

## Article

# The Mei Symmetries for the Lagrangian Corresponding to the Schwarzschild Metric and the Kerr Black Hole Metric

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**Abstract:** In this paper, the Mei symmetries for the Lagrangians corresponding to the spherically and axially symmetric metrics are investigated. For this purpose, the Schwarzschild and Kerr black hole metrics are considered. Using the Mei symmetries criterion, we obtained four Mei symmetries for the Lagrangian of Schwarzschild and Kerr black hole metrics. The results reveal that, in the case of the Schwarzschild metric, the obtained Mei symmetries are a subset of the Lie point symmetries of equations of motion (geodesic equations), while in the case of the Kerr black hole metric, the Noether symmetry set is a subset of the Mei symmetry set and that Mei symmetries and the Lie point symmetries of the equations of motion are same.

**Keywords:** Mei symmetries; Lagrangian; Schwarzschild metric; Kerr black hole metric



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## 1. Introduction

In mathematics and mechanics, the study of symmetry and conserved quantity is extremely significant [1,2]. Noether symmetries are the modern way of determining a mechanical system's conserved quantities [3,4]. The Noether symmetry is an invariance of the Euler Lagrange equation under infinitesimal transformations. In the last decade, several significant results in the study of the Lie point symmetry [5,6] and the Noether symmetry [7] have been obtained. In the year 2000, Mei Feng-Xiang [8] proposed a new symmetry called form invariance. Form invariance, also known as Mei symmetry, is an invariant property that states that the dynamical functions (such as Lagrangian) appearing in the mechanical system's dynamical equations still satisfy the original equations after the infinitesimal transformation. Mei symmetry, like Noether symmetry, admits first integrals known as Mei conserved quantities [9].

The infinitesimal transformation of a group discusses the form invariance of the Appell equations [10]. The Lagrangian of Appell equations is used to compute the Noether symmetries. Following that, Noether symmetries are compared to form invariance, and several conserved values are found. W Shu-Yong and M Feng-Xiang [11] provided non-holonomic systems' form invariance and Lie symmetries. Structure equations and form invariance, which are similar to Lie symmetries, are deduced in this study. F Jian-Hui [12] discussed the Mei symmetries of the rotational relativistic mass variable system, with a focus on the link between Lie and Mei symmetries. The Mei symmetries for non-material volumes were discovered by Jiang et al. [13]. To determine the conserved quantities, a single-degree-of-freedom non-material volume is used as an example. XH Zhai and Y Zhang [14] computed the Mei symmetries of the Lagrangian system on a time scale. Its relationship to the Noether symmetries is fully explained in this article. XH Zhai and Y Zhang [15] also analyzed the Mei symmetry and new conserved quantities of time-scale Birkhoff's equations as a special case of Hamiltonian canonical equations.

In this paper, Mei symmetries for the Lagrangian corresponding to the Schwarzschild metric and Kerr black hole metric are obtained. The first exact solution of the Einstein field

equation is by Karl Schwarzschild in 1915 [16]. The Schwarzschild metric describes the spherically symmetric, static, homogeneous and isotropic gravitational field without electric field and angular momentum. It is the most general vacuum solution. The generalization of the Schwarzschild metric describing the geometry of empty spacetime around a rotating uncharged axially symmetric black hole is the Kerr metric. It is the first ever exact solution with angular momentum “ $a$ ” of the Einstein field equations due to Roy Kerr and, therefore, named the Kerr metric [17].

Noether’s symmetry algebra for the Lagrangian of the Schwarzschild metric is five-dimensional, given by  $so(3) \oplus \mathbb{R} \oplus d1$  (where  $d1$  is the Lie algebra generated by  $\partial/\partial s$ ), and it contains the isometry algebra appropriately, but the symmetry algebra for the geodesic equations is  $so(3) \oplus \mathbb{R} \oplus d2$  (where  $d2$  is the dilation algebra with generators  $\partial/\partial s$  and  $s\partial/\partial s$ ). As a result, the set of Noether symmetry of the Schwarzschild metric is said to be a subset of the Lie point symmetry of the Schwarzschild metric [18]. The Kerr black hole, on the other hand, is a more realistic scenario that depicts the gravitational field outside of an uncharged spinning black hole and is no longer spherically symmetric in contrast to the Schwarzschild metric. The Kerr black hole metric is one of the well-known solutions to Einstein’s field equations. The nonlinearity of these equations makes precise solutions extremely difficult to obtain. The Kerr spacetime’s isometry algebra is two-dimensional, whereas Noether’s symmetry algebra is three-dimensional, i.e., the two Killing vectors  $(\partial/\partial t, \partial/\partial \phi)$  and the translation in the geodetic parameter  $\partial/\partial s$  [19]. In this spacetime, the only conserved quantities are energy and angular momentum.

The plan of the paper is as follows. In the next section, the mathematical formalism of Mei symmetry to be used is given. In Section 3, the Mei symmetries corresponding to Lagrangian of the Schwarzschild metric are considered. In Section 4, the Mei symmetries for the Lagrangian corresponding to the Kerr black hole metric are considered. Finally, a summary and conclusion are given in Section 5.

## 2. Mathematical Preliminaries

X.H. Zhai and Y. Zhang [14] discovered a methodology for determining the Mei symmetries of a Lagrangian system. Assume we have a Lagrangian

$$L = L(t, q^i, \dot{q}^i). \quad (1)$$

Consider the infinitesimal transformation group with a one-parameter

$$\begin{aligned} \hat{t} &= t + \varepsilon \xi(t, q^j), \\ \hat{q}^i &= q^i + \varepsilon \eta^i(t, q^j), \end{aligned} \quad (2)$$

where  $i, j = 1, \dots, n$  and  $\varepsilon \in \mathbb{R}$ . The associated infinitesimal generator is

$$X = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i}. \quad (3)$$

As a result of the transformation (2), the Lagrangian (1) becomes

$$\begin{aligned} \hat{L} &= \hat{L}(\hat{t}, \hat{q}^i, \dot{\hat{q}}^i), \\ &= \hat{L}\left(t + \varepsilon \xi, q^i + \varepsilon \eta^i, \frac{\dot{q}^i + \varepsilon \dot{\eta}^i}{1 + \varepsilon \dot{\xi}}\right). \end{aligned} \quad (4)$$

The Taylor series expansion of (4) with respect to  $\varepsilon = 0$  yields

$$\hat{L} = L(t, q^i, \dot{q}^i) + \varepsilon X^{[1]}(L) + O(\varepsilon^2), \quad (5)$$

where

$$X^{[1]} = \xi \frac{\partial}{\partial t} + \eta^i \frac{\partial}{\partial q^i} + (\dot{\eta}^i - \dot{\xi} \dot{q}^i) \frac{\partial}{\partial \dot{q}^i}, \quad (6)$$

is the first prolongation of the infinitesimal generator  $X$ . The Euler–Lagrange equations are written as

$$E_i(L) = 0, \quad (7)$$

where  $E_i$  denotes the Euler operator

$$E_i = \frac{d}{dt} \frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}. \quad (8)$$

If (7) remains unchanged when the new Lagrangian  $\hat{L}$  from (5) is substituted in place of the Lagrangian, i.e.,

$$E_i(\hat{L}) = 0, \quad (9)$$

this invariance is known as the Mei symmetries corresponding to the Lagrangian. As a result, we can present the method for determining Mei symmetries.

#### Method for Determining Mei Symmetries

If the infinitesimals  $\xi$  and  $\eta^i$  satisfy

$$E_i[X^{[1]}(L)] = 0, \quad i = 1, \dots, n. \quad (10)$$

then the corresponding invariance is the Mei symmetry for the Lagrangian. When the Euler operator  $E_i$  is applied to  $X^{[1]}(L)$ , an equation comprising various powers of  $\dot{q}$  is generated. A system of PDEs is produced by separating coefficients of various powers of  $\dot{q}$ . This system's solution yields  $\xi$  and  $\eta^i$ , which fulfills the specified requirement in Equation (10) of the Mei symmetries.

### 3. The Mei Symmetries Corresponding to Lagrangian of the Schwarzschild Metric

In the Schwarzschild coordinates  $(t, r, \theta, \phi)$ , with the signature convention  $(-, +, +, +)$ , Schwarzschild metric has the form [16]

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (11)$$

where  $r_s$  is the Schwarzschild radius of the massive body defined as  $r_s = \frac{2GM}{c^2}$ ,  $G$  is the gravitational constant,  $c$  is the speed of light,  $t$  is the time coordinate,  $r$  is the radial coordinate,  $\theta$  is the colatitude of a point on 2-sphere,  $\phi$  is the longitude of a point on 2-sphere.

Using these coordinates, the Lagrangian for the Schwarzschild metric reads:

$$L = -\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2, \quad (12)$$

we find the Euler–Lagrange Equation (8) for the Lagrangian given by Equation (12). These equations are

$$\ddot{t} = -2\left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r^2} \dot{t} \dot{r}, \quad (13)$$

$$\begin{aligned} \ddot{r} &= \left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r^2} \dot{r}^2 - \left(1 - \frac{2m}{r}\right) \frac{m}{r^2} \dot{t}^2 + \left(1 - \frac{2m}{r}\right) r \dot{\theta}^2 \\ &\quad + \left(1 - \frac{2m}{r}\right) r \sin^2 \theta \dot{\phi}^2, \end{aligned} \quad (14)$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \sin \theta \cos \theta \dot{\phi}^2, \quad (15)$$

$$\ddot{\phi} = -\frac{2\dot{r}\dot{\phi}}{r} - 2 \cot \theta \dot{\theta} \dot{\phi}. \quad (16)$$

Applying first extended generator on the Lagrangian given in Equation (12) gives

$$\begin{aligned} \mathbf{X}^{[1]}(L) = & \eta^2 \left[ -\frac{2mt^2}{r^2} - 2 \left(1 - \frac{2m}{r}\right)^{-2} \frac{mi^2}{r^2} + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2 \right] \\ & + 2\eta^3 r^2 \sin\theta \cos\theta\dot{\phi}^2 + [\dot{\eta}^1 - i\xi] \left[ -2 \left(1 - \frac{2m}{r}\right) \dot{t} \right] \\ & + [\dot{\eta}^2 - \dot{r}\xi] \left[ 2 \left(1 - \frac{2m}{r}\right)^{-1} \dot{r} \right] + [\dot{\eta}^3 - \dot{\theta}\xi] [2r^2\dot{\theta}] \\ & + [\dot{\eta}^4 - \dot{\phi}\xi] [2r^2 \sin^2\theta\dot{\phi}]. \end{aligned} \quad (17)$$

