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Minimum Superstability of Stochastic Ternary Antiderivations in Symmetric Matrix-Valued FB-Algebras and Symmetric Matrix-Valued FC-⋄-Algebras

Zahra Eidinejad [1], Reza Saadati [1,*], Donal O'Regan [2] and Fehaid Salem Alshammari [3]

1 School of Mathematics, Iran University of Science and Technology, Narmak, Tehran 13114-16846, Iran
2 School of Mathematical and Statistical Sciences, National University of Ireland, Galway, University Road, H91 TK33 Galway, Ireland
3 Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University, Riyadh 11432, Saudi Arabia
* Correspondence: rsaadati@eml.cc

Abstract: Our main goal in this paper is to investigate stochastic ternary antiderivatives (STAD). First, we will introduce the random ternary antiderivative operator. Then, by introducing the aggregation function using special functions such as the Mittag-Leffler function (MLF), the Wright function (WF), the H-Fox function (HFF), the Gauss hypergeometric function (GHF), and the exponential function (EXP-F), we will select the optimal control function by performing the necessary calculations. Next, by considering the symmetric matrix-valued FB-algebra (SMV-FB-A) and the symmetric matrix-valued FC-⋄-algebra (SMV-FC-⋄-A), we check the superstability of the desired operator. After stating each result, the superstability of the minimum is obtained by applying the optimal control function.

Keywords: stochastic ternary antiderivatives (STAD); symmetric matrix-valued FC-⋄-algebra (SMV-FC-⋄-A); matrix-valued FB-algebra; superstability; fixed point method (FTP); fuzzy inequality

MSC: 47C10

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1. Introduction

The study of the stability problem for group homomorphisms started with the famous Ulam question in 1940, and in 1941 Hyers established the stability of nonlinear functions in a particular case.In 1978, Rassias extended this problem and now it is known as the Hyers-Ulam-Rasias problem.Various aspects of the Hyers-Ulam stability for various functions and mappings have been investigated and among these equations and mappings, we refer the reader to Euler-Lagrange functional equations (E-L-FE), differential equations (DE), Navier-Stokes equations (N-SE), as well as mappings such as Cauchy-Jensen mappings, k-additive mappings, multiplicative mappings [1–4]. Most of the investigations have been carried out in Banach spaces and now research is conducted in other spaces as well [5–8]. Also, the stability of groups, Banach algebra, ternary Banach algebras, and C- ternary algebras have attracted the attention of many researchers [9–11]. Among the various applications of ternary algebras, we mention Nambu mechanics and quark in physics and mathematics and we also point to different applications of the additive principle in physics in the field of internal energy and the superposition principle. In addition, many physics problems can be considered as a linear system and can be solved [12,13].

In 2003, Radu-Mihet proposed a new method to obtain the exact solution and error estimation, which was based on the alternative fixed-point method. Note that fixed point theory arises in various fields such as dynamical systems, equilibrium problems, and differential equations because this theory provides basic tools for examining these problems [14,15].

Specific functions can be used to establish stability and the selection of the control function plays a special role. The Mittag-Leffler function (M-LF) is used in many fields of science and engineering and certain areas of physical, biological, and earth sciences [16]. Another important function that is particularly important in solving partial differential equations and fractional theory is the Wright function (WF) [17,18]. Two other types of functions used in this article are the *H*-Fox function (*H*-FF) and the Gauss hypergeometric function (GHF) [19,20]. We note that many special functions that we encounter in physics, engineering, and probability theory are special cases of Gauss hypergeometric functions. To determine the optimal control function, we introduce the aggregation functions (AF) and aggregation functions are rooted in different fields. Application areas of aggregation functions include applied mathematics, probability, statistics, decision theory, computer science, artificial intelligence, operations research, as well as many applied fields, economics and finance, pattern recognition and image processing, data fusion, multicriteria decision aid, automated reasoning, etc. [8,20].

In this article, we define a stochastic ternary antiderivative operator and consider the symmetric matrix-valued FB-algebra (SMV-FB-A) and the symmetric matrix-valued FC-⋄-algebra (SMV-FC-⋄-A), and we investigate the superstability and minimum superstability of this operator.

We assume that (Π, X, Ω_M) is a probability measure space. Considering the measurable space (M, G_M), we define the stochastic operator $\phi : \Pi \times M \to M$. In all of the proofs, we consider the (σ_1, σ_2)-functional inequality which is as follows

$$\begin{aligned} &\Omega_M\Big(\phi(z,u+v+w)-\phi(z,u+w)-\phi(z,u+v-w)-\phi(z,u-w),\eta\Big)\\ &\succeq \Omega_M\Big(\sigma_1(\phi(z,u+v-w)+\phi(z,u-w)-\phi(z,v)),\eta\Big) \circledast \Omega_M\Big(\sigma_2(\phi(z,u-w)+\phi(z,u)-\phi(z,w)),\eta\Big), \end{aligned} \tag{1}$$

where $\sigma_1, \sigma_2 \in \mathbb{C}$ and $|\sigma_1| + |\sigma_2| > 2$. The general structure of the article is as follows.

In the second section, all the required concepts including special functions and spaces used to prove the desired results are given. In the third section, after stating the necessary lemmas, the stability of the stochastic operator in SMVFB-A is investigated, and then the minimum stability of this operator is proved. In the fourth section, the superstability of stochastic ternary antiderivatives is proved by introducing stochastic ternary antiderivatives and considering T-SMVFB and T-SMVFC-⋄-A spaces. Also, in the form of an example, minimum stability is investigated. In the last part, the superstability of continuous stochastic ternary antiderivatives in the introduced spaces is proved.

2. Preliminaries

We denote the set of all $p \times p$ diagonal matrices by $\mathcal{D}_H = \mathrm{diagH}([0,1])$, and we consider this set as follows

$$\mathcal{D}_H = \mathrm{diagH}([0,1]) = \left\{ \begin{bmatrix} h_1 & & \\ & \ddots & \\ & & h_p \end{bmatrix} = \mathrm{diag}[h_1, \cdots, h_p],\ h_1, \ldots, h_p \in [0,1] \right\}.$$

For the above set, we have

- If $\mathbf{h}, \mathbf{g} \in \mathcal{D}_H$, then $\mathbf{h} = \mathrm{diag}[h_1, \cdots, h_p]$ and $\mathbf{g} = \mathrm{diag}[g_1, \cdots, g_p]$;
- $\mathbf{h} \preceq \mathbf{g}$ means that $h_\iota \le g_\iota$ for every $\iota = 1, \ldots, m$;
- $\mathbf{h} \prec \mathbf{g}$ denotes that $h_\iota < g_\iota$ for every $\iota = 1, \ldots, m$;
- $\mathrm{diag}[1, \ldots, 1] = \mathbf{1}$ and $\mathrm{diag}[0, \ldots, 0] = \mathbf{0}$.

Definition 1 ([5,9]). *A mapping $\circledast : \mathcal{D}_H \times \mathcal{D}_H \to \mathcal{D}_H$ is called a GTN (generalized t-norm) if the boundary condition, commutativity condition, associativity condition and monotonicity condition hold as follows:*

(I) $\mathbf{h} \circledast \mathbf{1} = \mathbf{h}$ *for all* $\mathbf{h} \in \mathcal{D}_H$;

(II) $\mathbf{h} \circledast \mathbf{g} = \mathbf{g} \circledast \mathbf{h}$ *for all* $\mathbf{h}, \mathbf{g} \in \mathcal{D}_{\mathrm{H}}$*;*
(III) $\mathbf{h} \circledast (\mathbf{g} \circledast \mathbf{k}) = (\mathbf{h} \circledast \mathbf{g}) \circledast \mathbf{k}$ *for all* $\mathbf{h}, \mathbf{g}, \mathbf{k} \in \mathcal{D}_{\mathrm{H}}$*;*
(IV) $\mathbf{h} \preceq \mathbf{g}$ *and* $\mathbf{k} \preceq \mathbf{s}$ *implies that* $\mathbf{h} \circledast \mathbf{k} \preceq \mathbf{g} \circledast \mathbf{s}$*, for all* $\mathbf{h}, \mathbf{k}, \mathbf{g}, \mathbf{s} \in \mathcal{D}_{\mathrm{H}}$*;*
For convergent sequences $\{\mathbf{h}_{\mathrm{p}}\}$ *and* $\{\mathbf{g}_{\mathrm{p}}\}$ *with convergence points* $\mathbf{h}$ *and* $\mathbf{g}$*, if we have*
(V) $\lim_{\mathrm{m}}(\mathbf{h}_{\mathrm{p}} \circledast \mathbf{g}_{\mathrm{p}}) = \mathbf{h} \circledast \mathbf{g}$*,*
then the GTN $\circledast$ *is a continuous GTN (CGTN).*

In the following, we define some examples of CGTN:

Definition 2. $\circledast_M : \mathcal{D}_{\mathrm{H}} \times \mathcal{D}_{\mathrm{H}} \to \mathcal{D}_{\mathrm{H}}$ *is called minimum CGTN (MIN-CGTN), which is defined as follows*

$$\mathbf{h} \circledast_M \mathbf{g} = diag[\mathrm{h}_1, \cdots, \mathrm{h}_{\mathrm{p}}] \circledast_M diag[\mathrm{g}_1, \cdots, \mathrm{g}_{\mathrm{p}}] = diag[\min\{\mathrm{h}_1, \mathrm{g}_1\}, \cdots, \min\{\mathrm{h}_{\mathrm{p}}, \mathrm{h}_{\mathrm{p}}\}].$$

Definition 3. $\circledast_P : \mathcal{D}_{\mathrm{H}} \times \mathcal{D}_{\mathrm{H}} \to \mathcal{D}_{\mathrm{H}}$ *is called product CGTN (P-CGTN), which is defined as follows*

$$\mathbf{h} \circledast_P \mathbf{g} = diag[\mathrm{h}_1, \cdots, \mathrm{h}_{\mathrm{p}}] \circledast_P diag[\mathrm{g}_1, \cdots, \mathrm{g}_{\mathrm{p}}] = diag[\mathrm{h}_1.\mathrm{g}_1, \cdots, \mathrm{h}_{\mathrm{m}}.\mathrm{g}_{\mathrm{p}}].$$

Definition 4. $\circledast_L : \mathcal{D}_{\mathrm{H}} \times \mathcal{D}_{\mathrm{H}} \to \mathcal{D}_{\mathrm{H}}$ *is called Lukasiewicz CGTN(L-CGTN), which is defined as follows*

$$\mathbf{h} \circledast_L \mathbf{g} = diag[\mathrm{h}_1, \cdots, \mathrm{h}_{\mathrm{p}}] \circledast_L diag[\mathrm{g}_1, \cdots, \mathrm{g}_{\mathrm{p}}] = diag[\max\{\mathrm{h}_1 + \mathrm{g}_1 - 1, 0\}, \cdots, \max\{\mathrm{h}_{\mathrm{p}} + \mathrm{g}_{\mathrm{p}} - 1, 0\}].$$

For MIN-CGTN, P-CGTN and L-CGTN introduced in the above examples, the following inequality always holds (we have considered the dimension of the matrix as 3)

$$\mathrm{diag}[\mathrm{h}, \mathrm{g}, \mathrm{k}] \circledast_M \mathrm{diag}[\mathrm{l}, \mathrm{m}, \mathrm{n}] \succeq \mathrm{diag}[\mathrm{h}, \mathrm{g}, \mathrm{k}] \circledast_P \mathrm{diag}[\mathrm{l}, \mathrm{m}, \mathrm{n}] \succeq \mathrm{diag}[\mathrm{h}, \mathrm{g}, \mathrm{k}] \circledast_L \mathrm{diag}[\mathrm{l}, \mathrm{m}, \mathrm{n}].$$

We refer to [5,6,9] to see more numerical examples of the introduced CGTNs.

