

Article

Iterative Borel Summation with Self-Similar Iterated Roots

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Abstract: Borel summation is applied iteratively in conjunction with self-similar iterated roots. In general form, the iterative Borel summation is presented in the form of a multi-dimensional integral. It can be developed only numerically and is rarely used. Such a technique is developed in the current paper analytically and is shown to be more powerful than the original Borel summation. The self-similar nature of roots and their asymptotic scale invariance allow us to find critical indices and amplitudes directly and explicitly. The locations of poles remain the same with the uncontrolled self-similar Borel summation. The number of steps employed in the course of iterations is used as a continuous control parameter. To introduce control into the discrete version of the iterative Borel summation, instead of the exponential function, we use a stretched (compactified) exponential function. For the poles, considering inverse quantities is prescribed. The simplest scheme of the iterative Borel method, based on averaging over the one-step and two-step Borel iterations, works well when lower and upper bounds are established by making those steps. In the situations when only a one-sided bound is found, the iterative Borel summation with the number of iterations employed as the control works best by extrapolating beyond the bound. Several key examples from condensed matter physics are considered. Iterative application of Borel summation leads to an improvement compared with a conventional, single-step application of the Borel summation.

Keywords: self-similar root approximants; critical amplitude; critical index; optimization; control parameters



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1. Preliminaries

Consider the case when a certain problem could be reduced to explicitly finding a real, sign-definite, positive-valued function $f(x)$ of a real variable x . Let the function possess the power-law asymptotic behavior characterized by the large-variable exponent β and amplitude A

$$f(x) \simeq Ax^\beta \quad (x \rightarrow \infty). \quad (1)$$

The class of power laws, $f(x) = x^\beta$, is scale-invariant, i.e.,

$$f(\lambda x) = \Lambda f(x), \quad (2)$$

where $\Lambda = \lambda^\beta$. The property (2) becomes an asymptotic scale invariance if it holds only in some limit, say of $x \rightarrow \infty$, as in relation (1). The most studied example of power laws is represented by critical phenomena in thermodynamic systems. That explains why we use terminology with critical index β and critical amplitude A . The approach to infinity is often of primary interest, and one should be able to calculate A and β .

To properly take into account the asymptotic scale invariance of the physical properties, one should think of approximation schemes that inherently possess such property. Such analytical self-similar Borel approximations with the asymptotic property of scale invariance (1), were first discussed in [1] and developed further in [2].

In practice, the equations defining $f(x)$ are very complicated. Even to such an extent that only truncated asymptotic expansion

$$f(x) \simeq f_k(x) \quad (x \rightarrow 0), \quad (3)$$

at small variables could be extracted in the form of finite expressions

$$f_k(x) = \sum_{n=0}^k a_n x^n, \quad (4)$$

where $a_0 > 0$. For simplicity, one may consider the function $f(x)/a_0$.

The truncations are reflections of the complete, divergent or convergent series. Often they are unbearably short. Together with the asymptotic conditions (1), they form the basis of resummation procedures. To this end, the Borel summation of different shades could be applied to define the effective sums of the functions with known truncation (4) [3–7]. Extensive references can be found in our recent paper [2].

Indubitably, the truncation with a finite number of coefficients should be extended to all a_n . Such extension is made either by means of Padé approximants [8–10] or from the knowledge of large- n asymptotics of a_n [5]. In many instances, only a pitiful number of terms in expansion is available, and no information on high-order a_n is given whatsoever. In addition, Padé approximants are not able to capture scale invariance with arbitrary β [11] and are used for extrapolation to finite values of variable x . Self-similar approximants, on the other hand, were designed to solve the problems with arbitrary β [11].

The method of iterative Borel summation, based on repeated application of the conventional Borel summation, was suggested in [4]. The result of iterative Borel summation is expressed in the form of a multi-dimensional integral. It is rarely used because of technical difficulties and can be developed only numerically. However, one can expect that such a technique, if developed analytically, can be more powerful than the original Borel summation.

- (1) We suggest below the practical way to combine iterative Borel summation with the self-similar approximants. The asymptotic scale invariance of the self-similar approximants leads to factorization of the integrals in the limit of large x and to the relatively simple analytical expressions for critical amplitudes A and critical indices β .
- (2) Combined with some optimization procedures, the found integrals are applied for the calculation of critical properties and indices. The controls are designed in order to improve the convergence of the summation methods. To this end, the minimal-difference conditions on critical amplitudes are imposed.
- (3) The number of steps in the course of Borel-iterations could be considered as a continuous control parameter. However, in order to introduce control into the discrete version of the iterative Borel summation, we suggest using instead of the exponential e^{-x} , a stretched (compacted) exponential function e^{-x^u} , with arbitrary positive parameters u .
- (4) Some physical cases with fast-growing a_n , exemplified by the Schwinger model energy gap (Example 1), anharmonic oscillator ground state energy (Example 7), and expansion factor of three-dimensional polymer (Example 6), are considered. The Bose condensation temperature (Example 4) and Schwinger model ground state energy (Example 5), with slow decay of the coefficients, are discussed as well. The situation with fast-decaying coefficients, such as of the Lieb–Liniger model ground state energy (Example 2), and optical polaron mass (Example 3), is discussed too.

