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$\mathcal{N}$-Structures Applied to Commutative Ideals of BCI-Algebras

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Abstract: The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Algebraic structures are useful structures in pure mathematics for learning a geometrical object's symmetries. In order to provide a mathematical tool for dealing with negative information, a negative-valued function came into existence along with $\mathcal{N}$-structures. In the present analysis, the notion of $\mathcal{N}$-structures is applied to the ideals, especially the commutative ideals of BCI-algebras. Firstly, several properties of $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCI-algebras are investigated. Furthermore, the notion of a commutative $\mathcal{N}$-ideal is defined, and related properties are investigated. In addition, useful characterizations of commutative $\mathcal{N}$-ideals are established. A condition for a closed $\mathcal{N}$-ideal to be a commutative $\mathcal{N}$-ideal is provided. Finally, it is proved that in a commutative BCI-algebra, every closed $\mathcal{N}$-ideal is a commutative $\mathcal{N}$-ideal.

Keywords: BCI-algebra; fuzzy ideal; $\mathcal{N}$-subalgebras; $\mathcal{N}$-ideal; commutative $\mathcal{N}$-ideal

MSC: 06D72; 06F35; 03G25

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1. Introduction

The idea of BCK/BCI-algebras came into existence in 1966 due to K. Iséki [1,2]. Since then, the notion and the generalizations have been studied in different ways. The various attributes of BCK/BCI-algebra are considered in [3–5]. First, the fuzzy sets were introduced in 1965 due to Zadeh [6]. To deal with negative information in the fuzzy set theory, Jun et al. [7] gave the notation of negative-valued function and constructed $\mathcal{N}$-structures (briefly, "$\mathcal{N}$-STRUC"). After the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields such as statistics, graph theory, medical and life science, engineering, business and social science, computer network, decision making, artificial intelligence, pattern recognition, robotics, and automata theory (see [8,9]). More concepts related to our study in different fields have been studied in [10,11].

In summary, in 2009, Jun et al. [7] applied "$\mathcal{N}$-STRUC" to BCK/BCI-algebras, and discussed $\mathcal{N}$-subalgebras (briefly, "$\mathcal{N}$-SUB") and $\mathcal{N}$-ideals (briefly, "$\mathcal{N}$-I") in BCK/BCI-algebras. Later, in 2010, "$\mathcal{N}$-SUB" in BCK/BCI-algebras based on point "$\mathcal{N}$-STRUC" were initiated in [12]. Jun et al. [13] considered closed ideals in BCH-algebras based on "$\mathcal{N}$-STRUC". In the same year, Jun et al. [14] gave the notion of "$\mathcal{N}$-I" in subtraction algebras and studied their characterizations. In 2012, Jun and Lee [15] applied the notion of "$\mathcal{N}$-STRUC" to the subtraction algebras and studied related properties. They also provided many examples in support of these notions. As a follow up, Jun et al. studied ideal theory in BCK/BCI-algebras based on soft sets and "$\mathcal{N}$-STRUC" [16]. In addition, Lee et al. worked on "$\mathcal{N}$-SUB" of type $(\in, \in \vee q)$ based on point "$\mathcal{N}$-STRUC" in BCK/BCI-algebras [17] while Jun and Kang have given the ideal theory of BE-algebras based on "$\mathcal{N}$-STRUC" [18]. Additionally, several notions based on fuzzy subalgebras and "ideals" in different algebras, are given in [19–22]. Recently, a number of research papers have appeared on different fuzzy algebras based on "$\mathcal{N}$-STRUC". Some published papers, related to the present work, are listed below.

- In 2014, Jun et al. applied the notion of N-structures in BCC-Algebras [23], and also applied the soft set theory and N-structures to BCH-algebras [24].
- In semigroups, Khan et al. [25] (2017) gave the notion of "Neutrosophic $\mathcal{N}$-STRUC and their applications" and Song et al. introduced "Neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras [26]".
- In 2018, Jun et al. inroduced the notion of Neutrosophic positive "implicative $\mathcal{N}$-I" in BCK-algebras [27].
- In 2019, Muhiuddin et al. studied "implicative $\mathcal{N}$-I" of BCK-algebras based on "Neutrosophic $\mathcal{N}$-STRUC" [28].
- Most recently, in 2021, Muhiuddin initiated the concept of "p-BCI-ideals" based on "Neutrosophic $\mathcal{N}$-STRUC" [29].

