symmetry MDPI

Article

Some New Generalizations of Reverse Hilbert-Type Inequalities via Supermultiplicative Functions

Haytham M. Rezk [1,*], Ahmed I. Saied [2], Ghada AlNemer [3] and Mohammed Zakarya [4,5]

1 Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Egypt
2 Department of Mathematics, Faculty of Science, Benha University, Benha 13511, Egypt
3 Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
4 Department of Mathematics, College of Science, King Khalid University, P.O. Box 9004, Abha 61413, Saudi Arabia
5 Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
* Correspondence: haythamrezk@azhar.edu.eg

Abstract: Our work in this paper is based on the reverse Hölder-type dynamic inequalities illustrated by El-Deeb in 2018 and the reverse Hilbert-type dynamic inequalities illustrated by Rezk in 2021 and 2022. With the help of Specht's ratio, the concept of supermultiplicative functions, chain rule, and Jensen's inequality on time scales, we can establish some comprehensive and generalize a number of classical reverse Hilbert-type inequalities to a general time scale space. In time scale calculus, results are unified and extended. At the same time, the theory of time scale calculus is applied to unify discrete and continuous analysis and to combine them in one comprehensive form. This hybrid theory is also widely applied on symmetrical properties which play an essential role in determining the correct methods to solve inequalities. As a special case of our results when the supermultiplicative function represents the identity map, we obtain some results that have been recently published.

Keywords: reverse Hilbert-type inequalities; Specht's ratio; time scales; reverse Hölder inequalities

MSC: 26D10; 26D15; 34N05; 47B38; 39A12

check for updates

Citation: Rezk, H.M.; Saied, A.I.; ALNemer, G.; Zakarya, M. Some New Generalizations of Reverse Hilbert-Type Inequalities via Supermultiplicative Functions. *Symmetry* **2022**, *14*, 2043. https://doi.org/10.3390/sym14102043

Academic Editors: Aviv Gibali and Ioan Raşa

Received: 29 July 2022
Accepted: 23 September 2022
Published: 30 September 2022

1. Introduction

In [1] (p. 253), Hardy established that

$$\sum_{l=1}^{\infty}\sum_{r=1}^{\infty}\frac{z_r F_l}{r+l} \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{r=1}^{\infty} z_r^p\right)^{\frac{1}{p}}\left(\sum_{l=1}^{\infty} F_l^{\delta}\right)^{\frac{1}{\delta}}, \tag{1}$$

where $z_r, F_l \geq 0$ with $0 < \sum_{r=1}^{\infty} z_r^p < \infty$, $0 < \sum_{l=1}^{\infty} F_l^{\delta} < \infty$ and $p > 1$, $p^{-1} + \delta^{-1} = 1$. The continuous shape (see [2]) of (1) is called Hardy–Hilbert's inequality and given by

$$\int_0^{\infty}\int_0^{\infty}\frac{Z(\vartheta)F(y)}{\vartheta+y}d\vartheta dy \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^{\infty} Z^p(\vartheta)d\vartheta\right)^{\frac{1}{p}}\left(\int_0^{\infty} F^{\delta}(y)dy\right)^{\frac{1}{\delta}}, \tag{2}$$

where $p > 1$, $p^{-1} + \delta^{-1} = 1$ and Z, F are measurable nonnegative functions such that $0 < \int_0^{\infty} Z^p(\vartheta)d\vartheta < \infty$ and $0 < \int_0^{\infty} F^{\delta}(y)dy < \infty$. The constant $\frac{\pi}{\sin\frac{\pi}{p}}$ in (1) and (2) sharp. In particular, when $p = \delta = 2$, the inequality (2) is reduced to the classical Hilbert integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{Z(\vartheta)F(y)}{\vartheta+y}d\vartheta dy \leq \pi\left(\int_0^{\infty} Z^2(\vartheta)d\vartheta\right)^{\frac{1}{2}}\left(\int_0^{\infty} F^2(y)dy\right)^{\frac{1}{2}}.$$

In [2] (p. 253), the author proved the following extension of Hilber's double-series (1). Let $p, \delta > 1$, $p^{-1} + \delta^{-1} \geq 1$ and $0 < \gamma = 2 - (p^{-1} + \delta^{-1}) = p^{-1} + \delta^{-1} \leq 1$. Then,

$$\sum_{l=1}^{\infty} \sum_{r=1}^{\infty} \frac{z_r b_l}{(r+l)^{\gamma}} \leq K(p,\delta) \left(\sum_{r=1}^{\infty} z_r^p \right)^{\frac{1}{p}} \left(\sum_{l=1}^{\infty} b_l^{\delta} \right)^{\frac{1}{\delta}}, \tag{3}$$

The following continuous shape of (3) is also given in [2] (p. 254). Under the same condition with (3), we have

$$\int_0^{\infty} \int_0^{\infty} \frac{Z(\vartheta)F(y)}{(\vartheta + y)^{\gamma}} d\vartheta dy \leq K(p,\delta) \left(\int_0^{\infty} Z^p(\vartheta) d\vartheta \right)^{\frac{1}{p}} \left(\int_0^{\infty} F^{\delta}(y) dy \right)^{\frac{1}{\delta}}, \tag{4}$$

where $K(p,\delta)$ in (3) and (4) depends on p and δ only.

As we all know, the classic Hölder inequality plays a very important and basic role in many areas of pure and applied mathematics. It is also a bridge to help solve problems in depth. In [3], Hölder established that

$$\sum_{l=1}^{r} \zeta_l y_l \leq \left(\sum_{l=1}^{r} \zeta_l^{\mu} \right)^{\frac{1}{\mu}} \left(\sum_{l=1}^{r} y_l^{\beta} \right)^{\frac{1}{\beta}}, \tag{5}$$

where $\zeta_l \geq 0$, $y_l \geq 0$ $(l = 1, 2, ..., r)$, $\mu \geq \beta > 0$ and $\mu^{-1} + \beta^{-1} = 1$. The continuous shape of (5) is

$$\int_z^r \phi(x)\varpi(x)dx \leq \left(\int_z^r \phi^{\mu}(x)dx \right)^{\frac{1}{\mu}} \left(\int_z^r \varpi^{\beta}(x)dx \right)^{\frac{1}{\beta}}, \tag{6}$$

where μ, $\beta > 1$ s.t $\mu^{-1} + \beta^{-1} = 1$ and $\phi, \varpi \in C((z,r), \mathbb{R}^+)$.

