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Diamond-α Hardy-Type Inequalities on Time Scales

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Abstract: In the present article, we prove some new generalizations of dynamic inequalities of Hardy-type by utilizing diamond-α dynamic integrals on time scales. Furthermore, new generalizations of dynamic inequalities of Hardy-type in two variables on time scales are proved. Moreover, we present Hardy inequalities for several functions by using the diamond-α dynamic integrals on time scales. The results are proved by using the dynamic Jensen inequality and the Fubini theorem on time scales. Our main results extend existing results of the integral and discrete Hardy-type inequalities. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Hardy's inequality; Jensen's inequality; diamond-α dynamic integrals; dynamic inequality; time scale

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1. Introduction

Hardy discrete inequality [1] states that:

Theorem 1. *If $\{a_\imath\}$ is a nonnegative real sequence and $\beta > 1$, then:*

$$\sum_{\imath=1}^{\infty}\left(\frac{1}{\imath}\sum_{\jmath=1}^{\imath}a_\jmath\right)^{\beta} \leq \left(\frac{\beta}{\beta-1}\right)^{\beta}\sum_{\imath=1}^{\infty}a_\imath^{\beta}, \quad \beta > 1. \tag{1}$$

Hardy himself, in [2], gave the following integral analog of inequality (1).

Theorem 2. *If ψ is a nonnegative integrable function over a finite interval $(0,\kappa)$, such that $\psi \in L^{\beta}(0,\infty)$ and $\beta > 1$, then:*

$$\int_0^{\infty}\left(\frac{1}{\kappa}\int_0^{\kappa}\psi(\varsigma)d\varsigma\right)^{\beta}d\kappa \leq \left(\frac{\beta}{\beta-1}\right)^{\beta}\int_0^{\infty}\psi^{\beta}(\kappa)d\kappa. \tag{2}$$

It is worth mentioning that inequalities (1) and (2) are sharp in the sense that the constant $(\beta/\beta-1)^{\beta}$ in each of them can not be replaced by a smaller one.

The study of Hardy-type inequalities attracted, and still attracts, the attention of many researchers. Over several decades, many generalizations, extensions and refinements have been made to the above inequalities; we refer the reader to the papers [2–12], the books [13,14] and the references cited therein.

Time scale calculus—with the objective to unify discrete and continuous analyses—was introduced by S. Hilger [15]. For additional subtleties on time scales, we allude to the peruser of the books by Bohner and Peterson [16,17]. If $\mathbb{T}$ is a time scale, and ψ is a function, which is delta and nabla differentiable on $\mathbb{T}$, then, for any $\varsigma \in \mathbb{T}$, the diamond-α dynamic derivative of ψ at ς, denoted by $\psi^{\diamond_\alpha}(\varsigma)$, is defined as follows:

$$\psi^{\diamond_\alpha}(\varsigma) = \alpha\psi^{\Delta}(\varsigma) + (1-\alpha)\psi^{\nabla}(\varsigma), \qquad 0 \leq \alpha \leq 1, \tag{3}$$

and the definite diamond-α integral of ψ is defined as follows:

$$\int_a^b \psi(\varsigma)\diamondsuit_\alpha \varsigma = \alpha \int_a^b \psi(\varsigma)\Delta\varsigma + (1-\alpha)\int_a^b \psi(\varsigma)\nabla\varsigma, \qquad 0 \le \alpha \le 1, \quad a, b \in \mathbb{T}. \tag{4}$$

For $\alpha = 1$, the diamond-α derivative boils down to a delta derivative, and for $\alpha = 0$, it boils down to a nabla derivative. We refer the interested reader to [18] for further details on diamond-α calculus.