The optimization conditions in the general form of minimal-difference or minimal-derivative conditions were derived by V.I. Yukalov (1976). Yukalov also pioneered the notion of self-similarity and development of the approximation theory along the lines of a field-theoretic renormalization group.

Below, the method of iterative Borel summation is combined with self-similar approximants. The property of asymptotic scale invariance pertinent to the self-similar approximations leads to factorization of the multi-dimensional integrals in the limit of large x . The critical properties, indices and amplitudes can be found analytically. We argue that the iterative application of Borel summation could lead to an improvement compared with a single-step Borel summation. The number of iterations by itself can serve as a control parameter. Alternatively, the control parameter could be introduced by modifying the form

of Borel-transform by putting the stretched (compact) exponential function in place of the exponential.

2. Discrete and Continuous Version of Iterative Borel Summation

The iterative Borel summation starts with the transformation of the series (4)

$$B_k(x, b) = \sum_{n=0}^k \frac{a_n}{(\Gamma(1+n))^b} x^n, \tag{5}$$

defined following [4]. In the book [4], only positive integers b are considered. We will also consider such a case but will not be restricted to it. In view of introducing control into the summation procedure, b can become real. With $b = 1$, we return to the conventional Borel transform. With a larger integer b , the transform is meant to tame the series with a_n growing as $(n!)^b$ (see, also, [5]). However, it also makes sense for arbitrary real and finite b .

The resulting series can be summed by means of self-similar approximants $B_k^*(x, b)$ [1,2,11]. Then, the sought function $f(x)$ is approximated by the expression

$$f_k^*(x, b) = \int_0^\infty \dots \int_0^\infty e^{-t_1} e^{-t_2} \dots e^{-t_b} B_k^*(xt_1 t_2 \dots t_b, b) dt_1 dt_2 \dots dt_b. \tag{6}$$

Assume that we know the value of the critical index β . The self-similar approximant at large x behaves as

$$B_k^*(x, b) \simeq C_k(b) x^\beta \quad (x \rightarrow \infty). \tag{7}$$

Therefore, the sought Function (6), in the limit $x \rightarrow \infty$, reduces to

$$f_k^*(x, b) \simeq C_k(b) x^\beta \int_0^\infty \dots \int_0^\infty e^{-t_1} t_1^\beta e^{-t_2} t_2^\beta \dots e^{-t_b} t_b^\beta dt_1 dt_2 \dots dt_b. \tag{8}$$

As a result, the large-variable behavior of the function acquires the form

$$f_k^*(x, b) \simeq A_k(b) x^\beta \quad (x \rightarrow \infty), \tag{9}$$

with the amplitude

$$A_k(b) = C_k(b) (\Gamma(1 + \beta))^b. \tag{10}$$

The exponent β appearing in the course of the iterative Borel transform is inherited by the approximation for the sought function (see also [5] for the case of Borel transform). In the course of calculations, we have to recognize that

$$\int_0^\infty e^{-t} t^\beta dt = \Gamma(1 + \beta).$$

Now, we have to specify the self-similar approximant $B_k^*(x, b)$ to be used in concrete calculations. The self-similar iterated root approximants [11] have the following form:

$$\mathcal{R}_k^*(x, b) = a_0 \left(\left(\left((1 + \mathcal{P}_1 x)^2 + \mathcal{P}_2 x^2 \right)^{3/2} + \mathcal{P}_3 x^3 \right)^{4/3} + \dots + \mathcal{P}_k x^k \right)^{\beta/k}, \tag{11}$$

defining the parameters $\mathcal{P}_j = \mathcal{P}_j(b)$, from the asymptotic equivalence with the transformed series (5), e.g., for $k = 4$

$$\mathcal{R}_4^*(x, b) = a_0 \left(\left(\left((1 + \mathcal{P}_1 x)^2 + \mathcal{P}_2 x^2 \right)^{3/2} + \mathcal{P}_3 x^3 \right)^{4/3} + \mathcal{P}_4 x^4 \right)^{\beta/4}.$$

This gives the large- x asymptotic form

$$\mathcal{R}_k^*(x, b) \simeq C_k(b) x^\beta \quad (x \rightarrow \infty), \tag{12}$$

where the marginal amplitudes $C_k(b)$ are

$$C_k(b) = a_0 \left(\left((\mathcal{P}_1(b)^2 + \mathcal{P}_2(b))^{3/2} + \mathcal{P}_3(b) \right)^{4/3} + \dots + \mathcal{P}_k(b) \right)^{\beta/k}. \quad (13)$$

e.g., for $k = 4$

$$C_4(b) = a_0 \left(\left((\mathcal{P}_1(b)^2 + \mathcal{P}_2(b))^{3/2} + \mathcal{P}_3(b) \right)^{4/3} + \mathcal{P}_4(b) \right)^{\beta/4}.$$

In the discrete case of positive integer b , we consider the most simple and natural sequences

$$A_k^* = \frac{A_k(1) + A_k(2)}{2}, \quad (14)$$

with positive integers $k = 1, 2, 3, \dots$. Of course, it could be easily extended to larger integers, but it seems that for small integers the approach already works well enough. The approach is also relatively easily amenable to optimization by means of some optimal control technique considered in Section 3.

The simplest technical scheme of iterative Borel summation, based on averaging over the one-step and two-step Borel iterations (14), works when lower and upper bounds are established by $A_k(1)$ and $A_k(2)$. The sought answer A_k^* is obtained by interpolation between the two values. With only the discrete, positive integer b , there is no way to control the convergence but only to observe it (or not).

