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Generalization of Two-Sided Length Biased Inverse Gaussian Distributions and Applications

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Abstract: The notion of length-biased distribution can be used to develop adequate models. Length-biased distribution was known as a special case of weighted distribution. In this work, a new class of length-biased distribution, namely the two-sided length-biased inverse Gaussian distribution (TS-LBIG), was introduced. The physical phenomenon of this scenario was described in a case of cracks developing from two sides. Since the probability density function of the original TS-LBIG distribution cannot be written in a closed-form expression, its generalization form was further introduced. Important properties such as the moment-generating function and survival function cannot be provided. We offered a different approach to solving this problem. Some distributional properties were investigated. The parameters were estimated by the method of the moment. Monte Carlo simulation studies were carried out to appraise the performance of the suggested estimators using bias, variance, and mean square error. An application of a real dataset was presented for illustration. The results showed that the suggested estimators performed better than the original study. The proposed distribution provided a more appropriate model than other candidate distributions for fitting based on Akaike information criterion.

Keywords: method of moment; lifetime distribution; parametrization; re-parameterized distribution; length-biased distribution

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1. Introduction

Recorded observations may not have original distributions when practitioners collect natural observations according to certain stochastic models. Each observation is taken with unequal probabilities of recording. Weighted distributions can be adopted in this situation for selecting appropriate models [1]. One of the most widely known for special cases of weighted distributions is length-biased distributions. Precisely, let X denote a non-negative random variable with a probability density function shortly called PDF or $f_X(x)$. The weighted version of X denoted by X_w has a PDF defined as

$$f_{X_w}(x) = \frac{w(x) f_X(x)}{E[w(x)]}, \quad x > 0, \tag{1}$$

where $w(x)$ is the weighted function and $0 < E[w(x)] < \infty$. Different weighted models are formulated depending on choices of the weight function $w(x)$. In cases of $w(x) = x$, the resulting distribution is called length-biased whose PDF is defined by

$$f(x) = \frac{x f_X(x)}{E[X]}, \quad x > 0, \tag{2}$$

Several versions of length-biased distributions are employed in various applications. For example, length-biased Birnbaum–Saunders distribution with an application in water quality was proposed by Leiva et al. [2]. Length-biased weighted Weibull distribution introduced by Das and Roy [3] was utilized in rainfall data. A generalization of length-biased Nakagami distribution offered by Abdullahi and Phaphan [4] was applied in heart attack data. Further, length-biased distributions can be used in percolation theory. Since percolation models are formulated from different weights and the distribution of a weight is a non-negative random variable, length-biased distributions can be employed as an alternative distribution. Some examples in this area were given in [5,6].

The length-biased inverse Gaussian (LBIG), one of special cases of the length-biased weighted distributions, is frequently used as a lifetime distribution. The LBIG distribution has been studied by many authors. In the early state, Khattree [7] presented a description of the inverse Gaussian (IG) and gamma distributions via their length-biased versions. Akman and Gupta [8] proposed a comparison of several estimators of the mean for IG and LBIG distributions. Akman and Gupta [9] offered statistical properties of the mixture of the IG and LBIG distributions. Recently, Naik [10] introduced a convoluted form of length-biased inverse Gaussian and gamma distributions. Budsaba and Phaphan [11] provided maximum likelihood estimation for re-parameterized LBIG distribution. The LBIG distribution has been utilized as a component of mixed distributions. For instance, it was used for constructing a mixture inverse Gaussian distribution [12], new parametrization of mixture inverse Gaussian distribution [13], weighted inverse Gaussian distribution [14], Birnbaum–Saunders distribution [15], re-parametrization of Birnbaum–Saunders distribution [16], three-parameter crack distribution [17], and two-parameter crack distribution [18].

In a reliability framework, a two-sided model can be described in a situation in which fatigue cracks evolve from two sides of the studied object. Lisawadi [19] early introduced two distributions using the parametrization suggested by Ahmed et al. [16], namely the two-sided Birnbaum–Saunders (TS-BS) and two-sided inverse Gaussian (TS-IG) distributions. Subsequently, Simmachan et al. [20] presented an alternative distribution applying the approach of Lisawadi [19] called two-sided length-biased inverse Gaussian (TS-LBIG) distribution. However, all of the two-sided versions have no closed-form PDFs. Important distributional properties such as a moment-generating function (MGF) and a survival function cannot be presented.