For  $q^1 = t$ , Equation (10) yields

$$\left[ \frac{d}{ds} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \right] [\mathbf{X}^{[1]}(L)] = 0. \quad (18)$$

Using Equation (17) in Equation (18) and substituting Equations (13)–(16), simplifying it further, and then the powers of  $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$  are compared to obtain a system of determining equations as follows:

$$(constant) : \eta_{ss}^1 = 0, \quad (19a)$$

$$(\dot{t}) : -\frac{m}{r^2} \eta_s^2 - \left(1 - \frac{2m}{r}\right) \eta_{st}^1 + \left(1 - \frac{2m}{r}\right) \xi_{ss} = 0, \quad (19b)$$

$$(\dot{r}) : \eta_{sr}^1 = 0, \quad (19c)$$

$$(\dot{\theta}) : \eta_{s\theta}^1 = 0, \quad (19d)$$

$$(\dot{\phi}) : \eta_{s\phi}^1 = 0, \quad (19e)$$

$$\begin{aligned} (\dot{t}^2) : & -\frac{2m}{r^2} \eta_t^2 - \left(1 - \frac{2m}{r}\right) \eta_{tt}^1 + \left(1 - \frac{2m}{r}\right)^2 \frac{m}{r^2} \eta_r^1 \\ & + 4 \left(1 - \frac{2m}{r}\right) \xi_{st} = 0, \end{aligned} \quad (19f)$$

$$(\dot{r}^2) : \left(1 - \frac{2m}{r}\right) \eta_{rr}^1 + \frac{3m}{r^2} \eta_r^1 = 0, \quad (19g)$$

$$(\dot{\theta}^2) : \eta_{\theta\theta}^1 + \left(1 - \frac{2m}{r}\right) r \eta_r^1 = 0, \quad (19h)$$

$$(\dot{\phi}^2) : \eta_{\phi\phi}^1 + \left(1 - \frac{2m}{r}\right) r \sin^2\theta \eta_r^1 + \sin\theta \cos\theta \eta_\theta^1 = 0, \quad (19i)$$

$$\begin{aligned} (\dot{t}\dot{r}) : & \frac{2m}{r^3} \eta^2 + 2 \left(1 - \frac{2m}{r}\right)^{-1} \frac{m^2}{r^4} \eta^2 - \frac{m}{r^2} \eta_r^2 - \frac{m}{r^2} \eta_s^1 \\ & - \left(1 - \frac{2m}{r}\right) \eta_{tr}^1 + 2 \left(1 - \frac{2m}{r}\right) \xi_{sr} = 0, \end{aligned} \quad (19j)$$

$$(\dot{t}\dot{\theta}) : -\frac{m}{r^2} \eta_\theta^2 - \left(1 - \frac{2m}{r}\right) \eta_{t\theta}^1 + 2 \left(1 - \frac{2m}{r}\right) \xi_{s\theta} = 0, \quad (19k)$$

$$(\dot{t}\dot{\phi}) : -\frac{m}{r^2} \eta_\phi^2 - \left(1 - \frac{2m}{r}\right) \eta_{t\phi}^1 + 2 \left(1 - \frac{2m}{r}\right) \xi_{s\phi} = 0, \quad (19l)$$

$$(\dot{r}\dot{\theta}) : -\left(1 - \frac{2m}{r}\right) \eta_{r\theta}^1 + \frac{1}{r} \left(1 - \frac{3m}{r}\right) \eta_\theta^1 = 0, \quad (19m)$$

$$(\dot{r}\dot{\phi}) : -\left(1 - \frac{2m}{r}\right) \eta_{r\phi}^1 + \frac{1}{r} \left(1 - \frac{3m}{r}\right) \eta_\phi^1 = 0, \quad (19n)$$

$$(\dot{\theta}\dot{\phi}) : -\eta_{\theta\phi}^1 + \cot\theta \eta_\phi^1 = 0, \quad (19o)$$

$$(t^3) : \xi_{tt} - \left(1 - \frac{2m}{r}\right) \frac{m}{r^2} \xi_r = 0, \quad (19p)$$

$$(t^2\dot{r}) : \left(1 - \frac{2m}{r}\right) \xi_{tr} - \frac{m}{r^2} \xi_t = 0, \quad (19q)$$

$$(t^2\dot{\theta}) : \xi_{t\theta} = 0, \quad (19r)$$

$$(t^2\dot{\phi}) : \xi_{t\phi} = 0, \quad (19s)$$

$$(t\dot{r}^2) : \left(1 - \frac{2m}{r}\right) \xi_{rr} + \frac{m}{r^2} \xi_r = 0, \quad (19t)$$

$$(t\dot{\theta}^2) : \xi_{\theta\theta} + \left(1 - \frac{2m}{r}\right) r \xi_r = 0, \quad (19u)$$

$$(t\dot{\phi}^2) : \xi_{\phi\phi} + \left(1 - \frac{2m}{r}\right) r \sin^2 \theta \xi_r \\ + \sin \theta \cos \theta \xi_\theta = 0, \quad (19v)$$

$$(t\dot{r}\dot{\theta}) : \xi_{r\theta} - \frac{1}{r} \xi_\theta = 0, \quad (19w)$$

$$(t\dot{r}\dot{\phi}) : \xi_{r\phi} - \frac{1}{r} \xi_\phi = 0, \quad (19x)$$

$$(t\dot{\theta}\dot{\phi}) : \xi_{\theta\phi} - \cot \theta \xi_\phi = 0, \quad (19y)$$

Following similar procedure for  $q^2 = r$ ,  $q^3 = \theta$ , and  $q^4 = \phi$ , Equation (10) yields

$$(constant) : \eta_{ss}^2 = 0, \quad (20a)$$

$$(t) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{st}^2 + \frac{m}{r^2} \eta_s^1 = 0, \quad (20b)$$

$$(\dot{r}) : -\left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r^2} \eta_s^2 + \eta_{sr}^2 - \xi_{ss} = 0, \quad (20c)$$

$$(\dot{\theta}) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{s\theta}^2 - r \eta_s^3 = 0, \quad (20d)$$

$$(\dot{\phi}) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{s\phi}^2 - r \sin^2 \theta \eta_s^4 = 0, \quad (20e)$$

$$(t^2) : 2 \left(1 - \frac{2m}{r}\right)^{-1} \frac{m^2}{r^4} \eta^2 + \left(1 - \frac{2m}{r}\right)^{-1} \eta_{tt}^2 \\ - \frac{m}{r^2} \eta_r^2 - 2 \frac{m}{r^3} \eta^2 + 2 \frac{m}{r^2} \eta_t^1 = 0, \quad (20f)$$

$$(\dot{r}^2) : 2 \left(1 - \frac{2m}{r}\right)^{-2} \frac{m^2}{r^4} \eta^2 + 2 \left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r^3} \eta^2 \\ - \left(1 - \frac{2m}{r}\right) \frac{m}{r^2} \eta_r^2 + \eta_{rr}^2 - 4 \xi_{sr} = 0, \quad (20g)$$

$$(\dot{\theta}^2) : -2 \left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r} \eta^2 + \left(1 - \frac{2m}{r}\right) \eta_{\theta\theta}^2 \\ + r \eta_r^2 - 2r \eta_\theta^3 - \eta^2 = 0, \quad (20h)$$

$$(\dot{\phi}^2) : -2 \left(1 - \frac{2m}{r}\right)^{-1} \frac{m}{r} \sin^2 \theta \eta^2 + r \sin^2 \theta \eta_r^2 \\ + \left(1 - \frac{2m}{r}\right)^{-1} \eta_{\phi\phi}^2 - \sin^2 \theta \eta^2 - 2r \sin \theta \cos \theta \eta^3 \\ + \left(1 - \frac{2m}{r}\right)^{-1} \sin \theta \cos \theta \eta_\theta^2 - 2r \sin^2 \theta \eta_\phi^4 = 0, \quad (20i)$$

$$(i\dot{r}) : -2\left(1 - \frac{2m}{r}\right)^{-2} \frac{m}{r^2} \eta_t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \eta_{tr}^2 - 2\left(1 - \frac{2m}{r}\right)^{-1} \xi_{st} + \frac{m}{r^2} \eta_r^1 = 0, \quad (20j)$$

$$(i\dot{\theta}) : -\left(1 - \frac{2m}{r}\right)^{-2} \frac{m}{r^2} \eta_\theta^2 + \left(1 - \frac{2m}{r}\right)^{-1} \eta_{r\theta}^2 - \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} \eta_\theta^2 - 2\left(1 - \frac{2m}{r}\right)^{-1} \xi_{s\theta} - r\eta_r^3 = 0, \quad (20k)$$

$$(i\dot{\phi}) : -\left(1 - \frac{2m}{r}\right)^{-2} \frac{m}{r^2} \eta_\phi^2 + \left(1 - \frac{2m}{r}\right)^{-1} \eta_{r\phi}^2 - \frac{1}{r} \left(1 - \frac{2m}{r}\right)^{-1} \eta_\phi^2 - 2\left(1 - \frac{2m}{r}\right)^{-1} \xi_{s\phi} - r\sin^2\theta\eta_r^4 = 0, \quad (20l)$$

$$(i\dot{\theta}) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{t\theta}^2 + \frac{m}{r^2} \eta_\theta^1 - r\eta_t^3 = 0, \quad (20m)$$

$$(i\dot{\phi}) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{t\phi}^2 + \frac{m}{r^2} \eta_\phi^1 - r\sin^2\theta\eta_t^4 = 0, \quad (20n)$$

$$(\dot{\theta}\dot{\phi}) : \left(1 - \frac{2m}{r}\right)^{-1} \eta_{\theta\phi}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \cot\theta\eta_\phi^2 - r\eta_\phi^3 - r\sin^2\theta\eta_\theta^4 = 0. \quad (20o)$$

$$(constant) : \eta_{ss}^3 = 0, \quad (21a)$$

$$(\dot{t}) : \eta_{st}^3 = 0, \quad (21b)$$

$$(\dot{r}) : \eta_s^3 + r\eta_{sr}^3 = 0, \quad (21c)$$

$$(\dot{\theta}) : \eta_s^2 + r\eta_{s\theta}^3 - r\xi_{ss} = 0, \quad (21d)$$

$$(\dot{\phi}) : \eta_{s\phi}^3 - \sin\theta\cos\theta\eta_s^4 = 0, \quad (21e)$$