In the situations when only a one-sided bound is established, we suggest the iterative Borel method with explicit optimization (15) and (16), with the number of iterations employed as the control. The method is expected to be able to extrapolate beyond the bound. Indeed, consider b , the number of iterations, as a continuous control parameter. As the integral (8) is easy to define for integer b , introducing continuous b means to interpolate smoothly between the values of integral for discrete b . Such an approach is similar to the way the Γ -function interpolates the factorial defined for discrete numbers [12]. It seems reasonable to think that we can, in principle, after making an integer number of iterations/steps, move backward (or forward) just a little bit, making only a part of a step.

Formally, with arbitrary b we are confronted with the enormously difficult problem of how to define the integral over the continuum of variables [13]. However, we are able to approach it constructively using explicit b analytical results for an integer number of iterations b to interpolate to continuous b and only in the limit-case of large x . By introducing the continuous b , we acquire a technical advantage since it becomes possible to find b from some optimization conditions, designed to impose convergence on the sequences of the $A_k(b)$ -type [14].

In the spirit of [14], the differences

$$\Delta_{k,k+1} = A_{k+1}(b_k) - A_k(b_k), \quad (15)$$

with positive integers $k = 1, 2, 3, \dots$, are going to be the object of optimization. A set b_k of control parameters is defined as follows,

$$| \Delta_{k,k+1}(b_k) | = \min_b | \Delta_{k,k+1}(b) |, \quad (16)$$

with positive integer $k = 1, 2, 3, \dots$

However, the problems of high interest for physical applications with $\beta = -1, -2$, appear to be divergent, or rather indeterminate, when treated by the self-similar Borel summation techniques, since at $\beta = -1, -2, \dots$, the Γ -function has poles. In such cases, some other types of summation could be used, as in the paper [2]. Otherwise, outside of such cases, we can speak about determinate problems and employ the techniques already discussed without reservations.

There is a rather simple way to bypass the divergence. Let us consider an inverse function to $f(x)$ so that for large x

$$f(x)^{-1} \simeq Y x^{-\beta}. \quad (17)$$

With such a critical index $-\beta$, there is no divergence in the formulas for negative integers. Now, the resummed value of the critical amplitude Y can be found by embracing the techniques developed for calculating the amplitude A in Sections 2 and 3.

Again, at small x , the inverse function $f(x)^{-1}$ could be represented as the truncated power series. After the resummation procedure with some self-similar approximants, applied to such series with the condition (17) at infinity, we will arrive at the resummed amplitude Y^* , and

$$A^* \equiv \frac{1}{Y^*},$$

giving the sought value of the critical amplitude.

Thus, taking the inverse of the sought quantity restores the finiteness of the sought quantity by discarding the poles. Consider that Weierstrass (1865) suggested defining the inverse as the way to avoid discussing poles in the Γ -function (see [12]). We adapt the same trick here but for purposes of resummation. A more sophisticated variant would include a power transform with the power to be determined from some additional optimization (see [15] and references therein).

We should also mention some valiant efforts by Luschny to find the definition of Γ -function in such a way that poles are erased from the start (see his website and the paper therein [12]). However, the standard definition through the integral is a part of the Borel technique employed in the current paper. However, in principle, one can avoid integration altogether if a Borel transform is corrected by means of Padé approximants [1], introduced to restore asymptotic equivalence with the original truncation (4). Then it is possible to change the definition of the Γ -function in the transformation of the series and perform an inverse transformation.

Example 1. *The massive Schwinger model in Hamiltonian lattice theory formulation [16,17] describes quantum electrodynamics in two space-time dimensions. It also mimics quantum chromodynamics by including such refined features as the confinement, chiral symmetry breaking, and a topological vacuum. It is also perhaps the simplest non-trivial gauge theory, and this makes it a touchstone for the new techniques in high-energy physics. The spectrum of bound states is of interest in the Schwinger model.*

Let us consider the energy gap between the lowest and second excited states of the scalar boson as a function $\Delta(z)$ of the variable $z = (1/ga)^4$, where g is a coupling parameter and a , lattice spacing. This energy gap at small z can be represented as a series

$$2\Delta(z) \simeq \sum_n a_n z^n \quad (z \rightarrow 0), \quad (18)$$

with a rapid increase using absolute value coefficients

$$a_0 = 1, \quad a_1 = 6, \quad a_2 = -26, \quad a_3 = 190.66667, \quad a_4 = -1756.66667,$$

$$a_5 = 18048.33651, \quad a_6 = -197905.20008, \quad a_7 = 2.267368 * 10^6,$$

and so on, up to the 13th order, as included in [17]. In the continuous limit, where the lattice spacing tends to zero, the variable z tends to infinity, and the gap acquires the limiting form of a power-law

$$\Delta(z) \simeq 1.1284 z^{1/4} \quad (z \rightarrow \infty), \quad (19)$$

with the large-variable critical amplitude $A = 1.1284$.