In [4], the researchers proved that, if $\phi(x)$ and $\varpi(x)$ are nonnegative continuous functions on $[z,r]$, then

$$\left(\int_z^r \phi^{\mu}(x)dx \right)^{\frac{1}{\mu}} \left(\int_z^r \varpi^{\beta}(x)dx \right)^{\frac{1}{\beta}} \leq \int_z^r S\left(\frac{\Upsilon \phi^{\mu}(x)}{\vartheta \varpi^{\beta}(x)} \right) \phi(x)\varpi(x)dx, \tag{7}$$

with

$$\vartheta = \int_z^r \phi^{\mu}(x)dx, \ \Upsilon = \int_z^r \varpi^{\beta}(x)dx, \ \alpha > 1 \text{ and } \mu^{-1} + \beta^{-1} = 1,$$

where $S(.)$ is the Specht's ratio function ([5]) and defined by

$$S(t) := \frac{t^{1/(t-1)}}{e \log t^{1/(t-1)}}, \ t \neq 1, \quad S(1) = 1.$$

In [4], the researchers established that, if $\psi, \varpi \in C((z,r), \mathbb{R}^+)$ and $q > 0$, then

$$\int_z^r \frac{\psi^{q+1}(x)}{\varpi^q(x)} dx \leq \frac{\left(\int_z^r S\left(\frac{G\psi^{q+1}(x)}{F\varpi^{q+1}(x)} \right) \psi(x)dx \right)^{q+1}}{\left(\int_z^r \varpi(x)dx \right)^q}, \tag{8}$$

where

$$G = \int_z^r \varpi(x)dx \text{ and } F = \int_z^r \frac{\psi^{q+1}(x)}{\varpi^q(x)} dx.$$

In addition, they established the discrete form of (8) as follows:

$$\sum_{i=1}^{n} \frac{z_i^{m+1}}{b_i^m} \leq \frac{\sum_{i=1}^{n} S\left(\frac{Bz_i^{m+1}}{Ab_i^{m+1}}\right) z_i}{\left(\sum_{i=1}^{n} b_i\right)^m}, \tag{9}$$

where $B = \sum_{i=1}^{n} b_i$ and $A = \sum_{i=1}^{n} z_i^{m+1}/b_i^m$.

In [6], the researchers proved that, if $0 < p, \delta \leq 1$, and $\{\lambda_i\}_{i=1}^{k}$, $\{\omega_j\}_{j=1}^{r}$ are nonnegative and decreasing sequences of real numbers with $k, r \in \mathbb{N}$, then

$$\begin{aligned} &\sum_{i=1}^{k} \sum_{j=1}^{r} \frac{S_{p,\delta,k,r,i,j} \left(\sum_{\mu=1}^{i} \lambda_\mu\right)^p \left(\sum_{t=1}^{j} \psi_t\right)^\delta}{(ij)^{\frac{1}{2}}} \\ &\geq p\delta (kr)^{\frac{1}{2}} \left(\sum_{i=1}^{k} \left[\lambda_i \left(\sum_{\mu=1}^{i} \lambda_\mu\right)^{p-1}\right]^2 (k-i+1)\right)^{\frac{1}{2}} \\ &\times \left(\sum_{j=1}^{r} \left[\psi_j \left(\sum_{t=1}^{j} \psi_t\right)^{\delta-1}\right]^2 (r-j+1)\right)^{\frac{1}{2}}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} S_{p,\delta,k,r,i,j} = {} & S\left(\frac{k\sum_{\mu=1}^{i}\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}{\sum_{\mu=1}^{k}(k-\mu+1)\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}\right) S\left(\frac{r\sum_{t=1}^{j}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}{\sum_{t=1}^{r}(r-t+1)\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}\right) \\ & \times S\left(\frac{i\left[\lambda_u\left(\sum_{\tau=1}^{u}\lambda_\tau\right)^{p-1}\right]^2}{\sum_{\mu=1}^{i}\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}\right) S\left(\frac{j\left[\psi_v\left(\sum_{\tau=1}^{v}\psi_\tau\right)^{\delta-1}\right]^2}{\sum_{t=1}^{j}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}\right), \end{aligned}$$

$$\begin{aligned} S\left(\frac{i\left[\lambda_u\left(\sum_{\tau=1}^{u}\lambda_\tau\right)^{p-1}\right]^2}{\sum_{\mu=1}^{i}\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}\right) = \max\Bigg\{ & S\left(\frac{i\left[\lambda_1\left(\sum_{\tau=1}^{1}\lambda_\tau\right)^{p-1}\right]^2}{\sum_{\mu=1}^{i}\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}\right) \\ & ; S\left(\frac{i\left[\lambda_i\left(\sum_{\tau=1}^{i}\lambda_\tau\right)^{p-1}\right]^2}{\sum_{\mu=1}^{i}\left[\lambda_\mu\left(\sum_{\tau=1}^{\mu}\lambda_\tau\right)^{p-1}\right]^2}\right)\Bigg\}, \end{aligned}$$

$$\begin{aligned} S\left(\frac{j\left[\psi_v\left(\sum_{\tau=1}^{v}\psi_\tau\right)^{\delta-1}\right]^2}{\sum_{t=1}^{j}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}\right) = \max\Bigg\{ & S\left(\frac{j\left[\psi_1\left(\sum_{\tau=1}^{1}\psi_\tau\right)^{\delta-1}\right]^2}{\sum_{t=1}^{j}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}\right) \\ & ; S\left(\frac{j\left[\psi_j\left(\sum_{\tau=1}^{j}\psi_\tau\right)^{\delta-1}\right]^2}{\sum_{t=1}^{j}\left[\psi_t\left(\sum_{\tau=1}^{t}\psi_\tau\right)^{\delta-1}\right]^2}\right)\Bigg\}. \end{aligned}$$