$$(t^2) : r^2\eta_{tt}^3 - \left(1 - \frac{2m}{r}\right)m\eta_r^3 = 0, \quad (21f)$$

$$(\dot{r}^2) : r^2\eta_{rr}^3 + \left[2r + \left(1 - \frac{2m}{r}\right)^{-1}m\right]\eta_r^3 = 0, \quad (21g)$$

$$(\dot{\theta}^2) : 2\eta_\theta^2 + r\eta_{\theta\theta}^3 + \left(1 - \frac{2m}{r}\right)r^2\eta_r^3 - 4r\xi_{s\theta} = 0, \quad (21h)$$

$$(\dot{\phi}^2) : \eta_{\phi\phi}^3 + \left(1 - \frac{2m}{r}\right)r\sin^2\theta\eta_r^3 + \sin\theta\cos\theta\eta_\theta^3 + (\sin^2\theta - \cos^2\theta)\eta_r^3 - 2\sin\theta\cos\theta\eta_\phi^4 = 0, \quad (21i)$$

$$(\dot{\theta}\dot{t}) : \eta_t^2 + r\eta_{t\theta}^3 - 2r\xi_{st} = 0, \quad (21j)$$

$$(\dot{\theta}\dot{r}) : -\eta^2 + r\eta_r^2 + r^2\eta_{r\theta}^3 - 2r^2\xi_{sr} = 0, \quad (21k)$$

$$(\dot{\theta}\dot{\phi}) : \eta_\phi^2 + r\eta_{\theta\phi}^3 - r\cot\theta\eta_\phi^3 - 2r\xi_{s\phi} - r\sin\theta\cos\theta\eta_\theta^4 = 0, \quad (21l)$$

$$(\dot{t}\dot{r}) : r^2\eta_{tr}^3 + \left[r - \left(1 - \frac{2m}{r}\right)^{-1}m\right]\eta_t^3 = 0, \quad (21m)$$

$$(\dot{t}\dot{\phi}) : \eta_{t\phi}^3 - \sin\theta\cos\theta\eta_t^4 = 0, \quad (21n)$$

$$(\dot{r}\dot{\phi}) : \eta_{r\phi}^3 - \sin\theta\cos\theta\eta_r^4 = 0. \quad (21o)$$

$$(constant) : \eta_{ss}^4 = 0, \quad (22a)$$

$$(t) : \eta_{st}^4 = 0, \quad (22b)$$

$$(\dot{r}) : \eta_s^4 + r\eta_{sr}^4 = 0, \quad (22c)$$

$$(\dot{\theta}) : \sin \theta \cos \theta \eta_s^4 + \sin^2 \theta \eta_{s\theta}^4 = 0, \quad (22d)$$

$$(\dot{\phi}) : \eta_s^2 + r \cot \theta \eta_s^3 + r\eta_{s\phi}^4 - r\xi_{ss} = 0, \quad (22e)$$

$$(\dot{t}^2) : r^2\eta_{tt}^4 - \left(1 - \frac{2m}{r}\right)m\eta_r^4 = 0, \quad (22f)$$

$$(\dot{r}^2) : r^2\eta_{rr} + \left[2r + \left(1 - \frac{2m}{r}\right)m\right]\eta_r^4 = 0, \quad (22g)$$

$$(\dot{\theta}^2) : 2\cot \theta \eta_\theta^4 + \eta_{\theta\theta}^4 + \left(1 - \frac{2m}{r}\right)r\eta_r^4 = 0, \quad (22h)$$

$$\begin{aligned} (\dot{\phi}^2) : & 2\eta_\phi^2 + 2r \cot \theta \eta_\phi^3 + r\eta_{\phi\phi}^4 + r \sin \theta \cos \theta \eta_\theta^4 \\ & + \left(1 - \frac{2m}{r}\right)r^2 \sin^2 \theta \eta_r^4 - 4r\xi_{s\phi} = 0, \end{aligned} \quad (22i)$$

$$(\dot{\phi}\dot{t}) : \eta_t^2 + r \cot \theta \eta_t^3 + r\eta_{t\phi}^4 - 2r\xi_{st} = 0, \quad (22j)$$

$$\begin{aligned} (\dot{\phi}\dot{r}) : & -\eta^2 + r\eta_r^2 + r^2 \cot \theta \eta_r^3 + r^2\eta_{r\phi}^4 \\ & - 2r^2\xi_{sr} = 0, \end{aligned} \quad (22k)$$

$$(\dot{\phi}\dot{\theta}) : \eta_\theta^2 - r \cot^2 \theta \eta^3 - r\eta^3 + r \cot \theta \eta_\theta^3 \quad (22l)$$

$$+ r\eta_{\theta\phi}^4 - 2r\xi_{s\theta} = 0, \quad (22m)$$

$$(\dot{t}\dot{r}) : r^2\eta_{tr}^4 + \left[r - \left(1 - \frac{2m}{r}\right)^{-1}m\right]\eta_t^4 = 0, \quad (22n)$$

$$(\dot{t}\dot{\theta}) : \cot \theta \eta_t^4 + \eta_{t\theta}^4 = 0, \quad (22o)$$

$$(\dot{r}\dot{\theta}) : \cot \theta \eta_r^4 + \eta_{r\theta}^4 = 0. \quad (22p)$$

Solving the above system of partial differential equations we obtain four Mei symmetries corresponding to the Lagrangian of the Schwarzschild metric

$$\begin{aligned} X_1 &= \frac{\partial}{\partial s}, & X_2 &= s\frac{\partial}{\partial s}, \\ X_3 &= \frac{\partial}{\partial t}, & X_4 &= \frac{\partial}{\partial \phi}. \end{aligned} \quad (23)$$

Three Mei symmetries of the Schwarzschild metric are same as three of the Noether symmetries of the Schwarzschild metric. In the case of the Schwarzschild metric, there is no explicit connection found between Noether symmetries and Mei symmetries. The obtained Mei symmetries of the Schwarzschild metric form a sub-algebra of the symmetries of the Euler–Lagrange (geodesic) equations given by the Equations (13)–(16).

#### 4. The Mei Symmetries for the Lagrangian Corresponding to the Kerr Black Hole Metric

In Boyer-Lindquist coordinates, the Kerr black hole metric [17] (or, equivalently, its line element for proper time) is given by

$$\begin{aligned} ds^2 &= -c^2 d\tau^2, \\ &= \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 + a^2)d\phi]^2 - \frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2, \end{aligned} \quad (24)$$

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2mr, \quad (25)$$

$m$  is the mass of the rotational object,  $a$  is the spin parameter or specific angular momentum and is related to the angular momentum  $J$  by  $a = \frac{J}{m}$ . When  $a = 0$ , this metric is reduced to the Schwarzschild metric and, therefore, it is a generalized form of the Schwarzschild metric.

To begin, we write the Lagrangian for the Kerr black hole metric as

$$L = - \left( 1 - \frac{2mr}{\rho^2} \right) \dot{t}^2 + \frac{\rho^2}{\Delta} \dot{r}^2 + \rho^2 \dot{\theta}^2 + \frac{\sin^2 \theta}{\rho^2} \Sigma \dot{\phi}^2 - \frac{4mar \sin^2 \theta}{\rho^2} \dot{t} \dot{\phi}, \quad (26)$$

where

$$\Sigma = [(r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta]. \quad (27)$$

The Euler Lagrange Equation (8) for the Lagrangian given by Equation (26) is

$$\begin{aligned} \ddot{t} &= - \frac{2m(r^2 + a^2)\Omega}{\rho^4 \Delta} \dot{t} \dot{r} + \frac{4ma^2 r \sin \theta \cos \theta}{\rho^4} \dot{t} \dot{\phi} - \frac{4ma^3 r \sin^3 \theta \cos \theta}{\rho^4} \dot{\theta} \dot{\phi} \\ &\quad + \frac{2ma \sin^2 \theta [(r^2 + a^2)\Omega + 2r\rho^2]}{\rho^4 \Delta} \dot{r} \dot{\phi}, \end{aligned} \quad (28)$$

$$\begin{aligned} \ddot{r} &= \frac{m\Omega - a^2 r \sin^2 \theta}{\rho^2 \Delta} \dot{r}^2 + \frac{2a^2 \sin \theta \cos \theta}{\rho^2} \dot{r} \dot{\theta} - \frac{m\Delta\Omega}{\rho^6} \dot{t}^2 + \frac{2ma \sin^2 \theta \Delta\Omega}{\rho^6} \dot{t} \dot{\phi} \\ &\quad - \frac{[ma^2 \sin^4 \theta \Delta\Omega - r \sin^2 \theta \Delta \rho^4]}{\rho^6} \dot{\phi}^2 + \frac{r\Delta}{\rho^2} \dot{\theta}^2, \end{aligned} \quad (29)$$

$$\begin{aligned} \ddot{\theta} &= \frac{a^2 \sin \theta \cos \theta}{\rho^2} \dot{\theta}^2 + \frac{2ma^2 r \sin \theta \cos \theta}{\rho^6} \dot{t}^2 - \frac{4mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \dot{t} \dot{\phi} \\ &\quad - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \dot{r}^2 + \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^6} \dot{\phi}^2 - \frac{2r}{\rho^2} \dot{r} \dot{\theta}, \end{aligned} \quad (30)$$

$$\begin{aligned} \ddot{\phi} &= - \frac{2ma\Omega}{\rho^4 \Delta} \dot{t} \dot{r} + \frac{4mar \cot \theta}{\rho^4} \dot{t} \dot{\theta} + \frac{2ma^2 \sin^2 \theta \Omega + 2r(-\rho^2 + 2mr)\rho^2}{\rho^4 \Delta} \dot{r} \dot{\phi} \\ &\quad - \frac{2 \cot \theta \rho^4 + 4ma^2 r \sin \theta \cos \theta}{\rho^4} \dot{\theta} \dot{\phi}, \end{aligned} \quad (31)$$