In the case of discrete iterations, using formula (14), we obtain for the large-variable critical amplitudes the following estimates,

$$\begin{aligned}
 A_1^* &= 0.9562, & A_2^* &= 1.015, & A_3^* &= 1.0481, & A_4^* &= 1.0679, & A_5^* &= 1.0818, \\
 A_6^* &= 1.0916, & A_7^* &= 1.0992, & A_8^* &= 1.1051, & A_9^* &= 1.1099, & A_{10}^* &= 1.1139, \\
 A_{11}^* &= 1.1172, & A_{12}^* &= 1.12, & A_{13}^* &= 1.1224.
 \end{aligned}$$

The above sequence shows a good numerical convergence to the value of $A^* = 1.1224$. The upper bound achieved in a conventional one-step Borel summation, $A_{13}(1) = 1.16796$, is sensible too. In two steps, we also find a reasonable lower bound, $A_{13}(2) = 1.07689$.

The best result $A \approx 1.268$ achieved by optimization, according to (15) and (16), is inferior. It appears to be close to the number 1.25(15) quoted in the paper [17]. We have also applied optimal Mittag–Leffler summation [2] and estimated in the tenth order that $A \approx 1.2476 \pm 0.0077$, also close to the estimates of [17]. The methods employed in [17] are very sophisticated, but the problem is notoriously hard, as is also pointed out in [17]. However, the application of the discrete version of iterative Borel summation works well—even better than the finite lattice result of 1.14(3) [17].

Example 2. Consider a one-dimensional Bose gas with contact interactions quantified by the non-dimensional coupling parameter g . The ground-state energy $E(g)$ of the Lieb and Liniger model [18] can be written according to the latest results of [19] in the variables $e(x) \equiv E(x^2)$, $g \equiv x^2$, as follows

$$\begin{aligned}
 e(x) \simeq & x^2 (1 - 0.4244131815783876 x + 0.06534548302432888 x^2 - \\
 & 0.001587699865505945 x^3 - 0.00016846018782773904 x^4 - \\
 & 0.00002086497335840174 x^5 - 3.1632142185373668 \cdot 10^{-6} x^6 - \\
 & 6.106860595675022 \cdot 10^{-7} x^7 - 1.4840346726187777 \cdot 10^{-7} x^8) .
 \end{aligned} \tag{20}$$

In the limit of very strong interactions $g \rightarrow \infty$, another exact result

$$E(\infty) = \frac{\pi^2}{3} \approx 3.289868, \tag{21}$$

was found by Tonks and Girardeau. In this case, $\beta = -2$, and we are confronted with the indeterminate problem while attempting to reconstruct the strong-coupling amplitude based on the information from the weak-coupling limit. Here, also $E(\infty) = A$.

Using the Formula (14), but in application to the inverse series and then taking the inverse as discussed above for the indeterminate case, we get rather poor results for the large-variable critical amplitudes, with the best number $A_8^* \approx 4.91$. The result achieved in conventional one-step Borel summation, $A_8(1) = 4.506$, is a little better. The sequence of one-step approximations seems to converge rapidly and monotonously, improving with increasing k . However, the number of terms in the truncated series is still not enough to closely approach the correct result.

Much better results are achieved by optimization according to (15) and (16). In higher orders, we arrive at

$$A_6 = 3.0642, \quad A_7 = 3.3192, \quad A_8 = 3.3898,$$

curiously corresponding to the negative optimal values of the parameter b , i.e.,

$$b_5 = -0.7588, \quad b_6 = -0.4483, \quad b_7 = -0.2875.$$

That means an increase in the coefficients a_n by the iterative Borel-transform. Thus, taking the inverse of the sought quantity restores the finiteness of the sought quantity at $\beta = -2$ by discarding the pole in the Γ -function.

Generally speaking, this particular problem of extrapolation is not simple at all. We found that conventional Padé approximants $P_{n,n+2}$ applied to the original series (21) failed completely. The same is true for the self-similar Borel method realized by means of factor approximants [1] but applied to the inverse of (21), as prescribed in the discussion above for the indeterminate case. The optimal Borel–Leroy method of [2] fails to produce an optimum.

However, the performance of the optimal Mittag–Leffler method, $A \approx 3.5195$, obtained from the minimal-derivative condition imposed in the eight-order following the techniques of [2] is better.

The latter result is close to results brought by application of the iterated roots to the original series (even without Borel transform),

$$A_6 = 3.8302, \quad A_7 = 3.6433, \quad A_8 = 3.527.$$

Example 3. Consider the problem of the effective mass of the Fröhlich optical polaron. A perturbation theory in powers of the electron-phonon coupling constant g is well developed [20]. It gives the following expansion at weak couplings,

$$m(g) \simeq 1 + \frac{1}{6}g + 0.0236276g^2 \quad (g \rightarrow 0). \tag{22}$$

For strong couplings, the effective mass behaves as a power law [20],

$$m(g) \simeq Ag^4 \quad (g \rightarrow \infty), \tag{23}$$

with the amplitude

$$A = 0.022702. \tag{24}$$

For amplitude A , we have a determinate problem with $\beta = 4$. Using formula (14), we obtain rather poor results for the large-variable critical amplitudes, with $A_2^* = 0.00014864$. The best result is achieved by optimization according to (15) and (16),

$$A_2(b_1) = A_1(b_1) = 0.019841 \quad (b_1 = 2.7654).$$

Again, the problem appears to be hard since all other methods of [1,2], and of the current paper, fail when applied to the same number of terms. Only by adding one more trial term to the truncation (22), and by applying the method of continued root approximants, did we obtain another reasonable estimate from above, $A \approx 0.0252$ [15].