This study aims to re-introduce the TS-LBIG distribution originally proposed by Simmachan et al. [20] in closed-form expression. The reciprocal property is employed for derivation of the MGF. The resulting MGF is compared to a known MGF. By uniqueness property, the PDF of the TS-LBIG distribution can be obtained.

The rest of the article is organized as follows: a review of IG and LBIG distributions is presented in Sections 2 and 3, respectively. The TS-LBIG random variable is described in Section 4. Reciprocal properties are provided in Section 5. In this section, four propositions are given. The MGF of TS-LBIG distribution is derived in Section 6. The PDF of TS-LBIG in closed-form expression is introduced in Section 7. Other distributional properties are established in Sections 8–11. Parameter estimation by the method of moment is provided in Section 12. Numerical results consisting of a simulation study and real data application are shown in Section 13. Finally, conclusions and discussion are reported in Section 14.

2. Inverse Gaussian Distribution

Chikara and Folk [21] studied the variables of the two-parameter inverse Gaussian distribution which is the continuous probability distribution $(0, \infty)$. Suppose X is a random variable with an inverse Gaussian distribution. Consequently, a PDF can be written in this formula:

$$f_{IG}(x;\mu,\beta) = \begin{cases} \sqrt{\frac{\beta}{2\pi}}x^{-\frac{3}{2}}exp\left(-\frac{\beta(x-\mu)^2}{2\mu^2 x}\right) & ;x>0 \\ 0 & ;otherwise, \end{cases} \tag{3}$$

where $\mu > 0$ is a location parameter or a mean, and $\beta > 0$ is a shape or scale parameter. The two parameters are called classical parameters. However, this research pays attention for studying the re-parameterized version of IG distribution. The parametrization was originally presented by Ahmed et al. [16] in the form of the Birnbaum–Saunders distribution (BS). The BS distribution was combined from IG and LBIG distributions. Precisely,

$$f_{BS}(x;\lambda,\theta) = \frac{1}{2}f_{IG}(x;\lambda,\theta) + \frac{1}{2}f_{LBIG}(x;\lambda,\theta), \tag{4}$$

where f_{BS}, f_{IG} and f_{LBIG} are the PDFs of the Birnbaum–Saunders, inverse Gaussian and length-biased inverse Gaussian distributions, respectively. The new form of the distribution parameters (λ and θ) is called non-classical parameters, where $\lambda > 0$ and $\theta > 0$ represent the thickness of the machine element and nominal treatment pressure on the machine element, respectively. The interrelations between (μ,β) and (λ,θ) are as follows.

$$\lambda = \frac{\beta}{\mu}, \theta = \frac{\mu^2}{\beta}, \mu = \lambda\theta, \text{ and } \beta = \lambda^2\theta. \tag{5}$$

From Equations (3) and (5), the PDF of non-classical IG distribution, denoted as $f_{IG}(\lambda,\theta)$, can be written in this form:

$$f_{IG}(x;\lambda,\theta) = \begin{cases} \frac{\lambda}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{3}{2}} exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right] & ; x > 0 \\ 0 & ; otherwise. \end{cases} \tag{6}$$

3. Length-Biased Inverse Gaussian Distribution

By the definition of a length-biased distribution defined in Equation (2), the length-biased inverse Gaussian distribution can be explained as follows. Let X be an inverse Gaussian random variable with parameters λ and θ or $X \sim IG(\lambda,\theta)$. The PDF of X is denoted as $f_{LBIG}(\lambda,\theta)$. As the relations between the classical parameters and non-classical parameters, the expected value or the first moment of X is $E[X] = \mu = \lambda\theta$. Therefore, the length-biased version of X can be expressed as

$$f_{LBIG}(x;\lambda,\theta) = \begin{cases} \frac{1}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} exp\left[-\frac{1}{2}\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right)^2\right] & ; x > 0 \\ 0 & ; otherwise. \end{cases} \tag{7}$$