In [19], Řehák has given the time scales version of Hardy inequality (2) as follows:

Theorem 3. *Let $\mathbb{T}$ be a time scale, and $\psi \in C_{rd}([a,\infty)_{\mathbb{T}}, [0,\infty))$, $\Lambda(\varsigma) = \int\limits_a^{\varsigma} \psi(\zeta)\Delta\zeta$, for $\varsigma \in [a,\infty)_{\mathbb{T}}$.*

$$\int\limits_a^{\infty} \left(\frac{\Lambda^\sigma(\varsigma)}{\sigma(\varsigma)-a}\right)^{\beta} \Delta\varsigma < \left(\frac{\beta}{\beta-1}\right)^{\beta} \int\limits_a^{\infty} \psi^{\beta}(\varsigma)\Delta\varsigma, \quad \beta > 1, \tag{5}$$

unless $\psi \equiv 0$.
Furthermore, if $\mu(\varsigma)/\varsigma \to 0$ as $\varsigma \to \infty$, then inequality (5) is sharp.

In [20], Saker and O'Regan established a generalization of Řehák's result in the following form.

Theorem 4. *Let $a \in [0,\infty)_{\mathbb{T}}$, λ, $g \in C_{rd}([a,\infty)_{\mathbb{T}}, [0,\infty))$, and define, for $\varsigma \in [0,\infty)_{\mathbb{T}}$,*

$$\Phi(\varsigma) := \int\limits_a^{\varsigma} \lambda(\zeta) g(\zeta)\Delta\zeta \qquad \textit{and} \qquad \Lambda(\varsigma) := \int\limits_a^{\varsigma} \lambda(\zeta)\Delta\zeta. \tag{6}$$

If $p \ge c > 1$, then:

$$\int\limits_a^{\infty} \lambda(\varsigma)\frac{(\Phi^\sigma(\varsigma))^p}{(\Lambda^\sigma(\varsigma))^q}\Delta\varsigma \le \left(\frac{p}{q-1}\right)^p \int\limits_a^{\infty} \lambda(\varsigma)\frac{(\Lambda^\sigma(\varsigma))^{q(p-1)}}{(\Lambda(\varsigma))^{p(q-1)}} g^p(\varsigma)\Delta\varsigma. \tag{7}$$

In [21], Ozkan and Yildirim gave the following result among many other results.

Theorem 5. *Let $a \in [0,\infty)_{\mathbb{T}}$ and $v \in C_{rd}\big([a,b]_{\mathbb{T}}, \mathbb{R}_+\big)$ such that the delta integral $\int\limits_{\kappa}^{b} \frac{v(\varsigma)}{(\varsigma-a)(\sigma(\varsigma)-a)}\Delta\varsigma$ converges. If $\psi \in C_{rd}\big([a,b]_{\mathbb{T}}, (\theta,\vartheta)\big)$ and $\Xi \in C\big((\theta,\vartheta), \mathbb{R}\big)$ is convex, then:*

$$\int\limits_a^b \frac{v(\kappa)}{\kappa-a}\Xi\left(\frac{\int\limits_a^{\sigma(\kappa)} \psi(\varsigma)\Delta\varsigma}{\sigma(\kappa)-a}\right)\Delta\kappa \le \int\limits_a^b \Xi\big(\psi(\kappa)\big)\left(\int\limits_{\kappa}^{b} \frac{v(\varsigma)}{(\varsigma-a)(\sigma(\varsigma)-a)}\Delta\varsigma\right)\Delta\kappa.$$

In this paper, by employing diamond-α integrals, we establish a few variations of Hardy-type dynamic inequalities that were given by Ozkan and Yildirim in [21]. The obtained results extend some known Hardy-type integral inequalities and unify some continuous inequalities and their corresponding discrete analogs. The paper is arranged as follows: In Section 2, we state and prove the main results. In Section 3, we state the conclusion.