In addition, they proved that

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{S_{k,r,i,j}\phi(\Lambda_i)\psi(\Omega_j)}{(ij)^{\frac{1}{2}}} \\
&\geq \left(\sum_{i=1}^{k}\left(\frac{\phi(P_i)}{P_i}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{r}\left(\frac{\psi(W_j)}{W_j}\right)^2\right)^{\frac{1}{2}} \\
&\times \left(\sum_{\mu=1}^{k}\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2(k-\mu+1)\right)^{\frac{1}{2}} \\
&\times \left(\sum_{t=1}^{r}\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2(r-t+1)\right)^{\frac{1}{2}},
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
S_{k,r,i,j} &= S\left(\frac{\left(\sum_{\mu=1}^{k}\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2(k-\mu+1)\right)\left(\frac{\phi(P_i)}{P_i}\right)^2}{\left(\sum_{i=1}^{k}\left(\frac{\phi(P_i)}{P_i}\right)^2\right)\left(\sum_{\mu=1}^{i}\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2\right)}\right) \\
&\times S\left(\frac{\left(\sum_{t=1}^{r}\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2(r-t+1)\right)\left(\frac{\psi(W_j)}{W_j}\right)^2}{\left(\sum_{j=1}^{r}\left(\frac{\psi(W_j)}{W_j}\right)^2\right)\left(\sum_{t=1}^{j}\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2\right)}\right),
\end{aligned}
$$

$$
\Lambda_i = \sum_{\mu=1}^{i} S\left(\frac{i\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2}{\sum_{\mu=1}^{i}\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2}\right)\lambda_s,
$$

$$
\Omega_j = \sum_{t=1}^{j} S\left(\frac{j\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2}{\sum_{t=1}^{j}\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2}\right)\omega_t,
$$

$$
P_i = \sum_{\mu=1}^{i} S\left(\frac{i\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2}{\sum_{\mu=1}^{i}\left[p_\mu\phi\left(\frac{\lambda_\mu}{p_\mu}\right)\right]^2}\right)p_s,
$$

$$
W_j = \sum_{t=1}^{j} S\left(\frac{j\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2}{\sum_{t=1}^{j}\left[\delta_t\psi\left(\frac{\omega_t}{\delta_t}\right)\right]^2}\right)\delta_t,
$$

and $\{\lambda_i\}_{i=1}^{k}$, $\{\omega_j\}_{j=1}^{r}$ are nonnegative sequences with k, $r \in \mathbb{N}$, $\{p_i\}$, $\{\delta_j\}$ are positive sequences ϕ, ψ are nonnegative, concave and supermultiplicative functions.

In [6], the authors proved that

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{r}\frac{S_{k,r,i,j}\Lambda_i\Omega_j}{(ij)^{\frac{1}{2}}} \\
&\geq (kr)^{\frac{1}{2}}\left(\sum_{i=1}^{k}\lambda_i^2(k-i+1)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{r}\omega_j^2(r-j+1)\right)^{\frac{1}{2}},
\end{aligned}
\tag{12}
$$

where

$$S_{k,r,i,j} = S\left(\frac{\sum_{\mu=1}^{k} \lambda_\mu^2 (k-\mu+1)}{k\left(\sum_{\mu=1}^{i} \lambda_\mu^2\right)}\right) S\left(\frac{\sum_{t=1}^{r} \omega_t^2 (r-t+1)}{r\left(\sum_{t=1}^{j} \omega_t^2\right)}\right),$$

$$\Lambda_i = \sum_{\mu=1}^{i} S\left(\frac{i\lambda_\mu^2}{\sum_{\mu=1}^{i} \lambda_\mu^2}\right)\lambda_\mu \quad \text{and} \quad \Omega_j = \sum_{t=1}^{j} S\left(\frac{j\omega_t^2}{\sum_{t=1}^{j} \omega_t^2}\right)\omega_t.$$

For some generalizations and extensions of reversed inequalities of Hilbert-type and Hölder-type on time scales, see ([7–14]).

The primary objective of this article is to develop some new generalisations of reverse Hilbert-type inequalities via supermultiplicative functions by using reverse Hölder inequalities with Specht's ratio on $\mathbb{T}$ (a time scale $\mathbb{T}$ is defined as an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$).

The structure of the paper is summarised below. Section 2 covers some of the fundamentals of time scale theory as well as several time scale lemmas that will be useful in Section 3, where we prove our findings. As particular examples (when $\mathbb{T} = \mathbb{N}$), our major findings are (10), as demonstrated by Zhao and Cheung [6].

2. Preliminaries

The forward jump operator is defined as

$$\sigma(c) := \inf\{u \in \mathbb{T} : u > c\}.$$

The set of all such rd-continuous functions is denoted by the space $C_{rd}(\mathbb{T}, \mathbb{R})$, and for any function $Z : \mathbb{T} \to \mathbb{R}$, the notation $Z^\sigma(c)$ denotes $Z(\sigma(c))$.

The derivatives of $Z\Omega$ and Z/Ω of two differentiable functions Z and Ω are given by

$$(Z\Omega)^\Delta = Z^\Delta\Omega + Z^\sigma\Omega^\Delta = Z\Omega^\Delta + Z^\Delta\Omega^\sigma, \quad \left(\frac{Z}{\Omega}\right)^\Delta = \frac{Z^\Delta\Omega - Z\Omega^\Delta}{\Omega\Omega^\sigma}, \quad \Omega\Omega^\sigma \neq 0. \tag{13}$$

The integration by parts formula on $\mathbb{T}$ is

$$\int_{v_0}^{v} \lambda(x) Z^\Delta(x)\Delta x = [\lambda(x)Z(x)]_{v_0}^{v} - \int_{v_0}^{v} \lambda^\Delta(x) Z^\sigma(x)\Delta x. \tag{14}$$

The time scales chain rule ([10] (Theorem 1.87)) is

$$(\Omega \circ Z)^\Delta(\tau) = \Omega'(Z(c))Z^\Delta(x), \text{ where } c \in [x, \sigma(x)], \tag{15}$$

where $\Omega : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and $\mathbb{Z} : \mathbb{T} \to \mathbb{R}$ is $\Delta-$differentiable. More information on time scale calculus can be found at ([10,11]).

Now, we will give some properties of multiplicative and supermultiplicative functions.