Definition 5 ([5])**.** *We say* Ψ *is a matrix-valued fuzzy function (MVFF) if* $\Psi : [0, \mathrm{a}] \times (0, +\infty) \to \mathcal{D}_{\mathrm{H}}$ *and*
(MF1) Ψ *is increasing and continuous;*
(MF2) $\lim_{\eta \to +\infty} \Psi(\mathrm{u}, \eta) = \mathbf{1}$ *for any* $\mathrm{u} \in [0, \mathrm{a}]$ *and* $\eta \in (0, +\infty)$*;*
(MF3) If $\mathcal{Z}$ *is another MVFF, the relationship of* $\preceq$ *for these functions is defined as* $\mathcal{Z} \precsim \Psi$ *if and only if* $\mathcal{Z}(\mathrm{u}, \eta) \preceq \mathcal{Z}(\mathrm{u}, \eta)$*, for all* $\eta \in (0, +\infty)$ *and* $\mathrm{u} \in [0, \mathrm{a}]$.

Definition 6 ([5,9])**.** *Let* $\circledast$ *be a CGTN,* M *be a vector space and* $\Omega_{\mathrm{M}} : \mathrm{X} \times (0, +\infty) \to \mathcal{D}_{\mathrm{H}}$ *be a matrix valued fuzzy set (MVFS). The triple* $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast)$ *is called a symmetric matrix valued fuzzy normed space (SMVFN-S) if*
(NORM1) $\Omega_{\mathrm{M}}(\mathrm{u}, \eta) = \mathbf{1}$ *if and only if* $\mathrm{u} = 0$ *for* $\eta \in (0, +\infty)$*;*
(NORM2) $\Omega_{\mathrm{M}}(\gamma \mathrm{u}, \eta) = \Omega_{\mathrm{M}}(\mathrm{u}, \frac{\eta}{|\gamma|})$ *for all* $\mathrm{u} \in \mathrm{M}$ *and* $\gamma \neq 0 \in \mathbb{C}$*;*
(NORM3) $\Omega_{\mathrm{M}}(\mathrm{u} + \mathrm{v}, \eta + \alpha) \succeq \Omega_{\mathrm{M}}(\mathrm{u}, \eta) \circledast \Omega_{\mathrm{M}}(\mathrm{u}, \alpha)$ *for all* $\mathrm{u} \in \mathrm{M}$ *and any* $\eta, \alpha \in (0, +\infty)$*;*
(NORM4) $\lim_{\eta \to +\infty} \Omega_{\mathrm{M}}(\mathrm{u}, \eta) = \mathbf{1}$ *for any* $\eta \in (0, +\infty)$.

When an SMVFN-S is complete it is called an SMVFB-S [3,5–7].

Definition 7 ([21])**.** *A symmetric matrix-valued fuzzy normed-algebra (SMVFN-A)* $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast, \circledstar)$ *is an SMVFN-S* $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast)$ *if the following condition holds*
(NORM5) $\Omega_{\mathrm{M}}(\mathrm{uv}, \eta\alpha) \succeq \Omega_{\mathrm{M}}(\mathrm{u}, \eta) \circledstar \Omega_{\mathrm{M}}(\mathrm{v}, \alpha)$*, for all* $\mathrm{u}, \mathrm{v} \in \mathrm{M}$ *and all* $\eta, \alpha \in (0, \infty)$ *in which* $\circledstar$ *is a CGTN.*

A complete SMVFN-A is called an SMVFB-A.

Definition 8 ([21]). *We consider an SMVFB-A* $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast, \circledstar)$. *We say that* M *is an SMVFB-*$\diamond$*-A if the mapping* $\delta \to \delta^{\diamond}$ *on* M *has the following conditions*

(1) $\delta^{\diamond\diamond} = \delta$ *for any* $\delta \in \mathrm{M}$;
(2) $(\beta_1\delta + \beta_2\omega)^{\diamond} = \bar{\beta_1}\delta^{\diamond} + \bar{\beta_2}\omega^{\diamond}$;
(3) $(\delta\omega)^{\diamond} = \omega^{\diamond}\delta^{\diamond}$ *for any* $\delta, \omega \in \mathrm{M}$.

Moreover, with the following condition
(4) $\Omega_{\mathrm{M}}(\delta^{\diamond}\delta, \eta) = \Omega_{\mathrm{M}}(\delta, \eta)$ *for each* $\delta \in \mathrm{M}$ *and* $\eta \in (0,1)$,
we say that M *is an SMVFC-*$\diamond$*-algebra (SMVFC-*$\diamond$*-A).*

Definition 9 ([11–13]). *We consider the complex SMVFB-S* **M** *(CSMVFB-S). If by using* **M**, *we can define the* $\mathbb{C}$*-linear ternary product (*$\mathbb{C}$*-LTP) of* $\Omega_{\mathrm{M}}\Big((x,y,z), t\Big) \longrightarrow \Omega_{\mathrm{M}}\Big([\mathrm{u},\mathrm{v},\mathrm{w}], \eta\Big)$, *which is from* $\mathbf{M}^3$ *to* **M**, *such that it has the following properties*

(TP1) $\Omega_{\mathrm{M}}\Big([\mathrm{u},\mathrm{v},[\mathrm{w},\mathrm{s},\mathrm{t}]], \eta\Big) = \Omega_{\mathrm{M}}\Big([\mathrm{u},[\mathrm{s},\mathrm{w},\mathrm{v}],\mathrm{t}], \eta\Big) = \Omega_{\mathrm{M}}\Big([\mathrm{u},\mathrm{v},\mathrm{w}],\mathrm{s},\mathrm{t}\Big)$ *(associative);*

(TP2) $\Omega_{\mathrm{M}}\Big([\mathrm{u},\mathrm{v},\mathrm{w}], \eta\alpha\gamma\Big) \succeq \Omega_{\mathrm{M}}(\mathrm{u},\eta) \circledast \Omega_{\mathrm{M}}(\mathrm{v},\alpha) \circledast \Omega_{\mathrm{M}}(\mathrm{w},\gamma)$ *for all* $\mathrm{u},\mathrm{v},\mathrm{w},\mathrm{s},\mathrm{t} \in \mathrm{M}$;

then, we say that **M** *is a ternary SMVFB-A (T-SMVFB-A).*

We assume that $(\mathbf{M}, [.,.,.])$ is T-SMVFB-A. If M has an identity member say such that $\Omega_{\mathrm{M}}(\mathrm{u},\eta) = \Omega_{\mathrm{M}}\Big([\mathrm{u},\mathbb{E},\mathbb{E}], \eta\Big) = \Omega_{\mathrm{M}}\Big([\mathbb{E},\mathbb{E},\mathrm{u}], \eta\Big)$ for all $\mathrm{u} \in \mathrm{M}$, then we have

$$\begin{aligned} \Omega_{\mathrm{M}}\Big(\mathrm{u}\circ\mathrm{v}, \eta\Big) &= \Omega_{\mathrm{M}}\Big([\mathrm{u},\mathbb{E},\mathrm{v}], \eta\Big) \\ \Omega_{\mathrm{M}}(\mathrm{u}^*, \eta) &= \Omega_{\mathrm{M}}\Big([\mathbb{E},\mathrm{u},\mathbb{E}], \eta\Big), \end{aligned} \tag{2}$$

where u^* is an unital algebra. If $(\mathbf{M}, \circ)$ is an unital algebra, then the relation

$$\Omega_{\mathrm{M}}\Big([\mathrm{u},\mathrm{v},\mathrm{w}], t\Big) = \Omega_{\mathrm{M}}\Big(\mathrm{u}\circ\mathrm{v}^*\circ\mathrm{w}, \eta\Big),$$

shows that **M** is a T-SMVFB-A. Also, **M** is an usual SMVFB-A (U-SMVFB-A) if for identity member $\mathbb{E} \in \mathrm{M}$, we have $\Omega_{\mathrm{M}}\Big(\mathrm{u}\circ\mathrm{v}, \eta\Big) = \Omega_{\mathrm{M}}\Big([\mathrm{u},\mathbb{E},\mathrm{v}], \eta\Big)$.

Definition 10 ([3,12,13]). *We assume that* **M** *and* **N** *are SMVFB-As. A* $\mathbb{C}$*-linear stochastic mapping* $\mathcal{Y} : \Pi \times \mathbf{M} \longrightarrow \mathbf{N}$, *which satisfies*

$$\mathcal{Y}(\mathrm{z},[\mathrm{u},\mathrm{v},\mathrm{w}]) = [\mathcal{Y}(\mathrm{z},\mathrm{u}), \mathcal{Y}(\mathrm{z},\mathrm{v}), \mathcal{Y}(\mathrm{z},\mathrm{w})] \;\; \textit{for all} \;\; \mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M},$$

is called a stochastic ternary homomorphism (STH).

Definition 11 ([12,13]). *We assume that* **M** *is an SMVFB-A. A* $\mathbb{C}$*-linear stochastic mapping* $\mathcal{Q} : \Pi \times \mathbf{M} \longrightarrow \mathbf{M}$, *which satisfies*

$$\mathcal{Q}(\mathrm{z},[\mathrm{u},\mathrm{v},\mathrm{w}]) = [\mathcal{Q}(\mathrm{z},\mathrm{u}),\mathrm{v},\mathrm{w}] + [\mathrm{u},\mathcal{Q}(\mathrm{z},\mathrm{v}),\mathrm{w}] + [\mathrm{u},\mathcal{Q}(\mathrm{z},\mathrm{v}),\mathrm{w}] \;\; \textit{for all} \;\; \mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M},$$

is called a stochastic ternary derivation (STD).

In the following, we introduce special functions that we need to select the optimal control function. We have also performed the necessary calculations on these functions and have shown a representation of these functions and calculations in the form of graphs.

Definition 12 ([7,9])**.** *Let* u *be a real number and consider the generic parameters* $a, b, c > 0$. *We define the GHF* ${}_2F_1 : \mathbb{R}^3 \times [0, a] \longrightarrow (0, \infty)$ *by the infinite sum (that is convergent)*

$$ {}_2F_1(a, b, c; u) = \sum_{p=0}^{\infty} \frac{(a)_p (b)_p}{(c)_p} \frac{u^p}{p!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{p=0}^{\infty} \frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)} \frac{u^p}{p!}. $$

Definition 13 ([9])**.** *One-parameter and two-parameter M-LF are defined as follows, respectively*

$$ E_\kappa(u) = \sum_{p=0}^{\infty} \frac{u^p}{\Gamma(p\kappa + 1)}, $$

$$ E_{\kappa,\mu}(u) = \sum_{p=0}^{\infty} \frac{u^p}{\Gamma(p\kappa + \mu)}, $$

where $\kappa, \mu \in \mathbb{C}$, $Re(\kappa), Re(\mu) > 0$ *and* $\Gamma(.)$ *used in the above functions is the gamma function.*

Definition 14 ([9])**.** *For* $0 \le \lambda_1 \le \lambda_2$, $1 \le \lambda_3 \le \lambda_4$, $\{x_\iota, y_\iota\} \in \mathbb{C}$, $\{u_\iota, v_\iota\} \in \mathbb{R}^+$, *we define the following functions*

- $\mathcal{L}_1(f) = \prod_{\iota=1}^{\lambda_1} \Gamma(y_\iota - v_\iota f)$,
- $\mathcal{L}_2(f) = \prod_{\iota=1}^{\lambda_3} \Gamma(1 - x_\iota + u_\iota f)$,
- $\mathcal{L}_3(f) = \prod_{\iota=\lambda_3+1}^{\lambda_3} \Gamma(1 - y_\iota + v_\iota f)$,
- $\mathcal{L}_4(f) = \prod_{\iota=\lambda_1+1}^{\lambda_2} \Gamma(x_\iota - u_\iota f)$.