4. TS-LBIG Random Variable

In this section, the TS-LBIG random variable (τ) introduced by Simmachan et al. [20] is described. Let X be a non-negative continuous random variable and let $F(x) = F_{LBIG}(x,\lambda,\theta)$ denote the distribution function of the breakdown time moment τ for one-sided loading. The parameters λ and θ were previously defined. Let $Y = k/\tau$ be the random variable denoted as a crack speed. Under the object consideration, a crack expands from two sides with the same distribution function of the time to approach the length k. The random variables from both sides, τ_1 and τ_2, are supposed to be independent and identically distributed. The crack speed for the two-sided situation is definded as

$$Y_1 + Y_2 = \frac{k}{\tau_1} + \frac{k}{\tau_2} = k\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right) = X. \tag{8}$$

The breakdown moment of the interested object is defined as the following random variable

$$\tau = \frac{k}{Y_1 + Y_2} = \frac{k}{k\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)} = \left[\left(\frac{1}{\tau_1} + \frac{1}{\tau_2}\right)\right]^{-1} = \frac{1}{X}. \tag{9}$$

5. Reciprocal Properties

Proposition 1. *If the random variable $\tau > 0$ has the probability density function $f_\tau(x)$, then the reciprocal random variable $1/\tau$ has the probability density function $f_{1/\tau}(x) = x^{-2} f_\tau(1/x)$.*

Proof of Proposition 1. For the reciprocal random variable $1/\tau$, the distribution function is given by

$$F_{1/\tau}(x) \; = \; P\left(\frac{1}{\tau} \leq x\right) \; = \; P\left(\tau \geq \frac{1}{x}\right) \; = \; 1 - P\left(\tau \leq \frac{1}{x}\right) \; = \; 1 - F_\tau\left(\frac{1}{x}\right),$$

and applying the chain rule, the density function is

$$f_{1/\tau}(x) \; = \; F^{'}_{1/\tau}(x) \; = \; -F^{'}_\tau\left(\frac{1}{x}\right) \; = \; -\left[f_\tau\left(\frac{1}{x}\right)\right] \cdot \left(-\frac{1}{x^2}\right) \; = \; x^{-2} f_\tau\left(\frac{1}{x}\right). \qquad \square$$

Proposition 2. *If random variable $\tau > 0$ has $LBIG(\lambda, \theta)$ distribution, then the reciprocal random variable $1/\tau$ is $IG[\lambda, 1/(\lambda^2\theta)]$ distributed.*

Proof of Proposition 2. By Proposition 1,

$$\begin{aligned}
f_{1/\tau}(x) \quad &= \quad x^{-2} f_{LBIG}\left(\frac{1}{x}; \lambda, \theta\right) \\
&= \quad \frac{x^{-2}\theta^{-1/2}x^{1/2}}{\sqrt{2\pi}} exp\left\{-\frac{1}{2}\left(\lambda\theta^{1/2}x^{1/2} - \theta^{-1/2}x^{-1/2}\right)^2\right\} \\
&= \quad \frac{x^{-3/2}}{\sqrt{2\pi\theta}} exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\theta x} - \frac{1}{\sqrt{\theta x}}\right)^2\right\} \\
&= \quad \frac{x^{-3/2}}{\sqrt{2\pi\theta}} exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}}\right)^2\right\} \\
&= \quad \frac{\lambda(1/\lambda^2\theta)^{1/2}}{\sqrt{2\pi}} x^{-3/2} exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}}\right)^2\right\} \\
&= \quad f_{IG}[x; \lambda, 1/(\lambda^2\theta)]. \qquad \square
\end{aligned}$$

Proposition 3. *If the random variable $\tau > 0$ has $IG(\lambda, \theta)$ distribution, then the reciprocal random variable $1/\tau$ is $LBIG[\lambda, 1/(\lambda^2\theta)]$ distributed.*

Proof of Proposition 3. By Proposition 1,

$$\begin{aligned} f_{1/\tau}(x) &= x^{-2} f_{IG}\left(\frac{1}{x}; \lambda, \theta\right) \\ &= \frac{\lambda\theta^2 x^{-1/2}}{\sqrt{2\pi}} exp\left\{-\frac{1}{2}\left(\lambda\theta^{1/2}x^{1/2} - \theta^{-1/2}x^{-1/2}\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi}(1/\lambda^2\theta)^{1/2}} x^{-1/2} exp\left\{-\frac{1}{2}\left(\lambda\sqrt{\frac{1/(\lambda^2\theta)}{x}} - \sqrt{\frac{x}{1/(\lambda^2\theta)}}\right)^2\right\} \\ &= f_{LBIG}[x; \lambda, 1/(\lambda^2\theta)]. \end{aligned}$$