Definition 1. *A function $L : I \to \mathbb{R}^+$ is multiplicative if*

$$L(\varkappa\zeta) = L(\varkappa)L(\zeta), \qquad \forall \varkappa, \zeta \in I \subset \mathbb{R}. \tag{16}$$

Definition 2 ([15])**.** *A function $L : I \to \mathbb{R}^+$ is supermultiplicative if*

$$L(\varkappa\zeta) \geq L(\varkappa)L(\zeta), \qquad \forall \varkappa, \zeta \in I \subset \mathbb{R}. \tag{17}$$

where L is the identity map (i.e., $L(\zeta) = \zeta$) and represents the multiplicative function. L is said to be a submultiplicative function if the last inequality has the opposite sign.

Lemma 1. *Let $z \in \mathbb{T}$, $\lambda \in C_{rd}(\mathbb{T}, \mathbb{R})$ be nonnegative and $0 \le \gamma \le 1$. Then,*

$$\left(\int_z^{\sigma(t)} \lambda(\tau)\Delta\tau \right)^{\gamma} \ge \gamma \int_z^{\sigma(t)} \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta. \tag{18}$$

Proof. By using (15) on $\int_z^{\vartheta} \lambda(\tau)\Delta\tau$, we obtain

$$\left[\left(\int_z^{\vartheta} \lambda(\tau)\Delta\tau \right)^{\gamma} \right]^{\Delta} = \gamma \left(\int_z^{\zeta} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \lambda(\vartheta), \quad \zeta \in [\vartheta, \sigma(\vartheta)]. \tag{19}$$

Since $\zeta \le \sigma(\vartheta)$, then we obtain (note $0 \le \gamma \le 1$) that

$$\left(\int_z^{\zeta} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \ge \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{\gamma-1}. \tag{20}$$

By substituting (20) into (19), we can observe that

$$\left[\left(\int_z^{\vartheta} \lambda(\tau)\Delta\tau \right)^{\gamma} \right]^{\Delta} \ge \gamma \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \lambda(\vartheta). \tag{21}$$

By integrating (21) from z to $\sigma(t)$, we obtain

$$\int_z^{\sigma(t)} \left[\left(\int_z^{\vartheta} \lambda(\tau)\Delta\tau \right)^{\gamma} \right]^{\Delta} \Delta\vartheta \ge \gamma \int_z^{\sigma(t)} \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta.$$

i.e.,

$$\left(\int_z^{\sigma(t)} \lambda(\tau)\Delta\tau \right)^{\gamma} \ge \gamma \int_z^{\sigma(t)} \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{\gamma-1} \lambda(\vartheta)\Delta\vartheta,$$

which is (18). □

Lemma 2 (Specht's ratio [5]). *Let c, d be positive numbers, $\gamma > 1$ and $1/\gamma + 1/\delta = 1$. Then,*

$$S\left(\frac{c}{d}\right) c^{1/\gamma} d^{1/\delta} \ge \frac{c}{\gamma} + \frac{d}{\delta},$$

where

$$S(l) = \frac{l^{1/(l-1)}}{e \log l^{1/(l-1)}}, l \ne 1, \ \ S(1) = 1.$$

Lemma 3 ([5]). *Let $S(.)$ be defined as in Lemma 2. Then, $S(l)$ is strictly decreasing for $0 < l < 1$ and strictly increasing for $l > 1$. In addition, the following equations are true:*

$$S(1) = 1 \text{ and } S(l) = S\left(\frac{1}{l}\right) \ \forall l > 0.$$

Lemma 4 ([12, when $\alpha = 1$]). *Let $g, h \in C([z, w]_{\mathbb{T}}, \mathbb{R}^+)$ s.t. g^{β}, h^{ν} be $\Delta-$integrable on $[z, w]_{\mathbb{T}}$. If $\mu > 1$ and $1/\mu + 1/\nu = 1$, then*

$$\begin{aligned} &\int_z^w S\left(\frac{Y g^{\mu}(\zeta)}{X h^{\nu}(\zeta)}\right) g(\zeta)h(\zeta)\Delta\zeta \\ \ge\ & \left(\int_z^w g^{\mu}(\zeta)\Delta\zeta \right)^{\frac{1}{\mu}} \left(\int_z^w h^{\nu}(\zeta)\Delta\zeta \right)^{\frac{1}{\nu}}, \end{aligned} \tag{22}$$

where $X = \int_z^w g^{\mu}(\zeta)\Delta\zeta$ and $Y = \int_z^w h^{\nu}(\zeta)\Delta\zeta$.

Lemma 5 (Jensen's inequality)**.** *Assume that $\zeta_0, \zeta \in \mathbb{T}$ and $r_0, r \in \mathbb{R}$. If $\lambda \in C_{rd}([\zeta_0, \zeta]_{\mathbb{T}}, \mathbb{R})$, $\varphi \in C_{rd}([\zeta_0, \zeta]_{\mathbb{T}}, (r_0, r))$ and $\Psi : (r_0, r) \to \mathbb{R}$ is continuous and convex, then*

$$\Psi\left(\frac{1}{\int_{\zeta_0}^{\zeta} \lambda(\tau)\Delta\tau} \int_{\zeta_0}^{\zeta} \lambda(\tau)\varphi(\tau)\Delta\tau\right) \leq \frac{1}{\int_{\zeta_0}^{\zeta} \lambda(\tau)\Delta\tau} \int_{\zeta_0}^{\zeta} \lambda(\tau)\Psi(\varphi(\tau))\Delta\tau. \quad (23)$$

The inequality (23) is reversed when Ψ is continuous and concave.