In the introduced functions $\lambda_1 = 0$ *if and only if* $\mathcal{L}_2(f) = 1$, $\lambda_3 = \lambda_4$ *if and only if* $\mathcal{L}_3(z) = 1$ *and* $\lambda_1 = \lambda_2$ *if and only if* $\lambda_4(f) = 1$. *According to the introduced functions, we consider* $\mathscr{H}^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}(f) = \frac{\mathcal{L}_1(f)\mathcal{L}_2(f)}{\mathcal{L}_3(f)\mathcal{L}_4(f)}$. *The Mellin-Barnes integral (M-BI) representation of the H-Fox function (H-FF) is shown below*

$$ H^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}(u) = \frac{1}{2\pi i} \int_{\mathcal{A}} \mathscr{H}^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}(f) u^f df, \tag{3} $$

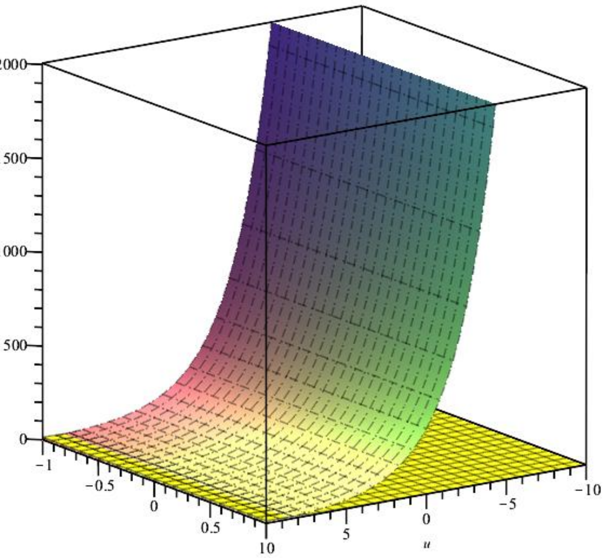
where $u^f = \exp\{f(\log|u| + i \arg u)\}$ *and* $\mathcal{A} \in \mathbb{C}$ *is a path that is deleted. Also, the symbol* $H^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}(u) = H^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}\left[u \middle| \begin{matrix} (x_\iota, \epsilon_\iota)_{\iota=1,\cdots,\lambda_2} \\ (y_\iota, \rho_\iota)_{\iota=1,\cdots,\lambda_2} \end{matrix} \right]$ *is considered for this integral.*

Definition 15 ([6,8,9])**.** *The generalized Bessel Maitland function (GBMF) or the Wright function (WF) of order* $1/(1+\sigma)$ *is represented by using the series*

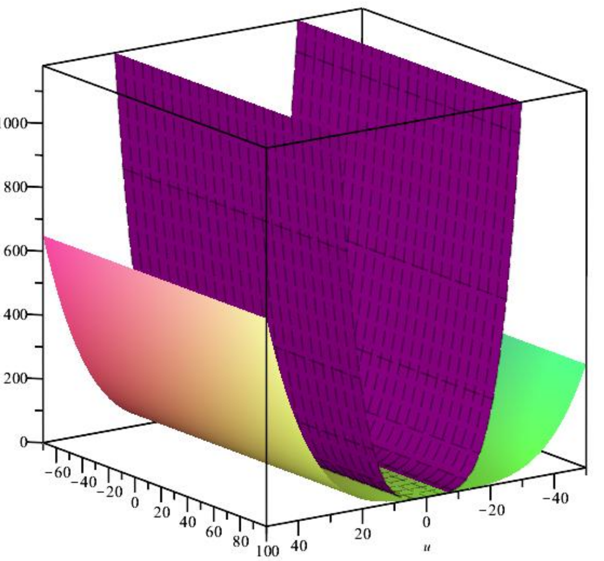
$$ W_{\kappa,\mu}(u) = \sum_{p=0}^{\infty} \frac{u^p}{p!\Gamma(\kappa p + \mu)}, $$

for $\kappa > -1, \mu > 0, u \in \mathbb{R}$.

In the Figures 1–3 we can see the values of introduced special functions in this paper.

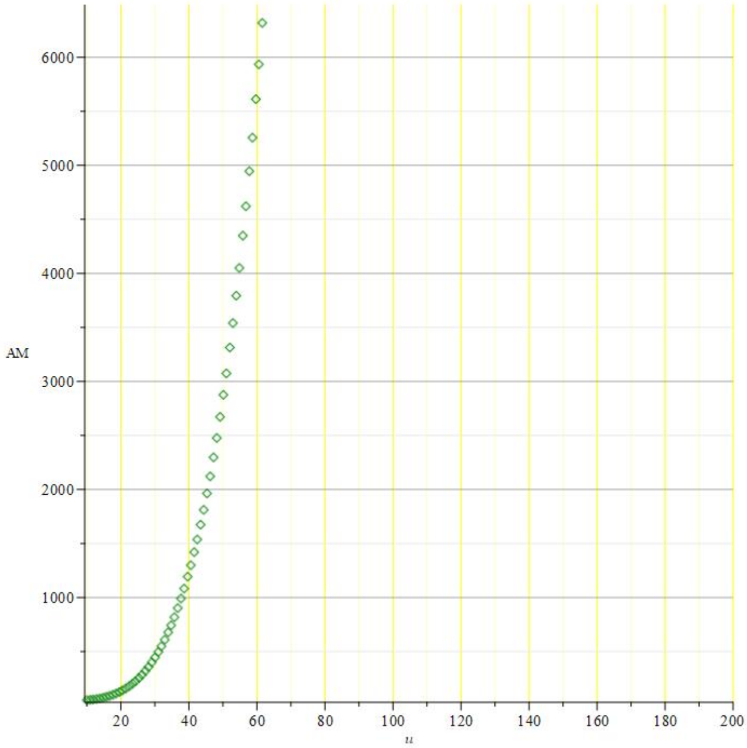


(**a**) Graph of Gauss hypergeometric and H-Fox functions for u $\in (-10, 10)$



(**b**) Graph of Mittag-Leffler and Bessel Maitland functions for u $\in (-50, 50)$

Figure 1. *Cont*.



(c) Graph of Gauss hypergeometric, H-Fox, Mittag-Leffler, Bessel Maitland and exponential functions for $u \in (-100, 10)$

Figure 1. 3D graphs of special functions Gauss hypergeometric and H-Fox functions for $\eta = 3$ and different values of u.

Definition 16 ([9]). *An* p*-ary* ($p \in \mathbb{N}$) *generalized aggregation function* (p*-AGAF*) $A^{(p)} : \mathbb{R}^p \longrightarrow \mathbb{R}$ *has the following property*

- $u_\iota \leq v_\iota \Longrightarrow A^{(p)}(u_1, \cdots, u_p) \leq A^{(p)}(v_1, \cdots, v_p)$,

for all $\iota \in \{1, \cdots, p\}$, *and for* $(u_1, \cdots, u_p), \ (v_1, \cdots, v_p) \in \mathbb{R}^p$. *For the sake of simplicity, we can remove the* m *number, which represents the number of variables of the aggregation function, and denote this function as* A. *Also, when* $p = 1$, *the aggregation function is shown as* $A^{(1)}(u) = u$ *for all* $\phi \in \mathbb{R}$.

Example 1. *(AMF)* $AM : \mathbb{R}^p \longrightarrow \mathbb{R}$ *is defined by*

$$AM(u) = \frac{1}{u} \sum_{\iota=1}^{p} u_\iota.$$

Example 2. *The geometric mean function (GMF)* $GM : \mathbb{R}^p \longrightarrow \mathbb{R}$ *is defined by*

$$GM(u) = (\prod_{\iota=1}^{p} u_\iota)^{\frac{1}{p}}.$$

Example 3. *The projection function (PF)* $\mathcal{P}_\beta : \mathbb{R}^p \longrightarrow \mathbb{R}$ *for* $\beta \in [p]$ *and* β*th argument is defined by*

$$\mathcal{P}_\beta(u) = u_\beta,$$

where $u_{(\beta)}$ *is the* β*th lowest coordinate of* u, *i.e.,* $u_{(1)} \leq \cdots \leq u_{(\beta)} \leq \cdots u_{(p)}$. *Also, the following functions show the PF in the first and last coordinates*

$$\begin{aligned} \mathcal{P}_F(u) &= \mathcal{P}_1(u) = u_1, \\ \mathcal{P}_L(u) &= \mathcal{P}_p(u) = u_p. \end{aligned}$$

Example 4. *The order statistic function (OSF) $OS_\beta : \mathbb{R}^{\mathrm{p}} \longrightarrow \mathbb{R}$ with the βth argument and βth lowest coordinate is defined by*

$$OS_\beta(\mathrm{u}) = \mathrm{u}_{(\beta)},$$

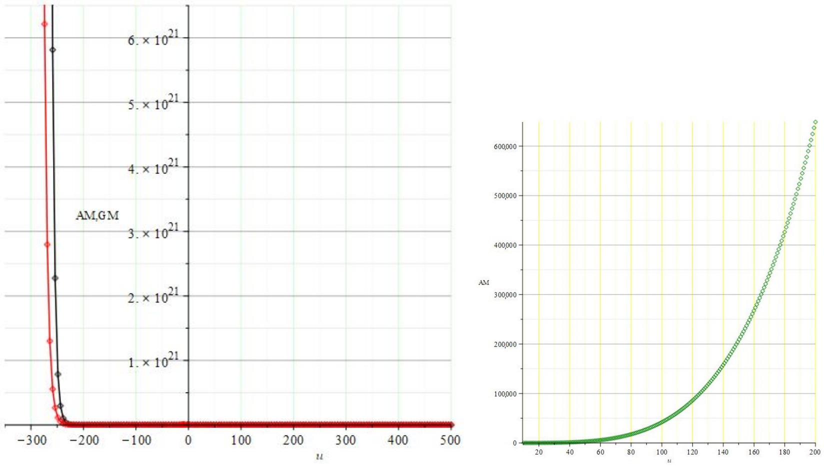
for any $\beta \in [\mathrm{p}]$.