3. Discrete Version of Iterative Borel Summation with Control

In order to introduce control into the discrete version of the iterative Borel summation, we suggest taking another function instead of the conventional e^{-t} . To this end, one can consider a stretched (for $0 < u < 1$), or compacted (for $u > 1$), exponential function e^{-t^u} , with arbitrary positive parameters u . The purpose of such substitution is to improve the convergence of the summation methods.

The functions of such type were mentioned by Hardy on page 85 of the book [3]. Even more generally, a two-parameter replacement $e^{-t^u} t^{a-1}$ was considered. For the purposes of control, the simpler form with $a = 1$ will suffice. With $u = 0$, we arrive at the Borel–Leroy summation already discussed in [1,2]. However, if we try the function from the exponential class e^{-ut} [21], the final results would not depend on u and can not be controlled.

The idea can be formalized if we consider the series with a_n behaving as $\left(\frac{\Gamma(\frac{1+n}{u})}{u}\right)^b$. In order to simplify the problem, we can remove such dependencies for any positive integer b . To this end, consider the following generalization of the iterative Borel summation. The generalized iterative Borel summation starts with the transformation of the series (4), defined as

$$B_{k,b}(x, u) = \sum_{n=0}^k \frac{a_n u^b}{\left(\Gamma(\frac{1+n}{u})\right)^b} x^n. \tag{25}$$

Compaction will tend to increase the coefficients and stretching will tend to decrease them. Such an effect is qualitatively different from the effect of the Borel–Leroy transform, which always decreases the coefficients a_n .

The resulting series can be summed by means of self-similar approximants $B_{k,b}^*(x, u)$ [1,2,11], designed as asymptotically equivalent to the transform (25). The sought function is approximated by the expression with a stretched (compacted) exponential in place of the familiar exponential, i.e.,

$$f_{k,b}^*(x, u) = \int_0^\infty \dots \int_0^\infty e^{-t_1^u} e^{-t_2^u} \dots e^{-t_b^u} B_{k,b}^*(xt_1t_2 \dots t_b, u) dt_1 dt_2 \dots dt_b. \tag{26}$$

The self-similar approximant at large x behaves as

$$B_{k,b}^*(x, u) \simeq C_{k,b}(u)x^\beta \quad (x \rightarrow \infty). \tag{27}$$

In addition, the sought function (26), in the limit $x \rightarrow \infty$, reduces to

$$f_{k,b}^*(x, u) \simeq C_{k,b}(u)x^\beta \int_0^\infty \dots \int_0^\infty e^{-t_1^u} t_1^\beta e^{-t_2^u} t_2^\beta \dots e^{-t_b^u} t_b^\beta dt_1 dt_2 \dots dt_b. \tag{28}$$

The large-variable behavior of the function acquires the form

$$f_{k,b}^*(x, u) \simeq A_{k,b}(u)x^\beta \quad (x \rightarrow \infty), \tag{29}$$

with the amplitude

$$A_{k,b}(u) = C_{k,b}(u) \left(\frac{\Gamma(\frac{1+\beta}{u})}{u} \right)^b. \tag{30}$$

In the course of calculations, say with $b = 1$, we have to recognize that

$$\int_0^\infty e^{-t^u} t^\beta dt = \frac{\Gamma(\frac{1+\beta}{u})}{u}.$$

By substituting the approximant or simply its asymptotic equivalent (25) into Formula (26), and also formally changing the order of summation and integration [6], we return to the asymptotic equivalence with the original series (4). Obviously, as $u = 1$ we recover the iterative Borel summation (6).

Assume that the critical index β is known. Let us once again employ for $f_{k,b}^*(x, u)$ the self-similar iterated root approximants [11], which have the following form

$$\mathcal{R}_{k,b}^*(x, u) = \frac{a_0 u^b}{\left(\Gamma(\frac{1}{u})\right)^b} \left(\left(\left((1 + \mathcal{P}_1 x)^2 + \mathcal{P}_2 x^2 \right)^{3/2} + \mathcal{P}_3 x^3 \right)^{4/3} + \dots + \mathcal{P}_k x^k \right)^{\beta/k}, \tag{31}$$

with the parameters $\mathcal{P}_j = \mathcal{P}_j(b, u)$, defined from the asymptotic equivalence with the transformed series (25). The large- x asymptotic form is easily found,

$$\mathcal{R}_{k,b}^*(x, u) \simeq C_{k,b}(u)x^\beta \quad (x \rightarrow \infty), \tag{32}$$

where the marginal amplitudes $C_{k,b}(u)$ are

$$C_{k,b}(u) = \frac{a_0 u^b}{\left(\Gamma(\frac{1}{u})\right)^b} \left(\left((\mathcal{P}_1(b, u)^2 + \mathcal{P}_2(b, u))^{3/2} + \mathcal{P}_3(b, u) \right)^{4/3} + \dots + \mathcal{P}_k(b, u) \right)^{\beta/k}. \tag{33}$$

In order to calculate the critical amplitudes, we analyze the unconventional differences for the critical amplitudes in k -th order, with $b = 1, 2$, in the spirit of [14],

$$\Delta_{k,2-1}(u_k) = A_{k,2}(u_k) - A_{k,1}(u_k). \tag{34}$$

The differences could measure the distance between approximations with different numbers of discrete iteration steps but based on the same number of terms k from the (25). Here, again, we already expect the convergence at the starting two steps of the procedure, and strive to control the convergence through the explicit control parameter. In the case of the discrete method (14), the control is implicit since it is expected that the second step is

already “optimal” as it should be able to bring an improvement. The role of further steps is yet to be investigated, but we first have to stumble on the proper example.