Lemma 1 (Fubini's Theorem on Time Scales, see [22], Theorem 3.10)**.** *Let ψ be bounded and Δ-integrable over $R = [a,b) \times [c,d)$ and suppose that the single integrals:*

$$I(\varsigma) = \int_c^d \psi(\varsigma,\zeta)\Delta\zeta \qquad \text{and} \qquad K(\zeta) = \int_a^b \psi(\varsigma,\zeta)\Delta\varsigma$$

exist for each $\varsigma \in [a,b)$ *and for each* $\zeta \in [c,d)$*, respectively. Then the iterated integrals:*

$$\int_a^b \Delta\varsigma \int_c^d \psi(\varsigma,\zeta)\Delta\zeta \qquad \text{and} \qquad \int_c^d \Delta\zeta \int_a^b \psi(\varsigma,\zeta)\Delta\varsigma$$

exist and the equality:

$$\int_a^b \Delta\varsigma \int_c^d \psi(\varsigma,\zeta)\Delta\zeta = \int_c^d \Delta\zeta \int_a^b \psi(\varsigma,\zeta)\Delta\varsigma$$

holds.

Lemma 2 (Dynamic Jensen's Inequality, see [23], Theorem 2.2.6)**.** *Suppose that* $a,b \in \mathbb{T}$ *with* $a < b$*. Further, let* $\psi \in C\big([a,b]_{\mathbb{T}},(\theta,\vartheta)\big)$ *and* $\varphi \in C\big([a,b]_{\mathbb{T}},\mathbb{R}_+\big)$*. If* $\Xi \in C\big((\theta,\vartheta),\mathbb{R}_+\big)$ *is convex, then:*

$$\Xi\left(\frac{\int_a^b \varphi(\zeta)\psi(\zeta)\diamondsuit_\alpha\zeta}{\int_a^b \varphi(\zeta)\diamondsuit_\alpha\zeta}\right) \le \int_a^b \frac{\varphi(\zeta)\Xi\big(\psi(\zeta)\big)\diamondsuit_\alpha\zeta}{\int_a^b \varphi(\zeta)\diamondsuit_\alpha\zeta}. \tag{8}$$

We will need the following lemma, which gives two-dimensional dynamic Jensen's inequality, in the proof of our main results.

Lemma 3. *Suppose that* $a,b,c,d \in \mathbb{T}$ *with* $a < b$ *and* $c < d$*. Further, let* $\psi \in C\big([a,b]_{\mathbb{T}} \times [c,d]_{\mathbb{T}},(\theta,\vartheta)\big)$*,* $\phi \in C\big([c,d]_{\mathbb{T}},\mathbb{R}_+\big)$ *and* $\varphi \in C\big([a,b]_{\mathbb{T}},\mathbb{R}_+\big)$*. If* $\Xi \in C\big((\theta,\vartheta),\mathbb{R}_+\big)$ *is convex, then:*

$$\Xi\left(\frac{\int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right) \le \int_a^b\int_c^d \frac{\phi(\varsigma)\varphi(\zeta)\Xi\big(\psi(\varsigma,\zeta)\big)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}. \tag{9}$$

Proof. This lemma is a direct extension of the [23] (Theorem 2.2.6). □

2. Main Results

Theorem 6. *Let* $a \in [0,\infty)_{\mathbb{T}}$ *and* $\upsilon,\delta \in C\big([a,b]_{\mathbb{T}},\mathbb{R}_+\big)$*, such that the delta integral* $\int_\kappa^b \frac{\upsilon(\varsigma)}{\int_a^\varsigma \delta(\kappa)\diamondsuit_\alpha\kappa \int_a^{\sigma(\varsigma)} \delta(\kappa)\diamondsuit_\alpha\kappa}\diamondsuit_\alpha\varsigma$ *converges. If* $\psi \in C\big([a,b]_{\mathbb{T}},(\theta,\vartheta)\big)$ *and* $\Xi \in C\big((\theta,\vartheta),\mathbb{R}\big)$ *is convex, then:*