Example 5. *The minimum function (MIN-F) and maximum function (MAX-F) are defined as follows respectively*

$$\begin{aligned} MIN(\mathrm{u}) &= OS_1(\mathrm{u}) = \min\{\mathrm{u}_1, \cdots, \mathrm{u}_{\mathrm{p}}\} = \textstyle\bigwedge_{\iota=1}^{\mathrm{p}} \mathrm{u}_\iota, \\ MAX(\mathrm{u}) &= OS_{\mathrm{p}}(\mathrm{u}) = \max\{\mathrm{u}_1, \cdots, \mathrm{u}_{\mathrm{p}}\} = \textstyle\bigvee_{\iota=1}^{\mathrm{p}} \mathrm{u}_\iota. \end{aligned}$$

Example 6. *The median function (MF) is defined as follows for odd and even values of $(\mathrm{u}_1, \cdots, \mathrm{u}_{2\beta-1})$ and $(\mathrm{u}_1, \cdots, \mathrm{u}_{2\beta})$, respectively*

$$\begin{aligned} MED(\mathrm{u}_1, \cdots, \mathrm{u}_{2\beta-1}) &= \mathrm{u}_{(\beta)}, \\ MED(\mathrm{u}_1, \cdots, \mathrm{u}_{2\beta}) = AM(\mathrm{u}_{(\beta)}, \mathrm{u}_{(\beta+1)}) &= \tfrac{\mathrm{u}_{(\beta)} + \mathrm{u}_{(\beta+1)}}{2}. \end{aligned}$$



(**a**) AM and GM functions for $\mathrm{u} \in (-500, 500)$ (**b**) The aggregation AM function for $\mathrm{u} \in (10, 200)$

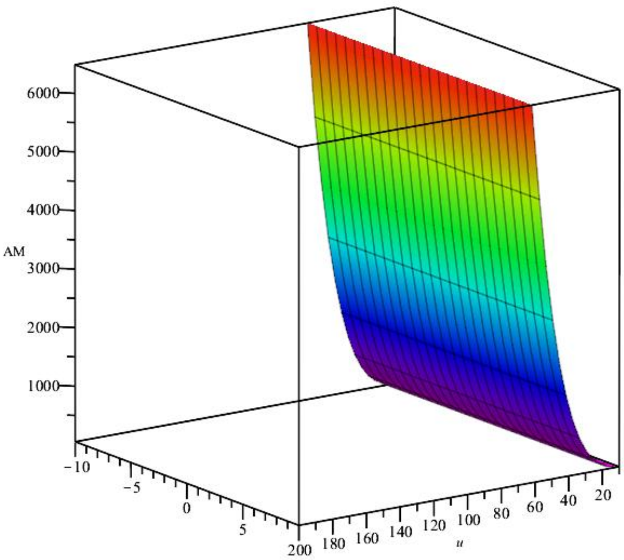
Figure 2. Two-dimensional graphs of aggregation functions for $\omega = 3$ and different values u.

By referring to [9] and studying the information in the presented table, we choose the minimum function as the control function. Also Figures 1–3 help us to choose this function. Consider the function

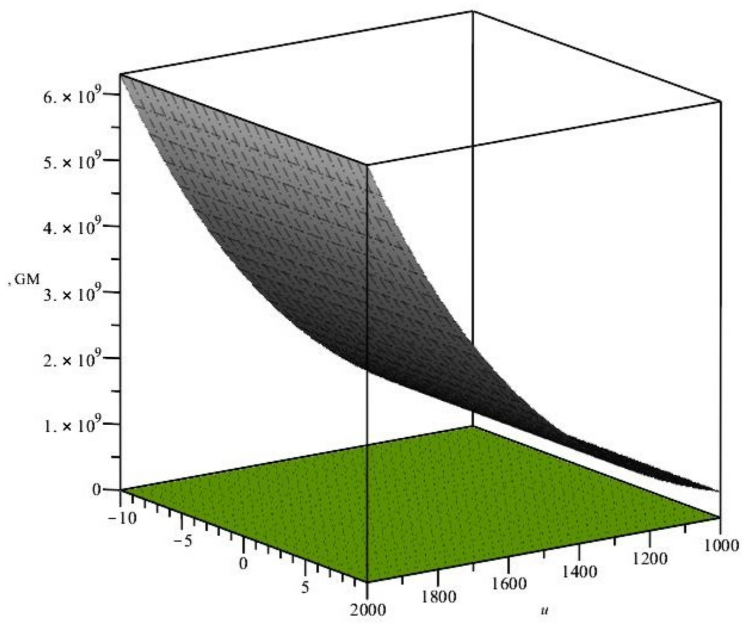
$$\Psi(\mathrm{u}, \eta) = \mathrm{diag}\Big[{}_2F_1(\mathrm{a}, \mathrm{b}, \mathrm{c}, \tfrac{-\|\mathrm{u}\|}{\eta}), \mathrm{E}_{\kappa,\mu}(\tfrac{-\|\mathrm{u}\|}{\eta}), \mathrm{W}_{\kappa,\mu}(\tfrac{-\|\mathrm{u}\|}{\eta}), \mathrm{H}^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}(\tfrac{-\|\mathrm{u}\|}{\eta}), \exp(\tfrac{-\|\mathrm{u}\|}{\eta})\Big], \tag{4}$$

and we choose the following function as a control function

$$\boldsymbol{MIN}\Big(\boldsymbol{\Psi}(\mathbf{u}, \boldsymbol{\eta})\Big) = \mathrm{diag} \quad \Big[\bigwedge\Big(\Psi(\mathrm{u}, \eta)\Big), \bigwedge\Big(\Psi(\mathrm{u}, \eta)\Big), \bigwedge\Big(\Psi(\mathrm{u}, \eta)\Big), \bigwedge\Big(\Psi(\mathrm{u}, \eta)\Big)\Big]. \tag{5}$$

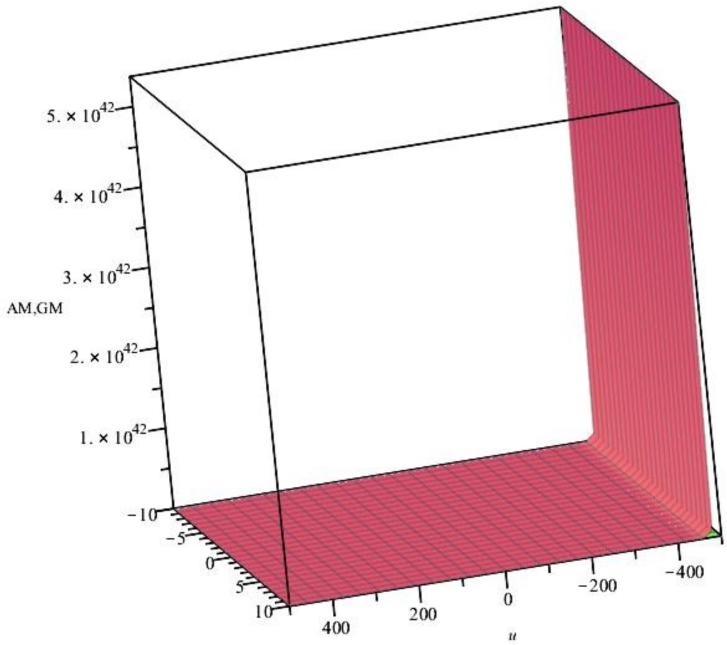


(**a**) The aggregation AM function for u $\in$ $(10, 200)$



(**b**) AM and GM functions for u $\in$ $(1000, 200)$

Figure 3. *Cont*.



(**c**) AM and GM functions for $u \in (-500, 500)$

Figure 3. Three-dimensional graphs of aggregation functions for $\omega = 3$ and different values of u.

If we assume that $(\mathrm{M}, \mathcal{T}_{\mathrm{M}})$ is a generalized complete metric space (GCMS), we define a set as follows

$$\mathcal{CON}(\mathrm{M}) := \left\{ \mathrm{J} : \mathrm{M} \to \mathrm{M} | \mathcal{T}_{\mathrm{M}}(\mathrm{Jv}_1, \mathrm{Jv}_2) \leq \epsilon_{\mathrm{J}} \mathcal{T}_{\mathrm{M}}(\mathrm{v}_1, \mathrm{v}_2), \quad \forall \mathrm{v}_1, \mathrm{v}_2 \in \mathrm{M}, \quad \epsilon_{\mathrm{J}} \in [0, 1) \right\}. \tag{6}$$

We call this set the set of all contraction mappings that is, every contractive function is located in this set. Next, we present the Diaz-Margolis theorem (FTP) [5,6,8,9].

Theorem 1. *We consider GCMS* $(\mathrm{M}, \mathcal{T}_{\mathrm{M}})$ *and assume that* $\mathrm{u}, \mathrm{v} \in \mathrm{M}$, *and also* $\mathrm{J} \in \mathcal{CON}(\mathrm{M})$ *such that* $\epsilon_{\mathrm{J}} < 1$. *With these assumptions, we assume that for every* $\mathbf{r}, \mathbf{r}_0 \in \mathbb{N}$ $(\mathbf{r} \geq \mathbf{r}_0)$ *and for* $\mathrm{u} \in \mathrm{M}$, $\mathcal{T}_{\mathrm{M}}\big(\mathrm{J}^{\mathbf{r}}\mathrm{u}, \mathrm{J}^{\mathbf{r}+1}\mathrm{v}\big) < \infty$,. *If this condition holds, we have*

(1) *The fixed point* j *of* J *is the convergence point of the sequence* $\{\mathrm{J}^{\mathbf{r}}\mathrm{u}\}$;
(2) *In the set* $\mathcal{K} = \{\mathrm{u} \in \mathrm{M} \mid \mathcal{T}_{\mathrm{M}}(\mathrm{J}^{\mathbf{r}_0}\mathrm{u}, \mathrm{v}) < \infty\}$, j *is the unique fixed point of* J;
(3) $(1 - \epsilon_{\mathrm{J}})\mathcal{T}_{\mathrm{M}}(\mathrm{v}, \mathrm{j}) \leq \mathcal{T}_{\mathrm{M}}(\mathrm{v}, \mathrm{Jv})$ *for every* $\mathrm{v} \in \mathrm{M}$.

Also, if the condition $\mathcal{T}_{\mathrm{M}}\big(\mathrm{J}^{\mathbf{r}}\mathrm{v}, \mathrm{J}^{\mathbf{r}+1}\mathrm{v}\big) < \infty$ *is not satisfied, we have* $\mathcal{T}_{\mathrm{M}}\big(\mathrm{J}^{\mathbf{r}}\mathrm{v}, \mathrm{J}^{\mathbf{r}+1}\mathrm{v}\big) = \infty$, *for all* $\mathbf{r} \in \mathbb{N}$.

3. Minimum Stability of Stochastic Operator on SMVFB-A

Here, considering Theorem 1, we investigate the stability of a stochastic operator according to the additive (σ_1, σ_2)-functional inequality (A-(σ_1, σ_2)-FI).

Lemma 1. *Assume that* $\Big(\mathbf{M}, \Omega_{\mathrm{M}}, \circledast, \circledast\Big)$ *is an SMVFB-S. We consider the stochastic mapping* $\phi : \Pi \times \mathbf{M} \to \mathbf{M}$ *in such a way that the following inequality holds*

$$\begin{aligned}\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{v}-\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}-\mathrm{w}),\eta\Big) \\ \succeq \Omega_{\mathrm{M}}\Big(\sigma_1(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}-\mathrm{w}) + \phi(\mathrm{z},\mathrm{u}-\mathrm{w}) - \phi(\mathrm{z},\mathrm{v})),\eta\Big) \\ \circledast\Omega_{\mathrm{M}}\Big(\sigma_2(\phi(\mathrm{z},\mathrm{u}-\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}) + \phi(\mathrm{z},\mathrm{w})),\eta\Big),\end{aligned}$$

for all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M}$. *Then the stochastic mapping* $\phi : \Pi \times \mathbf{M} \to \mathbf{M}$ *is additive.*