Motivated by previous work in this direction, in this paper, we applied the notion of "$\mathcal{N}$-STRUC" to the commutative ideals of BCI-algebras and provided many examples in support of these new findings. We organized this paper as follows: In Section 2, some fundamental notions of BCK/BCI-algebras are presented. In Section 3, several properties of "$\mathcal{N}$-SUB" and "$\mathcal{N}$-I" in BCI-algebras are investigated with examples. In Section 4, the notion of a commutative $\mathcal{N}$-ideal (briefly, COMMU $\mathcal{N}$-I) is defined, and related properties are investigated. Finally, we discuss the conclusions of this work with some future directions.

2. Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

Definition 1. *A system $\Omega := (\Omega, *, 0)$ is called a BCI-algebra, if satisfying the following axioms, for all $\varpi, \varrho, \eta \in \Omega$:*

(a1) $((\varpi * \varrho) * (\varpi * \eta)) * (\eta * \varrho) = 0,$
(a2) $(\varpi * (\varpi * \varrho)) * \varrho = 0,$
(a3) $\varpi * \varpi = 0,$
(a4) $\varpi * \varrho = 0 = \varrho * \varpi \;\Rightarrow\; \varpi = \varrho$

We can define a partial ordering $\preceq$ by

$$(\forall \varpi, \varrho \in \Omega)\,(\varpi \preceq \varrho \;\Leftrightarrow\; \varpi * \varrho = 0).$$

In a BCI-algebra Ω for all $\varpi, \varrho, \eta \in \Omega$, the following holds:

(b1) $\varpi * 0 = \varpi,$
(b2) $(\varpi * \varrho) * \eta = (\varpi * \eta) * \varrho,$
(b3) $0 * (0 * (0 * \varpi)) = 0 * \varpi,$
(b4) $0 * (\varpi * \varrho) = (0 * \varpi) * (0 * \varrho)$

If a BCI-algebra Ω satisfies

(a5) $0 * \varpi = 0$ for all $\varpi \in \Omega$,

then we say that Ω is a *BCK-algebra*.

A BCI-algebra Ω is said to be *commutative* (see [30]) if it satisfies:

(a6) $(\forall \varpi, \varrho \in \Omega)\,(\varpi \preceq \varrho \;\Rightarrow\; \varpi = \varrho * (\varrho * \varpi),$

A BCI-algebra Ω is commutative if and only if it satisfies (see [30]):

(a7) $(\forall \varpi, \varrho \in \Omega)\,(\varpi * (\varpi * \varrho) = \varrho * (\varrho * (\varpi * (\varpi * \varrho))).$

By a *subalgebra*, we mean a non-empty subset S of a BCK/BCI-algebra Ω if $\varpi * \varrho \in S$ for all $\varpi, \varrho \in S$.

Let Ω be a BCI-algebra. Then Ω is called a "*p-semisimple*" if $0 * (0 * \varpi) = \varpi$ for all $\varpi \in \Omega$.

A subset "A" of a BCK/BCI-algebra Ω is called an "*ideal*" of Ω if it satisfies:

(a8) $0 \in A$,

(a9) $(\forall \varpi, \varrho \in \Omega)\ (\varpi * \varrho \in A,\ \varrho \in A\ \Rightarrow\ \varpi \in A)$.

Let Ω be a BCI-algebra. Then an ideal A of Ω is said to be "*closed*" if A is a subalgebra of Ω. Note that an ideal A of a BCI-algebra Ω is closed if and only if $0 * \varpi \in A$ for all $\varpi \in A$.

A subset A of a BCI-algebra Ω is called a *commutative ideal* (*briefly COMMU I*) of Ω (see [31]) if it satisfies (a8) and

(a10) $(\varpi * \varrho) * \eta \in A,\ \eta \in A\ \Rightarrow\ x * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho)))) \in A$

for all $\varpi, \varrho, \eta \in \Omega$.

For more information regarding BCK/BCI-algebras, we refer the reader to [4,5].

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote, $\mathcal{F}(\Omega, [-1,0])$; the collection of functions from a set Ω to $[-1,0]$. Note that an element of $\mathcal{F}(\Omega, [-1,0])$ is a "*negative-valued function*" from Ω to $[-1,0]$ (briefly, $\mathcal{N}$*-function* on Ω). Also, by an "$\mathcal{N}$*-STRUC*" we mean an ordered pair (X, μ) of Ω and an $\mathcal{N}$-function μ on Ω.