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)} \delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)} \delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \le \int_a^b \delta(\kappa)\Xi\big(\psi(\kappa)\big)\left(\int_\kappa^b \frac{\upsilon(\varsigma)}{\int_a^\varsigma \delta(\kappa)\diamondsuit_\alpha\kappa \int_a^{\sigma(\varsigma)} \delta(\kappa)\diamondsuit_\alpha\kappa}\diamondsuit_\alpha\varsigma\right)\diamondsuit_\alpha\kappa. \tag{10}$$

Proof. Employing the dynamic Jensen inequality (8), we obtain:

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa$$
$$\leq \int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma \int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\left(\int_a^{\sigma(\kappa)}\delta(\varsigma)\Xi(\psi(\varsigma))\diamondsuit_\alpha\varsigma\right)\diamondsuit_\alpha\kappa \tag{11}$$

Applying Fubini's theorem on the right hand side of (11), we obtain:

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma \int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\left(\int_a^{\sigma(\kappa)}\delta(\varsigma)\Xi(\psi(\varsigma))\diamondsuit_\alpha\varsigma\right)\diamondsuit_\alpha\kappa$$
$$= \int_a^b \delta(\varsigma)\Xi(\psi(\varsigma))\left(\int_\varsigma^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma \int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\diamondsuit_\alpha\kappa\right)\diamondsuit_\alpha\varsigma. \tag{12}$$

From (11) and (12) we receive:

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \leq \int_a^b \delta(\kappa)\Xi(\psi(\kappa))\left(\int_\kappa^b \frac{\upsilon(\varsigma)}{\int_a^\varsigma \delta(\kappa)\diamondsuit_\alpha\kappa \int_a^{\sigma(\varsigma)}\delta(\kappa)\diamondsuit_\alpha\kappa}\diamondsuit_\alpha\varsigma\right)\diamondsuit_\alpha\kappa,$$

which is our desired result (10). □

Corollary 1. *Assuming $\alpha = 1$, we obtain the delta version of Theorem 6 as follows:*

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\Delta\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\Delta\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\Delta\varsigma}\right)\Delta\kappa \leq \int_a^b \delta(\kappa)\Xi(\psi(\kappa))\left(\int_\kappa^b \frac{\upsilon(\varsigma)}{\int_a^\varsigma \delta(\kappa)\Delta\kappa \int_a^{\sigma(\varsigma)}\delta(\kappa)\Delta\kappa}\Delta\varsigma\right)\Delta\kappa. \tag{13}$$

Corollary 2. *Assuming $\alpha = 0$, we obtain the nabla version of Theorem 6 as follows:*

$$\int_a^b \frac{\upsilon(\kappa)}{\int_a^\kappa \delta(\varsigma)\nabla\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\nabla\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\nabla\varsigma}\right)\nabla\kappa \leq \int_a^b \delta(\kappa)\Xi(\psi(\kappa))\left(\int_\kappa^b \frac{\upsilon(\varsigma)}{\int_a^\varsigma \delta(\kappa)\nabla\kappa \int_a^{\sigma(\varsigma)}\delta(\kappa)\nabla\kappa}\nabla\varsigma\right)\nabla\kappa.$$

Remark 1. *If we use $\delta(\varsigma) = 1$, in Corollary 1, then we recapture Theorem 5.*

Corollary 3. *In Theorem 6, if $\upsilon(\kappa) = 1$ and b is finite, then inequality (10) reads:*

$$\int_a^b \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \leq \int_a^b \delta(\kappa)\Xi(\psi(\kappa))\left(\frac{1}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\kappa, \tag{14}$$

while for $b \to \infty$ it becomes:

$$\int_a^\infty \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \le \int_a^\infty \frac{\delta(\kappa)\Xi(\psi(\kappa))}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\diamondsuit_\alpha\kappa. \tag{15}$$