Proof. As stated in the assumption of the theorem, $\phi : \Pi \times \mathbf{M} \to \mathbf{M}$ satisfies (7). Now, if we put $\mathrm{u} = \mathrm{v} = \mathrm{w} = 0$ in (7), we obtain

$$\Omega_{\mathrm{M}}\Big(2\phi(\mathrm{z},0),\eta\Big) \succeq \Omega_{\mathrm{M}}\Big((|\sigma_1| + |\sigma_2|)\phi(\mathrm{z},0),\eta\Big),$$

according to the assumption $|\sigma_1| + |\sigma_2| > 2$, we have $\phi(\mathrm{z},0) = 0$. Once again, putting $\mathrm{w} = \mathrm{u}$ in (7), we have

$$\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},2\mathrm{u}+\mathrm{v}) - \phi(\mathrm{z},2\mathrm{u}) - \phi(\mathrm{z},\mathrm{v}),\eta\Big) \succeq \mathbf{1},$$

then, according to (NORM1), we get

$$\phi(\mathrm{z},2\mathrm{u}+\mathrm{v}) = \phi(\mathrm{z},2\mathrm{u}) + \phi(\mathrm{z},\mathrm{v}),$$

for all $\mathrm{u},\mathrm{v} \in \mathrm{M}$. Hence ϕ is additive. □

Theorem 2. *In the SMVFB-S* $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast, \circledast)$, *we consider the MVF function* $\Psi : \mathrm{M} \times \mathrm{M} \times \mathrm{M} \to \mathcal{D}_{\mathrm{H}}$ *and the stochastic mapping of* $\phi : \Pi \times \mathrm{M} \to \mathrm{M}$ *along with the constant* $\theta < 1$ *in such a way that we have the following conditions:*

(S1) For all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M}$

$$\Psi\Big(\mathrm{u},\mathrm{v},\mathrm{w},\eta\Big) \succeq \Psi\Big(2\mathrm{u},2\mathrm{v},2\mathrm{w},\frac{2}{\theta}\eta\Big), \tag{7}$$

(S2) For all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M}$

$$\begin{aligned}\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{v}-\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}-\mathrm{v})\Big) \\ \succeq \Omega_{\mathrm{M}}\Big(\sigma_1(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}-\mathrm{w}) + \phi(\mathrm{z},\mathrm{u}-\mathrm{w}) - \phi(\mathrm{u},\mathrm{v})),\eta\Big) \\ \circledast\Omega_{\mathrm{M}}\Big(\sigma_2(\phi(\mathrm{z},\mathrm{u}-\mathrm{w}) + \phi(\mathrm{z},\mathrm{u}) - \phi(\mathrm{z},\mathrm{w})),\eta\Big) \circledast \Psi\Big(\mathrm{u},\mathrm{v},\mathrm{w},\eta\Big).\end{aligned}$$

Therefore, there exists a unique stochastic additive mapping (SAM) $\mathcal{E} : \Pi \times \mathrm{M} \to \mathrm{M}$ *such that*

$$\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}) - \mathcal{E}(\mathrm{z},\mathrm{u}),\eta\Big) \succeq \Psi\Big(\Big(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2}\Big),\frac{2(1-\theta)}{\theta}\eta\Big), \tag{8}$$

for all $\mathrm{u} \in \mathrm{M}$.

Proof. First, we assume that $u = v = w = 0$ and apply this assumption to condition (S2), which is inequality (8). We get

$$\Omega_{M}\Big(2\phi(z,0),\eta\Big) \succeq \Omega_{M}\Big((|\sigma_1| + |\sigma_2|)\phi(z,0),\eta\Big) \circledast \Psi\Big(0,0,0,\eta\Big),$$

according to $|\sigma_1| + |\sigma_2| > 2$ and condition (S1) means (7) , we have $\phi(z,0) = 0$ and $\Psi(0,0,0,\eta) = \mathbf{1}$. Now, let us assume $u = w = \frac{\zeta}{2}$ and $v = \zeta$. By placing the new assumption in condition (S2), we have

$$\Omega_{M}\Big(\phi(z,2\zeta) - 2\phi(z,\zeta),\eta\Big) \succeq \Psi\Big(\frac{\zeta}{2},\zeta,\frac{\zeta}{2},\eta\Big), \tag{9}$$

for all $\zeta \in M$. On the set $Y = \{\mathcal{R} : \Pi \times M \to M : \mathcal{R}(z,0) = 0\}$, we define the mapping $\mathcal{T}_M : Y \times Y \to Y$ as follows

$$\mathcal{T}_M(\mathcal{Q},\mathcal{R}) = \inf\Big\{\varsigma \in \mathbb{R}_+ : \Omega_{M}\Big(\mathcal{Q}(z,u) - \mathcal{R}(z,u),\eta\Big) \succeq \Psi\Big(\frac{u}{2},u,\frac{u}{2},\frac{\eta}{\varsigma}\Big), \forall u \in M\Big\}.$$

It is easy to see that $\mathcal{T}_M$ is a complete generalized metric space (CGMS) [5,6,8]. In the following, for all $u \in M$, we define a stochastic linear mapping $\mathcal{S} : Y \to Y$ as follows:

$$\mathcal{S}(\mathcal{Q}(z,u)) := 2\mathcal{Q}\Big(z,\frac{u}{2}\Big).$$

For $\mathcal{Q},\mathcal{R} \in Y$, we assume $\mathcal{T}_M(\mathcal{Q},\mathcal{R}) = \varkappa$. As a result, for all $u \in M$, we have

$$\Omega_{M}\Big(\mathcal{Q}(z,u) - \mathcal{R}(z,u),\eta\Big) \succeq \Psi\Big(\frac{u}{2},u,\frac{u}{2},\frac{\eta}{\varkappa}\Big).$$

Then, for all $u \in M$, we get

$$\begin{aligned}\Omega_{M}\Big(\mathcal{S}(\mathcal{Q}(z,u)) - \mathcal{S}(\mathcal{R}(z,u)),\eta\Big) &= \Omega_{M}\Big(2\mathcal{Q}(z,\tfrac{u}{2}) - 2\mathcal{R}(z,\tfrac{u}{2}),\eta\Big) \\ &\succeq \Psi\big(\tfrac{u}{4},\tfrac{u}{2},\tfrac{u}{4},\tfrac{\eta}{2\varkappa}\big) \\ &\succeq \Psi\big(\tfrac{u}{2},u,\tfrac{u}{2},\tfrac{\eta}{\theta\varkappa}\big),\end{aligned}$$

and this means $\mathcal{T}_M(\mathcal{S}(\mathcal{Q}(z,u)),\mathcal{S}(\mathcal{R}(z,u))) \leq \theta\varkappa$ or $\mathcal{T}_M(\mathcal{S}(\mathcal{Q}(z,u)),\mathcal{S}(\mathcal{R}\ (z,u))) \leq \theta\mathcal{T}_M(\mathcal{Q},\mathcal{R})$. In the sequel, due to (9), for all $u \in M$, we get

$$\begin{aligned}\Omega_{M}\Big(\phi(z,u) - 2\phi(z,\tfrac{u}{2})\Big) &\succeq \Psi\big(\tfrac{u}{4},\tfrac{u}{2},\tfrac{u}{4},\eta\big) \\ &\succeq \Psi\Big(\tfrac{u}{2},u,\tfrac{u}{2},\tfrac{2\eta}{\theta}\Big),\end{aligned}$$

and this means $\mathcal{T}_M(\phi,\mathcal{S}\phi) \leq \frac{\theta}{2}$. According to Theorem 1, there exists a unique fixed point such as the stochastic mapping $\mathcal{E} : Y \times M \to M$ for $\mathcal{S}$, which is defined as

$$\mathcal{E}(z,u) = 2\mathcal{E}\Big(z,\frac{u}{2}\Big).$$

Then, considering this fixed point, there is a $\varsigma \in (0,\infty)$ such that for all $u \in M$, we have

$$\Omega_{M}\Big(\phi(z,u) - \mathcal{E}(z,u),\eta\Big) \succeq \Psi\Big(\frac{u}{2},u,\frac{u}{2},\frac{\eta}{\varsigma}\Big).$$

On the other hand, because $\lim_{p\to\infty} \Omega_{M}\Big(\mathcal{S}^p\phi - \mathcal{E},\eta\Big) = \mathbf{1}$, then for all $u \in M$, we have

$$\lim_{p\to\infty} 2^p\phi\Big(z,\frac{u}{2^p}\Big) = \mathcal{E}(z,u).$$

Also, $\mathcal{T}_{\mathrm{M}}(\phi,\mathcal{E}) \leq \frac{1}{1-\theta}\mathcal{T}_{\mathrm{M}}(\phi,\mathcal{S}\phi)$ which implies for all $\mathrm{u} \in \mathrm{M}$

$$\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}) - \mathcal{E}(\mathrm{z},\mathrm{u}),\eta\Big) \succeq \Psi\Big(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2},\frac{2(1-\theta)\eta}{\theta}\Big).$$

According to conditions (S1) and (S2), that is, inequalities (7) and (8), for all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathscr{A}$, we have

$$\begin{aligned}
&\Omega_{\mathrm{M}}\Big(\mathcal{E}(\mathrm{z},\mathrm{u}+\mathrm{v}+\mathrm{w}) - \mathcal{E}(\mathrm{z},\mathrm{u}+\mathrm{w}) - \mathcal{E}(\mathrm{z},\mathrm{v}-\mathrm{u}+\mathrm{w}) - \mathcal{E}(\mathrm{z},\mathrm{u}-\mathrm{w})\,\eta\Big)\\
&= \Omega_{\mathrm{M}}\Big(\lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\big(\phi(\mathrm{z},\tfrac{\mathrm{u}+\mathrm{v}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{u}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{v}-\mathrm{u}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{u}-\mathrm{w}}{2^{\mathrm{p}}})\big),\eta\Big)\\
&= \lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\tfrac{\mathrm{u}+\mathrm{v}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{u}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{v}-\mathrm{u}+\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{u}-\mathrm{w}}{2^{\mathrm{p}}}),\eta\Big)\\
&\succeq \lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\Omega_{\mathrm{M}}\Big(\sigma_1\big(\phi(\mathrm{z},\tfrac{\mathrm{u}+\mathrm{v}-\mathrm{w}}{2^{\mathrm{p}}}) + \phi(\mathrm{z},\tfrac{\mathrm{u}-\mathrm{w}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{v}}{2^{\mathrm{p}}})\big),\eta\Big)\\
&\circledast \lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\Omega_{\mathrm{M}}\Big(\beta\big(\phi(\mathrm{z},\tfrac{\mathrm{u}-\mathrm{w}}{2^{\mathrm{p}}}) + \phi(\mathrm{z},\tfrac{\mathrm{u}}{2^{\mathrm{p}}}) - \phi(\mathrm{z},\tfrac{\mathrm{w}}{2^{\mathrm{p}}})\big),\eta\Big) \circledast \lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\Psi\Big(\tfrac{\mathrm{u}}{2^{\mathrm{p}}},\tfrac{\mathrm{v}}{2^{\mathrm{p}}},\tfrac{\mathrm{w}}{2^{\mathrm{p}}},\eta\Big)\\
&= \Omega_{\mathrm{M}}\Big(\sigma_1(\Delta(\mathrm{z},\mathrm{u}+\mathrm{v}-\mathrm{w}) + \mathcal{E}(\mathrm{z},\mathrm{v}-\mathrm{w}) - \mathcal{E}(\mathrm{z},\mathrm{v})),\eta\Big) \circledast \Omega_{\mathrm{M}}\Big(\sigma_2(\mathcal{E}(\mathrm{z},\mathrm{u}-\mathrm{w}) + \mathcal{E}(\mathrm{z},\mathrm{u}) - \mathcal{E}(\mathrm{z},\mathrm{w})),\eta\Big).
\end{aligned}$$

Therefore, according to Lemma 1, $\mathcal{E}$ is a stochastic additive mapping (SAM). □

Example 7. *We consider the stochastic mapping* $\phi : \Pi \times \mathrm{M} \to \mathrm{M}$ *such that for all* $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M}$ *the following inequality holds*

$$\begin{aligned}
&\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{v}-\mathrm{u}+\mathrm{w}) - \phi(\mathrm{z},\mathrm{u}-\mathrm{w}),\eta\Big)\\
&\succeq \Omega_{\mathrm{M}}\Big(\sigma_1(\phi(\mathrm{z},\mathrm{u}+\mathrm{v}-\mathrm{w}) + \phi(\mathrm{z},\mathrm{u}-\mathrm{w}) - \phi(\mathrm{z},\mathrm{v})),\eta\Big)\\
&\circledast\Omega_{\mathrm{M}}\Big(\sigma_2(\phi(\mathrm{z},\mathrm{u}-\mathrm{w}) + \phi(\mathrm{z},\mathrm{u}) - \phi(\mathrm{z},\mathrm{w})),\eta\Big) \circledast \boldsymbol{MIN}\Big(\boldsymbol{\Phi}([\mathbf{u},\mathbf{v},\mathbf{w}]),\boldsymbol{\eta}\Big).
\end{aligned}$$

Then, there exists a unique SAM $\mathcal{E} : \Pi \times \mathrm{M} \to \mathrm{M}$ *such that*

$$\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z},\mathrm{u}) - \mathcal{E}(\mathrm{z},\mathrm{u}),\eta\Big) \succeq \boldsymbol{MIN}\Big(\boldsymbol{\Phi}([\mathbf{u},\mathbf{u},\mathbf{u}]),\boldsymbol{\eta}\Big),$$

for all $\mathrm{u} \in \mathrm{M}$.