Let (Ω, μ) be an "$\mathcal{N}$-STRUC" and $t \in [-1,0]$,, then

$$C(\mu; t) := \{\varpi \in \Omega \mid \mu(\varpi) \leq t\}$$

is said to be a *closed* (μ, t)*-cut* of (Ω, μ).

Following the transfer principle in [32,33], transfer principle in "$\mathcal{N}$-STRUC" can be considered as follows: suppose $A \subseteq \Omega$, satisfying the following property $\mathcal{P}$, given by

$$\mathcal{P} : \frac{t_1(\varpi, \cdots, \varrho) \in A, \cdots, t_n(\varpi, \cdots, \varrho) \in A}{t(\varpi, \cdots, \varrho) \in A},$$

where $t_1(\varpi, \cdots, \varrho), \cdots, t_n(\varpi, \cdots, \varrho)$ and $t(\varpi, \cdots, \varrho)$ are terms of Ω constructed by variables $\varpi, \cdots, \varrho$. We say that $A \subseteq \Omega$ satisfies the property $\mathcal{P}$ if, for all elements $\varpi, \cdots, \varrho \in \Omega$, $t(\varpi, \cdots, \varrho) \in A$ whenever "$t_1(\varpi, \cdots, \varrho), \cdots, c_n(\varpi, \cdots, \varrho) \in A$." For the subset A we define an "$\mathcal{N}$-STRUC" (Ω, μ_A) satisfying

$$\overline{\mathcal{P}} : \mu_A(t(\varpi, \cdots, \varrho) \leq \vee\{\mu_A(t_1(\varpi, \cdots, \varrho)), \cdots \mu_A(t_n(\varpi, \cdots, \varrho))\}.$$

Theorem 1 ([13])**.** *An "$\mathcal{N}$-STRUC" (Ω, μ) satisfies the property $\overline{\mathcal{P}} \Leftrightarrow \forall\ \alpha \in [-1,0]$, $C(\phi; \alpha)$ is not empty implies $C(\phi; \alpha)$ satisfies the property $\mathcal{P}$. This is called $\mathcal{N}$-transfer principle (briefly, $\mathcal{N}TP$).*

In what follows let Ω denote a BCI-algebra unless otherwise specified.

Definition 2 ([7])**.** *By a subalgebra of Ω based on an $\mathcal{N}$-function μ ("$\mathcal{N}$-SUB" of Ω), we mean an $\mathcal{N}$-structure (Ω, μ) in which μ satisfies the following assertion:*

(c1) $(\forall \varpi, \varrho \in \Omega)\ (\mu(\varpi * \varrho) \leq \vee\{\mu(\varpi), \mu(\varrho)\})$.

Definition 3 ([7])**.** *An "$\mathcal{N}$-STRUC" (Ω, μ) is called "$\mathcal{N}$-I" of Ω if*

(c2) $(\forall \varpi, \varrho \in \Omega)\ (\mu(0) \leq \mu(\varpi) \leq \vee\{\mu(\varpi * \varrho), \mu(\varrho)\})$.

Lemma 1 ([7])**.** *In any "$\mathcal{N}$-SUB" (Ω, μ) of Ω, the following conditions hold:*

(c3) $(\forall \varpi \in \Omega)\ (\mu(0) \leq \mu(\varpi))$.

Using the *NTP*, we have the following.

Lemma 2 ([7])**.** *Let (Ω, μ) be an"$\mathcal{N}$-STRUC". Then the following are equivalent:*

(a) *(Ω, μ) is an "$\mathcal{N}$-SUB" of Ω.*
(b) *$(C(\mu;t) \neq \emptyset$ implying $C(\mu;t)$ is a subalgebra of $\Omega)$ $\forall\ t \in [-1, 0]$.*

Lemma 3 ([7])**.** *Let (Ω, μ) be an"$\mathcal{N}$-STRUC". Then the following are equivalent:*

(a) *(Ω, μ) is an "$\mathcal{N}$-I" of Ω.*
(b) *$(C(\mu;t) \neq \emptyset$ implying $C(\mu;t)$ is an ideal of $\Omega)$ $\forall\ t \in [-1, 0]$.*

3. BCI-Commutative $\mathcal{N}$-Ideals

In this section, we obtain our main findings.