where

$$\Omega = (r^2 - a^2 \cos^2 \theta). \quad (32)$$

Applying first prolonged generator on the Lagrangian given in (26) yields

$$\begin{aligned} X^{[1]} L &= (\dot{\eta}^1 - \dot{t} \dot{\xi}) \left[ -2 \left( 1 - \frac{2mr}{\rho^2} \right) \dot{t} - \frac{4mar \sin^2 \theta}{\rho^2} \dot{\phi} \right] + \eta^2 \left[ - \frac{2m\Omega}{\rho^4} \dot{t}^2 \right. \\ &\quad \left. - \frac{2m\Omega - a^2 r \sin^2 \theta}{\Delta^2} \dot{r}^2 + 2r \dot{\theta}^2 - \frac{2ma^2 \sin^4 \theta \Omega - 2r \sin^2 \theta}{\rho^4} \dot{\phi}^2 \right. \\ &\quad \left. + \frac{4ma \sin^2 \theta \Omega}{\rho^4} \dot{t} \dot{\phi} \right] + (\dot{\eta}^2 - \dot{r} \dot{\xi}) \left[ \frac{2\rho^2}{\Delta} \dot{r} \right] + \eta^3 \left[ - \frac{2a^2 \sin \theta \cos \theta}{\Delta} \dot{r}^2 \right. \\ &\quad \left. + \frac{4ma^2 r \sin \theta \cos \theta}{\rho^4} \dot{t}^2 + \frac{8mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^4} \dot{t} \dot{\phi} + 2 \sin \theta \right. \\ &\quad \left. \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2] \dot{\phi}^2 - 2a^2 \sin \theta \cos \theta \dot{\theta}^2 \right] + (\dot{\eta}^3 \\ &\quad - \dot{\theta} \dot{\xi}) \left[ 2\rho^2 \dot{\theta} \right] + (\dot{\eta}^4 - \dot{\phi} \dot{\xi}) \left[ \frac{2 \sin^2 \theta}{\rho^2} \Sigma \dot{\phi} - \frac{4mar \sin^2 \theta}{\rho^2} \dot{t} \right]. \end{aligned} \quad (33)$$

Following the procedure mentioned in Section 3 to obtain the following set of partial differential equations.

$$(constant) : (-\rho^2 + 2mr)\eta_{ss}^1 - 2mar \sin^2 \theta \eta_{ss}^4 = 0, \quad (34a)$$

$$\begin{aligned} (\dot{t}) : & (-\rho^2 + 2mr)\xi_{ss} - (-\rho^2 + 2mr)\eta_{st}^1 + 2mar \sin^2 \theta \eta_{ss}^4 \\ & + \frac{m\Omega}{\rho^2} \eta_s^2 - \frac{2ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_s^3 = 0, \end{aligned} \quad (34b)$$

$$\begin{aligned} (\dot{r}) : & (-\rho^2 + 2mr)\eta_{sr}^1 - 2mar \sin^2 \theta \eta_{sr}^4 - \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_s^4 \\ & - \frac{m\Omega}{\rho^2} \eta_s^1 = 0, \end{aligned} \quad (34c)$$

$$\begin{aligned} (\dot{\theta}) : & (-\rho^2 + 2mr)\eta_{s\theta}^1 - 2mar \sin^2 \theta \eta_{s\theta}^4 + \frac{2ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_s^1 \\ & - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_s^4 = 0, \end{aligned} \quad (34d)$$

$$\begin{aligned} (\dot{\phi}) : & (2mar \sin^2 \theta)\xi_{ss} + (-\rho^2 + 2mr)\eta_{s\phi}^1 - 2mar \sin^2 \theta \eta_{s\phi}^4 \\ & + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_s^2 - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_s^3 = 0, \end{aligned} \quad (34e)$$

$$\begin{aligned} (\dot{t}^2) : & 4(-\rho^2 + 2mr)\xi_{st} - (-\rho^2 + 2mr)\eta_{tt}^1 + 2mar \sin^2 \theta \eta_{tt}^4 + \\ & \Delta \frac{m(-\rho^2 + 2mr)\Omega}{\rho^6} \eta_r^1 - \frac{2ma^2 r \sin \theta \cos \theta (-\rho^2 + 2mr)}{\rho^6} \eta_\theta^1 \\ & + \frac{2m\Omega}{\rho^2} \eta_t^2 - \frac{4ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_t^3 - \frac{2m^2 ar \sin^2 \theta \Delta \Omega}{\rho^6} \eta_r^4 \\ & + \frac{4ma^2 a^3 r^2 \sin^3 \theta \cos \theta}{\rho^6} \eta_\theta^4 = 0, \end{aligned} \quad (34f)$$

$$\begin{aligned} (\dot{r}^2) : & (-\rho^2 + 2mr)\eta_{rr}^1 - 2mar \sin^2 \theta \eta_{rr}^4 + \frac{(-\rho^2 + 2mr)}{\rho^2 \Delta} \\ & [m\Omega - a^2 r \sin^2 \theta] \eta_r^1 - \frac{2m\Omega}{\rho^2} \eta_r^1 - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} (-\rho^2 \\ & + 2mr) \eta_\theta^1 + \frac{2ma^3 r \sin^3 \theta \cos \theta}{\rho^2 \Delta} \eta_\theta^4 + \frac{2ma \sin^2 \theta \Omega}{\rho^2} \eta_r^4 \\ & - \frac{2mar \sin^2 \theta [m\Omega - a^2 r \sin^2 \theta]}{\rho^2 \Delta} \eta_r^4 = 0, \end{aligned} \quad (34g)$$

$$\begin{aligned} (\dot{\theta}^2) : & (-\rho^2 + 2mr)\eta_{\theta\theta}^1 + \frac{2a^2 \sin \theta \cos \theta (-\rho^2 + 4mr)}{\rho^2} \eta_\theta^1 + \frac{r\Delta}{\rho^2} \\ & (-\rho^2 + 2mr)\eta_r^1 - \frac{4mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_\theta^4 - 2ma^3 r \\ & [\frac{\sin^3 \theta \cos \theta (-\rho^2 + 2mr)}{\rho^2}] \eta_\theta^4 - \frac{2mar^2 \sin^2 \theta \Delta}{\rho^2} \eta_r^4 - 2mar \\ & \sin^2 \theta \eta_{\theta\theta}^4 = 0, \end{aligned} \quad (34h)$$

$$\begin{aligned} (\dot{\phi}^2) : & (8mar \sin^2 \theta)\xi_{s\phi} + \frac{2ma \sin^2 \theta \Omega}{\rho^2} \eta_\phi^2 - 2mar \sin^2 \theta \eta_{\phi\phi}^4 + \\ & \frac{(-\rho^2 + 2mr)}{\rho^6} [\sin \theta \cos \theta (r^2 + a^2) \Sigma - a^2 \sin^3 \theta \cos \theta \Delta \rho^2] \eta_\theta^1 \\ & - \frac{(-\rho^2 + 2mr)\Delta}{\rho^6} [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4] \eta_r^1 - \frac{2mar}{\rho^6} \end{aligned}$$

$$\begin{aligned} & \sin^2 \theta [\sin \theta \cos \theta (r^2 + a^2) \Sigma - a^2 \sin^3 \theta \cos \theta \Delta \rho^2] \eta_\theta^4 + \Delta \\ & \frac{(2mr \sin^2 \theta)}{\rho^6} [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4] \eta_r^4 + (-\rho^2 + 2 \\ & mr) \eta_{\phi\phi}^1 - \frac{4mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_\phi^3 = 0, \end{aligned} \quad (34i)$$

$$\begin{aligned} (\dot{t}\dot{r}) : & (-\rho^2 + 2mr) \xi_{sr} - (-\rho^2 + 2mr) \eta_{tr}^1 + 2mar \sin^2 \theta \eta_{tr}^4 \\ & + \frac{m\Omega}{\rho^2} \eta_t^1 + \frac{(-\rho^2 + 2mr)}{\rho^4 \Delta} [m(r^2 + a^2) \Omega] \eta_t^1 + \frac{(-\rho^2 + 2mr)}{\rho^4 \Delta} \\ & [ma\Omega] \eta_\phi^1 + \frac{m\Omega}{\rho^2} \eta_r^2 - \frac{2mr}{\rho^4} [-3a^2 \cos^2 \theta + r^2] \eta_r^2 - \frac{2m^2 \Omega^2}{\rho^4 \Delta} \eta_r^2 \\ & - 2ma^2 r \left[ \frac{\sin \theta \cos \theta}{\rho^2} \right] \eta_r^3 - \left[ \frac{2mr(r^2 + a^2)}{\rho^2 \Delta} + 1 \right] \left[ \frac{ma \sin^2 \theta \Omega}{\rho^2} \right] \\ & \eta_t^4 - \frac{2m^2 a^2 r \sin^2 \theta \Omega}{\rho^4 \Delta} \eta_\phi^4 + \frac{2ma^2 \sin \theta \cos \theta (3r^2 - a^2 \cos^2 \theta)}{\rho^4} \eta_r^3 \\ & = 0, \end{aligned} \quad (34j)$$

$$\begin{aligned} (\dot{t}\dot{\theta}) : & 2(-\rho^2 + 2mr) \xi_{s\theta} - (-\rho^2 + 2mr) \eta_{t\theta}^1 + 2mar \sin^2 \theta \eta_{t\theta}^4 + 2 \\ & mar \cot \theta \frac{(-\rho^2 + 2mr)}{\rho^4} \eta_\phi^1 - \frac{4m^2 a^2 r \sin \theta \cos \theta}{\rho^4} \eta_t^1 + \frac{m}{\rho^2} \Omega \eta_\theta^2 \\ & - \frac{2ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_\theta^3 + \frac{2ma^2 r \cos^2 \theta (4mr\rho^2 - 1)}{\rho^2} \eta_r^3 + 2 \\ & \left( \frac{ma^2 r \sin^2 \theta \Omega}{\rho^4} \right) \eta_r^3 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_t^4 + \frac{4m^2 a^3 r^2}{\rho^4} \\ & (\sin^3 \theta \cos \theta) \eta_t^4 + \frac{4m^2 a^2 r^2 \sin \theta \cos \theta}{\rho^4} \eta_\phi^4 - \frac{2ma^2 \sin \theta \cos \theta}{\rho^4} \\ & (3r^2 - a^2 \cos^2 \theta) \eta_r^2 = 0, \end{aligned} \quad (34k)$$