Proof. From the proof of Theorem 2, assuming $\theta = \frac{1}{100}$ and

$$\begin{aligned}
&\boldsymbol{MIN}\Big(\boldsymbol{\Phi}([\mathbf{u},\mathbf{v},\mathbf{w}],\boldsymbol{\eta})\Big) =\\
&\mathrm{diag}\Big[MIN\Big(\Phi([\mathrm{u},\mathrm{u},\mathrm{u}]),\eta\Big), MIN\Big(\Phi([\mathrm{u},\mathrm{u},\mathrm{u}],\eta)\Big), MIN\Big(\Phi([\mathrm{u},\mathrm{u},\mathrm{u}],\eta)\Big),\\
&MIN\Big(\Phi([\mathrm{u},\mathrm{u},\mathrm{u}],\eta)\Big), MIN\Big(\Phi([\mathrm{u},\mathrm{u},\mathrm{u}],\eta)\Big)\Big],
\end{aligned}$$

where for all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathscr{M}$

$$\begin{aligned}
\Phi([\mathrm{u},\mathrm{v},\mathrm{w}],\eta) &= \mathrm{diag}\Big[\mathrm{E}_{\kappa,\mu}\Big(\tfrac{\|[\mathrm{u},\mathrm{u},\mathrm{u}]\|}{\eta}\Big), \mathrm{W}_{\kappa,\mu}\Big(\tfrac{\|[\mathrm{u},\mathrm{u},\mathrm{u}]\|}{\eta}\Big), {}_2\mathrm{F}_1\Big(\mathrm{a},\mathrm{b},\mathrm{c},\tfrac{\|[\mathrm{u},\mathrm{u},\mathrm{u}]\|}{\eta}\Big),\\
&\mathrm{H}^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}\Big(\tfrac{\|[\mathrm{u},\mathrm{u},\mathrm{u}]\|}{\eta}\Big), \exp\Big(\tfrac{\|[\mathrm{u},\mathrm{u},\mathrm{u}]\|}{\eta}\Big)\Big],
\end{aligned}$$

and the proof is complete. □

4. Stochastic Ternary Antiderivations in T-SMVFB-A and T-SMVFC-⋄-A

In this section, we first define the stochastic ternary antiderivatives in T-SMVFB-A and T-SMVFC-⋄-A and we prove the minimum superstability. As we mentioned before, all the results are proved by considering the (α, β)- functional inequality.

Definition 17. *[12,13] Consider the T-SMVFB-S* M*. A stochastic ternary antiderivative is a stochastic $\mathbb{C}$-linear mapping $\psi : \Pi \times \mathrm{M} \to \mathrm{M}$ with*

$$[\psi(\mathrm{z},\mathrm{u}), \psi(\mathrm{z},\mathrm{v}), \psi(\mathrm{z},\mathrm{w})] = \psi[\psi(\mathrm{z},\mathrm{u}), \mathrm{v}, \mathrm{w}] + \psi[\mathrm{u}, \psi(\mathrm{z},\mathrm{v}), \mathrm{w}] + \psi[\mathrm{z}, \mathrm{v}, \psi(\mathrm{z},\mathrm{w})],$$

for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$.

Lemma 2. *We consider a CSMVFB-A* M *and a stochastic additive mapping $\delta : \Pi \times \mathrm{M} \to \mathrm{M}$ such that for all $\varrho \in \mathbb{B}^1 := \{\omega \in \mathbb{C} : |\omega| = 1\}$ and all $\mathrm{u} \in \mathrm{M}$, $\delta(\mathrm{z}, \omega\mathrm{u}) = \omega\delta(\mathrm{z},\mathrm{u})$. Then δ is $\mathbb{C}$-linear.*

Theorem 3. *In the SMVFB-S $(\mathrm{M}, \Omega_{\mathrm{M}}, \circledast, \circledast)$, we consider the MVF function $\Psi : \mathrm{M} \times \mathrm{M} \times \mathrm{M} \to \mathcal{D}_{\mathrm{H}}$ and the stochastic mapping of $\phi : \Pi \times \mathrm{M} \to \mathrm{M}$ along with the constant $\theta < 1$ in such a way that we have the following conditions:*

(S3) For all $\omega \in \mathbb{B}^1$ and all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$

$$\Psi\left(\frac{\mathrm{u}}{\omega}, \frac{\mathrm{v}}{\omega}, \frac{\mathrm{w}}{\omega}, \eta\right) \succeq \Psi\left(2\mathrm{u}, 2\mathrm{v}, 2\mathrm{w}, \frac{8}{\theta}\eta\right), \tag{10}$$

(S4) For all $\omega \in \mathbb{T}^1$ and all $x, y, z \in \mathcal{A}$

$$\begin{aligned}\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z}, \omega(\mathrm{u}+\mathrm{v}+\mathrm{w})) - \omega\phi(\mathrm{z}, \mathrm{u}+\mathrm{w}) - \omega\phi(\mathrm{z}, \mathrm{v}-\mathrm{u}+\mathrm{w}) - \omega\phi(\mathrm{z}, \mathrm{u}-\mathrm{w}), \eta\Big) \\ \succeq \Omega_{\mathrm{M}}\Big(\sigma_1(\phi(\mathrm{z}, \mathrm{u}+\mathrm{v}-\mathrm{w}) + \phi(\mathrm{z}, \mathrm{u}-\mathrm{w}) - \phi(\mathrm{z}, \mathrm{v})), \eta\Big) \\ \circledast \Omega_{\mathrm{M}}\Big(\sigma_2(\phi(\mathrm{z}, \mathrm{u}-\mathrm{w}) + \phi(\mathrm{z}, \mathrm{u}) - \phi(\mathrm{z}, \mathrm{w})), \eta\Big) \circledast \Psi\Big(\mathrm{u}, \mathrm{v}, \mathrm{w}, \eta\Big),\end{aligned}$$

(S5) For every $\omega \in \mathbb{B}^1$ and all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in \mathrm{M}$

$$\begin{aligned}\Omega_{\mathrm{M}}\Big([\phi(\mathrm{z},\mathrm{u}), \phi(\mathrm{z},\mathrm{v}), \phi(\mathrm{z},\mathrm{w})] - \phi[\phi(\mathrm{z},\mathrm{u}), \mathrm{v}, \mathrm{w}] + \phi[\mathrm{u}, \phi(\mathrm{z},\mathrm{v}), \mathrm{w}] + \phi[\mathrm{u}, \mathrm{v}, \phi(\mathrm{z},\mathrm{w})], \eta\Big) \\ \succeq \Psi\Big(\mathrm{u}, \mathrm{v}, \mathrm{w}, \eta\Big).\end{aligned}$$

If ϕ is continuous and $\phi(\mathrm{z}, 2\mathrm{u}) = 2\phi(\mathrm{z},\mathrm{u})$ for all $\mathrm{u} \in \mathrm{M}$, then the stochastic mapping $\phi : \Pi \times \mathrm{M} \to \mathrm{M}$ is a stochastic ternary antiderivation.

Proof. First, we assume that $\mathrm{u} = \mathrm{v} = \mathrm{w} = 0$ and $\omega = 1$. Then, inequality (8) will be

$$\Omega_{\mathrm{M}}\Big(2\phi(\mathrm{z}, 0), \eta\Big) \succeq \Omega_{\mathrm{M}}\Big((|\sigma_1| + |\sigma_2|)\phi(\mathrm{z}, 0), \eta\Big) \circledast \Psi\Big(0, 0, 0, \eta\Big),$$

according to $|\sigma_1| + |\sigma_2| > 2$ and condition (S1) means (7) , we have $\phi(\mathrm{z}, 0) = 0$ and $\Psi(0, 0, 0, \eta) = \mathbf{1}$. Now, let us assume $\mathrm{u} = \mathrm{w} = \frac{\zeta}{2}$ and $\mathrm{v} = \zeta$. By placing the new assumption in condition (S2), for all $\omega \in \mathbb{B}^1$ and all $\eta \in \mathrm{M}$, we have

$$\Omega_{\mathrm{M}}\Big(\phi(\mathrm{z}, 2\omega\zeta) - 2\omega\phi(\mathrm{z}, \zeta), \eta\Big) \succeq \Psi\left(\frac{\zeta}{2}, \zeta, \frac{\zeta}{2}, \eta\right). \tag{11}$$

On the set $\mathrm{Y} = \{\mathcal{R} : \Pi \times \mathrm{M} \to \mathrm{M} : \mathcal{R}(\mathrm{z}, 0) = 0\}$, we define the mapping $\mathcal{T}_{\mathrm{M}} : \mathrm{Y} \times \mathrm{Y} \to \mathrm{Y}$ as follows

$$\mathcal{T}_{\mathrm{M}}(\mathcal{K},\mathcal{R}) = \inf\left\{\varsigma \in \mathbb{R}_{\geq 0} : \Omega_{\mathrm{M}}\left(\mathcal{K}(\mathrm{z},\mathrm{u}) - \mathcal{R}(\mathrm{z},\mathrm{u}),\eta\right) \succeq \Psi\left(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2},\frac{\eta}{\varsigma}\right), \forall \mathrm{u} \in \mathrm{M}\right\}.$$

In the following, for all $\mathrm{u} \in \mathrm{M}$, we define a stochastic linear mapping $\mathcal{S} : \mathrm{Y} \to \mathrm{Y}$ as follows

$$\mathcal{S}(\mathcal{K}(\mathrm{z},\mathrm{u})) = 2\omega\mathcal{K}\left(\mathrm{z},\frac{\mathrm{u}}{2\omega}\right).$$

For $\mathcal{K},\mathcal{R} \in \mathrm{Y}$, we assume $\mathcal{T}_{\mathrm{M}}(\mathcal{K},\mathcal{R}) = \varkappa$. As a result, for all $\mathrm{u} \in \mathrm{M}$, we have

$$\Omega_{\mathrm{M}}\left(\mathcal{K}(\mathrm{z},\mathrm{u}) - \mathcal{R}(\mathrm{z},\mathrm{u}),\eta\right) \succeq \Psi\left(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2},\frac{\eta}{\varkappa}\right).$$

