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Confidence Intervals for Common Coefficient of Variation of Several Birnbaum–Saunders Distributions

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Abstract: The Birnbaum–Saunders (BS) distribution, also known as the fatigue life distribution, is right-skewed and used to model the failure times of industrial components. It has received much attention due to its attractive properties and its relationship to the normal distribution (which is symmetric). Furthermore, the coefficient of variation (CV) is commonly used to analyze variation within a dataset. In some situations, the independent samples are collected from different instruments or laboratories. Consequently, it is of importance to make inference for the common CV. To this end, confidence intervals based on the generalized confidence interval (GCI), method of variance estimates recovery (MOVER), large-sample (LS), Bayesian credible interval (BayCrI), and highest posterior density interval (HPDI) methods are proposed herein to estimate the common CV of several BS distributions. Their performances in terms of their coverage probabilities and average lengths were investigated by using Monte Carlo simulation. The simulation results indicate that the HPDI-based confidence interval outperformed the others in all of the investigated scenarios. Finally, the efficacies of the proposed confidence intervals are illustrated by applying them to real datasets of PM10 (particulate matter $\leq$ 10 µm) concentrations from three pollution monitoring stations in Chiang Mai, Thailand.

Keywords: confidence interval; common coefficient of variation; Birnbaum–Saunder distribution; Bayesian

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1. Introduction

The original idea behind the Birnbaum–Saunders (BS) distribution lies in an investigation of vibrations in commercial aircraft that cause material fatigue. Fatigue is a type of structural deterioration that happens when a material is subjected to fluctuating stress and tension [1]. To address these problems, Birnbaum and Saunders [2] proposed the fatigue life distribution, which is commonly known as the BS distribution to describe the failure time of materials and equipment subjected to dynamic loads where failure is caused by the initiation and growth of a dominant fracture. The BS distribution is positively asymmetric and unimodal with two positive parameters: α, the shape parameter, and β, which is both the scale parameter and the median of the distribution. In addition, it has many attractive properties and has a close relationship with the normal distribution. The BS distribution is very effective for fitting data that are all positive. Despite its origins in materials science, the BS distribution has recently been applied to various other fields, including the environment, business, industry, finance, and medical sciences [3–6].

The coefficient of variation (CV) is an important descriptive statistic for analyzing the variability of data. In particular, it is a measure of variability relative to the mean. The CV is defined as a ratio of the standard deviation (σ) to the mean (μ), namely CV = σ/μ. It is free from the unit of measurement, and, thus, it has been preferentially used for comparing relative variability between two or more populations rather than the variance or standard deviation [7]. In many situations, independent samples are collected from methods involving different instruments, methodologies, and/or laboratories, and so estimating the common CV of these related populations is of great interest. Many researchers have developed confidence intervals for estimating the common CV of several populations from various distributions using several methods. For example, Tian [8] used the concept of the generalized confidence interval (GCI) to construct the confidence interval for the common CV of several independent normal samples. Verrill and Johnson [9] proposed a likelihood ratio-based confidence interval for a common CV of several normal distributions. Behboodian and Jafari [10] utilized the concept of generalized p-values and GCI to develop a new method for estimating the confidence interval for the common CV of several normal populations. Ng [11] suggested a method for estimating the confidence interval for the common CV of several lognormal samples by utilizing the concept of the generalized variable. Thangjai and Niwitpong [12] developed the adjusted method of variance estimates recovery (MOVER) for constructing the confidence interval for the common CV of two-parameter exponential distributions and then compared its performance with GCI and large-sample (LS) confidence intervals. Liu and Xu [13] introduced a new confidence interval for the common CV of several normal distributions based on the concept of the confidence distribution interval. Recently, Yosboonruang et al. [14] constructed confidence intervals for the common CV of delta-lognormal distributions using the fiducial GCI (FGCI), equal-tailed Bayesian credible intervals (BayCrI) based on the independent Jeffreys or uniform priors, and MOVER.

Estimating the parameters of a BS distribution is of significant interest to many researchers and has recently garnered much attention in the literature. For instance, the maximum likelihood estimation (MLE) of α and β were introduced in Birnbaum and Saunders [15] and Engelhardt et al. [16]. Ng et al. [17] presented modified moment estimators (MMEs) for α and β and a bias reduction method with Jackknife resampling to reduce the biases of the MMEs and MLEs. Wu and Wong [18] improved the confidence interval for the two-parameter BS distribution based on a high-order likelihood asymptotic procedure. Xu and Tang [19] explored Bayesian estimators for α and β under the reference prior by using Lindley's method and Gibbs' sampling to obtain approximate Bayesian estimators for these two parameters. Wang [20] examined GCI for α, as well as some important reliability quantities, such as mean, quantiles, and a reliability function. Wang et al. [21] considered Bayesian estimators under inverse-gamma priors for α and β to compute their Bayesian estimates and credible intervals. Guo et al. [22] applied a hybrid of the generalized inference method and the LS theory for interval estimation and hypothesis testing of the common mean of several BS distributions. Puggard et al. [23] proposed confidence intervals for the CV and the difference between the CVs of BS distributions based on GCI, the bootstrap confidence interval, BayCrI, and the highest posterior density interval (HPDI). Recently, Puggard et al. [24] presented confidence intervals for the ratio of the variances of two independent BS distributions using the generalized fiducial confidence interval, BayCrI, and HPDI based on a prior distribution with partial information and a proper prior with known hyperparameters. However, estimating the common CV of two or more independent BS distributions has not previously been reported. Therefore, the goal of this study is to estimate confidence intervals for the common CV of several BS distributions based on the concepts of GCI, MOVER, LS, BayCrI, and HPDI.

The remainder of this study is organized as follows. Section 2 provides the methodologies for constructing confidence intervals for the common CV of several BS distributions. Section 3 covers the methodology and results of an extensive Monte Carlo simulation study to compare the performances of the proposed confidence intervals. An illustration of the proposed confidence intervals with datasets of PM10 (particulate matter (PM) $\leq$ 10 µm) concentrations collected in March 2019 from three pollution monitoring stations in Chiang Mai, Thailand, is presented in Section 4. Finally, conclusions are covered in Section 5.