$$\begin{aligned} (\dot{t}\dot{\phi}) : & (-\rho^2 + 2mr) \eta_{t\phi}^1 + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_t^2 + \frac{(-\rho^2 + 2mr)}{\rho^6} [ma \Delta \\ & \sin^2 \theta \Omega] \eta_r^1 - \frac{(-\rho^2 + 2mr)}{\rho^6} [2mar \sin \theta \cos \theta (r^2 + a^2)] \eta_\theta^1 \\ & - \frac{m\Omega}{\rho^2} \eta_\phi^2 - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_t^3 + 4mar \sin^2 \theta \xi_{st} \\ & + \frac{2ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_\phi^3 - \frac{2m^2 a^2 r \sin^4 \theta \Delta \Omega}{\rho^6} \eta_r^4 - 2(-\rho^2 \\ & + 2mr) \xi_{s\phi} + \frac{4m^2 a^2 r^2 \sin^3 \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\theta^4 + 2mar \\ & \sin^2 \theta \eta_{t\phi}^4 = 0, \end{aligned} \quad (34l)$$

$$\begin{aligned} (\dot{r}\dot{\theta}) : & (-\rho^2 + 2mr) \eta_{r\theta}^1 - 2mar \sin^2 \theta \eta_{r\theta}^4 - 2mar \sin \theta \cos \theta \eta_r^4 \\ & + \frac{a^2 \sin \theta \cos \theta (-\rho^2 + 4mr)}{\rho^2} \eta_r^1 - \frac{1}{\rho^2} [m\Omega + r(-\rho^2 + 2 \\ & mr)] \eta_\theta^1 + \frac{ma \sin^2 \theta \Omega + 2mar^2 \sin^2 \theta}{\rho^2} \eta_\theta^4 = 0, \end{aligned} \quad (34m)$$

$$\begin{aligned} (\dot{r}\dot{\phi}) : & +4mar \sin^2 \theta \xi_{sr} + (-\rho^2 + 2mr) \eta_{r\phi}^1 - 2mar \sin^2 \theta \eta_{r\phi}^4 \\ & + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_r^2 + \frac{ma \sin^2 \theta (-\rho^2 + 2mr)}{\rho^4 \Delta} [(r^2 + a^2) \Omega \end{aligned}$$

$$\begin{aligned}
& + 2r^2\rho^2] \eta_t^1 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_r^3 - \frac{m\Omega}{\rho^2} \eta_\phi^1 \\
& + \frac{(-\rho^2 + 2mr)}{\rho^4 \Delta} [ma^2 \sin^2 \theta \Omega + r(-\rho^2 + 2mr)\rho^2] \eta_\phi^1 \\
& + \frac{8ma^3 \sin \theta \cos \theta \Omega}{\rho^4} \eta_r^3 - \frac{2m^2 a^2 r \sin^4 \theta}{\rho^4 \Delta} [(r^2 + a^2) \Omega \\
& + 2r^2 \rho^2] \eta_t^4 - \frac{(2mar \sin^2 \theta)}{\rho^4 \Delta} [ma^2 \sin^2 \theta \Omega + r(-\rho^2 \\
& + 2mr)\rho^2] \eta_\phi^4 - \frac{2ma \sin^2 \theta \Omega}{\rho^4 \Delta} [m\Omega + r\rho^2] \eta_r^2 - \frac{2mar}{\rho^4} \\
& \sin^2 \theta (r^2 - 3a^2 \cos^2 \theta) \eta_r^2 + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_\phi^4 + (r^2 + a^2) \\
& \frac{2mar \sin \theta \cos \theta}{\rho^2} \eta_r^3 = 0,
\end{aligned} \tag{34n}$$

$$\begin{aligned}
(\dot{\theta}\dot{\phi}) : & 4mar \sin^2 \theta \xi_{s\theta} + (-\rho^2 + 2mr) \eta_{\theta\phi}^1 - 2mar \sin^2 \theta \eta_{t\theta}^4 \\
& - \left( \frac{\cot \theta \rho^4 - 2ma^2 r \sin \theta \cos \theta}{\rho^4} \right) (-\rho^2 + 2mr) \eta_\phi^1 + 2 \\
& \frac{ma^2 r \sin \theta \cos \theta}{\rho^2} \eta_\phi^1 - \frac{2ma^3 r \sin^3 \theta \cos \theta (-\rho^2 + 2mr)}{\rho^4} \eta_t^1 \\
& + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_\theta^2 + \frac{4ma^3 r^2 \sin^3 \theta \cos \theta}{\rho^4} \eta_r^2 + 2mar \sin^2 \theta \\
& \frac{(r^2 - 3a^2 \cos^2 \theta)}{\rho^4} \eta_r^3 + \frac{8m^2 a^3 r^2 \sin^2 \theta \cos^2 \theta}{\rho^4} \eta_r^3 + 2mar \\
& \frac{\cos^2 \theta (r^2 + a^2)}{\rho^2} \eta_r^3 - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_\theta^3 + 4m^2 \\
& \frac{a^4 r^2 \sin^5 \theta \cos \theta}{\rho^4} \eta_t^4 + \frac{2ma^3 r \sin^3 \theta \cos \theta (-\rho^2 + 2mr)}{\rho^2} \eta_\phi^4 \\
& = 0,
\end{aligned} \tag{34o}$$

$$(t^2 \dot{r}) : \xi_{tt} - \frac{m\Delta\Omega}{\rho^6} \xi_r + \frac{2ma^2 r \sin \theta \cos \theta}{\rho^6} \xi_\theta = 0, \tag{34p}$$

$$(\dot{r}^3) : \xi_{rr} + \frac{m\Omega - a^2 r \sin^2 \theta}{\rho^2 \Delta} \xi_r - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \xi_\theta = 0, \tag{34q}$$

$$(\dot{r}\dot{\theta}^2) : \xi_{\theta\theta} + \frac{r\Delta}{\rho^2} \xi_r + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \xi_\theta = 0, \tag{34r}$$

$$\begin{aligned}
(\dot{r}\dot{\phi}^2) : & +\xi_{\phi\phi} + \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^6} \xi_\theta \\
& - \frac{\Delta [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4]}{\rho^6} \xi_r = 0,
\end{aligned} \tag{34s}$$

$$(\dot{t}\dot{r}^2) : \xi_{tr} - \frac{m(r^2 + a^2)\Omega}{\rho^4 \Delta} \xi_t - \frac{ma\Omega}{\rho^4 \Delta} \xi_\phi = 0, \tag{34t}$$

$$(\dot{t}\dot{r}\dot{\theta}) : \xi_{t\theta} + \frac{2ma^2 r \sin \theta \cos \theta}{\rho^4} \xi_t + \frac{2mar \cot \theta}{\rho^4} \xi_\phi = 0, \tag{34u}$$

$$(\dot{t}\dot{r}\dot{\phi}) : \xi_{t\phi} + \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \xi_r - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \xi_\theta = 0, \tag{34v}$$

$$(\dot{r}^2 \dot{\theta}) : \xi_{r\theta} + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \xi_r - \frac{r}{\rho^2} \xi_\theta = 0, \tag{34w}$$

$$(r^2\phi) : \xi_{r\phi} + \frac{ma \sin^2 \theta [(r^2 + a^2)\Omega + 2r^2\rho^2]}{\rho^4\Delta} \xi_t \\ + \frac{[ma^2 \sin^2 \theta \Omega + r(-\rho^2 + 2mr)\rho^2]}{\rho^4\Delta} \xi_\phi = 0, \quad (34x)$$

$$(r\dot{\theta}\phi) : \xi_{\theta\phi} - \frac{2ma^3r\sin^3\theta\cos\theta}{\rho^4}\xi_t - \frac{[\cot\theta\rho^4 - 2ma^2r\sin\theta\cos\theta]}{\rho^4}\xi_\phi = 0. \quad (34y)$$

$$(constant) : \eta_{ss}^2 = 0, \quad (35a)$$

$$(t) : \eta_{st}^2 + \frac{m\Delta\Omega}{\rho^6}\eta_s^1 + \frac{ma \sin^2 \theta \Delta\Omega}{\rho^6}\eta_s^4 = 0, \quad (35b)$$

$$(\dot{r}) : \xi_{ss} - \eta_{sr}^2 + \frac{m\Omega - a^2 r \sin^2 \theta}{\rho^2 \Delta} \eta_s^2 + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_s^3 = 0, \quad (35c)$$

$$(\dot{\theta}) : \eta_{s\theta}^2 - \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_s^2 - \frac{r\Delta}{\rho^2} \eta_s^3 = 0, \quad (35d)$$

$$(\dot{\phi}) : \eta_{s\phi}^2 + \Delta \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^6} \eta_s^4 - ma \sin^2 \theta \frac{\Delta \Omega}{\rho^6} \eta_s^1 = 0, \quad (35e)$$

$$(t^2) : \eta_{tt}^2 + \frac{2m\Delta\Omega}{\rho^6}\eta_t^1 - \frac{2m\Delta\Omega}{\rho^6}\eta_r^2 + \frac{2m\Omega[m\Omega - a^2r\sin^2\theta]}{\rho^8}\eta^2 - 2 \left[ \frac{m\Delta(r^2 - 3a^2\cos^2\theta)}{\rho^8} \right]\eta^2 + \frac{4ma^2\sin\theta\cos\theta\Delta(2r^2 - a^2\cos^2\theta)}{\rho^6}\eta^3 + \frac{2ma^2r\sin\theta\cos\theta}{\rho^6}\eta_\theta^2 - \frac{2ma\sin^2\theta\Delta\Omega}{\rho^2}\eta_t^4 = 0, \quad (35f)$$

$$\begin{aligned}
(\dot{r}^2) : & 4\xi_{sr} - \eta_{rr}^2 + \frac{m\Omega - a^2 r \sin^2 \theta}{\rho^2 \Delta} \eta_r^2 + \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_\theta^2 + \frac{2a^2}{\rho^2} \\
& \sin \theta \cos \theta \eta_r^3 - \frac{(2r - 2m)[m\Omega - a^2 r \sin^2 \theta]}{\rho^2 \Delta} \eta^2 - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \\
& [2r - 2m] \eta^3 + \frac{a^2 \sin \theta \Omega + 4ma^2 r \cos^2 \theta}{\rho^2 \Delta} \eta^2 + \frac{2a^2 \sin \theta \cos \theta}{\rho^4 \Delta} \\
& [m\Omega - a^2 r \sin^2 \theta] \eta^3 = 0,
\end{aligned} \tag{35g}$$