Lemma 6. *Let $z \in \mathbb{T}$, λ, ψ be positive and decreasing functions, f, g are positive and nondecreasing functions and $0 \leq p, \delta \leq 1$. Furthermore, assume that ϕ, φ are positive, increasing, concave and supermultiplicative functions. If $\beta > 1$, $\nu > 1$ with $1/\beta + 1/\nu = 1$, then*

$$\begin{aligned} & S\left(\frac{\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)\phi^\beta\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\zeta)\int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\ = \ & \max\left\{ S\left(\frac{\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)\phi^\beta\left[\lambda(z)\left(\int_z^{\sigma(z)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(z)\int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right. \\ & \left. ; S\left(\frac{\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)\phi^\beta\left[\lambda(t)\left(\int_z^{\sigma(t)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(t)\int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right\}, \end{aligned} \quad (24)$$

and

$$\begin{aligned} & S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(\eta)\left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(\eta)\int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right) \\ = \ & \max\left\{ S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(z)\left(\int_z^{\sigma(z)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(z)\int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\right. \\ & \left. ; S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(\xi)\left(\int_z^{\sigma(\xi)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(\xi)\int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\right\}. \end{aligned} \quad (25)$$

Proof. For $\vartheta \leq y$, we have

$$\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \leq \int_z^{\sigma(y)} \lambda(\tau)\Delta\tau,$$

and then (where $0 \leq p \leq 1$)

$$\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1} \geq \left(\int_z^{\sigma(y)} \lambda(\tau)\Delta\tau\right)^{p-1}. \quad (26)$$

Because λ is decreasing and $\vartheta \leq y$, we can deduce from (26) that

$$\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1} \geq \lambda(y)\left(\int_z^{\sigma(y)} \lambda(\tau)\Delta\tau\right)^{p-1}. \quad (27)$$

Based on the knowledge $\beta > 1$, ϕ is an increasing function and (27), we can conclude that

$$\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right] \geq \phi^{\beta}\left[\lambda(y)\left(\int_z^{\sigma(y)}\lambda(\tau)\Delta\tau\right)^{p-1}\right].$$

Then, we obtain (where $\vartheta \leq y$ and f is nondecreasing) that

$$\begin{aligned}&\frac{1}{f^{\nu}(\vartheta)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\\ &\geq \frac{1}{f^{\nu}(y)}\phi^{\beta}\left[\lambda(y)\left(\int_z^{\sigma(y)}\lambda(\tau)\Delta\tau\right)^{p-1}\right],\end{aligned}$$

thus the function $\frac{1}{f^{\nu}(\vartheta)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]$ is decreasing. Therefore, we have for $z \leq \vartheta$ that

$$\begin{aligned}&\frac{1}{f^{\nu}(z)}\phi^{\beta}\left[\lambda(z)\left(\int_z^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\\ \geq\ &\frac{1}{f^{\nu}(\vartheta)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\end{aligned}$$

and then

$$\begin{aligned}&f^{\nu}(\vartheta)\phi^{\beta}\left[\lambda(z)\left(\int_z^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\\ &\geq f^{\nu}(z)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right].\end{aligned} \tag{28}$$

Integrating (28) over ϑ from z to $\sigma(t)$, we obtain

$$\begin{aligned}&\phi^{\beta}\left[\lambda(z)\left(\int_z^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\\ &\geq f^{\nu}(z)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta,\end{aligned}$$

and then

$$\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(z)\left(\int_z^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(z)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta} \geq 1. \tag{29}$$

Since the function $\frac{1}{f^{\nu}(\vartheta)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]$ is decreasing and $\vartheta \leq t$, we obtain

$$\begin{aligned}&\frac{1}{f^{\nu}(\vartheta)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\\ &\geq \frac{1}{f^{\nu}(t)}\phi^{\beta}\left[\lambda(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right],\end{aligned}$$

and then

$$f^{\nu}(t)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\geq f^{\nu}(\vartheta)\phi^{\beta}\left[\lambda(t)\left(\int_{z}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]. \tag{30}$$

Integrating (30) over ϑ from z to $\sigma(t)$, we obtain

$$\begin{aligned}&f^{\nu}(t)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta\\&\geq\phi^{\beta}\left[\lambda(t)\left(\int_{z}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta,\end{aligned}$$

thus

$$\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(t)\left(\int_{z}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\leq 1. \tag{31}$$

Based on (29) and (31), we can see that

$$\begin{aligned}&\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(z)\left(\int_{z}^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(z)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\geq\ldots\geq 1\\&\geq\ldots\geq\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(t)\left(\int_{z}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}.\end{aligned}$$

Because $S(.)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$, we have that one of

$$S\left(\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(z)\left(\int_{z}^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(z)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)$$

and

$$S\left(\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(t)\left(\int_{z}^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)$$

is maximum (where $S(1)=1$), and it takes the shape

$$\begin{aligned}&S\left(\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_{z}^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\\&=\max\left\{S\left(\frac{\left(\int_{z}^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(z)\left(\int_{z}^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(z)\int_{z}^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_{z}^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.\end{aligned}$$

$$;S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\Bigg\},$$

and that is (24). In a similar manner, for ψ and $0 \le \delta \le 1$, we obtain

$$\begin{aligned}
&S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\\
&=\max\Bigg\{S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(z)\left(\int_z^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(z)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right);\\
&S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\xi)\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\Bigg\},
\end{aligned}$$

which is (25). □

Throughout the article, we will assume that the functions are nonnegative rd-continuous functions on $[z,\infty)_{\mathbb{T}} := [z,\infty)\cap\mathbb{T}$.

3. Principal Findings

Theorem 1. *Let $z \in \mathbb{T}$, $0 \le p, \delta \le 1$, λ, ψ be positive and decreasing functions and ϕ, φ are positive, increasing, concave and supermultiplicative functions. If f, g are positive and nondecreasing functions and $\beta > 1$, $\nu > 1$ with $1/\beta + 1/\nu = 1$, then*

$$\begin{aligned}
&\int_z^{\sigma(s)}\int_z^{\sigma(r)}\Phi(t,\xi)P(t)W(\xi)\\
&\times\phi\left[\frac{1}{P(t)}S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.\\
&\left.\times\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p}f(t)\right]\\
&\times\varphi\left[\frac{1}{W(\xi)}S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\right.\\
&\left.\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta}g(\xi)\right]\Delta t\Delta\xi\\
&\ge \nu C(p,\delta,r,s,\nu)\left(\int_z^{\sigma(r)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}}\\
&\times\left(\int_z^{\sigma(s)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}}, \qquad (32)
\end{aligned}$$

holds for all $r, s \in [z, \infty)_{\mathbb{T}}$, *where*

$$\begin{aligned}\Phi(t,\xi) &= \frac{f(t)g(\xi)}{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}}\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)^{\frac{1}{\nu}}} \\ &\times S\left(\frac{\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right)\int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_z^{\sigma(r)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta}\right) \\ &\times S\left(\frac{\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right)\int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi)\int_z^{\sigma(s)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y}\right),\end{aligned}$$

with

$$C(p,\delta,r,s,\nu) = \frac{\phi(p)\varphi(\delta)}{\nu}\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}}\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}},$$

$$P(t) = \int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\vartheta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) f(\vartheta)\Delta\vartheta,$$

$$W(\xi) = \int_z^{\sigma(\xi)} S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(y)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right) g(y)\Delta y,$$

such that

$$\begin{aligned}&S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\ &= \max\left\{S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(z)\left(\int_z^{\sigma(z)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(z)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right. \\ &\left.;S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right\},\end{aligned}$$

and

$$\begin{aligned}&S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right) \\ &= \max\left\{S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(z)\left(\int_z^{\sigma(z)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(z)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right);\right.\end{aligned}$$

$$S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\xi)\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\Bigg\}.$$