2. Methods

Let $X_{ij} = (X_{i1}, X_{i2}, \ldots, X_{in_i})$ be random samples of size n_i drawn from a BS distribution, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$. The cumulative distribution function (cdf) of random variable X_{ij} can be written as:

$$F(x_{ij}) = \Phi\left[\frac{1}{\alpha_i}\left(\sqrt{\frac{x_{ij}}{\beta_i}} - \sqrt{\frac{\beta_i}{x_{ij}}}\right)\right], x_{ij} > 0, \alpha_i, \beta_i > 0, \tag{1}$$

where $\Phi(\cdot)$ is the standard normal cdf and α_i and β_i are the shape and the scale parameters, respectively. Thus, the probability density function (pdf) of X_{ij} is given by:

$$f(x_{ij}, \alpha_i, \beta_i) = \frac{1}{2\alpha_i\beta_i\sqrt{2\pi}}\left\{\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right\} exp\left[-\frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta} + \frac{\beta_i}{x_{ij}} - 2\right)\right]. \tag{2}$$

The expected value and variance of X_{ij} are defined as:

$$E(X_{ij}) = \beta_i(1 + \frac{1}{2}\alpha_i^2) \tag{3}$$

and

$$V(X_{ij}) = (\alpha_i\beta_i)^2(1 + \frac{5}{4}\alpha_i^2), \tag{4}$$

respectively. Therefore, the CV of X_{ij} can be easily obtained as:

$$\lambda_i = \frac{\alpha_i\sqrt{1 + \frac{5}{4}\alpha_i^2}}{1 + \frac{1}{2}\alpha_i^2}. \tag{5}$$

According to Ng et al. [17], the MMEs of (α_i, β_i) are given by:

$$\hat{\alpha}_i = \left\{2\left(\left[\bar{x}_i \sum_{j=1}^{n_i} x_{ij}^{-1}/n_i\right]^{1/2} - 1\right)\right\}^{1/2} \quad \text{and} \quad \hat{\beta}_i = \left\{\bar{x}_i\left(\sum_{j=1}^{n_i} x_{ij}^{-1}/n_i\right)^{-1}\right\}^{1/2},$$

where $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$. In addition, it has been shown in the study of Ng et al. [17] that the asymptotic joint distribution of α_i and β_i is bivariate normal, which is given by:

$$\begin{pmatrix}\hat{\alpha}_i \\ \hat{\beta}_i\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha_i \\ \beta_i\end{pmatrix}, \begin{pmatrix}\frac{\alpha_i^2}{2n_i} & 0 \\ 0 & \frac{(\alpha_i\beta_i)^2}{n_i}\left(\frac{1+\frac{3}{4}\alpha_i^2}{(1+\frac{1}{2}\alpha_i^2)^2}\right)\end{pmatrix}\right]. \tag{6}$$

By applying the delta method, it follows that:

$$\sqrt{n_i}(\hat{\lambda}_i - \lambda_i) \xrightarrow{d} N\left[0, \frac{\alpha_i^2}{2}\left(\frac{8(2\alpha_i^2 + 1)}{(\alpha_i^2 + 2)^2\sqrt{5\alpha_i^2 + 4}}\right)^2\right], \tag{7}$$

where $\hat{\lambda}_i = \frac{\hat{\alpha}_i\sqrt{1+\frac{5}{4}\hat{\alpha}_i^2}}{1+\frac{1}{2}\hat{\alpha}_i^2}$. By applying Equation (7), the variance of $\hat{\lambda}_i$ becomes:

$$\delta_i = V(\hat{\lambda}_i) = \frac{\alpha_i^2}{2n_i}\left(\frac{8(2\alpha_i^2 + 1)}{(\alpha_i^2 + 2)^2\sqrt{5\alpha_i^2 + 4}}\right)^2. \tag{8}$$

According to Thangjai and Niwitpong [12] and Yosboonruang et al. [14], the common CV of several BS distributions can be written as:

$$\lambda = \frac{\sum_{i=1}^{k} w_i \lambda_i}{\sum_{i=1}^{k} w_i}, \tag{9}$$

where $w_i = 1/V(\hat{\lambda}_i)$. The following proposed methods are used to construct the confidence intervals for the common CV of several BS distributions.

2.1. The GCI Approach

Weerahandi [25] introduced the concept of the generalized pivotal quantity (GPQ) and deduced the GCI as an extension of the classical confidence interval. In contrast to a traditional pivotal quantity, the GPQ can be a function of the nuisance parameters and has a distribution that is independent of the unknown parameter and an observed value that is independent of the nuisance parameters. Therefore, the GCI is useful in situations when the traditional pivot quantity is either unavailable or difficult to obtain. A full detailed discussion, as well as several applications can be found in Weerahandi [25,26], Tian [8], Behboodian and Jafari [10], Chen and Ye [27–29], and Luo et al. [30].

Consider k independent random samples $X_{i1}, X_{i2}, \ldots, X_{in_i}$ from BS distributions. According to Sun [31] and Wang [20], the GPQ for β_i can be defined as:

$$T_{\beta_i} := T_{\beta_i}(\boldsymbol{x_{ij}}; T_i) = \begin{cases} max(\beta_{i1}, \beta_{i2}), & if \quad T_i \leq 0 \\ min(\beta_{i1}, \beta_{i2}), & if \quad T_i > 0, \end{cases} \tag{10}$$

where $x_{ij} = (x_{i1}, x_{i2}, \ldots, x_{in})$ are the observed values of X_{ij} and T_i follow a t-distribution with $n_i - 1$ degrees of freedom (denoted as $T_i \sim t(n_i - 1)$). By applying Equation (10), β_{i1} and β_{i2} are the two solutions for:

$$\left[(n_i - 1)B_i^2 - \frac{1}{n_i} D_i T_i^2\right]\beta_i^2 - 2\left[(n_i - 1)A_i B_i - (1 - A_i B_i) T_i^2\right]\beta_i + (n_i - 1)A_i^2 - \frac{1}{n_i} C_i T_i^2 = 0, \tag{11}$$

where $A_i = n_i^{-1} \sum_{j=1}^{n_i} \sqrt{X_{ij}}$, $B_i = n_i^{-1} \sum_{j=1}^{n_i} 1/\sqrt{X_{ij}}$, $C_i = \sum_{j=1}^{n_i} (\sqrt{X_{ij}} - A_i)^2$ and $D_i = \sum_{j=1}^{n_i} (1/\sqrt{X_{ij}} - B_i)^2$. Subsequently, Wang [20] also established the GPQ for α_i which is derived as:

$$T_{\alpha_i} := T_{\alpha_i}(\boldsymbol{x_{ij}}; v_i, T_i) = \left[\frac{S_{i2} T_{\beta_i}^2 - 2n_i T_{\beta_i} + S_{i1}}{T_{\beta_i} v_i}\right]^{1/2}, \tag{12}$$