Corollary 4. *If we take $\Xi(v) = v^\beta$, where $\beta > 1$ is a constant, then inequality (14) and (14) will, respectively, be:*

$$\int_a^b \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)^\beta\diamondsuit_\alpha\kappa \le \int_a^b \delta(\kappa)\psi^\beta(\kappa)\left(\frac{1}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\kappa \tag{16}$$

and:

$$\int_a^\infty \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)^\beta\diamondsuit_\alpha\kappa \le \int_a^\infty \frac{\delta(\kappa)\psi^\beta(\kappa)}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\diamondsuit_\alpha\kappa. \tag{17}$$

Corollary 5. *If we take $\Xi(v) = \exp(v)$ and replace ψ by $\ln\psi$, then inequalities (14) and (15) will, respectively, be:*

$$\int_a^b \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\exp\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\ln\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \le \int_a^b \delta(\kappa)\psi(\kappa)\left(\frac{1}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\kappa$$

and:

$$\int_a^\infty \frac{1}{\int_a^\kappa \delta(\varsigma)\diamondsuit_\alpha\varsigma}\exp\left(\frac{\int_a^{\sigma(\kappa)}\delta(\varsigma)\ln\psi(\varsigma)\diamondsuit_\alpha\varsigma}{\int_a^{\sigma(\kappa)}\delta(\varsigma)\diamondsuit_\alpha\varsigma}\right)\diamondsuit_\alpha\kappa \le \int_a^\infty \frac{\delta(\kappa)\psi(\kappa)}{\int_a^\kappa \delta(\zeta)\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\delta(\zeta)\diamondsuit_\alpha\zeta}\diamondsuit_\alpha\kappa.$$

Corollary 6. *If $\mathbb{T} = \mathbb{R}$, in Corollary 1, then we obtain the continuous version of inequality (13) as follows:*

$$\int_a^b \frac{v(\kappa)}{\int_a^\kappa \delta(\varsigma)d\varsigma}\Xi\left(\frac{\int_a^\kappa \delta(\varsigma)\psi(\varsigma)d\varsigma}{\int_a^\kappa \delta(\varsigma)d\varsigma}\right)d\kappa \le \frac{1}{2}\int_a^b \delta(\kappa)\Xi(\psi(\kappa))\left(\int_\kappa^b \frac{v(\varsigma)}{\int_a^\varsigma \delta(\kappa)d\kappa}d\varsigma\right)d\kappa.$$

Corollary 7. *If $\mathbb{T} = \mathbb{Z}$, in Corollary 1, then we acheive the discrete version of inequality (13) as follows:*

$$\sum_{\kappa=a}^{b-1}\frac{v(\kappa)}{\sum_{\varsigma=a}^{\kappa-1}\delta(\varsigma)}\Xi\left(\frac{\sum_{\varsigma=a}^{\kappa}\delta(\varsigma)\psi(\varsigma)}{\sum_{\varsigma=a}^{\kappa}\delta(\varsigma)}\right) \le \sum_{\kappa=a}^{b-1}\delta(\kappa)\Xi(\psi(\kappa))\left(\sum_{\varsigma=\kappa}^{b-1}\frac{v(\varsigma)}{\sum_{\kappa=a}^{\varsigma}\delta(\kappa)\sum_{\kappa=a}^{\varsigma-1}\delta(\kappa)}\right).$$

Theorem 7. *Suppose that $a, c \in [0,\infty)_{\mathbb{T}}$, $\psi \in C\big([a,b]_{\mathbb{T}} \times [c,d]_{\mathbb{T}}, \mathbb{R}\big)$, $\phi \in C\big([c,d]_{\mathbb{T}}, \mathbb{R}_{+}\big)$ and $\varphi \in C\big([a,b]_{\mathbb{T}}, \mathbb{R}_{+}\big)$. If $\Xi \in C\big((\theta,\vartheta), \mathbb{R}_{+}\big)$ is convex, then:*

$$\begin{aligned}
&\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\,\Xi\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa \\
&\leq \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\Xi\big(\psi(\varsigma,\zeta)\big)\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.
\end{aligned} \tag{18}$$