Proof. Denote

$$\begin{aligned}
&\Lambda(t,\xi)\\
&= P(t)\phi\left[S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.\\
&\left.\times\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} f(t)\right]\\
&\times W(\xi)\varphi\left[S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\right.\\
&\left.\times\frac{1}{W(\xi)}\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta} g(\xi)\right],
\end{aligned}$$

and

$$\begin{aligned}
\Omega(t,\xi) = {}& P(t)\phi\left[S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.\\
&\left.\times\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} f(t)\right]\\
&\times W(\xi)\varphi\left[S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\right.\\
&\left.\times\frac{1}{W(\xi)}\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta} g(\xi)\right].
\end{aligned}$$

Applying (18) with $\gamma = p$ gives us

$$\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} \geq p\int_z^{\sigma(t)}\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\Delta\vartheta. \tag{33}$$

By multiplying the previous inequality by

$$S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)f(t),$$

we have

$$\begin{aligned}
& S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} \\
\geq \quad & p\int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(t)\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\Delta\vartheta.
\end{aligned}$$

Using the fact that f is nondecreasing and $t \leq \vartheta$, we obtain

$$\begin{aligned}
& S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} \\
& \geq p\int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(\vartheta)\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\Delta\vartheta.
\end{aligned} \tag{34}$$

By Lemma 6, inequality (34) is

$$\begin{aligned}
& S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(t)\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} \\
& \geq p\int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\vartheta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right) \\
& \times f(\vartheta\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1})\Delta\vartheta.
\end{aligned} \tag{35}$$

Likewise, for the decreasing function ψ, the nondecreasing function g and $0 \leq \delta \leq 1$, we obtain

$$S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)$$

$$\times\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta}g(\xi)$$
$$\geq\delta\int_z^{\sigma(\xi)}S\left(\frac{\left(\int_z^{\sigma(\xi)}g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(y)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)$$
$$\times\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}g(y)\Delta y. \tag{36}$$

From (35), we deduce that

$$\phi\left[S\left(\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.$$
$$\left.\times\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p}f(t)\right]$$
$$\geq\ \phi\left[\frac{p}{P(t)}\int_z^{\sigma(t)}S\left(\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\vartheta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.$$
$$\left.\times\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}f(\vartheta)\Delta\vartheta\right].$$

Since ϕ is a positive, increasing and super-multiplicative function, we have

$$\phi\left[S\left(\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.$$
$$\left.\times\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p}f(t)\right]$$
$$\geq\phi(p)\phi\left[\frac{1}{P(t)}\int_z^{\sigma(t)}S\left(\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\vartheta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.$$
$$\left.\times\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}f(\vartheta)\Delta\vartheta\right]. \tag{37}$$

Then, by using the Jensen inequality on the right-hand side of (37) (where ϕ is a concave function), we obtain

$$\phi\left[S\left(\frac{\left(\int_z^{\sigma(t)}f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\right.$$
$$\left.\times\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p}f(t)\right]$$

$$\geq \frac{\phi(p)}{P(t)} \int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)\phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\vartheta) \int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta} \right)$$
$$\times f(\vartheta)\phi\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta. \tag{38}$$

Analogously, in the case of (36), we can see (where φ is a positive, increasing, concave and supermultiplicative function) that

$$\varphi\left[S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(\eta)\left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(\eta) \int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y} \right)\right.$$
$$\left.\times \frac{1}{W(\xi)}\left(\int_z^{\sigma(\xi)} \psi(\tau)\Delta\tau\right)^{\delta} g(\xi)\right]$$
$$\geq \frac{\varphi(\delta)}{W(\xi)} \int_z^{\sigma(\xi)} S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(y) \int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y} \right)$$
$$\times g(y)\varphi\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y. \tag{39}$$

Multiplying (38) and (39), we obtain

$$\Lambda(t,\xi) \geq \phi(p) \int_z^{\sigma(t)} S\left(\frac{\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)\phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^\nu(\vartheta) \int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta} \right)$$
$$\times f(\vartheta)\phi\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta$$
$$\times \varphi(\delta) \int_z^{\sigma(\xi)} S\left(\frac{\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)\varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^\nu(y) \int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y} \right)$$
$$\times g(y)\varphi\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y. \tag{40}$$

Applying (22) on the right-hand side of (40), we obtain

$$\Omega(t,\xi) \geq \phi(p)\left(\int_z^{\sigma(t)} f^\nu(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}} \left(\int_z^{\sigma(t)} \phi^\beta\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta\right)^{\frac{1}{\beta}}$$
$$\times \varphi(\delta)\left(\int_z^{\sigma(\xi)} g^\nu(y)\Delta y\right)^{\frac{1}{\nu}} \left(\int_z^{\sigma(\xi)} \varphi^\beta\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y\right)^{\frac{1}{\beta}}. \tag{41}$$

Multiplying (41) by

$$\Phi(t,\xi) = \frac{f(t)g(\xi)}{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}} \left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)^{\frac{1}{\nu}}} \times S\left(\frac{\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t) \int_z^{\sigma(r)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta}\right) \times S\left(\frac{\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi) \int_z^{\sigma(s)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y}\right),$$

we have

$$\begin{aligned} \Phi(t,\xi)\Omega(t,\xi) \geq\; & \phi(p)\varphi(\delta)\left(\int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta\right)^{\frac{1}{\beta}} f(t) \\ & \times S\left(\frac{\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t) \int_z^{\sigma(r)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta}\right) \\ & \times \left(\int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y\right)^{\frac{1}{\beta}} g(\xi) \\ & \times S\left(\frac{\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi) \int_z^{\sigma(s)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y}\right). \end{aligned}$$