Then, for all $\mathrm{u} \in \mathscr{M}$, we get

$$\begin{aligned}\Omega_{\mathrm{M}}\left(\mathcal{S}(\mathcal{K}(\mathrm{z},\mathrm{u})) - \mathcal{S}(\mathcal{R}(\mathrm{z},\mathrm{u})),\eta\right) &= \Omega_{\mathrm{M}}\left(2\omega\theta(\mathrm{z},\tfrac{\mathrm{u}}{2\omega}) - 2\omega\omega(\mathrm{z},\tfrac{\mathrm{u}}{2\omega}),\eta\right)\\ &\succeq \Psi\left(\tfrac{\mathrm{u}}{4\omega},\tfrac{\mathrm{u}}{2\omega},\tfrac{\mathrm{u}}{4\omega},\tfrac{\eta}{2\varkappa}\right)\\ &\succeq \Psi\left(\tfrac{\mathrm{u}}{2},\mathrm{u},\tfrac{\mathrm{u}}{2},\tfrac{4}{\theta}\varkappa\eta\right),\end{aligned}$$

and this means $\mathcal{T}_{\mathrm{M}}(\mathcal{S}(\mathcal{K}(\mathrm{z},\mathrm{u})),\mathcal{S}(\mathcal{R}(\mathrm{z},\mathrm{u}))) \leq \frac{\theta}{4}\varkappa$ or $\mathcal{T}_{\mathrm{M}}(\mathcal{S}(\mathcal{K}(\mathrm{z},\mathrm{u})),\mathcal{S}(\mathcal{R}(\mathrm{z},\mathrm{u}))) \leq \frac{\theta}{4}\mathcal{T}_{\mathrm{M}}\,(\mathcal{K},\mathcal{R})$. In the sequel, due to (11), for all $\mathrm{u} \in \mathrm{M}$, we get

$$\Omega_{\mathrm{M}}\left(\phi(\mathrm{z},\mathrm{x}) - 2\omega\phi\left(\mathrm{z},\frac{\mathrm{u}}{2\omega}\right),\eta\right) \succeq \Psi\left(\frac{\mathrm{u}}{4\omega},\frac{\mathrm{u}}{2\omega},\frac{\mathrm{u}}{4\omega},\eta\right) \succeq \Psi\left(\frac{x}{\mathrm{u}},\mathrm{u},\frac{\mathrm{u}}{2},\frac{8}{\theta}\eta\right),$$

and this means $\mathcal{T}_{\mathrm{M}}(\phi,\mathcal{S}\phi) \leq \frac{\theta}{8}$. According to Theorem 1, there exists a unique fixed point such as the stochastic mapping $\psi : \Pi \times \mathrm{M} \to \mathrm{M}$ for $\mathcal{S}$, which is defined as

$$\psi(\mathrm{z},\mathrm{u}) = 2\omega\psi\left(\mathrm{z},\frac{\mathrm{u}}{2\omega}\right).$$

Then, considering this fixed point, there is a $\varsigma \in (0,\infty)$ such that for all $\mathrm{u} \in \mathrm{M}$, we have

$$\Omega_{\mathrm{M}}\left(\phi(\mathrm{z},\mathrm{u}) - \psi(\mathrm{z},\mathrm{u}),\eta\right) \succeq \Psi\left(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2},\frac{\eta}{\varsigma}\right).$$

On the other hand, because $\lim_{\mathrm{p}\to\infty}\Omega_{\mathrm{M}}\left(\mathcal{S}^{\mathrm{p}}\phi - \psi,\eta\right) = \mathbf{1}$, then for all $\mathrm{u} \in \mathrm{M}$, we have

$$\lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\omega^{\mathrm{p}}\phi\left(\mathrm{z},\frac{\mathrm{u}}{2^{\mathrm{p}}\omega^{\mathrm{p}}}\right) = \psi(\mathrm{z},\mathrm{u}),$$

and in particular for all $\mathrm{u} \in \mathrm{M}$ since $\phi(\mathrm{z},2\mathrm{u}) = 2\phi(\mathrm{z},\mathrm{u})$, we have

$$\psi(\mathrm{z},\mathrm{u}) = \lim_{\mathrm{p}\to\infty} 2^{\mathrm{p}}\phi\left(\mathrm{z},\frac{\mathrm{u}}{2^{\mathrm{p}}}\right) = \phi(\mathrm{z},\mathrm{u}).$$

Also, $\mathcal{T}_{\mathrm{M}}(\phi,\psi) \leq \frac{1}{1-\frac{\theta}{4}}\mathcal{T}_{\mathrm{M}}(\phi,\mathcal{S}\phi)$ which implies for all $\mathrm{u} \in \mathrm{M}$

$$\Omega_{\mathrm{M}}\left(\phi(\mathrm{z},\mathrm{u}) - \psi(\mathrm{z},\mathrm{u}),\eta\right) \succeq \Psi\left(\frac{\mathrm{u}}{2},\mathrm{u},\frac{\mathrm{u}}{2},\frac{2(4-\theta)}{\theta}\eta\right). \tag{12}$$

According to conditions (S3) and (S4), that is, inequalities (10) and (11), for all $\mathrm{u},\mathrm{v},\mathrm{w} \in \mathrm{M}$, we have

$$\begin{aligned}
&\Omega_M\Big(\psi(z,u+v+w)-\psi(z,u+w)-\psi(z,v-u+w)-\psi(z,u-w),\eta\Big)\\
&=\Omega_M\Big(\lim_{p\to\infty}(2^p\omega^p(\phi(z,\tfrac{u+v+w}{2^p\omega^p})-\phi(z,\tfrac{u+w}{2^p\omega^p})-\phi(z,\tfrac{v-u+w}{2^p\omega^p})-\phi(z,\tfrac{u-w}{2^p\omega^p}))),\eta\Big)\\
&=\lim_{p\to\infty}\Omega_M\Big(2^p\omega^p(\phi(z,\tfrac{u+v+w}{2^p\omega^p})-\phi(z,\tfrac{u+w}{2^p\omega^p})-\phi(z,\tfrac{v-u+w}{2^p\omega^p})-\phi(z,\tfrac{u-w}{2^p\omega^p})),\eta\Big)\\
&\succeq\lim_{p\to\infty}2^p|\omega|^p\Omega_M\Big(\sigma_1(\phi(z,\tfrac{u+v-w}{2^p\omega^p})+\phi(z,\tfrac{u-w}{2^p\omega^p})-\phi(z,\tfrac{v}{2^p\omega^p})),\eta\Big)\\
&\circledast\lim_{p\to\infty}2^p|\omega|^p\Omega_M\Big(\sigma_2(\phi(z,\tfrac{u-w}{2^p\omega^p})+\phi(z,\tfrac{u}{2^p\omega^p})-\phi(z,\tfrac{w}{2^p\omega^p})),\eta\Big)\\
&\circledast\lim_{p\to\infty}2^p|\omega|^p\Psi(\tfrac{u}{2^p\omega^p},\tfrac{v}{2^p\omega^p},\tfrac{w}{2^p\omega^p},\eta)\\
&=\Omega_M\Big(\sigma_1(\psi(z,u+v-w)+\psi(z,u-w)-\psi(z,v)),\eta\Big)\circledast\Omega_M\Big(\sigma_2(\psi(z,u-w)+\psi(z,u)-\psi(z,w)),\eta\Big).
\end{aligned}$$

Therefore, according to Lemma 1, ψ is a stochastic additive.

Let us once again assume $u=w=\frac{\zeta}{2}$ and $v=0$. Considering condition (S4), which means inequality (11), and putting in these new assumptions, for all $\omega\in\mathbb{B}^1$ and all $\zeta\in M$, we have

$$\Omega_M\Big(\phi(z,\omega\zeta)-\omega\phi(z,\zeta),\eta\Big)\succeq\Psi\Big(\frac{\zeta}{2},0,\frac{\zeta}{2},\eta\Big).$$

Therefore

$$\begin{aligned}
\Omega_M\Big(\psi(z,\omega u)-\omega\psi(z,u),\eta\Big)\quad&=\lim_{p\to\infty}2^p|\omega|^p\Omega_M\Big(\phi(z,\omega\tfrac{u}{2^p\omega^p})-\omega\phi(z,\tfrac{u}{2^p\omega^p}),\eta\Big)\\
&\succeq\lim_{p\to\infty}2^p\Psi\Big(\tfrac{u}{2^{p+1}\omega^p},0,\tfrac{u}{2^{p+1}\omega^p},\eta\Big)\\
&\succeq\lim_{p\to\infty}\Psi\Big(\tfrac{u}{2},0,\tfrac{u}{2},\Big(\tfrac{4}{\theta}\Big)^p\eta\Big).
\end{aligned}$$

Since when $p\to\infty$, it tends to **1**, then with respect to (NORM1), we conclude $\psi(z,\omega u)=\omega\psi(z,u)$, for all $\omega\in\mathbb{B}^1$ and all $u\in M$. Therefore, by Lemma 2, ψ is stochastic $\mathbb{C}$-linear. Considering that ψ is continuous and stochastic $\mathbb{C}$-linear and also considering $\phi=\psi$, from conditions (S3) and (S5), which are the same as inequalities (10) and (11), we have

$$\begin{aligned}
&\Omega_M\Big([\psi(z,u),\psi(z,v),\psi(z,w)]-\psi[\psi(z,u),v,w]-\psi[u,\psi(z,v),w]-\psi[u,v,\psi(z,w)],\eta\Big)\\
&=\lim_{p\to\infty}\Omega_M\Big(2^{3p}\omega^{3p}[\phi(z,\tfrac{u}{2^p\omega^p}),\phi(z,\tfrac{v}{2^p\omega^p}),\phi(z,\tfrac{w}{2^p\omega^p})]-2^p\omega^p\psi[\phi(z,\tfrac{u}{2^p\omega^p}),v,w]\\
&\qquad-2^p\omega^p\psi[u,\phi(z,\tfrac{v}{2^p\omega^p}),w]-2^p\omega^p\psi[u,v,\phi(z,\tfrac{w}{2^p\omega^p})],\eta\Big)\\
&=\lim_{p\to\infty}2^{3p}\Omega_M\Big([\phi(z,\tfrac{u}{2^p\omega^p}),\phi(z,\tfrac{v}{2^p\omega^p}),\phi(z,\tfrac{w}{2^p\omega^p})]-\phi[\phi(z,\tfrac{u}{2^p\omega^p}),\tfrac{v}{2^p\omega^p},\tfrac{w}{2^p\omega^p}]\\
&\qquad-\phi[\tfrac{u}{2^p\omega^p},\phi(z,\tfrac{v}{2^p\omega^p}),\tfrac{w}{2^p\omega^p}]-\phi[\tfrac{u}{2^p\omega^p},\tfrac{v}{2^p\omega^p},\phi(z,\tfrac{w}{2^p\omega^p})],\eta\Big)\\
&\succeq\lim_{p\to\infty}2^{3p}\Psi(\tfrac{u}{2^p\omega^p},\tfrac{v}{2^p\omega^p},\tfrac{w}{2^p\omega^p},\eta)\\
&\succeq\lim_{p\to\infty}\Psi(u,v,w,\tfrac{\eta}{\theta^p}),
\end{aligned}$$

for all $\omega\in\mathbb{B}^1$ and all $u,v,w\in M$. Since $\theta<1$, ψ is a stochastic ternary antiderivative (STAD). Thus, ϕ is a STAD. □

Example 8. *We consider the stochastic mapping* $\phi:\Pi\times M\to M$ *such that for all* $u,v,w\in M$

$$\Omega_M\Big(\phi(z,\omega(u+v+w))-\omega\phi(z,u+w)-\omega\phi(v-u+w)-\omega\phi(z,u-w),\eta\Big)$$
$$\succeq \Omega_M\Big(\sigma_1(\phi(z,u+v-w)+\phi(z,u-w)-\phi(z,v)),\eta\Big)\circledast\Omega_M\Big(\sigma_2(\phi(z,u-w)+\phi(z,u)-\phi(z,w)),\eta\Big)$$
$$\circledast\boldsymbol{MIN}\Big(\boldsymbol{\Psi}\Big(\|[[\mathbf{u},\mathbf{u},\mathbf{v}],\mathbf{v},\mathbf{w}]\|,\boldsymbol{\eta}\Big)\Big),$$

and

$$\Omega_M\Big([\phi(z,u),\phi(z,v),\phi(z,w)]-\phi[\phi(z,u),v,w]+\phi[u,\phi(z,v),w]+\phi[u,v,\phi(z,w)],\eta\Big)$$
$$\succeq\boldsymbol{MIN}\Big(\boldsymbol{\Psi}\Big(\|[[\mathbf{u},\mathbf{u},\mathbf{v}],\mathbf{v},\mathbf{w}]\|,\boldsymbol{\eta}\Big)\Big).$$

If ϕ is continuous and $\phi(z,2u)=2\phi(z,u)$ for all $u\in M$, then ϕ is a stochastic ternary antiderivation (STAD).