A set u_k of control parameters is defined as follows,

$$|\Delta_{k,2-1}(u_k)| = \min_u |\Delta_{k,2-1}(u)|, \tag{35}$$

with positive integers $k = 1, 2, 3, \dots$. The controls are devised to minimize the differences. Composing the sequences prescribed by such formulas, we find the related approximate values u_k for the control parameters.

It is possible to investigate different, conventional sequences of $\Delta_{k,n}$

$$\Delta_{k,k+1} = A_{k+1,1}(u_k) - A_{k,1}(u_k), \tag{36}$$

with positive integers $k = 1, 2, 3, \dots$, as described in [14]. Such differences could measure the distance between approximations with the same number of discrete iteration steps but based on the different number of terms from the (25).

A set u_k of control parameters is defined as follows,

$$|\Delta_{k,k+1}(u_k)| = \min_u |\Delta_{k,k+1}(u)|, \tag{37}$$

with positive integers $k = 1, 2, 3, \dots$

Example 4. Below, we calculate the critical amplitude for the ground state of the Schwinger model from the expansions given dependent on the dimensionless variable $x = m/g$. Here m stands for electron mass and g is the dimensional coupling parameter. The energy of a vector boson f mass $M(x)$ is $E = M - 2m$.

The expansion at small- x for the ground-state energy is given by the following truncation [22–25],

$$E(x) \simeq 0.5642 - 0.219x + 0.1907x^2 \quad (x \rightarrow 0). \tag{38}$$

In the large- x limit [25–27], there is another truncation

$$E(x) \simeq 0.6418x^{-1/3} \quad (x \rightarrow \infty). \tag{39}$$

Let us add to the expansion (38) one more trial, cubic term a_3x^3 , with a very small a_3 set to zero. Using Formula (14), we obtain an, at most, reasonable estimate for the large-variable critical amplitude $A_3^* = 0.7436$.

The optimization according to (34) and (35) fails altogether with no non-trivial solutions to the problem. One-step Borel summation, however, gives a rather good number, $A_{3,1}(1) = 0.6562$. Good results are also achieved by the optimization according to (15) and (16)

$$A_2 = 0.564, \quad A_3 = 0.6333.$$

An even better result is achieved by optimization according to (36) and (37). In the third order, we arrive at

$$A_{3,1}(u_2) = A_{2,1}(u_2) = 0.6442 \quad (u_2 = 1.5449).$$

The latter result is almost as good as our previous estimate 0.6426 [28], which, in addition, had taken into account the known values of the correction-to-scaling exponents.

For comparison, we also calculated A by the optimal methods of [2]. The results from the new methods appear to be better than one can find from the optimal Borel–Leroy summation, $A \approx 0.6$. It is also better than $A \approx 0.62$ from the optimal Mittag–Leffler summation.

Example 5. The ideal uniform Bose gas is unstable below the condensation temperature T_0 . The repulsive atomic interactions stabilize the system. Introducing interactions shifts the transition temperature T_0 to the value of T_c . The shift $\Delta T_c \equiv T_c - T_0$ is considered as linear in the parameter $\gamma \equiv \rho^{1/3}a_s$. Namely,

$$\frac{\Delta T_c}{T_0} \simeq c_1 \gamma \quad (\gamma \rightarrow 0),$$

where a_s stands for atomic scattering length, and ρ is gas density. Monte Carlo simulations (see [29,30] and multiple references therein), give $c_1 = 1.3 \pm 0.05$. The coefficient c_1 can be defined [31–33] as the strong-coupling limit

$$c_1 = \lim_{g \rightarrow \infty} c_1(g) \equiv A. \tag{40}$$

of a function $c_1(g)$ with the expansion

$$c_1(g) \simeq 0.223286g - 0.0661032g^2 + 0.026446g^3 - 0.0129177g^4 + 0.00729073g^5, \tag{41}$$

as the effective coupling parameter $g \rightarrow 0$. The problem of finding A is undetermined.

Using Formula (14), but in application to the inverse series and then taking the inverse as discussed above for the indeterminate case, we get rather reasonable results for the large-variable critical amplitudes, with the number $A_4^* \approx 1.30288$, obtained after resummation in the fourth order of perturbation theory.

A close result is achieved by optimization according to (34) and (35), as in the fourth order, we arrive at

$$A_{4,2}(u_4) = A_{4,1}(u_4) = 1.33898, \quad (u_4 = 1.34757).$$

The results obtained above by the two methods agree with Monte–Carlo simulations and appear to be of the same quality as the results from the optimal Borel–Leroy summation, $A \approx 1.28676$ [2]. It is particularly close to $A \approx 1.33967$, obtained by the optimal Mittag–Leffler summation [2].

4. From Amplitude to Index

The title of Section 4 is meant to say that the critical indices can be found by using the very same techniques that are already developed for critical amplitudes.