where $S_{i1} = \sum_{j=1}^{n_i} X_{ij}$, $S_{i2} = \sum_{j=1}^{n_i} 1/X_{ij}$ and v_i follow a Chi-squared distribution with n_i degrees of freedom (denoted as $v_i \sim \chi^2_{(n_i)}$). By substituting T_{α_i} into Equations (5) and (8), the respective GPQs of λ_i and the variance of $\hat{\lambda}_i$ become:

$$T_{\lambda_i} = \frac{T_{\alpha_i}\sqrt{1 + \frac{5}{4} T_{\alpha_i}^2}}{1 + \frac{1}{2} T_{\alpha_i}^2} \tag{13}$$

and

$$T_{\delta_i} = \frac{T_{\alpha_i}^2}{2n_i} \left(\frac{8(2T_{\alpha_i}^2 + 1)}{(T_{\alpha_i}^2 + 2)^2 \sqrt{5T_{\alpha_i}^2 + 4}}\right)^2. \tag{14}$$

Consequently, the GPQ for the common CV of several BS distributions is the weighted average of GPQ T_{λ_i} based on k individual samples as follows:

$$T_{\lambda} = \frac{\sum_{i=1}^{k} T_{w_i} T_{\lambda_i}}{\sum_{i=1}^{k} T_{w_i}}, \tag{15}$$

where $T_{w_i} = 1/T_{\delta_i}$. It follows that the $100(1-\gamma)\%$ GCI for λ can be constructed as $[T_\lambda(\gamma/2), T_\lambda(1-\gamma/2)]$, where $T_\lambda(\gamma/2)$ and $T_\lambda(1-\gamma/2)$ denote the $100(\gamma/2)$th and $100(1-\gamma/2)$th percentiles of T_λ, respectively. Algorithm 1 summarizes the computational steps for constructing GCI.

Algorithm 1 : GCI approach

1. Generate datasets x_{ij}, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ from a BS distribution.
2. Compute $A_i, B_i, C_i, D_i, S_{i1}$ and S_{i2}, respectively.
3. For $m = 1$ to M
4. Generate $T_i \sim t(n_i - 1)$, and then compute T_{β_i} by using Equation (10).
5. If $T_{\beta_i} < 0$, regenerate $T_i \sim t(n_i - 1)$.
6. Generate $v_i \sim \chi^2_{(n_i)}$, and then compute T_{α_i} by using Equation (12).
7. Compute T_{λ_i} and T_{w_i} to obtain T_λ.
8. (End M loops)
9. Compute $T_\lambda(\gamma/2)$ and $T_\lambda(1-\gamma/2)$.

2.2. The MOVER Approach

The original concept behind MOVER is to estimate a closed-form confidence interval for the sum or difference between two independent parameters based on the confidence intervals of the individual parameters [32,33]. The MOVER technique was recently applied to a linear combination of parameters $\theta_1, \theta_2, \ldots, \theta_k$ [34]. Suppose $\sum_{i=1}^k c_i\theta_i$ is a linear combination of parameters $\theta_1, \theta_2, \ldots, \theta_k$, where c_i are known constants. Assume that $\hat{\theta}_i$ is an unbiased estimate of θ_i. In addition, let (l_i, u_i) denote the $100(1-\gamma)\%$ confidence interval for θ_i, for $i = 1, 2, \ldots, k$. Hence, the $100(1-\gamma)\%$ MOVER confidence interval for $\sum_{i=1}^k c_i\theta_i$ can be written as:

$$L = \sum_{i=1}^{k} c_i\hat{\theta}_i - \sqrt{\sum_{i=1}^{k} c_i^2(\hat{\theta}_i - l_i^*)^2}\,; \quad l_i^* = \begin{cases} l_i & \text{if} \quad c_i > 0 \\ u_i & \text{if} \quad c_i < 0 \end{cases} \tag{16}$$

and

$$U = \sum_{i=1}^{k} c_i\hat{\theta}_i + \sqrt{\sum_{i=1}^{k} c_i^2(\hat{\theta}_i - u_i^*)^2,}\,; \quad u_i^* = \begin{cases} u_i & \text{if} \quad c_i > 0 \\ l_i & \text{if} \quad c_i < 0. \end{cases} \tag{17}$$

By applying Equation (13), the $100(1-\gamma)\%$ confidence interval for λ_i based on the GPQs becomes

$$[L_i, U_i] = [T_{\lambda_i}(\gamma/2), T_{\lambda_i}(1-\gamma/2)], \tag{18}$$

where $T_{\lambda_i}(\gamma/2)$ and $T_{\lambda_i}(1-\gamma/2)$ denote the $100(\gamma/2)$th and $100(1-\gamma/2)$th percentiles of T_{λ_i}, respectively. Therefore, the $100(1-\gamma)\%$ MOVER confidence interval for the common CV of several BS distributions can be expressed as

$$L = \sum_{i=1}^{k} c_i^*\hat{\lambda}_i - \sqrt{\sum_{i=1}^{k} c_i^{*2}(\hat{\lambda}_i - L_i^*)^2}\,; \quad L_i^* = \begin{cases} L_i & \text{if} \quad c_i^* > 0 \\ U_i & \text{if} \quad c_i^* < 0 \end{cases} \tag{19}$$

and

$$U = \sum_{i=1}^{k} c_i^*\hat{\lambda}_i + \sqrt{\sum_{i=1}^{k} c_i^{*2}(\hat{\lambda}_i - U_i^*)^2}\,; \quad U_i^* = \begin{cases} U_i & \text{if} \quad c_i^* > 0 \\ L_i & \text{if} \quad c_i^* < 0, \end{cases} \tag{20}$$

where $c_i^* = w_i / \sum_{j=1}^k w_j$. The confidence interval based on MOVER can be easily constructed using Algorithm 2.

Algorithm 2 : MOVER approach

1. Generate datasets x_{ij}, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ from a BS distribution.
2. Compute c_i^* and $\hat{\lambda}_i$.
3. Compute the $100(1-\gamma)\%$ GCI for λ_i by applying Equation (18).
4. Compute L and U, by using Equations (19) and (20), respectively, leading to obtain the 95% confidence interval based on MOVER.