$$(\theta^2) : \eta_{\theta\theta}^2 + \frac{r\Delta}{\rho^2}\eta_r^2 - \frac{a^2 \sin \theta \cos \theta}{\rho^2}\eta_\theta^2 - \frac{2r\Delta}{\rho^2}\eta_\theta^3 - \frac{2a^2 r \sin \theta \cos \theta \Delta}{\rho^4}\eta^3 - \frac{2r[m\Omega - a^2 r \sin^2 \theta] + \Delta \rho^2}{\rho^4}\eta^2 = 0, \quad (35h)$$

$$(\phi^2) : \eta_{\phi\phi}^2 - \frac{\Delta[m a^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4]}{\rho^6} \eta_r^2 + \frac{\sin \theta \cos \theta}{\rho^6} [(r^2 + a^2)$$

$$\Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_\theta^2 - \frac{2ma^2 \sin^2 \theta \Delta \Omega}{\rho^6} \eta_\phi^1 - \frac{4mar \sin \theta \cos \theta}{\rho^2}$$

$$(r^2 + a^2)\eta_\phi^3 + \frac{2[m\Omega - a^2 r \sin^2 \theta]}{\rho^8} [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4] \eta^2$$

$$+ \frac{2\Delta[m a^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4]}{\rho^6} \eta_\phi^4 + \frac{2 \cot \theta \Delta(r^2 + a^2)}{\rho^8} [ma^2$$

$$\sin^4 \theta \Omega - r \sin^2 \theta \rho^4] \eta^3 - \frac{2ma^2 \sin^3 \theta \Delta}{\rho^8} [\cos \theta [\rho^4 - a^2 \sin^2$$

$$\Omega] \eta^3 + \frac{2a^2r \sin^2 \theta \Sigma}{a^8} \eta^3 - 2ma^2r \sin^4 \theta \left[ \frac{(r^2 - 3a^2 \cos^2 \theta)}{a^8} + \right.$$

$$\frac{\sin^2 \theta \Delta}{\rho^2} \eta^2 = 0, \quad (35i)$$

$$(i\dot{r}) : \xi_{st} - \eta_{tr}^2 - \frac{m\Delta\Omega}{\rho^6} \eta_r^1 + \frac{m(r^2 + a^2)\Omega}{\rho^4\Delta} \eta_t^2 + \frac{m\Omega - a^2r \sin^2 \theta}{\rho^2\Delta} \eta_t^2 \\ + \frac{ma\Omega}{\rho^4\Delta} \eta_\phi^2 + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_t^3 + \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_r^4 = 0, \quad (35j)$$

$$(i\dot{\theta}) : \eta_{t\theta}^2 + \frac{m\Delta\Omega}{\rho^6} \eta_\theta^1 + \frac{a^2 \sin \theta \cos \theta (-\rho^2 + 2mr)}{\rho^2} \eta_t^2 - \frac{r\Delta}{\rho^2} \eta_t^3 \\ - \frac{2mar \cot \theta}{\rho^4} \eta_\phi^2 - \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_\theta^4 = 0, \quad (35k)$$

$$(i\dot{\phi}) : \eta_{t\phi}^2 - \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_t^1 + \frac{m\Delta\Omega}{\rho^6} \eta_\phi^1 + \frac{\Delta}{\rho^6} [ma^2 \sin^4 \theta \Omega - r \\ \sin^2 \theta \rho^4] \eta_t^4 - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\theta^2 - \frac{2ma \sin^2 \theta \Delta}{\rho^8} \\ [-r(r^2 - 3a^2 \cos^2 \theta) + \Omega(m\Omega - a^2r \sin^2 \theta)] \eta^2 - 8mar \\ \sin \theta \Delta \cos \theta [\frac{(r^2 + a^2)\Omega}{\rho^8}] \eta^3 - 2ma^3r \sin^3 \theta \cos^2 \theta \Delta \frac{\Omega}{\rho^8} \eta^3 \\ + \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_r^2 - ma\Delta \frac{\sin^2 \theta \Omega}{\rho^6} \eta_\phi^4 = 0, \quad (35l)$$

$$(i\dot{\theta}) : 2\xi_{s\theta} - \eta_{r\theta}^2 - \frac{r}{\rho^2} \eta_\theta^2 + \frac{m\Omega - a^2r \sin^2 \theta}{\rho^2} \eta_\theta^1 + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_r^3 \\ - \frac{a^2 \sin \theta \cos \theta (2r - 2m)}{\rho^2\Delta} \eta^2 - \frac{a \sin^2 \theta \Omega - a^2 \cos^2 \theta \rho^2}{\rho^4} \eta^3 - r \\ \frac{\Delta}{\rho^2} \eta_r^3 = 0, \quad (35m)$$

$$(i\dot{\phi}) : 2\xi_{s\phi} - \eta_{r\phi}^2 + \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_r^1 + \frac{m\Omega - a^2r \sin^2 \theta}{\rho^2\Delta} \eta_\phi^2 + a^2 \\ \sin \theta \cos \theta \eta_\phi^3 - \frac{ma^2 \sin^2 \theta \Omega + r(-\rho^2 + 2mr)\rho^2}{\rho^4\Delta} \eta_\phi^2 - \frac{\Delta}{\rho^6} \\ [ma^2 \sin^2 \theta \Omega - r \sin^2 \theta \rho^4] \eta_r^4 - \frac{ma \sin^2 \theta}{\rho^4\Delta} ((r^2 + a^2)\Omega + 2 \\ r^2\rho^2) \eta_t^2 = 0, \quad (35n)$$

$$(\dot{\theta}\phi) : \eta_{\theta\phi}^2 - \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_\theta^1 - \frac{2ma^3r \sin^3 \theta \cos \theta}{\rho^4} \eta_\phi^2 + \frac{\Delta}{\rho^6} [ma^2 \\ \sin^4 \theta \Omega - r \sin^2 \theta \rho^4] \eta_\theta^4 + \frac{\cot \theta \rho^4 - 2ma^2r \sin \theta \cos \theta - r\Delta \rho^2}{\rho^4} \\ \eta_\phi^3 = 0. \quad (35o)$$

$$(constant) : \eta_{ss}^3 = 0, \quad (36a)$$

$$(i) : \eta_{st}^3 - \frac{2ma^2r \sin \theta \cos \theta}{\rho^6} \eta_s^1 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_s^4 = 0, \quad (36b)$$

$$(\dot{r}) : \eta_{sr}^3 + \frac{a^2 \sin \theta \cos \theta}{\rho^2\Delta} \eta_s^2 + \frac{r}{\rho^2} \eta_s^3 = 0, \quad (36c)$$

$$(\dot{\theta}) : \xi_{ss} - \eta_{s\theta}^3 - \frac{r}{\rho^2} \eta_s^2 + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_s^3 = 0, \quad (36d)$$

$$(\dot{\phi}) : \eta_{s\phi}^3 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_s^1 - \frac{\sin \theta \cos \theta}{\rho^6} [(r^2 + a^2)\Sigma$$

$$-a^2 \sin^2 \theta \Delta \rho^2] \eta_s^4 = 0, \quad (36e)$$

$$(i^2) : -\frac{4ma^2r \sin \theta \cos \theta}{\rho^6} \eta_t^1 - \frac{m\Delta\Omega}{\rho^6} \eta_r^3 + \frac{2ma^2r}{\rho^6} \sin \theta \cos \theta \eta_\theta^3 \\ + \frac{4mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_t^4 + \frac{2ma^2 \sin \theta \cos \theta}{\rho^8} (5r^2 - a^2 \cos^2 \theta) \eta^2 + \frac{2ma^2r [\sin^2 \theta (r^2 - 5a^2 \cos^2 \theta) - \cos^2 \theta \rho^2]}{\rho^8} \eta^3 \\ + \eta_{tt}^3 = 0, \quad (36f)$$

$$(r^2) : \frac{2a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_r^2 - \frac{a^2 \sin \theta \cos \theta [2r\Delta + (2r - 2m)\rho^2]}{\rho^4 \Delta^2} \eta^2 \\ + \frac{[m\Omega - a^2r \sin^2 \theta] + 2r\Delta}{\rho^2 \Delta} \eta_r^3 - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_\theta^3 + \eta_{rr}^3 \\ + \frac{a^2 \rho^2 (\cos^2 \theta - \sin^2 \theta) + 2a^4 \sin^2 \theta \cos^2 \theta}{\rho^2 \Delta} \eta^3 = 0, \quad (36g)$$

$$(\theta^2) : -\eta_{\theta\theta}^3 - \frac{2r}{\rho^2} \eta_\theta^2 - \frac{r\Delta}{\rho^2} \eta_r^3 + \frac{a^2 \sin \theta \cos \theta}{\rho^2} \eta_\theta^3 - \frac{2a^2r \sin \theta \cos \theta}{\rho^4} \eta^2 \\ + \frac{a^2 \rho^2 (\cos^2 \theta - \sin^2 \theta) + 2a^4 \sin^2 \theta \cos^2 \theta}{\rho^4} \eta^3 + 4\xi_{s\theta} = 0, \quad (36h)$$

$$(\phi^2) : \eta_{\phi\phi}^3 - \frac{\Delta [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4]}{\rho^6} \eta_r^3 + \frac{\sin \theta \cos \theta}{\rho^6} [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \\ \Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_\theta^3 - \frac{2 \sin \theta \cos \theta}{\rho^6} [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \\ \rho^2] \eta_\phi^4 + \frac{2r \sin \theta \cos \theta [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^8} \eta^2 + 2 \sin \theta \\ \frac{\cos \theta}{\rho^8} [-r\rho^6 + 2ma \sin^2 \theta (r^2 + a^2) \Omega + ma^3 \sin^4 \theta] \eta^2 + \sin \theta \\ \frac{(r^2 - 5a^2 \cos^2 \theta)}{\rho^6} [-a^2 \sin^2 \theta \Delta \rho^2 + (r^2 + a^2) \Sigma] \eta^3 - \frac{\cos^2 \theta}{\rho^6} \\ [[(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2] - 4a^2 \sin^2 \theta \Delta \rho^2] \eta^3 + 4mar \sin \theta \\ \frac{\cos \theta (r^2 + a^2)}{\rho^4} \eta_\phi^1 = 0, \quad (36i)$$

$$(i\dot{r}) : \eta_{tr}^3 - \frac{2ma^2r \sin \theta \cos \theta}{\rho^6} \eta_r^1 + \frac{m(r^2 + a^2)\Omega - r\rho^2\Delta}{\rho^4 \Delta} \eta_t^3 + \frac{ma\Omega}{\rho^4 \Delta} \\ \eta_\phi^3 + \frac{2a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_t^2 + \frac{2mar \sin \theta (r^2 + a^2)}{\rho^6} \cos \theta \eta_r^4 = 0, \quad (36j)$$