Let us deal now with a function with power-law asymptotic behavior

$$f(x) \simeq Ax^\beta \quad (x \rightarrow \infty),$$

but with unknown critical index β and amplitude A . The critical exponent can be expressed as the limit of a diff-log function $\psi(x)$

$$\beta = \lim_{x \rightarrow \infty} x \frac{d}{dx} \ln f(x) \equiv \lim_{x \rightarrow \infty} x\psi(x), \tag{42}$$

where $\psi(x) \equiv \frac{d}{dx} \ln f(x)$, as shown, e.g., in [11,34,35].

When the small-variable expansion for the original function is given by the sum $f_k(x)$, we can find the small-variable expression for the diff-log function

$$\psi_k(x) = \frac{d}{dx} \ln f_k(x), \quad (x \rightarrow 0),$$

The expansion of $\psi_k(x)$ leads to the truncated series

$$\psi_k(x) = \sum_{n=0}^k d_n x^n. \tag{43}$$

However, for $x \rightarrow \infty$, we have to require

$$\psi(x) \simeq \beta x^\delta,$$

where the “critical amplitude” is the sought critical index β , and the “critical index” is fixed to $\delta \equiv -1$, to make the limit (42) existent.

In such formulation, we can calculate the critical index β similar to some peculiar amplitude and apply the technique of self-similar approximants described above for the critical amplitudes. However, the case of $\delta = -1$, appears to be divergent, or indeterminate, when treated by the self-similar Borel summation technique of [1], since at $\delta = -1$, the Γ -function has a pole. In such a case, some other types of summation could be used, as in the paper [2]. There is also some simple way to avoid the divergence.

To this end, let us consider an inverse diff-log function, so that for large x

$$\psi(x)^{-1} \simeq \gamma x^{-\delta}. \tag{44}$$

With such a critical index, there is no divergence in the formulas of [1]. Now, the resummed value of the critical amplitude γ can be found by fully embracing the techniques developed for calculating amplitude A in Sections 2 and 3. We will literally follow the same formulas with A faithfully replaced by β and the known critical index set to unity.

Of course, at small x , the inverse diff-log function could be represented as the truncated power-series. After the resummation procedure with some self-similar approximants, applied to such series with the condition (44) at infinity, we will arrive at the resummed amplitude γ^* , and

$$\beta^* \equiv \frac{1}{\gamma^*},$$

where β^* gives the sought value of the critical index β . The iterative Borel summation is fully applicable now as well. For instance, the program just sketched could be adapted for the factor approximants [1], giving even more methods of self-similar Borel summation, in addition to the four methods of finding the critical indices discussed in [1].

Example 6. Consider the problem of critical index calculation for the much-revered quantum anharmonic oscillator. The expansion of ground-state energy $E(g)$ in powers of the coupling g results in a divergent series. The coefficients a_n in very high orders can be found in Refs. [36,37]. The strong-coupling behavior is

$$E(g) \simeq 0.667986 g^{1/3} \quad (g \rightarrow \infty). \tag{45}$$

First, let us discuss the results of a discrete iteration. Using formula (14), but in application to the inverse series and then taking the inverse as discussed above for the indeterminate case, we get rather reasonable results for the large-variable critical amplitudes, with the best number

$$\beta_4^* = 0.334432,$$

obtained in the fourth order of perturbation theory. The calculations stopped naturally at $k = 4$ because for $k > 4$ in two-step Borel summation, only complex solutions were obtained. The result achieved in conventional one-step Borel summation, $\beta_4(1) = 0.382946$, is sensible too. In two steps, we also find a reasonable number, $\beta_4(2) = 0.296829$. Together, they provide an upper and lower bound on the energy.

A close result is achieved by optimization according to (34) and (35). In the fourth order, we arrive at the two very close solutions,

$$\beta_{4,2}(u_4) = \beta_{4,1}(u_4) = 0.3394 \quad (u_4 = 1.14768),$$

and

$$\beta_{4,2}(u_4) = \beta_{4,1}(u_4) = 0.32377 \quad (u_4 = 5.76971).$$

Their average equals 0.33162 ± 0.00785 .

The results obtained above by the two methods agree well with the exact $1/3$. The results from the two methods appear to be of the same quality as achieved by the optimal Borel–Leroy summation, $\beta \approx 0.324663$ [2]. It is particularly close to $\beta \approx 0.330762$ obtained by the optimal Mittag–Leffler summation [2].

Rather sophisticated and painstaking Borel summation technique of [5] based on the adaptation of Lipatov’s asymptotic expression for the coefficients a_n at infinity, and on a very large number of the weak-coupling coefficients, gives comparable (but, nevertheless, to our surprise, worse) results of $\beta \approx 0.317 \pm 0.032$.

One can think that precise knowledge of large n coefficients a_n is somewhat redundant when the critical index at infinity is concerned. Such redundancy also holds for some other examples presented above when critical amplitudes are of main interest. That is not to say that knowledge of large n coefficients a_n is not relevant at all, in particular when one is concerned with achieving the quality of exact solutions numerically. For accurate calculations with a small number of a_n and incomplete pre-knowledge of their large n behavior, it is more important to be able to take the sought limit as $x \rightarrow \infty$ explicitly, obtain the critical amplitudes (indices) in analytical form, and analyze/optimize it further if needed.