2.3. The LS Approach

A large sample is a set of values that are used to estimate the true value of a population parameter. For the BS distribution, the LS estimate of the CV is a pooled estimate of it, as defined in Equation (9). Therefore, the $100(1-\gamma)\%$ LS confidence interval for the common CV can be derived as:

$$[L_{LS}, U_{LS}] = \left[\hat{\lambda} - z_{1-\frac{\gamma}{2}}\sqrt{1/\sum_{i=1}^{k} w_i}, \hat{\lambda} + z_{1-\frac{\gamma}{2}}\sqrt{1/\sum_{i=1}^{k} w_i}\right]. \tag{21}$$

Algorithm 3 was applied to obtain the LS confidence interval.

Algorithm 3 : LS approach

1. Generate datasets x_{ij}, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ from a BS distribution.
2. Compute λ_i to obtain $\hat{\lambda}$.
3. Compute the 95% LS confidence interval for λ by using Equation (21).

2.4. The BayCrI Approach

The Bayesian method involves making statistical inferences about a parameter based on two sources of information: experimental data via its likelihood function and judgment based on previous knowledge via its prior distribution. Combining these data sources results in uncovering the posterior distribution.

For the BS distribution, the likelihood function for the parameters (α_i, β_i) from random sample $x_{ij} = (x_{i1}, x_{i2}, \ldots, x_{in_i})$ can be written as:

$$L(x_{ij}|\alpha_i, \beta_i) \propto \frac{1}{\alpha_i{}^{n_i}\beta_i{}^{n_i}} \prod_{j=1}^{n_i} \left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right] exp\left[-\sum_{j=1}^{n_i} \frac{1}{2\alpha_i^2}\left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2\right)\right]. \tag{22}$$

The reference (independent Jeffreys') prior of a BS distribution can lead to an improper posterior distribution [35], so a suitable prior with known hyperparameters is needed to ensure that a proper one is obtained. By utilizing useful reparameterization $\eta_i = \alpha_i^2$, an inverse-gamma (IG) distribution with parameters a_i and b_i is a suitable prior for η_i (denoted as $IG(\eta_i|a_i, b_i)$). In addition, an IG distribution with parameters c_i and d_i is a suitable prior for β_i (denoted as $IG(\beta_i|c_i, d_i)$) [21]. Hence, the joint posterior density function of (η_i, β_i) can be obtained by combining the likelihood function from Equation (22) with the IG prior distributions for η_i and β_i as follows:

$$\begin{aligned} p(\eta_i, \beta_i|\boldsymbol{x}_{ij}) &\propto L(\boldsymbol{x}_{ij}|\alpha_i, \beta_i)\pi(\eta_i|a_i, b_i)\pi(\beta_i|c_i, d_i) \\ &\propto \frac{1}{(\eta_i)^{\frac{n_i}{2}}\beta_i^{n_i}} \prod_{j=1}^{n_i} \left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right] exp\left[-\sum_{j=1}^{n_i} \frac{1}{2\eta_i}\left(\frac{x_{ij}}{\beta_i} + \frac{\beta_i}{x_{ij}} - 2\right)\right] \\ &\times (\eta_i)^{-a_i-1} exp\left(-\frac{b_i}{\eta_i}\right)\beta_i^{-c_i-1} exp\left(-\frac{d_i}{\beta_i}\right). \end{aligned} \tag{23}$$

Subsequently, the marginal posterior distribution of β_i can be written as:

$$\pi(\beta_i|\boldsymbol{x}_{ij}) \propto \beta_i^{-(n_i+c_i+1)} exp\left(-\frac{d_i}{\beta_i}\right)\prod_{j=1}^{n_i}\left[\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{1}{2}}+\left(\frac{\beta_i}{x_{ij}}\right)^{\frac{3}{2}}\right] \times \left[\sum_{j=1}^{n_i}\frac{1}{2}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)+b_i\right]^{\frac{-(n_i+1)}{2-a_i}}. \tag{24}$$

Moreover, the conditional posterior distribution of η_i given β_i can be derived as:

$$\eta_i|\beta_i\boldsymbol{x}_{ij}\sim IG\left(\frac{n_i}{2}+a_i, \frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_i}+\frac{\beta_i}{x_{ij}}-2\right)+b_i\right). \tag{25}$$

Since the marginal posterior in Equation (24) is mathematically intractable, the Markov Chain–Monte Carlo method can be utilized to draw posterior samples to be used for inference. According to Wang et al. [21], the posterior sample of β_i (β_i^*) can be generated by applying the generalized ratio-of-uniforms method [36] as follows.

Let

$$A(r_i) = \left\{(u_i, v_i) : 0 < u_i \le \left[\pi\left(\frac{v_i}{u_i^{r_i}}|\boldsymbol{x}_{ij}\right)\right]^{1/(r_i+1)}\right\}, \tag{26}$$

where $\pi(\cdot|\boldsymbol{x}_{ij})$ is defined as in Equation (24) and $r_i \ge 0$ is a constant value. If (u_i, v_i) is a random vector uniformly distributed over $A(r_i)$, then $\beta_i = v_i/u_i^{r_i}$ has probability density function $\pi(\beta_i|\boldsymbol{x}_{ij})/\int \pi(\beta_i|\boldsymbol{x}_{ij})d\beta_i$. In general, directly generating (u_i, v_i) uniformly over $A(r_i)$ is not possible, so the accept–reject method from minimal bounding rectangle $[0, a(r_i)] \times [b^-(r_i), b^+(r_i)]$ is applied, where

$$a(r_i) = \sup_{\beta_i>0}\{[\pi(\beta_i|\boldsymbol{x}_{ij})]^{1/(r_i+1)}\}, \tag{27}$$

$$b^-(r_i) = \inf_{\beta_i>0}\{\beta_i[\pi(\beta_i|\boldsymbol{x}_{ij})]^{r_i/(r_i+1)}\}, \tag{28}$$

and

$$b^+(r_i) = \sup_{\beta_i>0}\{\beta_i[\pi(\beta_i|\boldsymbol{x}_{ij})]^{r_i/(r_i+1)}\}. \tag{29}$$

As in Wang et al. [21], $a(r_i)$ and $b^+(r_i)$ are finite, whereas $b^-(r_i) = 0$. The principal steps of the generalized ratio-of-uniforms method for generating the posterior sample of β_i from Equation (24) can be summarized as follows:

1. Calculate $a(r_i)$ and $b^+(r_i)$.
2. Generate u_i and v_i from $U(0, a(r_i))$ and $U(0, b^+(r_i))$, where $U(v, w)$ is a uniform distribution with parameters v and w.
3. Calculate $\rho_i = v_i/u_i^{r_i}$.
4. If $u_i \le [\pi(\rho_i|\boldsymbol{x}_{ij})]^{1/(r_i+1)}$, set $\beta_i^* = \rho_i$; otherwise repeat the procedure.