Lemma 4 ([7])**.** *In any "$\mathcal{N}$-I" (Ω, μ) of Ω, we have*

$$(\forall \varpi, \varrho, \eta \in \Omega)\ (\varpi * \varrho \preceq \eta\ \Rightarrow\ \mu(\varpi) \leq \vee\{\mu(\varrho), \mu(\eta)\}). \tag{1}$$

Theorem 2. *Let (Ω, μ) be an "$\mathcal{N}$-STRUC" satisfying the conditions* (c3) *and* (1)*. Then (Ω, μ) is an "$\mathcal{N}$-I" of Ω.*

Proof. Combining (a2) and (1), we have $\mu(\varpi) \leq \vee\{\mu(\varpi * \varrho), \mu(\varrho)\}$ for all $\varpi, \varrho \in \Omega$. Hence (Ω, μ) is an "$\mathcal{N}$-I" of Ω. □

Let $a \in \Omega$. Then consider

$$\Omega_a := \{\varpi \in \Omega \mid \mu(\varpi) \leq \mu(a)\}.$$

Obviously, $a \in \Omega_a,\ \Longrightarrow\ \Omega_a \neq \emptyset$ and a subset of Ω (see [7]).

Proposition 1. *Let (Ω, μ) be an "$\mathcal{N}$-STRUC" such that Ω_a is an ideal of Ω $\forall\ a \in \Omega$. Then (Ω, μ) satisfying,*

(c4) $(\forall \varpi, \varrho, \eta \in \Omega)\ (\mu(\varpi) \geq \vee\{\mu(\varrho * \eta), \mu(\eta)\}\ \Rightarrow\ \mu(\varpi) \geq \mu(\varrho))$.

Proof. Let $\varpi, \varrho, \eta \in \Omega$ be such that $\mu(\varpi) \geq \vee\{\mu(\varrho * \eta), \mu(\eta)\}$. Then $\varrho * \eta \in \Omega_x$ and $\eta \in \Omega_x$. Since Ω_x is an ideal of Ω, then $\varrho \in \Omega_x\ \Longrightarrow\ \mu(\varrho) \leq \mu(\varpi)$. □

If an "$\mathcal{N}$-STRUC" (Ω, μ) is an "$\mathcal{N}$-I" of Ω in which μ is a 0-negative function, that is, $\mu(0 * \varpi) \leq \mu(\varpi)$ for all $\varpi \in \Omega$, we say (Ω, μ) is a "*closed $\mathcal{N}$-ideal*(briefly, C-$\mathcal{N}$-I)" of Ω (see [7]).

Theorem 3. *For any "$\mathcal{N}$-STRUC" (Ω, μ) in which μ satisfies the condition* (c3)*, if the condition* (c4) *is valid then Ω_a is an ideal of Ω for all $a \in \Omega$.*

Proof. For each $a \in \Omega$, let $\varpi, \varrho \in \Omega$ be such that $\varpi * \varrho \in \Omega_a$ and $\varrho \in \Omega_a$. Then $\mu(\varpi * \varrho) \leq \mu(a)$ and $\mu(\varrho) \leq \mu(a)\ \Longrightarrow\ \vee\{\mu(\varpi * \varrho), \mu(\varrho)\} \leq \mu(a)$. By (c4), we have $\mu(\varpi) \leq \mu(a)$ $\Longrightarrow\ x \in \Omega_a$. By the condition (c3), we have $0 \in \Omega_a$. Therefore, Ω_a is an ideal of Ω for all $a \in \Omega$. □

Now, we prove the following theorem.

Theorem 4. *For any "p-semisimple" BCI-algebra Ω, every "$\mathcal{N}$-SUB" of Ω is an "N-I" of Ω.*

Proof. Let (Ω, μ) be an "$\mathcal{N}$-SUB" of a p-semisimple BCI-algebra Ω. Let $t \in [-1, 0]$ be such that $C(\mu;t) \neq \emptyset$. Lemma 1 induces $0 \in C(\mu;t)$. Let $\varpi, \varrho \in \Omega$ be such that $\varrho * \varpi \in C(\mu;t)$

and $\varpi \in C(\mu;t)$. Since $C(\mu;t)$ is a subalgebra of Ω by Lemma 2, we have $0 * \varpi \in C(\mu;t)$ and $(\varrho * \varpi) * (0 * \varpi) \in C(\mu;t)$. Since ϱ is a minimal element of Ω, it follows from $(\varrho * \varpi) * (0 * \varpi) \preceq \varrho$ that $\varrho = (\varrho * \varpi) * (0 * \varpi) \in C(\mu;t)$. Consequently, $C(\mu;t)$ is an ideal of X, and so (Ω, μ) is an "$\mathcal{N}$-I" of Ω by Lemma 3. □