Proof. Using the two-dimensional dynamic Jensen inequality (9) and Fubini's theorem on time scales, we obtain:

$$\begin{aligned}
&\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\,\Xi\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa \\
&\leq \int_a^b\int_c^d \frac{1}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\left(\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\Xi\big(\psi(\varsigma,\zeta)\big)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta\right)\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa \\
&= \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\Xi\big(\psi(\varsigma,\zeta)\big)\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.
\end{aligned}$$

This concludes the proof. □

Corollary 8. *Assuming $\alpha = 1$, we obtain the delta version of Theorem 7 as follows:*

$$\begin{aligned}
&\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\Delta\varsigma\Delta\zeta}\,\Xi\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\Delta\varsigma\Delta\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\Delta\varsigma\Delta\zeta}\right)\Delta\eta\Delta\kappa \\
&\leq \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\Xi\big(\psi(\varsigma,\zeta)\big)\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta \int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)} \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.
\end{aligned} \tag{19}$$

Corollary 9. *Assuming $\alpha = 0$, we get the nabla version of Theorem 7 as follows:*

$$\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\nabla\varsigma\nabla\zeta}\Xi\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\nabla\varsigma\nabla\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\nabla\varsigma\nabla\zeta}\right)\nabla\eta\nabla\kappa$$
$$\le \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\Xi(\psi(\varsigma,\zeta))\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.$$

Remark 2. *If we use $\phi(\varsigma)=\varphi(\varsigma)=1$, in Corollary 8, then we recapture [21] (Theorem 3.2).*

Corollary 10. *In Theorem 7, if we take $\Xi(v)=v^\beta$, where $\beta>1$ is a constant, then we have:*

$$\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)^\beta\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa$$
$$\le \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\psi^\beta(\varsigma,\zeta)\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.$$

Corollary 11. *In Theorem 7, if we take $\Xi(v)=\exp(v)$ and replace ψ by $\ln\psi$, then we have:*

$$\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\exp\left(\frac{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\ln\psi(\varsigma,\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}{\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa$$
$$\le \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)\left(\int_\zeta^b\int_\varsigma^d \frac{\diamondsuit_\alpha\eta\diamondsuit_\alpha\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta}\right)\diamondsuit_\alpha\varsigma\diamondsuit_\alpha\zeta.$$

Corollary 12. *If $\mathbb{T}=\mathbb{R}$, in Corollary 8, then we obtain the continuous version of inequality (19) as follows:*

$$\int_a^b\int_c^d \frac{1}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)dtds}\Xi\left(\frac{\int_a^\kappa\int_c^\eta\phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)dtds}{\int_a^\kappa\int_c^\eta\phi(\varsigma)\varphi(\zeta)dtds}\right)dydx$$
$$\le \int_a^b\int_c^d \phi(\varsigma)\varphi(\zeta)\Xi(\psi(\varsigma,\zeta))\left(\int_\zeta^b\int_\varsigma^d \frac{d\eta d\kappa}{\int_a^\kappa\int_c^\eta \phi(\varsigma)\varphi(\zeta)d\varsigma d\zeta\int_a^{\sigma(\kappa)}\int_c^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)d\varsigma d\zeta}\right)d\varsigma d\zeta.$$

Corollary 13. *If $\mathbb{T}=\mathbb{Z}$, in Corollary 8, then we obtain the discrete version of inequality (19) as follows:*

