



Figure 8. Experimental result of the MM_1 map, which is captured from the microcontroller.

4. Conclusions

The ability to connect the memristor and nonlinear term to propose chaotic maps was examined in this work. Our finding reveals that constructed maps depend on the selected discrete memristance and nonlinearity. We study a typical map as an example and observe its dynamical behaviors. Numerical simulations demonstrate the chaos, symmetry, and coexistence of iterative plots in the map. The implementation of the MM₁ map with microcontroller confirms its feasibility. We believe that memristive maps can provide advantages in practical applications such as random signal generators.

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