The results obtained by the two other optimization techniques discussed in the current paper, are still sensible but inferior. Optimization according to (15) and (16) gives at best case, $\beta \approx 0.306$. Similar quality is achieved by optimization according to (36) and (37). In the fourth order, we find at best, $\beta \approx 0.302$.

Example 7. A perturbation theory for the expansion factor $\alpha(g)$ of three-dimensional polymer leads to a series in a single dimensionless interaction parameter g . The parameter quantifies the repulsive interaction between segments of the polymer chain [38,39]. As $g \rightarrow 0$, the expansion factor can be presented as the truncated series of the same type as for the anharmonic oscillator, with the coefficients

$$a_0 = 1, \quad a_1 = \frac{4}{3}, \quad a_2 = -2.075385396 \quad a_3 = 6.296879676, \\ a_4 = -25.05725072, \quad a_5 = 116.134785, \quad a_6 = -594.71663.$$

The strong-coupling behavior of the expansion factor as $g \rightarrow \infty$ is power-law

$$\alpha(g) \propto g^\beta.$$

It was found numerically in the paper [40], $\beta \approx 0.3508$, and a slightly lower result, $\beta \approx 0.3504$, was obtained in [41]. Earlier, in [39], it was also found numerically that $\beta \approx 0.3544$.

Again, we have to consider the indeterminate case. The quite sensible upper bound, $\beta_5(1) = 0.3596$, is achieved in conventional one-step Borel summation applied to the inverse series and by taking the inverse of the result. In two-steps we also find a reasonable number for a lower bound, $\beta_5(2) = 0.3405$. Their average gives $\beta_5^* = 0.35002$. The corresponding sequence

$$\beta_1^* = 0.2999, \quad \beta_2^* = 0.3147, \quad \beta_3^* = 0.3281, \quad \beta_4^* = 0.3402, \quad \beta_5^* = 0.35002,$$

is numerically convergent.

A good result is achieved by optimization according to (34) and (35), as in the fifth order

$$\beta_{5,2}(u_5) = \beta_{5,1}(u_5) = 0.3553 \quad (u_5 = 1.87665).$$

The method of (36) and (37) fails to produce the solutions with positive u . Optimization according to (15) and (16) in the fifth-order finds three very close solutions

$$0.3496, \quad 0.3504, \quad 0.3531.$$

Their average and variance give the estimate of 0.351 ± 0.00151 .

The results from the three variants of iterative Borel methods appear to be of the same quality as achieved by the optimal Borel–Leroy summation, $\beta \approx 0.3512$ [2]. They are close also to $\beta \approx 0.3504$ obtained by the optimal Mittag–Leffler summation [2]. The results of self-similar Borel summation with factor approximants $\beta \approx 0.3726$ [1] are higher than those obtained from all other approaches.

5. Concluding Remarks

The method of iterative Borel summation [4] is combined with self-similar approximants [1,2]. The property of asymptotic scale invariance pertinent to the self-similar approximants leads to factorization of the multi-dimensional integrals in the limit of large x . The critical properties, indices and amplitudes are considered as a limit at infinity and can be found analytically.

The iterative application of Borel summation leads to an improvement compared with a conventional, single-step Borel summation technique. Novel methods of control are introduced to accelerate the convergence of the iterative Borel summation. The number of iterations by itself can serve as an explicit (or implicit) control parameter. Furthermore, the control parameter could be introduced by more traditional means by modifying the form of Borel transformation by putting the stretched (compacted) exponential function in place of the exponential.

In all cases, the controls are introduced in such a way that the locations of poles remain the same with the original self-similar Borel method [1]. For such special points, considering inverse quantities is recommended so that the original schemes remain applicable. Such an approach is different from the regularization schemes of Borel–Leroy and Mittag–Leffler summations [2], where the control parameters eliminate poles at negative integers.

The simplest technical scheme of iterative Borel summation based on averaging over the one-step and two-step Borel iterations (14), works well when lower and upper bounds are established. The scheme works well, as could be expected, for the cases with fast-growing a_n , exemplified by the Schwinger model energy gap (Example 1), anharmonic oscillator ground state energy (Example 7), and expansion factor of three-dimensional polymer (Example 6). The Bose condensation temperature (Example 4) and Schwinger model ground state energy (Example 5), with slow decay of the coefficients, are still covered by the very same method. The quality of approximations appears to be better or not worse compared to some other versions of Borel summation and also to various advanced resummation and numerical techniques. In particular, good results are obtained for the very hard problem of the Schwinger model energy gap.

However, in the situations with fast-decaying coefficients, when only a one-sided bound is established, the iterative Borel summation with optimization (15) and (16), with the number of iterations employed as the control, still works by extrapolating beyond the bound. Specifically, the cases of Lieb–Liniger model ground state energy (Example 2) and optical polaron mass (Example 3) are covered by the method. For the former problem, we verify the good quality of the novel expansions in the weak-coupling limit by a successful extrapolation to the strong-coupling limit. Thus, the methods of iterative Borel summation of the current paper can be adapted to various types of a_n and applied not only to the cases with factorial growth of a_n .

For future work, we have in mind the following. Iterative versions of the Borel–Leroy and Mittag–Leffler transforms deserve to be studied as well and compared with the techniques presented in the paper. Furthermore, it is of interest to study Padé approximants modified for criticality, as well as factor approximants in place of iterated roots. Such techniques have a certain advantage over the iterated roots in allowing to also cover the case of $\beta = 0$.

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