For the posterior sample of α_i (denoted as α_i^*), a value for η_i from Equation (25) is generated by using the *LearnBayes* package from the R software, then $\alpha_i^* = \sqrt{\eta_i}$. By Equations (5) and (8), the Bayesian estimator for the CV and variance of CV become

$$\lambda_i^* = \frac{\alpha_i^*\sqrt{1+\frac{5}{4}\alpha_i^{*2}}}{1+\frac{1}{2}\alpha_i^{*2}} \tag{30}$$

and

$$\delta_i^* = \frac{\alpha_i^{*2}}{2n_i}\left(\frac{8(2\alpha_i^{*2}+1)}{(\alpha_i^{*2}+2)^2\sqrt{5\alpha_i^{*2}+4}}\right)^2, \tag{31}$$

respectively. Consequently, the Bayesian estimator for the common CV of several BS distributions can be derived as

$$\lambda^* = \frac{\sum_{i=1}^{k} w_i^* \lambda_i^*}{\sum_{i=1}^{k} w_i^*}, \tag{32}$$

where $w_i^* = 1/\delta_i^*$. Finally, the $100(1-\gamma)\%$ BayCrI for λ can be constructed as $[\lambda^*(\gamma/2), \lambda^*(1-\gamma/2)]$, where $\lambda^*(\gamma/2)$ and $\lambda^*(1-\gamma/2)$ denote the $100(\gamma/2)$th and $100(1-\gamma/2)$th percentiles of λ^*, respectively. Therefore, BayCrI for λ can be estimated via Algorithm 4.

Algorithm 4 : BayCrI approach

1. Generate datasets x_{ij}, for $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, n_i$ from a BS distribution.
2. Set the values for a_i, b_i, c_i, d_i, and r_i.
3. Compute $a(r_i)$ and $b^+(r_i)$.
4. At the hth step,
   (a) Generate $u_i \sim U(0, a(r_i))$ and $v_i \sim U(0, b^+(r_i))$, independently, and then compute $\rho_i = v_i / u_i^{r_i}$.
   (b) If $u_i \leq [\pi(\rho_i | \boldsymbol{x}_{ij})]^{1/(r_i+1)}$, accept ρ_i and set $\beta_{i,(h)}^* = \rho_i$; otherwise, repeat step (a).
   (c) Generate $\tilde{\eta}_{i,(h)} \sim IG\left(\frac{n_i}{2} + a_i, \frac{1}{2}\sum_{j=1}^{n_i}\left(\frac{x_{ij}}{\beta_{i,(h)}^*} + \frac{\beta_{i,(h)}^*}{x_{ij}} - 2\right) + b_i\right)$ and then $\alpha_{i,(h)}^* = \sqrt{\tilde{\eta}_{i,(h)}}$.
   (d) Compute $\lambda_{i,(h)}^*$ and $w_{i,(h)}^*$ to obtain $\lambda_{(h)}^*$.
5. Repeat step (4) H times.
6. Compute the $100(1-\gamma)\%$ BayCrI for λ.

2.5. The HPDI Approach

The Bayesian estimation has already been produced in the previous subsection, but in most cases, we have to construct an interval containing the estimated values of parameters with a high probability. HPDI has the property that the probability density of each point inside the interval is higher than that of every point outside it, and so the intervals of the former are the shortest given probability level $(1-\gamma)$ [37]. The *HDInterval* package (version 0.2.2) from the R software was applied at step (6) in Algorithm 4 to calculate the HPDI for λ.

3. Simulation Study and Results

Since a theoretical comparison of the confidence intervals is not possible, a Monte Carlo simulation study was conducted to assess their performances by comparing their coverage probabilities and average lengths. Throughout the simulation study, the nominal confidence level was set at 0.95. The best-performing method for a particular scenario is the one with a coverage probability greater than or close to the nominal confidence level and the shortest average length. Since β_i, $i = 1, 2, \ldots, k$ is the scale parameter, its value was fixed as $\beta_i = 1.0$ without losing any generality. The settings for the sample size and shape parameter are provided in Table 1. The number of simulation runs was 1000 replications with 3000 pivotal quantities for GCI. The following settings were used for BayCrI and HPDI: $H = 1000$; hyperparameters $a_i = b_i = c_i = d_i = 10^{-4}$; and $r_i = 2$ [21].

Table 1. The parameter settings for $k = 3, 5$, and 10.