Corollary 1. *If a BCI-algebra Ω satisfies one of the following conditions:*

(1) $(\forall \varpi, \varrho \in \Omega)\ (\varpi * (0 * \varrho) = \varrho * (0 * \varpi))$,
(2) $(\forall \varpi, \varrho, \eta \in \Omega)\ ((\varpi * \varrho) * (\varpi * \eta) = \eta * \varrho)$,
(3) $(\forall \varpi, \varrho, \eta, u \in \Omega)\ ((\varpi * \varrho) * (\eta * u) = (\varpi * \eta) * (\varrho * u))$,
(4) $(\forall \varpi, \varrho \in \Omega)\ (0 * (\varrho * \varpi) = \varpi * \varrho)$,
(5) $(\forall \varpi \in \Omega)\ (0 * \varpi = 0 \ \Rightarrow\ \varpi = 0)$,
(6) $(\forall \varpi, \varrho, \eta \in \Omega)\ (\eta * \varpi = \eta * \varrho \ \Rightarrow\ \varpi = \varrho)$,
(7) $\Omega = \{0 * \varpi \mid \varpi \in \Omega\}$,
(8) *the BCK-part of* Ω *is* $\{0\}$,

then every "$\mathcal{N}$-SUB" of Ω is an "$\mathcal{N}$-I of Ω.

Proof. Straightforward. □

Next, we will define our main results. For this purpose, we introduce the following definition.

Definition 4. *An "$\mathcal{N}$-STRUC" (Ω, μ) is called "COMMU $\mathcal{N}$-I" of Ω if μ satisfies the condition* (c3) *and*

(d1) $\mu(\varpi * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho))))) \leq \vee\{\mu((\varpi * \varrho) * \eta), \mu(\eta)\}$

for all $\varpi, \varrho, \eta \in \Omega$.

Example 1. *Let $\Omega := \{0, a, \hbar, \vartheta, \omega\}$ be a set with $*$ operation given by Table 1.*

Table 1. Cayley table for $*$ operation.

$*$	0	a	$\hbar$	ϑ	ω
0	0	0	ω	ϑ	$\hbar$
a	a	0	ω	ϑ	$\hbar$
$\hbar$	$\hbar$	$\hbar$	0	ω	ϑ
ϑ	ϑ	ϑ	$\hbar$	0	ω
ω	ω	ω	ϑ	$\hbar$	0

Clearly, $(\Omega, *, 0)$ is a BCI-algebra (see [34]). Let μ be an $\mathcal{N}$-function on Ω defined by

$$\mu = \begin{pmatrix} 0 & a & \hbar & \vartheta & \omega \\ t_0 & t_\hbar & t_\vartheta & t_\vartheta & t_\vartheta \end{pmatrix}.$$

where $t_0, t_\hbar, t_\vartheta \in [-1, 0]$ with $t_0 < t_\hbar < t_\vartheta$. It is easily verified that (Ω, μ) is a "COMMU $\mathcal{N}$-I" of Ω.

By using "$\mathcal{N}$TP", we present the following Theorem based on "COMMU $\mathcal{N}$-I".

Theorem 5. *Let (Ω, μ), be an "$\mathcal{N}$-STRUC". Then the following are equivalent:*

(1) *(Ω, μ) is a "COMMU $\mathcal{N}$-I" of Ω.*
(2) $(\forall t \in [-1, 0])\ (C(\mu;t) \neq \emptyset \ \Rightarrow\ C(\mu;t)$*is a "COMMU I" of* $\Omega)$.