□

Proposition 4. *If the random variable $\tau > 0$ has $IG[2\lambda, 1/(\lambda^2\theta)]$ distribution, then the reciprocal random variable $1/\tau$ is $LBIG[2\lambda, \theta/4)]$ distributed.*

Proof of Proposition 4. Applying Proposition 1,

$$\begin{aligned} f_{1/\tau}(x) &= x^{-2} f_{IG}\left(\frac{1}{x}; 2\lambda, 1/(\lambda^2\theta)\right) \\ &= \frac{2\lambda}{\sqrt{2\pi}(\lambda^2\theta)^{1/2}} x^{-1/2} exp\left\{-\frac{1}{2}\left(2\lambda(1/\lambda^2\theta)^{1/2}x^{1/2} - (1/\lambda^2\theta)^{-1/2}x^{-1/2}\right)^2\right\} \\ &= \frac{2\lambda}{\sqrt{2\pi}(\lambda^2\theta)^{1/2}} x^{-1/2} exp\left\{-\frac{1}{2}\left(2\lambda\sqrt{\frac{x}{\lambda^2\theta}} - \sqrt{\frac{\lambda^2\theta}{x}}\right)^2\right\} \\ &= \frac{(\theta/4)^{1/2}}{\sqrt{2\pi}} x^{-1/2} exp\left\{-\frac{1}{2}\left(\sqrt{\frac{x}{\theta/4}} - 2\lambda\sqrt{\frac{\theta/4}{x}}\right)^2\right\} \\ &= f_{LBIG}[x; 2\lambda, \theta/4]. \end{aligned}$$

□

6. Moment-Generating Function for TS-LBIG Distribution

Theorem 1. *If the random variables $\tau_1, \tau_2 \sim LBIG(\lambda, \theta)$, then the moment-generating function of $X = \tau_1^{-1} + \tau_2^{-1}$ is given as*

$$M_X(t) = \exp\left\{2\lambda\left[1 - \sqrt{1 - \frac{2t}{\lambda^2\theta}}\right]\right\},$$

where $0 < t < \frac{\lambda^2\theta}{2}$.

Proof of Theorem 1. We know that a moment-generating function (MGF) of $IG(\lambda,\theta)$ of a random variable τ is defined as

$$\psi_{IG}(t) = M_\tau(t) = \exp\left\{\lambda\left[1-\sqrt{1-2\theta t}\right]\right\}.$$

Now, we have two independent LBIG random variables τ_1 and τ_2. That is,

$$\tau_1 \sim LBIG(\lambda,\theta) \text{ and } \tau_2 \sim LBIG(\lambda,\theta).$$

Initially, we find the MGF of the random variable $X = \tau_1^{-1} + \tau_2^{-1}$. According to Proposition 2, if the random variable $\tau > 0$ has $LBIG(\lambda,\theta)$ distribution, then the reciprocal random variable $1/\tau$ is $IG[\lambda, 1/(\lambda^2\theta)]$ distributed. Therefore,

$$M_{\frac{1}{\tau_1}}(t) = M_{\frac{1}{\tau_2}}(t) = M_{IG}(t) = \exp\left\{\lambda\left[1-\sqrt{1-2\left(\frac{1}{\lambda^2\theta}\right)t}\right]\right\}.$$

Hence,

$$\begin{aligned}
M_X(t) &= E\left(e^{tX}\right) = E\left(e^{t\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)}\right) \\
&= E\left(e^{\frac{t}{\tau_1}}\right)E\left(e^{\frac{t}{\tau_2}}\right) \\
&= M_{\frac{1}{\tau_1}}(t) \times M_{\frac{1}{\tau_2}}(t) \\
&= \exp\left\{\lambda\left[1-\sqrt{1-\frac{2t}{\lambda^2\theta}}\right]\right\} \times \exp\left\{\lambda\left[1-\sqrt{1-\frac{2t}{\lambda^2\theta}}\right]\right\} \\
&= \exp\left\{2\lambda\left[1-\sqrt{1-\frac{2t}{\lambda^2\theta}}\right]\right\}.
\end{aligned}$$

□

By uniqueness property, it is implied that $X = \tau_1^{-1} + \tau_2^{-1} \sim IG[2\lambda, 1/(\lambda^2\theta)]$.