Scenarios	$(n_1, n_2, \ldots, n_k)$	$(\alpha_1, \alpha_2, \ldots, \alpha_k)$
$k = 3$		
1–6	(30_3)	(0.5_3), $(0.5, 1.0_2)$, (1.0_3), $(1.0_2, 2.0)$, (1.0, 1.5, 2.0), $(1.5, 2.0_2)$
7–12	$(30_2, 50)$	(0.5_3), $(0.5, 1.0_2)$, (1.0_3), $(1.0_2, 2.0)$, (1.0, 1.5, 2.0), $(1.5, 2.0_2)$
13–18	(50_3)	(0.5_3), $(0.5, 1.0_2)$, (1.0_3), $(1.0_2, 2.0)$, (1.0, 1.5, 2.0), $(1.5, 2.0_2)$
19–24	$(50_2, 100)$	(0.5_3), $(0.5, 1.0_2)$, (1.0_3), $(1.0_2, 2.0)$, (1.0, 1.5, 2.0), $(1.5, 2.0_2)$
25–30	(100_3)	(0.5_3), $(0.5, 1.0_2)$, (1.0_3), $(1.0_2, 2.0)$, (1.0, 1.5, 2.0), $(1.5, 2.0_2)$
$k = 5$		
31–36	$(30_2, 50_3)$	$(0.5_3, 1.0, 2.0)$, $(0.5_2, 1.0_2, 1.5)$, $(0.5, 1.0_3, 1.5)$, $(0.5, 1.0_2, 2.0_2)$, $(1.0_3, 1.5_2)$, $(1.0, 1.5, 2.0_3)$
37–42	$(30_2, 50_2, 100)$	$(0.5_3, 1.0, 2.0)$, $(0.5_2, 1.0_2, 1.5)$, $(0.5, 1.0_3, 1.5)$, $(0.5, 1.0_2, 2.0_2)$, $(1.0_3, 1.5_2)$, $(1.0, 1.5, 2.0_3)$
43–48	$(30, 50_2, 100_2)$	$(0.5_3, 1.0, 2.0)$, $(0.5_2, 1.0_2, 1.5)$, $(0.5, 1.0_3, 1.5)$, $(0.5, 1.0_2, 2.0_2)$, $(1.0_3, 1.5_2)$, $(1.0, 1.5, 2.0_3)$
49–54	(50_5)	$(0.5_3, 1.0, 2.0)$, $(0.5_2, 1.0_2, 1.5)$, $(0.5, 1.0_3, 1.5)$, $(0.5, 1.0_2, 2.0_2)$, $(1.0_3, 1.5_2)$, $(1.0, 1.5, 2.0_3)$
55–60	$(50_2, 100_3)$	$(0.5_3, 1.0, 2.0)$, $(0.5_2, 1.0_2, 1.5)$, $(0.5, 1.0_3, 1.5)$, $(0.5, 1.0_2, 2.0_2)$, $(1.0_3, 1.5_2)$, $(1.0, 1.5, 2.0_3)$
$k = 10$		
61–66	$(30_5, 50_5)$	$(0.5_3, 1.0_7)$, $(0.5_3, 1.0_4, 1.5_3)$, $(0.5_3, 1.0_2, 1.5_3, 2.0_2)$, $(1.0_4, 1.5_3, 2.0_3)$, $(1.0_3, 1.5_3, 2.0_4)$, $(1.0_2, 1.5_2, 2.0_6)$
67–72	$(30_5, 50_3, 100_2)$	$(0.5_3, 1.0_7)$, $(0.5_3, 1.0_4, 1.5_3)$, $(0.5_3, 1.0_2, 1.5_3, 2.0_2)$, $(1.0_4, 1.5_3, 2.0_3)$, $(1.0_3, 1.5_3, 2.0_4)$, $(1.0_2, 1.5_2, 2.0_6)$
73–78	$(30_3, 50_4, 100_3)$	$(0.5_3, 1.0_7)$, $(0.5_3, 1.0_4, 1.5_3)$, $(0.5_3, 1.0_2, 1.5_3, 2.0_2)$, $(1.0_4, 1.5_3, 2.0_3)$, $(1.0_3, 1.5_3, 2.0_4)$, $(1.0_2, 1.5_2, 2.0_6)$
79–84	$(50_6, 100_4)$	$(0.5_3, 1.0_7)$, $(0.5_3, 1.0_4, 1.5_3)$, $(0.5_3, 1.0_2, 1.5_3, 2.0_2)$, $(1.0_4, 1.5_3, 2.0_3)$, $(1.0_3, 1.5_3, 2.0_4)$, $(1.0_2, 1.5_2, 2.0_6)$

The simulation results for $k = 3, 5$, and 10 are reported in Tables 2, 3 and 4, respectively. It can be seen that they are similar for these three scenarios, and, thus, we can draw the following conclusions. The coverage probabilities of the GCI, BayCrI, and HPDI confidence intervals were greater than or close to the nominal confidence level of 0.95 under most circumstances whereas those for the MOVER and LS confidence intervals were under in all of the scenarios. As the sample sizes were increased, the coverage probabilities of the MOVER and LS confidence intervals performed better but were still under the nominal confidence level of 0.95. Note that both are based on the MME of α_i, which is highly biased when the sample size is small and α_i is large [17]. When considering the average lengths, those of the LS confidence interval were the shortest under most circumstances, followed by MOVER. However, the coverage probabilities of these two confidence intervals were lower than the nominal confidence level of 0.95 for all cases, and so they failed to meet the requirements. Among the remainder, the average lengths of HPDI were the shortest in all of the circumstances tested whereas those of GCI were the longest. When the sample sizes were increased, the average lengths of all of the confidence intervals became shorter, whereas when the shape parameter was increased, the average lengths of all of the confidence intervals became longer. Overall, HPDI performed the best in the simulation study because it fulfilled the requirements for both criteria.

Table 2. The coverage probabilities and the average lengths of the 95% confidence intervals for the common CV of several BS distributions when $k = 3$.

Scenarios	Coverage Probability					Average Length				
	GCI	MOVER	LS	BayCrI	HPDI	GCI	MOVER	LS	BayCrI	HPDI
1	0.931	0.936	0.851	0.922	0.912	0.1598	0.1588	0.1447	0.1590	0.1566
2	0.956	0.848	0.799	0.953	0.948	0.2887	0.2105	0.1948	0.2873	0.2835
3	0.948	0.942	0.878	0.949	0.943	0.2817	0.2657	0.2516	0.2795	0.2761
4	0.957	0.816	0.727	0.951	0.946	0.4024	0.2583	0.2478	0.3902	0.3844
5	0.961	0.872	0.812	0.958	0.955	0.3815	0.2615	0.2535	0.3703	0.3661
6	0.947	0.910	0.879	0.945	0.950	0.3090	0.2540	0.2501	0.2986	0.2957
7	0.936	0.935	0.877	0.933	0.922	0.1429	0.1420	0.1314	0.1424	0.1404
8	0.949	0.826	0.774	0.947	0.942	0.2788	0.1976	0.1843	0.2775	0.2743
9	0.952	0.944	0.887	0.951	0.943	0.2510	0.2394	0.2291	0.2488	0.2461
10	0.952	0.837	0.781	0.954	0.950	0.3468	0.2308	0.2229	0.3416	0.3381
11	0.946	0.840	0.806	0.947	0.943	0.3222	0.2328	0.2270	0.3165	0.3138
12	0.953	0.925	0.896	0.950	0.948	0.2670	0.2270	0.2242	0.2602	0.2579
13	0.950	0.951	0.897	0.949	0.944	0.1209	0.1206	0.1142	0.1205	0.1191
14	0.953	0.837	0.789	0.952	0.943	0.2237	0.1597	0.1527	0.2231	0.2207
15	0.946	0.945	0.907	0.946	0.939	0.2109	0.2038	0.1975	0.2094	0.2073
16	0.941	0.834	0.783	0.940	0.936	0.2980	0.1981	0.1935	0.2925	0.2889
17	0.949	0.843	0.806	0.950	0.943	0.2800	0.2008	0.1974	0.2758	0.2733
18	0.937	0.906	0.894	0.937	0.932	0.2254	0.1948	0.1932	0.2214	0.2195
19	0.926	0.930	0.889	0.924	0.918	0.1042	0.1037	0.0995	0.1037	0.1025
20	0.948	0.827	0.795	0.950	0.952	0.2101	0.1460	0.1404	0.2091	0.2072
21	0.951	0.942	0.925	0.946	0.941	0.1805	0.1760	0.1720	0.1793	0.1776
22	0.952	0.815	0.765	0.948	0.946	0.2483	0.1685	0.1655	0.2463	0.2441
23	0.946	0.855	0.814	0.944	0.942	0.2286	0.1702	0.1680	0.2263	0.2244
24	0.940	0.917	0.903	0.941	0.937	0.1854	0.1661	0.1652	0.1832	0.1817
25	0.940	0.943	0.909	0.936	0.926	0.0840	0.0837	0.0816	0.0838	0.0828
26	0.948	0.834	0.807	0.949	0.946	0.1581	0.1117	0.1093	0.1572	0.1558
27	0.959	0.959	0.933	0.957	0.954	0.1455	0.1432	0.1411	0.1446	0.1433
28	0.946	0.824	0.812	0.953	0.943	0.2036	0.1389	0.1374	0.2018	0.1999
29	0.951	0.847	0.837	0.950	0.946	0.1919	0.1410	0.1398	0.1897	0.1881
30	0.954	0.932	0.924	0.948	0.948	0.1519	0.1368	0.1363	0.1504	0.1491