$$(i\dot{\theta}) : 2\xi_{st} - \eta_{t\theta}^3 + \frac{2ma^2r \sin \theta \cos \theta}{\rho^6} \eta_\theta^1 - \frac{r}{\rho^2} \eta_t^2 - \frac{2mar \cot \theta}{\rho^4} \eta_\phi^3 \\ - \frac{a^2 \sin \theta \cos \theta (-\rho^2 + 2mr)}{\rho^2} \eta_t^3 - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\theta^4 \\ = 0, \quad (36k)$$

$$(i\dot{\phi}) : \eta_{t\phi}^3 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_t^1 - \frac{2ma^2r \sin \theta \cos \theta}{\rho^6} \eta_\phi^1 \\ - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\theta^3 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\phi^4 \\ - \frac{\sin \theta (r^2 - 5a^2 \cos^2 \theta)}{\rho^6} [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_t^4 - 2$$

$$\begin{aligned} & \frac{ma \sin \theta \cos \theta}{\rho^8} [2a^2 r^2 \sin^2 \theta + (r^2 + a^2)(3r^2 - a^2 \cos^2 \theta \rho^2)] \eta^2 \\ & - \frac{2mar}{\rho^8} [-\cos^2 \theta (r^2 + a^2) + \sin^2 \theta (r^2 + a^2)(r^2 - 5a^2 \cos^2 \theta) \\ & )] \eta^3 = 0, \end{aligned} \quad (36l)$$

$$\begin{aligned} (\dot{r}\dot{\theta}) : 2\xi_{sr} - \eta_{r\theta}^3 - \frac{2a^2 r \sin \theta \cos \theta}{\rho^4} \eta^3 + \frac{\Omega}{\rho^4} \eta^2 + \frac{ma \sin^2 \theta \Delta \Omega}{\rho^6} \eta_r^3 \\ - \frac{r}{\rho^2} \eta_\theta^3 - \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_\theta^2 = 0, \end{aligned} \quad (36m)$$

$$\begin{aligned} (\dot{r}\dot{\phi}) : \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_r^1 + \frac{a^2 \sin \theta \cos \theta}{\rho^2 \Delta} \eta_\phi^2 + \frac{r(\rho^2 + 2mr)}{\Delta} \eta_\phi^3 \\ + \eta_{r\phi}^3 - \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^6} \eta_r^4 + \frac{1}{\rho^4 \Delta} [ma \sin^2 \theta \\ [(r^2 + a^2)\Omega] + 2r^2 \rho^2] \eta_t^3 = 0, \end{aligned} \quad (36n)$$

$$\begin{aligned} (\dot{\theta}\dot{\phi}) : +2\xi_{s\phi} - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^6} \eta_\theta^1 - \frac{2ma^3 r \sin^3 \theta \cos \theta}{\rho^4} \eta_t^3 \\ - \frac{r}{\rho^2} \eta_\phi^2 - \eta_{\theta\phi}^3 - \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^6} \eta_t^4 \\ + \frac{\cot \theta (r^2 + a^2) \rho^2 + 2ma^2 r \sin \theta \cos \theta}{\rho^4} \eta_\phi^3 = 0. \end{aligned} \quad (36o)$$

$$(constant) : 2mar \sin^2 \theta \eta_{ss}^1 - \sin^2 \theta \Sigma \eta_{ss}^4 = 0, \quad (37a)$$

$$\begin{aligned} (\dot{t}) : 2mar \sin^2 \theta \xi_{ss} - 2mar \sin^2 \theta \eta_{st}^1 + \sin^2 \theta \Sigma \eta_{ss}^4 + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_s^2 \\ - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_s^3 = 0, \end{aligned} \quad (37b)$$

$$\begin{aligned} (\dot{r}) : 2mar \sin^2 \theta \eta_{sr}^1 + \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^2} \eta_s^4 - \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_s^1 \\ - \sin^2 \theta \Sigma \eta_{sr}^4 = 0, \end{aligned} \quad (37c)$$

$$\begin{aligned} (\dot{\theta}) : +2mar \sin^2 \theta \eta_{s\theta}^1 - \sin^2 \theta \Sigma \eta_{s\theta}^4 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_s^1 \\ - \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \rho^2]}{\rho^2} \eta_s^4 = 0, \end{aligned} \quad (37d)$$

$$\begin{aligned} (\dot{\phi}) : \sin^2 \theta \Sigma \xi_{ss} - \sin^2 \theta \Sigma \eta_{s\phi}^4 + \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^2} \eta_s^2 + 2mar \\ \sin^2 \theta \eta_{s\phi}^1 - \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^2} \eta_s^3 = 0, \end{aligned} \quad (37e)$$

$$\begin{aligned} (\dot{t}^2) : -2mar \sin^2 \theta \eta_{tt}^1 + 8mar \sin^2 \theta \xi_{st} + \sin^2 \theta \Sigma \eta_{tt}^4 + \frac{2ma \sin^2 \theta \Omega}{\rho^2} \eta_t^2 \\ + \frac{2mar \sin^2 \theta \Delta (-\rho^2 + 2mr) \Omega}{\rho^6} \eta_r^1 - \frac{4m^2 a^3 r^2 \sin^3 \theta \cos \theta}{\rho^6} \eta_\theta^1 - 4m \\ \left( \frac{ar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \right) \eta_t^3 - \frac{m \sin^2 \theta \Delta \Omega \Sigma}{\rho^6} \eta_r^4 + \frac{2ma^2 r \sin^3 \theta \cos \theta}{\rho^6} \\ \Sigma \eta_\theta^4 = 0, \end{aligned} \quad (37f)$$

$$(\dot{r}^2) : -\frac{2ma^3 r \sin^3 \theta \cos \theta}{\rho^2 \Delta} \eta_\theta^1 + \frac{2mar \sin^2 \theta [m\Omega - a^2 r \sin^2 \theta]}{\rho^2 \Delta} \eta_r^1 + 2m$$

$$\begin{aligned} ar \sin^2 \theta \eta_{rr}^1 - \frac{2ma \sin^2 \theta \Omega}{\rho^2} \eta_r^1 - \frac{\sin^2 \theta}{\rho^2 \Delta} [\Sigma(m\Omega - a^2 r \sin^2 \theta)] \eta_r^4 \\ + \frac{2ma^2 \sin^4 \theta \Omega - 2r \sin^2 \theta \rho^4}{\rho^2} \eta_r^4 + \frac{a^2 \sin^3 \theta}{\rho^2 \Delta} [\cos \theta \Sigma] \eta_\theta^4 = 0, \end{aligned} \quad (37g)$$

$$\begin{aligned} (\dot{\theta}^2) : 2mar \sin^2 \theta \eta_{\theta\theta}^1 + \frac{2a^2 \sin \theta \cos \theta (-\rho^2 + 2mr + mar \sin^2 \theta)}{\rho^2} \eta_\theta^1 \\ + \frac{2mar^2 \sin^2 \theta \Delta}{\rho^2} \eta_r^1 - \frac{\sin \theta \cos \theta \Sigma [2(r^2 + a^2) + a^2 \sin^2 \theta]}{\rho^2} \eta_\theta^4 \\ - \frac{r \sin^2 \theta \Delta \Sigma}{\rho^2} \eta_r^4 - \sin^2 \theta \Sigma \eta_{\theta\theta}^4 = 0, \end{aligned} \quad (37h)$$

$$\begin{aligned} (\dot{\phi}^2) : 4 \sin^2 \theta \Sigma \xi_{s\phi} + 2mar \sin^2 \theta \eta_{\phi\phi}^1 + \frac{2ma^2 \sin^4 \theta \Omega - 2r \sin^2 \theta \rho^4}{\rho^2} \eta_\phi^2 \\ - \sin^2 \theta \Sigma \eta_{\phi\phi}^4 - \frac{2mar \sin^2 \theta \Delta [ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4]}{\rho^6} \eta_r^1 + 2 \\ \left( \frac{mar \sin^2 \theta \cos \theta}{\rho^6} \right) [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_\theta^1 - \frac{2 \sin \theta \cos \theta}{\rho^2} \\ [-a^2 \sin^2 \theta \Delta \rho^2 + (r^2 + a^2) \Sigma] \eta_\phi^3 + \frac{\Delta \sin^2 \theta \Sigma}{\rho^6} [ma^2 \sin^4 \theta \Omega \\ - r \sin^2 \theta \rho^4] \eta_r^4 - [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2] \left( \frac{\sin^3 \theta \cos \theta \Sigma}{\rho^6} \right) \eta_\theta^4 \\ = 0, \end{aligned} \quad (37i)$$

$$\begin{aligned} (\dot{t}\dot{r}) : 4mar \sin^2 \theta \xi_{sr} - \frac{2mar \sin^2 \theta (r^2 - 3a^2 \cos^2 \theta)}{\rho^4} \eta^2 - 2mar \\ \sin^2 \theta \eta_{tr}^1 + \frac{ma \sin^2 \theta \Omega [\rho^2 \Delta + 2mr(r^2 + a^2)]}{\rho^2} \eta_t^1 + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_r^2 \\ + \frac{2m^2 a^2 r \sin^2 \theta \Omega}{\rho^4 \Delta} \eta_\phi^1 - 2ma \sin^2 \theta \left[ \frac{\Omega [m\Omega - \rho^2]}{\rho^4 \Delta} \right] \eta^2 + 2ma^2 \\ \sin \theta \cos \theta \left[ \frac{3r^2 - a^2 \cos^2 \theta}{\rho^4} \right] \eta^3 - \frac{2ma \sin \theta \cos \theta \Omega}{\rho^4} \eta^3 - 2mar \\ \left[ \frac{\sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \right] \eta_r^3 - \left( \frac{\Omega - r \sin^2 \theta \rho^4}{\rho^2} \right) ma^2 \sin^4 \theta \eta_t^4 - \Omega \Sigma \\ \frac{m \sin^2 \theta (r^2 + a^2)}{\rho^4 \Delta} \eta_t^4 - \frac{ma \sin^2 \theta \Omega \Sigma}{\rho^4 \Delta} \eta_\phi^4 + \sin^2 \theta \Sigma \eta_{tr}^4 = 0, \end{aligned} \quad (37j)$$