Using the integration over t from z to $\sigma(r)$ and the integration over ξ from z to $\sigma(s)$, respectively, we arrive at

$$\begin{aligned} & \int_z^{\sigma(s)} \int_z^{\sigma(r)} \Phi(t,\xi)\Omega(t,\xi)\Delta t\Delta\xi \\ \geq\; & \phi(p)\varphi(\delta) \int_z^{\sigma(r)} \left(\int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta\right)^{\frac{1}{\beta}} f(t) \\ & \times S\left(\frac{\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(t)} \phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(t) \int_z^{\sigma(r)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta}\right)\Delta t \\ & \times \int_z^{\sigma(s)} \left(\int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y\right)^{\frac{1}{\beta}} g(\xi) \\ & \times S\left(\frac{\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right) \int_z^{\sigma(\xi)} \varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\xi) \int_z^{\sigma(s)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y}\right)\Delta\xi. \end{aligned} \tag{42}$$

Applying Formula (14) on the term

$$\int_z^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r)-\vartheta)\Delta\vartheta,$$

with $u(\vartheta) = (\sigma(r)-\vartheta)$ and $v^\Delta(\vartheta) = \phi^\beta \left[\lambda(\vartheta) \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]$, we obtain

$$\begin{aligned} &\int_z^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r)-\vartheta)\Delta\vartheta \\ &= (\sigma(r)-\vartheta)v(\vartheta)|_z^{\sigma(r)} + \int_z^{\sigma(r)} v^\sigma(\vartheta)\Delta\vartheta, \end{aligned}$$

where $v(\vartheta) = \int_z^\vartheta \left[\lambda(\theta) \left(\int_z^{\sigma(\theta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right]^\beta \Delta\theta$ and then (where $v(z) = 0$)

$$\begin{aligned} &\int_z^{\sigma(r)} \phi^\beta \left[\lambda(\vartheta) \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] (\sigma(r)-\vartheta)\Delta\vartheta \\ &= \int_z^{\sigma(r)} \int_z^{\sigma(\vartheta)} \phi^\beta \left[\lambda(\theta) \left(\int_z^{\sigma(\theta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\theta\Delta\vartheta. \end{aligned} \tag{43}$$

In a similar vein, we note

$$\begin{aligned} &\int_z^{\sigma(s)} \varphi^\beta \left[\psi(y) \left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{\delta-1} \right] (\sigma(s)-y)\Delta y \\ &= \int_z^{\sigma(s)} \int_z^{\sigma(y)} \varphi^\beta \left[\psi(\theta) \left(\int_z^{\sigma(\theta)} \psi(\tau)\Delta\tau \right)^{\delta-1} \right] \Delta\theta\Delta y. \end{aligned} \tag{44}$$

Substituting (43) and (44) into (42), we have

$$\begin{aligned} &\int_z^{\sigma(s)} \int_z^{\sigma(r)} \Phi(t,\xi)\Omega(t,\xi)\Delta t\Delta\xi \\ \geq\ & \phi(p)\varphi(\delta) \int_z^{\sigma(r)} \left(\int_z^{\sigma(t)} \phi^\beta \left[\lambda(\vartheta) \left(\int_z^{\sigma(\vartheta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\vartheta \right)^{\frac{1}{\beta}} f(t) \\ & \times S\left(\frac{\left(\int_z^{\sigma(r)} f^\nu(\vartheta)\Delta\vartheta \right) \int_z^{\sigma(t)} \phi^\beta \left[\lambda(\zeta) \left(\int_z^{\sigma(\zeta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\zeta}{f^\nu(t) \int_z^{\sigma(r)} \int_z^{\sigma(\vartheta)} \phi^\beta \left[\lambda(\theta) \left(\int_z^{\sigma(\theta)} \lambda(\tau)\Delta\tau \right)^{p-1} \right] \Delta\theta\Delta\vartheta} \right) \Delta t \\ & \times \int_z^{\sigma(s)} \left(\int_z^{\sigma(\xi)} \varphi^\beta \left[\psi(y) \left(\int_z^{\sigma(y)} \psi(\tau)\Delta\tau \right)^{\delta-1} \right] \Delta y \right)^{\frac{1}{\beta}} g(\xi) \\ & \times S\left(\frac{\left(\int_z^{\sigma(s)} g^\nu(\vartheta)\Delta\vartheta \right) \int_z^{\sigma(\xi)} \varphi^\beta \left[\psi(\eta) \left(\int_z^{\sigma(\eta)} \psi(\tau)\Delta\tau \right)^{\delta-1} \right]}{g^\nu(\xi) \int_z^{\sigma(s)} \int_z^{\sigma(y)} \varphi^\beta \left[\psi(\theta) \left(\int_z^{\sigma(\theta)} \psi(\tau)\Delta\tau \right)^{\delta-1} \right] \Delta\theta\Delta y} \right) \Delta\xi. \end{aligned}$$

Then, by applying (22) on the previous inequality, using (43) and (44), we obtain

$$\begin{aligned}
&\int_z^{\sigma(s)} \int_z^{\sigma(r)} \Phi(t,\xi)\Omega(t,\xi)\Delta t \Delta \xi \\
\geq\ & \phi(p)\varphi(\delta)\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}}\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}} \\
&\times\left(\int_z^{\sigma(r)}\int_z^{\sigma(\vartheta)} \phi^{\beta}\left[\lambda(\theta)\left(\int_z^{\sigma(\theta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\theta\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times\left(\int_z^{\sigma(s)}\int_z^{\sigma(y)} \varphi^{\beta}\left[\psi(\theta)\left(\int_z^{\sigma(\theta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta\theta\Delta y\right)^{\frac{1}{\beta}} \\
=\ & \phi(p)\varphi(\delta)\left(\int_z^{\sigma(r)} f^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}}\left(\int_z^{\sigma(s)} g^{\nu}(\vartheta)\Delta\vartheta\right)^{\frac{1}{\nu}} \\
&\times\left(\int_z^{\sigma(r)} \phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times\left(\int_z^{\sigma(s)} \varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}},
\end{aligned}$$