$$\sum_{\kappa=a}^{b-1}\sum_{\eta=c}^{d-1}\frac{1}{\sum_{\zeta=a}^{\kappa-1}\sum_{\varsigma=c}^{\eta-1}\phi(\varsigma)\varphi(\zeta)}\Xi\left(\frac{\sum_{\zeta=a}^{\kappa}\sum_{\varsigma=c}^{\eta}\phi(\varsigma)\varphi(\zeta)\psi(\varsigma,\zeta)}{\sum_{\zeta=a}^{\kappa}\sum_{\varsigma=c}^{\eta}\phi(\varsigma)\varphi(\zeta)}\right)$$
$$\leq \sum_{\zeta=a}^{b-1}\sum_{\varsigma=c}^{d-1}\phi(\varsigma)\varphi(\zeta)\Xi(\psi(\varsigma,\zeta))\left(\sum_{\kappa=\zeta}^{b}\sum_{\kappa=\varsigma}^{d}\frac{1}{\sum_{\varsigma=a}^{\kappa}\sum_{\zeta=c}^{\eta}\phi(\varsigma)\varphi(\zeta)d\varsigma d\zeta\sum_{\varsigma=a}^{\sigma(\kappa)}\sum_{\zeta=c}^{\sigma(\eta)}\phi(\varsigma)\varphi(\zeta)}\right).$$

Our aim in the following theorem is to establish a dynamic Hardy inequality for several functions.

Theorem 8. *Assume that $a \in [0,\infty)_{\mathbb{T}}$ and $\delta, \psi_1, \psi_2, \ldots, \psi_n \in C([a,\infty)_{\mathbb{T}}, \mathbb{R}_+)$. Define $\Lambda(\varsigma) := \int_a^{\varsigma}\delta(\zeta)\diamondsuit_\alpha\zeta$ and $F_k(\varsigma) := \int_a^{\varsigma}\delta(\zeta)\psi_k(\zeta)\diamondsuit_\alpha\zeta$ for $k = 1,2,\ldots,n$. If $\beta \geq \gamma > 1$, then:*

$$\int_a^\infty \delta(\varsigma)\frac{\left(F_1^\sigma(\varsigma)F_2^\sigma(\varsigma)\ldots F_n^\sigma(\varsigma)\right)^{\beta/n}}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma}\diamondsuit_\alpha\varsigma$$
$$\leq \frac{1}{n^\beta}\int_a^\infty \delta(\varsigma)\frac{\left(\sum_{k=1}^n F_k^\sigma(\varsigma)\right)^\beta}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma}\diamondsuit_\alpha\varsigma. \tag{20}$$

Proof. Utilizing the arithmetic geometric mean inequality, we have:

$$\left(F_1^\sigma(\varsigma)F_2^\sigma(\varsigma)\ldots F_n^\sigma(\varsigma)\right)^{1/n} \leq \frac{\sum_{k=1}^n F_k^\sigma(\varsigma)}{n},$$

and thus:

$$\left(F_1^\sigma(\varsigma)F_2^\sigma(\varsigma)\ldots F_n^\sigma(\varsigma)\right)^{\beta/n} \leq \frac{\left(\sum_{k=1}^n F_k^\sigma(\varsigma)\right)^\beta}{n^\beta}. \tag{21}$$

Multiplying both sides of (21) by $\delta(\varsigma)/\left(\Lambda^\sigma(\varsigma)\right)^\gamma$ and integrating the resulting inequality over ς from a to ∞ yields:

$$\int_a^\infty \delta(\varsigma)\frac{\left(F_1^\sigma(\varsigma)F_2^\sigma(\varsigma)\ldots F_n^\sigma(\varsigma)\right)^{\beta/n}}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma}\diamondsuit_\alpha\varsigma \leq \frac{1}{n^\beta}\int_a^\infty \delta(\varsigma)\frac{\left(\sum_{k=1}^n F_k^\sigma(\varsigma)\right)^\beta}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma}\diamondsuit_\alpha\varsigma.$$

The proof is complete. □

Corollary 14. *Assuming $\alpha = 1$, and using the result of inequality (7), we obtain the delta version of Theorem 8 as follows.*

$$\int_a^\infty \delta(\varsigma)\frac{\left(F_1^\sigma(\varsigma)F_2^\sigma(\varsigma)\ldots F_n^\sigma(\varsigma)\right)^{\beta/n}}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma}\Delta\varsigma$$
$$\leq \left(\frac{\beta}{nq-n}\right)^\beta\int_a^\infty \frac{\delta(\varsigma)\left(\Lambda^\sigma(\varsigma)\right)^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(\varsigma)}\left(\psi_1(\varsigma)+\psi_2(\varsigma)+\cdots+\psi_n(\varsigma)\right)^\beta\Delta\varsigma. \tag{22}$$