Table 3. The coverage probabilities and the average lengths of the 95% confidence intervals for the common CV of several BS distributions when $k = 5$.

Scenarios	Coverage Probability					Average Length				
	GCI	MOVER	LS	BayCrI	HPDI	GCI	MOVER	LS	BayCrI	HPDI
31	0.953	0.706	0.612	0.948	0.942	0.2375	0.1225	0.1147	0.2334	0.2293
32	0.945	0.750	0.664	0.943	0.932	0.2365	0.1417	0.1325	0.2347	0.2323
33	0.944	0.749	0.684	0.947	0.948	0.2569	0.1560	0.1475	0.2550	0.2526
34	0.959	0.631	0.590	0.957	0.950	0.3481	0.1498	0.1427	0.3431	0.3399
35	0.954	0.900	0.840	0.951	0.948	0.2165	0.1767	0.1711	0.2143	0.2124
36	0.958	0.891	0.858	0.960	0.963	0.2198	0.1629	0.1602	0.2159	0.2140
37	0.959	0.609	0.544	0.953	0.943	0.2511	0.1147	0.1080	0.2483	0.2453
38	0.931	0.717	0.634	0.927	0.924	0.2376	0.1327	0.1248	0.2367	0.2346
39	0.953	0.726	0.686	0.952	0.945	0.2477	0.1438	0.1368	0.2460	0.2441
40	0.945	0.568	0.529	0.949	0.938	0.3138	0.1361	0.1304	0.3109	0.3084
41	0.947	0.879	0.828	0.950	0.947	0.1926	0.1594	0.1551	0.1910	0.1893
42	0.954	0.897	0.858	0.954	0.951	0.1874	0.1455	0.1435	0.1846	0.1830

Table 3. *Cont.*

Scenarios	Coverage Probability					Average Length				
	GCI	MOVER	LS	BayCrI	HPDI	GCI	MOVER	LS	BayCrI	HPDI
43	0.949	0.632	0.580	0.948	0.931	0.2153	0.1029	0.0980	0.2137	0.2111
44	0.961	0.748	0.704	0.963	0.955	0.1998	0.1157	0.1104	0.1989	0.1970
45	0.949	0.747	0.688	0.942	0.938	0.2118	0.1292	0.1240	0.2105	0.2086
46	0.945	0.591	0.556	0.946	0.941	0.2746	0.1221	0.1178	0.2726	0.2703
47	0.941	0.891	0.851	0.943	0.941	0.1679	0.1413	0.1384	0.1668	0.1654
48	0.956	0.888	0.868	0.956	0.960	0.1605	0.1291	0.1277	0.1586	0.1573
49	0.941	0.741	0.678	0.939	0.937	0.1964	0.1079	0.1028	0.1931	0.1897
50	0.942	0.787	0.722	0.940	0.932	0.1870	0.1201	0.1148	0.1860	0.1839
51	0.958	0.787	0.740	0.951	0.946	0.2218	0.1357	0.1304	0.2201	0.2181
52	0.953	0.634	0.599	0.958	0.952	0.3062	0.1313	0.1268	0.3014	0.2982
53	0.939	0.878	0.831	0.937	0.934	0.1930	0.1608	0.1568	0.1911	0.1893
54	0.942	0.839	0.806	0.947	0.948	0.2112	0.1509	0.1490	0.2073	0.2054
55	0.958	0.690	0.651	0.954	0.952	0.1704	0.0881	0.0852	0.1687	0.1666
56	0.961	0.744	0.682	0.957	0.952	0.1746	0.1036	0.1000	0.1740	0.1725
57	0.948	0.765	0.728	0.946	0.939	0.1850	0.1133	0.1101	0.1841	0.1826
58	0.950	0.597	0.573	0.950	0.949	0.2480	0.1079	0.1052	0.2464	0.2445
59	0.944	0.897	0.866	0.950	0.945	0.1502	0.1273	0.1252	0.1496	0.1483
60	0.951	0.876	0.849	0.952	0.944	0.1470	0.1168	0.1159	0.1456	0.1444

Table 4. The coverage probabilities and the average lengths of the 95% confidence intervals for the common CV of several BS distributions when $k = 10$.