Theorem 6. *Every COMMU $\mathcal{N}$-I is an $\mathcal{N}$-I.*

Proof. Let (Ω, μ) be a COMMU $\mathcal{N}$-I of Ω and let $\varpi, \varrho, \eta \in \Omega$. Then

$$\begin{aligned}&\vee\{\mu(\varpi * \eta), \mu(\eta)\} = \vee\{\mu((\varpi * 0) * \eta), \mu(\eta)\}\\ &\geq \mu(\varpi * (0 * (0 * \varpi)) * (0 * (0 * (\varpi * 0)))) = \mu(\varpi * 0) = \mu(\varpi).\end{aligned}$$

Therefore (Ω, μ) is an $\mathcal{N}$-I of Ω. □

Open question: What about the converse of Theorem 6? The answer is negative, that is, it is not true in general. To support this assertion, we construct the following example.

Example 2. *Let $\Omega := \{0, a, \hbar, \vartheta, \omega\}$ be a set with $*$ operation given by Table 2.*

Table 2. Cayley table for $*$ operation.

$*$	0	$\hbar$	ϑ	ω	κ
0	0	0	0	0	0
$\hbar$	$\hbar$	0	$\hbar$	0	0
ϑ	ϑ	ϑ	0	0	0
ω	ω	ω	ω	0	0
κ	κ	κ	κ	ω	0

Then $(\Omega, *, 0)$ is a BCI-algebra (see [31]). Let μ be an $\mathcal{N}$-function on Ω defined by

$$\mu = \begin{pmatrix} 0 & \hbar & \vartheta & \omega & \kappa \\ -0.8 & -0.5 & -0.3 & -0.3 & -0.3 \end{pmatrix}.$$

It is easily verified that (Ω, μ) is an "$\mathcal{N}$-I" of Ω. But it is not a "COMMU $\mathcal{N}$-I" of Ω since

$$\mu(\vartheta * ((\omega * (\omega * \vartheta)) * (0 * (0 * (\vartheta * \omega))))) \nleq \vee\{\mu((\vartheta * \omega) * 0), \mu(0)\}.$$

Open problem: Under what condition are we able to transform an "$\mathcal{N}$-I" into a "COMMU $\mathcal{N}$-I". Next, we provide a condition for this.

Theorem 7. *An "$\mathcal{N}$-STRUC" (Ω, μ) is a "COMMU $\mathcal{N}$-I" of $\Omega \Leftrightarrow$ it is an "$\mathcal{N}$-I" of Ω satisfying the following condition:*

(d2) $(\forall \varpi, \varrho \in \Omega)\ (\mu(\varpi * (\varrho * (\varrho * \varpi))) * (0 * (0 * (\varpi * \varrho)))) \leq \mu(\varpi * \varrho))$.

Proof. Let (Ω, μ) be a "COMMU $\mathcal{N}$-I" of Ω. Taking $\eta = 0$ in (d1) and using (b1) and (c3), we get (d2).

Conversely, let (Ω, μ) be an "$\mathcal{N}$-I" of Ω satisfying the condition (d2). Then $\mu(\varpi * \varrho) \leq \vee\{\mu((\varpi * \varrho) * \eta), \mu(\eta)\}\ \forall\, \varpi, \varrho, \eta \in \Omega$. By (d2), we have

$$\mu(\varpi * (\varrho * (\varrho * \varpi))) * (0 * (0 * (\varpi * \varrho))) \leq \vee\{\mu((\varpi * \varrho) * \eta), \mu(\eta)\}$$

$\forall\, \varpi, \varrho, \eta \in \Omega$. Therefore (Ω, μ) is a "COMMU $\mathcal{N}$-I" of Ω. □

Note that the "$\mathcal{N}$-SRTUC" (Ω, μ) which is given in Example 2 is a "C $\mathcal{N}$-I" of Ω which is not a "COMMU $\mathcal{N}$-I" of Ω. Also, in Example 1, the "$\mathcal{N}$-SRTUC" (Ω, μ) is a "$\mathcal{N}$-SRTUC", but it is not a "C-$\mathcal{N}$-I".

Open problem: Under what condition are we able to transform an "C-$\mathcal{N}$-I" into a "COMMU $\mathcal{N}$-I".

Next, we provide a condition for this.