$$\begin{aligned} (\dot{t}\dot{\theta}) : 4mar \sin^2 \theta \xi_{s\theta} - 2mar \sin^2 \theta \eta_{t\theta}^1 + \sin^2 \theta \Sigma \eta_{t\theta}^4 + \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_\theta^2 \\ - \frac{2mar \sin \theta \cos \theta [\rho^2 (r^2 + a^2) + 2ma^2 r \sin^2 \theta]}{\rho^4} \eta_t^1 - \left( \frac{4m^2 a^2 r^2}{\rho^4} \right) \\ \sin \theta \cos \theta \eta_\phi^1 + \frac{2ma \sin \theta \cos \theta (r^2 + a^2) (3r^2 - a^2 \cos^2 \theta)}{\rho^4} \eta^2 + 2 \\ \frac{mar \sin^2 \theta [(r^2 + a^2) (r^2 - 3a^2 \cos^2 \theta) + 4ma^2 r \cos^2 \theta]}{\rho^4} \eta^3 - 2ma \\ \frac{r \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_\theta^3 + \frac{\sin \theta \cos \theta [(r^2 + a^2) \Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^2} \eta_t^4 \\ + \frac{2ma^2 r^2 \sin^3 \theta \cos \theta \Sigma}{\rho^4} \eta_t^4 + \frac{2mar \cos^2 \theta (r^2 + a^2)}{\rho^2} \eta^3 + 2mar \\ \sin \theta \cos \theta \Sigma \eta_\phi^4 = 0, \end{aligned} \quad (37k)$$

$$(i\dot{\phi}) : 4mar \sin^2 \theta \xi_{s\phi} - 2 \sin^2 \theta \Sigma \xi_{st} + \sin^2 \theta \Sigma \eta_{t\phi}^4 - 2mar \sin^2 \theta \eta_{t\phi}^1 \\ - \frac{2m^2 a^2 r \sin^4 \theta \Delta \Omega}{\rho^6} \eta_r^1 + 4m^2 a^2 r^2 \cos \theta \left( \frac{\sin^3 \theta (r^2 + a^2)}{\rho^6} \right) \eta_r^1 \\ - \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_\phi^3 - \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^2} \eta_t^2 \\ + \frac{\sin \theta \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^2} \eta_t^3 + \frac{ma \sin^4 \theta \Delta \Omega \Sigma}{\rho^6} \eta_r^4 \\ - 2mar \left[ \frac{\sin^3 \theta \cos \theta (r^2 + a^2)\Sigma}{\rho^6} \right] \eta_\theta^4 + ma \sin^2 \theta \frac{\Omega}{\rho^2} \eta_\phi^2 = 0, \quad (37l)$$

$$(r\dot{\theta}) : 2mar \sin^2 \theta \eta_{r\theta}^1 + \frac{2mar \sin \theta \cos \theta (r^2 + a^2)}{\rho^2} \eta_r^1 - \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_\theta^1 \\ - \sin^2 \theta \Sigma \eta_{r\theta}^4 + \frac{2ma^3 r \sin^3 \theta \cos \theta}{\rho^2} \eta_r^1 - \frac{a^2 \sin^3 \theta \Sigma}{\rho^2} \eta_r^4 - \sin \theta \cos \theta \\ \frac{[(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^2} \eta_r^4 + \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^2} \eta_\theta^4 \\ + \frac{r \sin^2 \theta \Sigma}{\rho^2} \eta_\theta^4 = 0, \quad (37m)$$

$$(r\dot{\phi}) : 2 \sin^2 \theta \Sigma \xi_{sr} + 2mar \sin^2 \theta \eta_{r\phi}^1 - \sin^2 \theta \Sigma \eta_{tr}^4 - \frac{ma \sin^2 \theta \Omega}{\rho^2} \eta_\phi^1 \\ + \frac{2mar \sin^2 \theta [ma^2 \sin^2 \theta \Omega + r(-\rho^2 + 2mr)\rho^2]}{\rho^4 \Delta} \eta_\phi^1 - \sin \theta \cos \theta \\ \frac{[(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2]}{\rho^2} \eta_r^3 + \frac{2a^2 \sin^3 \theta \cos \theta}{\rho^4} [r\Sigma + m\rho^2 \\ (-\rho^2 + 2mr)] \eta^3 - \frac{ma \sin^4 \theta \Sigma}{\rho^4 \Delta} [(r^2 + a^2)\Omega + 2r^2 \rho^2] \eta_t^4 + \frac{\Sigma}{\rho^4 \Delta} \\ [ma^2 \sin^4 \theta \Omega + r\rho^2 \sin^2 \theta (-\rho^2 + 2mr)] \eta_\phi^4 - \frac{2ma^2 \sin^4 \theta \Omega}{\rho^4 \Delta} [r\rho^2 \\ + m\Omega] \eta^2 - \frac{2ma^2 r \sin^4 \theta (r^2 - 3a^2 \cos^2 \theta) + \sin^2 \theta \rho^6}{\rho^4} \eta^2 + ma^2 r \\ \frac{\sin^4 \theta}{\rho^2} [(r^2 + a^2)\Omega + 2r^2 \rho^2] \eta_r^2 + \frac{2m^2 a^3 r}{\rho^4 \Delta} [\sin^4 \theta (r^2 + a^2)\Omega + 2r^2 \\ \sin^4 \theta \rho^2] \eta_t^1 - ma^2 \sin^4 \theta \left( \frac{\Omega + r \sin^2 \theta \rho^4}{\rho^4 \Delta} \right) \eta_\phi^4 = 0, \quad (37n)$$

$$(\dot{\theta}\dot{\phi}) : 2 \sin^2 \theta \Sigma \xi_{s\theta} + 2mar \sin^2 \theta \eta_{\theta\phi}^1 - \sin^2 \theta \Sigma \eta_{t\theta}^4 + \frac{\sin \theta \cos \theta \Sigma}{\rho^2} [(r^2 \\ + a^2) + (-\rho^2 + 2mr)a^2 \sin^2 \theta] \eta_\phi^4 - \frac{\sin \theta \cos \theta}{\rho^2} [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_\phi^4 \\ - \frac{4m^2 a^4 r^2 \sin^5 \theta \cos \theta}{\rho^4} \eta_t^1 - \frac{2ma^3 r \sin^3 \theta \cos \theta}{\rho^4} \\ (-\rho^2 + 2mr) \eta_\phi^1 + \frac{ma^2 \sin^4 \theta \Omega - r \sin^2 \theta \rho^4}{\rho^2} \eta_\theta^2 + \frac{2ma^3 r \sin^5 \theta}{\rho^4} \\ \cos \theta \Sigma \eta_t^4 - \frac{2 \sin \theta \cos \theta [ma^2 \sin^2 \theta (r^2 - a^2) + r\rho^4]}{\rho^2} \eta^2 - \sin \theta \\ \cos \theta [(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta_\theta^3 + \frac{\sin^2 \theta (r^2 - 3a^2 \cos^2 \theta)}{\rho^4}$$

$$[(r^2 + a^2)\Sigma - a^2 \sin^2 \theta \Delta \rho^2] \eta^3 + \frac{2a^2 \sin^2 \theta \cos^2 \theta \Delta}{\rho^4} (\rho^4 + 2ma^2 r \sin^2 \theta) \eta^3 = 0. \quad (37o)$$

Solving the above system of partial differential equations yields the isometries for the Kerr black hole metric along with the dilatation algebra  $d_2 = \langle \frac{\partial}{\partial s}, s \frac{\partial}{\partial s} \rangle$  generated by the geodetic parameter by translation and re-scaling, i.e.,

$$\begin{aligned} X_1 &= \frac{\partial}{\partial s}, & X_2 &= s \frac{\partial}{\partial s}, \\ X_3 &= \frac{\partial}{\partial t}, & X_4 &= \frac{\partial}{\partial \phi}. \end{aligned} \quad (38)$$

These four symmetries are the required Mei symmetries corresponding to the Lagrangian of the Kerr black hole metric. Thus, in the case of the Kerr black hole metric, isometries and Noether symmetries form a sub-algebra of the Mei symmetries. One can also observe that all four Mei symmetries of the Kerr black hole metric are also Lie point symmetries of the system of equations of motion given by Equations (28)–(31), i.e., Mei symmetries are subset of Lie point symmetries.

## 5. Summary

This paper focuses on Mei symmetries for the Lagrangians given by Equations (12) and (26). The set of Noether symmetries of the geodetic Lagrangian contains three elements, *two* KVs and *one* additional symmetry  $\partial/\partial s$  and the Noether symmetries of the Lagrangian for the Schwarzschild metric has a *five* dimensional algebra, which contains the *four* KVs of this metric and  $\partial/\partial s$ . This implies that the isometries are a sub-algebra of the Noether symmetries. For both the Schwarzschild and Kerr spacetimes, Noether symmetries form a subset of the Lie point symmetries [18,19].

Using the Mei symmetries criteria, the Euler Lagrange equations in the Boyer–Lindquist coordinates  $(t, r, \theta, \phi)$  are compiled one by one. The infinitesimal generator is extended and the system of determining equations for all dependent variables is attained. The system is then solved to determine the values of the infinitesimals.

Comparing the resultant Mei symmetries to the Lie point symmetries and the Noether symmetries. In the case of the Schwarzschild metric, it is found that the set of Mei symmetries consists of four elements, three Noether symmetries and  $s\partial/\partial s$ . There is no explicit relationship between Noether and Mei symmetries. However, the obtained Mei symmetries of the Schwarzschild metric are found to be the subset of Lie point symmetries of the Schwarzschild metric. In the case of the Kerr black hole metric, the set of Mei symmetries consists of four elements, three Noether symmetries and  $s\partial/\partial s$ . All four Mei symmetries of the Kerr black hole metric are the Lie point symmetries of the system of equations of motion of the Kerr black hole metric. Therefore, for these two metrics, the isometries form a subalgebra of the Noether symmetries [18,19] and the Mei symmetries form a subalgebra of the Lie point symmetries. Notice that the Mei symmetries are the same for both the Schwarzschild and the Kerr black hole metrics.

A general result will be worth exploring to see if the Mei symmetries of all the spherically symmetric spacetimes form a subalgebra of the Lie symmetries. Further, it can also be seen if there is a general relation between Noether symmetries and Mei symmetries of the axially symmetric spacetimes.

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