Theorem 2. *If a random variable $Y \sim TS\text{-}LBIG(\lambda,\theta)$, the moment generating function of Y is given as*

$$M_Y(t) = \left(1-2\left(\frac{\theta}{4}\right)t\right)^{-1/2}\exp\left\{2\lambda\left[1-\sqrt{1-2\left(\frac{\theta}{4}\right)t}\right]\right\}.$$

Proof of Theorem 2. By Theorem 1, it is known that $X \sim IG[2\lambda, 1/(\lambda^2\theta)]$. To find the MGF of the TS-LBIG random variable Y, the reciprocal of X is considered. Let $Y = 1/X = [\tau_1^{-1} + \tau_2^{-1}]^{-1}$. We know that a MGF of $LBIG[\lambda,\theta]$ distribution is defined as

$$M_{LBIG}(t;\lambda,\theta) = (1-2\theta t)^{-1/2}\exp\left\{\lambda[1-(1-2\theta t)^{1/2}]\right\}. \tag{10}$$

According to Proposition 4, $Y \sim LBIG[2\lambda, \theta/4]$. Therefore, the moment-generating function of Y is given by

$$M_Y(t) = \left(1-2\left(\frac{\theta}{4}\right)t\right)^{-1/2}\exp\left\{2\lambda\left[1-\sqrt{1-2\left(\frac{\theta}{4}\right)t}\right]\right\}.$$

□

Most importantly, by uniqueness property, it is indicated that $Y \sim TS\text{-}LBIG(\lambda, \theta) \sim LBIG[2\lambda, \theta/4]$.

7. The Probability Density Function of TS-LBIG Distribution

Theorem 3. *Define X as a random variable of the TS-LBIG distribution. Then, the corresponding probability density function (PDF) of X is given by*

$$f_{TS\text{-}LBIG}(x;\lambda,\theta) = \frac{2}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}} - 2\sqrt{\frac{x}{\theta}}\right)^2\right]. \tag{11}$$

Proof of Theorem 3. By Theorem 2, we know that $TS\text{-}LBIG(\lambda,\theta) \sim LBIG[2\lambda, \theta/4]$. We start with Equation (7), which is the original form of LBIG distribution, $f_{LBIG}(x;\lambda,\theta)$. Substituting Equation (7) by the parameter λ to be 2λ and θ to be $\theta/4$, the probability density function of the TS-LBIG distribution is

$$\begin{aligned} f_{TS\text{-}LBIG}(x;\lambda,\theta) &= \frac{1}{\frac{\theta}{4}\sqrt{2\pi}}\left(\frac{\frac{\theta}{4}}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(2\lambda\sqrt{\frac{\frac{\theta}{4}}{x}} - \sqrt{\frac{x}{\frac{\theta}{4}}}\right)^2\right] \\ &= \frac{4}{\theta\sqrt{2\pi}}\left(\frac{\theta}{4x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(2\lambda\sqrt{\frac{\theta}{4x}} - \sqrt{\frac{4x}{\theta}}\right)^2\right] \\ &= \frac{2}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}} - 2\sqrt{\frac{x}{\theta}}\right)^2\right]. \end{aligned} \tag{12}$$

□

Several shapes of the PDF for the TS-LBIG distribution are illustrated in Figures 1 and 2 for various parameter values. The different shapes indicate that the TS-LBIG distribution is right-skewed and unimodal. Moreover, this distribution is a family of asymmetric distributions which are useful for skewed data analysis.