Scenarios	Coverage Probability					Average Length				
	GCI	MOVER	LS	BayCrI	HPDI	GCI	MOVER	LS	BayCrI	HPDI
61	0.921	0.736	0.634	0.921	0.910	0.1594	0.1060	0.0992	0.1587	0.1573
62	0.932	0.654	0.546	0.925	0.923	0.1975	0.1071	0.1004	0.1954	0.1938
63	0.945	0.534	0.456	0.936	0.930	0.2508	0.1056	0.0994	0.2471	0.2448
64	0.951	0.774	0.695	0.948	0.943	0.1839	0.1229	0.1197	0.1800	0.1783
65	0.958	0.822	0.753	0.955	0.958	0.1773	0.1213	0.1186	0.1736	0.1721
66	0.937	0.840	0.781	0.944	0.948	0.1679	0.1185	0.1164	0.1637	0.1622
67	0.944	0.747	0.617	0.938	0.934	0.1481	0.0976	0.0920	0.1470	0.1459
68	0.933	0.623	0.534	0.931	0.923	0.1908	0.0993	0.0937	0.1899	0.1882
69	0.945	0.516	0.454	0.942	0.937	0.2369	0.0959	0.0909	0.2348	0.2328
70	0.951	0.812	0.760	0.952	0.947	0.1537	0.1081	0.1058	0.1519	0.1506
71	0.953	0.843	0.787	0.957	0.955	0.1482	0.1068	0.1049	0.1460	0.1447
72	0.955	0.870	0.824	0.960	0.958	0.1411	0.1049	0.1034	0.1379	0.1367
73	0.919	0.736	0.648	0.920	0.916	0.1383	0.0913	0.0867	0.1374	0.1363
74	0.931	0.631	0.539	0.929	0.923	0.1785	0.0933	0.0886	0.1777	0.1762
75	0.955	0.582	0.499	0.954	0.944	0.2147	0.0906	0.0864	0.2133	0.2115
76	0.956	0.828	0.788	0.957	0.955	0.1381	0.0992	0.0975	0.1368	0.1356
77	0.953	0.820	0.774	0.948	0.945	0.1324	0.0985	0.0970	0.1308	0.1297
78	0.941	0.874	0.840	0.940	0.937	0.1242	0.0964	0.0953	0.1224	0.1214
79	0.940	0.751	0.689	0.938	0.937	0.1190	0.0797	0.0767	0.1185	0.1174
80	0.956	0.652	0.585	0.956	0.952	0.1494	0.0807	0.0778	0.1489	0.1476
81	0.941	0.557	0.515	0.937	0.934	0.1857	0.0791	0.0766	0.1843	0.1827
82	0.949	0.835	0.786	0.947	0.949	0.1288	0.0916	0.0903	0.1275	0.1265
83	0.944	0.811	0.788	0.942	0.937	0.1233	0.0902	0.0891	0.1221	0.1210
84	0.954	0.868	0.836	0.958	0.960	0.1170	0.0888	0.0879	0.1155	0.1145

4. Application of the Confidence Interval Methods with Real Data

Air pollution is currently one of the most important public health concerns since it causes mortality and morbidity. Of the various air pollutants, PM10 and PM2.5 (PM $\leq$ 2.5 µm) are widely considered to be the most damaging and important. In Chiang Mai, agricultural burning and forest fires during the dry season caused a haze of predominantly PM10 and PM2.5 each year. It begins in early February, peaks in March, and subsides by the end of April. During this time period, the population is significantly impacted by PM2.5 and PM10 pollution, with concentrations substantially above the World Health Organization's recommended levels. The average daily PM10 concentrations from three pollution monitoring stations located in Chiang Mai province: (1) Chang Phueak, (2) Si Phum, and (3) Changkerng were obtained from the Pollution Control Department [38] and selected to assess the performances of the proposed confidence intervals. Since the concentrations of PM10 are always positive and vary depending on factors, such as source, local topography, and local meteorology, they are positively skewed and suitable for fitting to a lognormal, BS, exponential, gamma, or Weibull distribution. It is important to check the suitability of the distribution for the datasets, and so minimum Akaike information criterion (AIC) and Bayesian information criterion (BIC) analyses were conducted.

As reported in Tables 5 and 6, it can be concluded that the BS distribution is suitable for fitting these datasets. The summary statistics for the PM10 concentrations data from the three pollution monitoring stations located in Chiang Mai are provided in Table 7. The estimated common CV was 0.4453. Note that we set $r_i = 2$ and $a_i = b_i = c_i = d_i = 10^{-4}$; $i = 1, 2, \ldots, k$ for BayCrI and HPDI. Table 8 reports the 95% confidence intervals for the common CV of PM10 concentration data from three pollution monitoring stations in Chiang Mai, Thailand. Similar to the simulation results when $(n_1, n_2, n_3) = (30, 30, 30)$, the average length of the LS confidence interval was the shortest, followed by MOVER. However, their coverage probabilities were under the nominal confidence level of 0.95, and so they are not recommended for constructing the confidence interval for the common CV of these datasets. When comparing GCI, BayCrI, and HPDI, although all three provided coverage probabilities greater than or close to the nominal confidence level of 0.95, the latter provided the shortest average length. Hence, HPDI is the most suitable method when considering the coverage probability and the average length together.

Table 5. AIC results for the fitting of five tested distributions.

Distributions	Lognormal	BS	Exponential	Gamma	Weibull
Chang Phueak	334.4613	333.9956	366.2775	335.8974	339.0671
Si Phum	326.8568	326.2713	351.1782	328.6437	331.6056
Changkerng	301.5002	301.1981	337.6783	303.2755	307.9442

Table 6. BIC results for the fitting of five tested distributions.

Distributions	Lognormal	BS	Exponential	Gamma	Weibull
Chang Phueak	337.3293	336.8636	367.7115	338.7653	341.9351
Si Phum	329.7248	329.1392	352.6122	331.5116	334.4735
Changkerng	304.3682	304.1661	339.1123	306.1435	310.8121

Table 7. Summary statistics for the PM10 data.

Area	n	Min.	Median	Mean	Max.	Variance	CV
Chang Phueak	31	61	122	131.0323	282	3310.556	0.4391
Si Phum	31	42	85	102.7097	248	2798.680	0.5151
Changkerng	31	43	81	82.6129	182	1191.045	0.4177

Table 8. The 95% confidence interval for the common CV of PM10 data from three pollution monitoring stations in Chiang Mai, Thailand.

Methods	Interval	Length
GCI	0.3788–0.5163	0.1375
MOVER	0.3860–0.5212	0.1352
LS	0.3698–0.4972	0.1274
BayCrI	0.3796–0.5160	0.1364
HPDI	0.3727–0.5059	0.1332

5. Conclusions

Herein, we propose confidence intervals for the common CV of several BS distributions constructed by using the GCI, MOVER, LS, BayCrI, and HPDI approaches. Their performances were studied numerically through Monte Carlo simulation in terms of their coverage probabilities and average lengths. The simulation results indicate that the coverage probabilities for GCI, BayCrI, and HPDI were greater than or close to the nominal confidence level, while HPDI produced the shortest average length for all cases. Therefore, HPDI is appropriate for constructing the confidence interval for the common CV of several BS distributions. Meanwhile, the coverage probabilities of the MOVER and LS confidence intervals were under the nominal confidence level, and so neither can be recommended as a solution for this scenario. Furthermore, when applying the methods to analyze PM10 concentrations from three pollution monitoring stations in Chiang Mai, Thailand, the results are in accordance with those from the simulation study.

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