Theorem 8. *For a "C-$\mathcal{N}$-I" (Ω, μ) of Ω, the following are equivalent:*

(a) (Ω, μ) *is a "COMMU $\mathcal{N}$-I" of Ω.*

(b) (Ω, μ) *satisfies the following inequality:*

$$(\forall \varpi, \varrho \in \Omega)\ (\mu(\varpi * (\varrho * (\varrho * \varpi)))) \leq \mu(\varpi * \varrho)).$$

Proof. (a) $\Rightarrow$ (b). Let $\varpi, \varrho \in \Omega$. Using (a1), (b2), (a3) and (b3), we have

$$\begin{aligned}&(\varpi * (\varrho * (\varrho * \varpi))) * (\varpi * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho)))))\\ &\preceq ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho)))) * (\varrho * (\varrho * \varpi))\\ &= ((\varrho * (\varrho * \varpi)) * (\varrho * (\varrho * \varpi))) * (0 * (0 * (\varpi * \varrho)))\\ &= 0 * (0 * (0 * (\varpi * \varrho))) = 0 * (\varpi * \varrho).\end{aligned}$$

If follows from Lemma 4 and Theorem 7 that

$$\begin{aligned}&\mu(\varpi * (\varrho * (\varrho * \varpi))) \leq \vee\{\mu(\varpi * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho)))), \mu(0 * (\varpi * \varrho))\}\\ &\leq \vee\{\mu(\varpi * \varrho), \mu(0 * (\varpi * \varrho))\} = \mu(\varpi * \varrho).\end{aligned}$$

(b) $\Rightarrow$ (a). For any $\varpi, \varrho \in \Omega$, we get

$$\begin{aligned}&(\varpi * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho))))) * (\varpi * (\varrho * (\varrho * \varpi)))\\ &\preceq (\varrho * (\varrho * \varpi)) * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho))))\\ &\preceq 0 * (0 * (\varpi * \varrho))\end{aligned}$$

by (a1) and (a2). By Lemma 4 and (2), we get

$$\begin{aligned}&\mu(\varpi * ((\varrho * (\varrho * \varpi)) * (0 * (0 * (\varpi * \varrho)))))\\ &\leq \vee\{\mu(\varpi * (\varrho * (\varrho * \varpi))), \mu(0 * (0 * (\varpi * \varrho)))\}\\ &\leq \vee\{\mu(\varpi * \varrho), \mu(0 * (0 * (\varpi * \varrho)))\}\\ &= \mu(\varpi * \varrho)\end{aligned}$$

so that (d2) is valid. Hence (Ω, μ) is a "COMMU $\mathcal{N}$-I" of Ω. □

Theorem 9. *Let Ω. be a commutative BCI-algebra. Then every "C-$\mathcal{N}$-I" is a "COMMU $\mathcal{N}$-I" of Ω.*

Proof. Let (Ω, μ) be an "C-$\mathcal{N}$-I" of Ω. Again, let $\varpi, \varrho \in \Omega$. Using (a7), we have

$$\begin{aligned}&(\varpi * (\varrho * (\varrho * \varpi))) * (\varpi * \varrho) = (\varpi * (\varpi * \varrho)) * (\varrho * (\varrho * \varpi)) = (\varrho * (\varrho * (\varpi * (\varpi * \varrho)))) * (\varrho * (\varrho * \varpi))\\ &= (\varrho * (\varrho * (\varrho * \varpi))) * (\varrho * (\varpi * (\varpi * \varrho))) = (\varrho * \varpi) * (\varrho * (\varpi * (\varpi * \varrho)))\\ &\preceq (\varpi * (\varpi * \varrho)) * \varpi = 0 * (\varpi * \varrho).\end{aligned}$$

Since μ is 0-negative function, it follows from Lemma 4 that

$$\mu(\varpi * (\varrho * (\varrho * \varpi))) \leq \vee\{\mu(\varpi * \varrho), \mu(0 * (\varpi * \varrho))\} \leq \mu(\varpi * \varrho)$$

Therefore, from Theorem 8, we get (Ω, μ) is a "COMMU $\mathcal{N}$-I". □

4. Conclusions

In the present paper, we apply the notion of "$\mathcal{N}$-SRTUC" to the ideals, especially the commutative ideals of BCI-algebras. We provided different characterizations of "$\mathcal{N}$-SUB" and "$\mathcal{N}$-I" in BCI-algebras. Furthermore, the notion of a "COMMU $\mathcal{N}$-I" is defined, and related properties are investigated. In addition, we provided a condition for a "C-$\mathcal{N}$-I" to be a "COMMU $\mathcal{N}$-I". It has been shown that in a commutative BCI-algebra, every "C-$\mathcal{N}$-I" is a "COMMU $\mathcal{N}$-I".

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