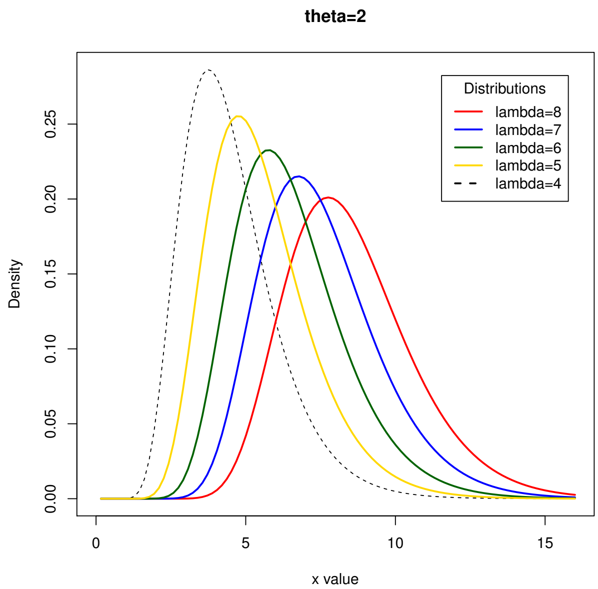

Figure 1. PDFs for the TS-LBIG distribution for several values of λ(*lambda*).

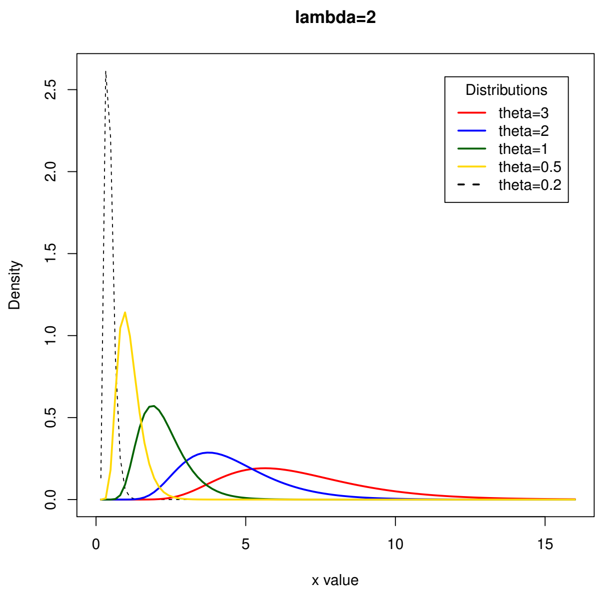

Figure 2. PDFs for the TS-LBIG distribution for several values of θ(*theta*).

8. The Cumulative Density Function of TS-LBIG Distribution

Theorem 4. *Let X be a random variable of the TS-LBIG distribution. The cumulative density function (CDF) of X is given by*

$$F_{TS\text{-}LBIG}(x) = \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \exp(4\lambda)\Phi\left[-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right], \tag{13}$$

where $\Phi(x)$ is the standard normal distribution function.

Proof of Theorem 4. The CDF of LBIG distribution is

$$F_{LBIG}(x) = \Phi\left(\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \exp(2\lambda)\Phi\left[-\left(\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right]. \tag{14}$$

Hence, the CDF of TS-LBIG is

$$\begin{aligned} F_{TS\text{-}LBIG}(x) &= \Phi\left(\sqrt{\frac{x}{\frac{\theta}{4}}} - 2\lambda\sqrt{\frac{\frac{\theta}{4}}{x}}\right) - \exp(4\lambda)\Phi\left[-\left(\sqrt{\frac{x}{\frac{\theta}{4}}} + 2\lambda\sqrt{\frac{\frac{\theta}{4}}{x}}\right)\right], \\ &= \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) - \exp(4\lambda)\Phi\left[-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right], \end{aligned} \tag{15}$$

where $\Phi(x)$ is the standard normal distribution function. □

9. The Survival Function of TS-LBIG Distribution

Theorem 5. *Let X be a random variable of the TS-LBIG distribution with parameters λ and θ. The survival function of X is obtained as:*

$$S_{TS\text{-}LBIG}(x) = 1 - \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \exp(4\lambda)\Phi\left[-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right], \tag{16}$$

where $\Phi(x)$ is the standard normal distribution function.