and thus we obtain

$$\begin{aligned}
&\int_z^{\sigma(s)} \int_z^{\sigma(r)} \Phi(t,\xi)P(t)W(\xi) \\
&\times \phi S\left(\frac{\left(\int_z^{\sigma(t)} f^{\nu}(\vartheta)\Delta\vartheta\right)\phi^{\beta}\left[\lambda(\zeta)\left(\int_z^{\sigma(\zeta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]}{f^{\nu}(\zeta)\int_z^{\sigma(t)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right]\Delta\vartheta}\right)\frac{1}{P(t)}\left(\int_z^{\sigma(t)}\lambda(\tau)\Delta\tau\right)^{p} f(t) \\
&\times \varphi S\left(\frac{\left(\int_z^{\sigma(\xi)} g^{\nu}(y)\Delta y\right)\varphi^{\beta}\left[\psi(\eta)\left(\int_z^{\sigma(\eta)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]}{g^{\nu}(\eta)\int_z^{\sigma(\xi)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right]\Delta y}\right)\frac{g(\xi)}{W(\xi)}\left(\int_z^{\sigma(\xi)}\psi(\tau)\Delta\tau\right)^{\delta}\Delta t\Delta\xi \\
&\geq \nu C(p,\delta,r,s,\nu)\left(\int_z^{\sigma(r)}\phi^{\beta}\left[\lambda(\vartheta)\left(\int_z^{\sigma(\vartheta)}\lambda(\tau)\Delta\tau\right)^{p-1}\right](\sigma(r)-\vartheta)\Delta\vartheta\right)^{\frac{1}{\beta}} \\
&\times\left(\int_z^{\sigma(s)}\varphi^{\beta}\left[\psi(y)\left(\int_z^{\sigma(y)}\psi(\tau)\Delta\tau\right)^{\delta-1}\right](\sigma(s)-y)\Delta y\right)^{\frac{1}{\beta}},
\end{aligned}$$

which is (32). The proof is complete. □

Remark 1. *As a special case of Theorem 1, when $\mathbb{T} = \mathbb{N}$, $\varphi(\vartheta) = \phi(\vartheta) = \vartheta$, $f(\vartheta) = g(\vartheta) = 1$ and $\beta = \nu = 2$, we can obtain (10) demonstrated in [6].*

Remark 2. *As a special case of Theorem 1, when $\varphi(\vartheta) = \phi(\vartheta) = \vartheta$, and $\beta = \nu = 2$, we can obtain the results demonstrated in [8].*

Remark 3. *As a special case of Theorem 1, when $\varphi(\vartheta) = \phi(\vartheta) = \vartheta$, we can obtain the results demonstrated in [16].*

4. Conclusions

In this paper, we use reverse Hölder inequalities with Specht's ratio on time scales to develop the study of reversed Hilbert-type inequalities. This aim holds by a study on some new generalizations of reversed Hilbert-type inequalities via supermultiplicative functions. In the future work, we can generalize dynamic inequalities of this article using a fractional Riemann–Liouville integral on time scale calculus, and we can present some of these dynamic inequalities on quantum calculus. It will be interesting to present dynamic inequalities in two or more dimensions.

Author Contributions: Software and Writing—original draft, H.M.R., G.A. and A.I.S.; Writing—review and editing H.M.R. and M.Z. All authors have read and agreed to the published version of the manuscript.

Funding: Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2022R45), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hardy, G.H. Note on a theorem of Hilbert concerning series of positive term. *Proc. Lond. Math. Soc.* **1925**, *23*, 45–46.
2. Hardy, G.H.; Littlewood, J.E.; Pólya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1934.
3. Hölder, O. Uber einen Mittelwerthssatz. *Nachr. Ges. Wiss. Gott.* **1889**, 38–47.
4. Zhao, C.J.; Cheung, W.S. Hölder's reverse inequality and its applications. *Publ. L'Inst. Math.* **2016**, *99*, 211–216. [CrossRef]
5. Tominaga, M. Specht's ratio in the Young inequality. *Sci. Math. Jpn.* **2002**, *55*, 583–588.
6. Zhao, C.J.; Cheung, W.S. Reverse Hilbert type inequalities. *J. Math. Inequalities* **2019**, *13*, 855–866. [CrossRef]
7. Ahmed, A.M.; AlNemer, G.; Zakarya, M.; Rezk, H.M. Some dynamic inequalities of Hilbert's type. *J. Funct. Spaces* **2020**, *2020*, 4976050. [CrossRef]
8. Al Nemer, G.; Saied, A.I.; Zakarya, M.; El-Hamid, H.A.; Bazighifan, O.; Rezk, H.M. Some new reverse Hilbert's inequalities on time scales. *Symmetry* **2021**, *13*, 2431. [CrossRef]
9. Al Nemer, G.; Zakarya, M.; El-Hamid, H.A.A.; Agarwal, P.; Rezk, H. Some Dynamic Hilbert-type inequality on time scales. *Symmetry* **2020**, *12*, 1410.
10. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, NA, USA, 2001.
11. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
12. El-Deeb, A.A.; Elsennary, H.A.; Cheung, Wi. Some reverse Hölder inequalities with Specht's ratio on time scales. *J. Nonlinear Sci. Appl.* **2018**, *11*, 444–455. [CrossRef]
13. Saker, S.H.; Mahmoud, R.R.; Peterson, A. Weighted Hardy-type inequalities on time scales with applications. *Mediterr. J. Math.* **2016**, *13*, 585–606. [CrossRef]
14. Saker, S.H.; Ahmed, A.M.; Rezk, H.M.; O'Regan, D.; Agarwal, R.P. New Hilbert's dynamic inequalities on time scales. *J Inequalities Appl.* **2017**, *20*, 1017–1039. [CrossRef]
15. Sandor, J. Inequalities for multiplicative arithmetic functions. *arXiv* **2011** arXiv:1105.0292.
16. Zakarya, M.; Saied, A.I.; AlNemer, G.; Rezk, H.M. A study on some new reverse Hilbert-Type inequalities and its generalizations on time scales. *J. Math.* **2022**, *2022*, 6285367.