Corollary 15. *Assuming $\alpha = 0$, we obtain the nabla version of Theorem 8 as follows.*

$$\int_a^\infty \delta(\varsigma) \frac{\left(F_1^\sigma(\varsigma) F_2^\sigma(\varsigma) \dots F_n^\sigma(\varsigma)\right)^{\beta/n}}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma} \nabla\varsigma$$
$$\leq \frac{1}{n^\beta} \int_a^\infty \delta(\varsigma) \frac{\left(\sum_{k=1}^n F_k^\sigma(\varsigma)\right)^\beta}{\left(\Lambda^\sigma(\varsigma)\right)^\gamma} \nabla\varsigma.$$

Remark 3. *If we use $\delta(\varsigma) = 1$, in Corollary 14, then we recapture [21] (Theorem 1.4).*

Corollary 16. *If $\mathbb{T} = \mathbb{R}$, in Corollary 14, then we obtain the continuous version of inequality (22) as follows:*

$$\int_a^\infty \delta(\varsigma) \frac{\left(F_1(\varsigma) F_2(\varsigma) \dots F_n(\varsigma)\right)^{\beta/n}}{\Lambda^\gamma(\varsigma)} d\varsigma$$
$$\leq \left(\frac{\beta}{nq - n}\right)^\beta \int_a^\infty \delta(\varsigma) \Lambda^{\beta-\gamma}(\varsigma) \left(\psi_1(\varsigma) + \psi_2(\varsigma) + \dots + \psi_n(\varsigma)\right)^\beta d\varsigma,$$

where $\Lambda(\varsigma) := \int_a^\varsigma \delta(\zeta) ds$ and $F_k(\varsigma) = \int_a^\varsigma \delta(\zeta)\psi_k(\zeta) ds$ for $k = 1, 2, \dots, n$.

Corollary 17. *If $\mathbb{T} = \mathbb{Z}$, in Corollary 14, then we obtain the discrete version of inequality (22) as follows:*

$$\sum_{\varsigma=a}^\infty \delta(\varsigma) \frac{\left(F_1(\varsigma+1) F_2(\varsigma+1) \dots F_n(\varsigma+1)\right)^{\beta/n}}{\Lambda^\gamma(\varsigma+1)}$$
$$\leq \left(\frac{\beta}{nq - n}\right)^\beta \sum_{\varsigma=a}^\infty \frac{\delta(\varsigma) \left(\Lambda(\varsigma+1)\right)^{\gamma(\beta-1)}}{\Lambda^{\beta(\gamma-1)}(\varsigma)} \left(\psi_1(\varsigma) + \psi_2(\varsigma) + \dots + \psi_n(\varsigma)\right)^\beta,$$

where $\Lambda(\varsigma) := \sum_{\zeta=a}^{\varsigma-1} \delta(\zeta)$ and $F_k(\varsigma) = \sum_{\zeta=a}^{\varsigma-1} \delta(\zeta)\psi_k(\zeta)$ for $k = 1, 2, \dots, n$.

3. Conclusions

In this work, with the help of Jensen inequality and Fubini's theorem on time scales, we proved a number of new diamond-α Hardy-type inequalities on time scales. Furthermore, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus. As future work, we will try to develop the right hand side of Inequality (20) to obtain a simple form with a sharp constant by using diamond-α calculus. Furthermore, Saker et al. [20], discussed this term and calculated a sharp inequality (see Inequality (7) above) by using delta calculus. Our question now is: What if we use diamond-α calculi? This is an open question waiting for an affirmative answer.

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