Proof. In the proof of Theorem 3, we set $\theta=\frac{2}{1000}$ and get

$$\boldsymbol{MIN}\Big(\boldsymbol{\Psi}([\mathbf{u},\mathbf{v},\mathbf{w}],\boldsymbol{\eta})\Big)=$$
$$\mathrm{diag}\Big[MIN\Big(\Psi([[u,u,v],v,w],\eta)\Big),MIN\Big(\Psi([[u,u,v],v,w],\eta)\Big),MIN\Big(\Psi([[u,u,v],v,w],\eta)\Big),$$
$$MIN\Big(\Psi([[u,u,v],v,w],\eta)\Big),MIN\Big(\Psi([[u,u,v],v,w],\eta)\Big)\Big],$$

where

$$\begin{aligned}\Psi([u,v,w],\eta)\ &=\mathrm{diag}\Big[E_{\kappa,\mu}\Big(\tfrac{\|[[u,u,v],v,w]\|}{\eta}\Big),W_{\kappa,\mu}\Big(\tfrac{\|[[u,u,v],v,w]\|}{\eta}\Big),{}_2F_1\Big(a,b,c,\tfrac{\|[[u,u,v],v,w]\|}{\eta}\Big),\\&\quad H^{\lambda_3,\lambda_1}_{\lambda_2,\lambda_4}\Big(\tfrac{\|[[u,u,v],v,w]\|}{\eta}\Big),\exp\Big(\tfrac{\|[[u,u,v],v,w]\|}{\eta}\Big)\Big],\end{aligned}$$

for all $u,v,w\in M$, and the proof is complete. □

Now, we consider the unital SMVFMC-⋄-A $\Big(M,\Omega_M,\circledast,\circledast\Big)$ with unit $\mathbb{E}$ and the unitary group $I(M)=\{\delta\in M:\delta^\diamond\delta=\delta\delta^\diamond=\mathbb{E}\}$.

Theorem 4. *We consider the SMVFC-⋄-A $\Big(M,\Omega_M,\circledast,\circledast\Big)$ and the MVF function $\Psi:M\times M\times M\times(0,\infty)\to\mathcal{D}_H$ such that Ψ satisfies (10). If we consider the stochastic operator $\phi:\Pi\times M\to M$ such that for every $u\in M$ and $z\in\Pi$, we have $\phi(z,0)=0$ and $\phi(z,2u)=2\phi(z,u)$ and it also satisfies inequality (11), then $\phi:\Pi\times M\to M$ is a stochastic ternary antiderivation (STAD).*

Proof. According Theorem 3, we have a unique $\mathbb{C}$-linear stochastic operator $\psi:\Pi\times M\to M$ that satisfies inequality (12) and is defined as follows:

$$\psi(z,u)=\lim_{p\to\infty}2^p\phi\Big(z,\frac{u}{2^p}\Big),\tag{13}$$

for all $u\in M$ and also $\psi(z,u)=\phi(z,u)$. Again, considering Theorem 3, for all $u\in I(M)$ and $z\in\Pi$, we have

$$[\psi(z,u),\psi(z,v),\psi(z,w)]=\psi[\psi(z,u),v,w]+\psi[u,\psi(z,v),w]+\psi[u,v,\psi(z,w)].\tag{14}$$

We assume that $u = \sum_{\iota=1}^{p} m_i u_\iota$ for all $m_i \in \mathbb{C}$, $u_i \in I(M)$ and $u \in M$. Since ψ in the second variable is $\mathbb{C}$-bilinear, for all $u, v, w \in M$, we get

$$\begin{aligned}[\psi(z,u),\psi(z,v),\psi(z,w)] &= [\psi(z,\textstyle\sum_{\iota=1}^{p} m_\iota u_\iota),\psi(z,\sum_{\iota=1}^{p} m_\iota v_\iota),\psi(z,\sum_{\iota=1}^{p} m_\iota w_\iota)] \\ &= \textstyle\sum_{\iota=1}^{p} m_\iota[\psi(z,u_\iota),\psi(z,v_\iota),\psi(z,w_\iota)] \\ &= \textstyle\sum_{\iota=1}^{p} m_\iota\Big[\psi[\psi(z,u_\iota),v_\iota,w_\iota]+\psi[u_\iota,\psi(z,v_\iota),w_\iota]+\psi[u_\iota,v_\iota,\psi(z,w_\iota)]\Big] \\ &= \psi[\psi(z,u),v,w]+\psi[u,\psi(z,v),w]+\psi[u,v,\psi(z,w)].\end{aligned}$$

Then, $\psi : \Pi \times M \to M$ is a stochastic ternary antiderivation (STAD). □

5. Superstability of Continuous Stochastic Ternary Antiderivations in T-SMVFB-A and T-SMVFC-⋄-A

Here, we show the superstability of continuous ternary antiderivatives ternary SMVFB-As and T-SMVFC-⋄-As.

Theorem 5. *We consider the MVF function of* $\Psi : M \times M \times \mathcal{A} \to \mathcal{D}_H$ *and the stochastic function of* $\phi : \Pi \times M \to M$ *such that*

(S6) For every $u, v, w \in M$

$$\Psi\left(\frac{u}{\omega},\frac{v}{\omega},\frac{w}{\omega},\eta\right) \succeq \Psi\left(2u,2v,2w,\frac{8}{\theta}\eta\right), \tag{15}$$

(S7)

$$\begin{aligned}&\Omega_M\Big(\phi(z,\omega(u+v+w)) - \omega\phi(z,u+w) - \omega\phi(z,v-u+w) - \omega\phi(z,u-w),\eta\Big) \\ &\succeq \Omega_M\Big(\sigma_1(\phi(z,u+v-w)+\phi(z,u-w)-\phi(z,v)),\eta\Big) \\ &\circledast\Omega_M\Big(\sigma_2(\phi(z,u-w)+\phi(z,w)-\phi(z,w)),\eta\Big) \circledast \Psi\Big(u,v,w,\eta\Big),\end{aligned}$$

(S8)

$$\begin{aligned}&\Omega_M\Big([\phi(z,u),\phi(z,v),\phi(z,w)] - \phi[\phi(z,u),v,w] + \psi[u,\phi(z,v),w] + \phi[u,v,\phi(z,w)],\eta\Big) \\ &\succeq \Psi\Big(u,v,w,\eta\Big),\end{aligned}$$

for each $|\omega| < 1$ *and constant* $\theta < 1$*. If for all* $u \in M$*,* ϕ *is continuous and* $\phi(z,2u) = 2\phi(z,u)$*, then the* ϕ *is a stochastic ternary antiderivation (STAD).*

Proof. If we assume that $\Lambda \in \mathbb{B}^1$, then for $|\Lambda_p| < 1$ there is a sequence $\{\Lambda_p\}_{p=1}^{\infty}$ such that $\lim_{p\to\infty}\Lambda_p = \mu$. According to conditions (S6) and (S7), i.e., inequalities (15) and (16), for every Λ_p with $|\Lambda_p| < 1$ and all $u, v, w \in \mathscr{M}$, we have

$$\Psi\left(\frac{u}{\Lambda_p},\frac{v}{\Lambda_p},\frac{w}{\Lambda_p},\eta\right) \succeq \Psi(2u,2v,2w,\frac{8}{\theta}\eta),$$

and for positive integers p

$$\begin{aligned}&\Omega_M\Big(\phi(z,\Lambda_p(u+v+w)) - \Lambda_p\phi(z,u+w) - \Lambda_p\phi(v-u+w) - \Lambda_p\phi(z,u-w),\eta\Big) \\ &\succeq \Omega_M\Big(\sigma_1(\phi(z,u+v-w)+\phi(z,u-w)-\phi(z,v)),\eta\Big) \\ &\circledast\Omega_M\Big(\sigma_2(\phi(z,u-w)+\phi(z,u)-\phi(z,w)),\eta\Big) \circledast \Psi\Big(u,v,w,\eta\Big).\end{aligned}$$

Using the continuity Ψ, ϕ and $\Omega_M(u,\eta)$ and considering the limit when $p \to \infty$, we have

$$\Psi\left(\frac{u}{\Lambda},\frac{v}{\Lambda},\frac{w}{\Lambda},\eta\right) \succeq \Psi\left(2u,2v,2w,\frac{8}{\theta}\eta\right),$$

and for $\Lambda \in \mathbb{B}^1$ and for every $u,v,w \in \mathscr{M}$

$$\begin{aligned}\Omega_M\Big(\phi(z,\Lambda(u+v+w)) - \Lambda\phi(z,u+w) - \Lambda\phi(z,v-u+w) - \Lambda\phi(z,u-w),\eta\Big)\\ \succeq \Omega_M\Big(\sigma_1(\phi(z,u+v-w)+\phi(z,u-w)-\phi(z,v)),\eta\Big)\\ \circledast\Omega_M\Big(\sigma_2(\phi(z,u-w)+\phi(z,u)-\phi(z,w)),\eta\Big) \circledast \Psi\Big(u,v,w,\eta\Big).\end{aligned}$$

Then, as stated in Theorem 3, the stochastic mapping $\phi : \Pi \times M \to M$ is a stochastic ternary antiderivation (STAD). □

Theorem 6. *We consider the SMVFC-⋄-A* $\Big(M,\Omega_M,\circledast,\circledast\Big)$ *and the MVF function* $\Psi : M \times M \times M \times (0,\infty) \to \mathcal{D}_H$ *such that* Ψ *satisfies* (15)*. If we consider the stochastic operator* $\phi : \Pi \times M \to M$ *such that for every* $u \in M$ *and* $z \in \Pi$*, we have* $\phi(z,0) = 0$ *and* $\phi(z,2u) = 2\phi(z,u)$ *and it also satisfies inequality* (16)*, then* $\phi : \Pi \times M \to M$ *is a stochastic ternary antiderivation (STAD).*

Proof. The proof is similar to the proof of Theorem 4. □

6. Conclusions

In this paper, we studied the concept of ternary antiderivatives in SMVFB-As and their multi-super-stability. We introduced the aggregation function using special functions such as the Mittag-Leffler function (MLF), the Wright function (WF), the *H*-Fox function (*H*FF), the Gauss hypergeometric function (GHF), and the exponential function (EXP-F), and we obtained the optimal control function by performing the necessary calculations. First, we have checked the minimum stability of the stochastic operator on SMVFB-A. Then, by introducing stochastic ternary antiderivatives, the superstability of stochastic ternary antiderivatives and continuous stochastic ternary antiderivatives in T-SMVFB-A and T-SMVFC-⋄-A spaces has been investigated.

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