Proof of Theorem 5. Let X be a continuous random variable with a cumulative density function $F(x)$ on the interval $[0,\infty)$. The survival function of X can be written in this form:

$$S(x) = \int_x^\infty f(t)dt \quad = \quad 1 - F(x) \tag{17}$$

Inserting Equation (13) into Equation (17) leads to the survival function of TS-LBIG distribution in equation:

$$S_{TS\text{-}LBIG}(x) = 1 - \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \exp(4\lambda)\Phi\left[-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right], \tag{18}$$

where $\Phi(x)$ is the standard normal distribution function. □

10. The Hazard Rate Function of TS-LBIG Distribution

Theorem 6. *Let X be a random variable of the TS-LBIG distribution with parameters λ and θ. The hazard rate function of X is given by*

$$h_{TS\text{-}LBIG}(x) = \frac{\frac{2}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}} - 2\sqrt{\frac{x}{\theta}}\right)^2\right]}{1 - \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \exp(4\lambda)\Phi\left(-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right)}, \tag{19}$$

where $\Phi(x)$ is the standard normal distribution function.

Proof of Theorem 6. Let X be an absolutely continuous non-negative random variable with the probability density function $f(x)$ and the survival function $S(x)$; then, the hazard rate function of X can be defined as:

$$\begin{aligned} h(x) &= \frac{f(x)}{S(x)}, \\ &= \frac{\frac{2}{\theta\sqrt{2\pi}}\left(\frac{\theta}{x}\right)^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\lambda\sqrt{\frac{\theta}{x}} - 2\sqrt{\frac{x}{\theta}}\right)^2\right]}{1 - \Phi\left(2\sqrt{\frac{x}{\theta}} - \lambda\sqrt{\frac{\theta}{x}}\right) + \exp(4\lambda)\Phi\left(-\left(2\sqrt{\frac{x}{\theta}} + \lambda\sqrt{\frac{\theta}{x}}\right)\right)}, \end{aligned} \tag{20}$$

where $\Phi(x)$ is the standard normal distribution function. □

11. The Mean and the Variance of TS-LBIG Distribution

From [22], let $Y \sim IG(x;\lambda,\theta)$ and $Z \sim LB(x;\lambda,\theta)$; then, the rth moment of Z for $r = 1,2,3,\ldots$ is given by

$$E[Z^r] = \frac{1}{\lambda\theta}E[Z^{r+1}]. \tag{21}$$

Hence, the first four raw moments of the LBIG distribution are

$$\begin{aligned} E[Z] &= \theta(\lambda+1) \\ E[Z^2] &= \theta^2(\lambda^2+3\lambda+3) \\ E[Z^3] &= \theta^3(\lambda^3+6\lambda^2+15\lambda+15) \\ E[Z^4] &= \theta^4(\lambda^4+10\lambda^3+45\lambda^2+105\lambda+105). \end{aligned} \tag{22}$$

Let $X \sim TS\text{-}LBIG$. Therefore, the first four raw moments of TS-LBIG distribution are

$$\begin{aligned} E[X] &= \tfrac{\theta}{4}(2\lambda+1) \\ E[X^2] &= \left(\tfrac{\theta}{4}\right)^2(4\lambda^2+6\lambda+3) \\ E[X^3] &= \left(\tfrac{\theta}{4}\right)^3(8\lambda^3+24\lambda^2+30\lambda+5) \\ E[X^4] &= \left(\tfrac{\theta}{4}\right)^4(16\lambda^4+80\lambda^3+180\lambda^2+210\lambda+105). \end{aligned} \tag{23}$$

Therefore, the mean of TS-LBIG distribution is

$$E[X] = \frac{\theta}{4}(2\lambda+1), \tag{24}$$

and the variance of TS-LBIG distribution is

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= \tfrac{\theta^2}{16}(4\lambda^2+6\lambda+3) - \left[\tfrac{\theta^2}{16}(2\lambda+1)^2\right] \\ &= \tfrac{4\lambda^2\theta^2+6\lambda\theta^2+3\theta^2}{16} - \tfrac{4\lambda^2\theta^2+4\lambda\theta^2+\theta^2}{16} \\ &= \tfrac{2\lambda\theta^2+2\theta^2}{16} \\ &= \tfrac{\lambda\theta^2+\theta^2}{8}. \end{aligned} \tag{25}$$

12. Parameter Estimation by the Method of Moments for the TS-LBIG Distribution

Recall that the rth raw population moment is equal to the rth raw sample moment.

$$\frac{\theta}{4}(2\lambda+1) = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{26}$$

$$\left(\frac{\theta}{4}\right)^2(4\lambda^2+6\lambda+3) = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{27}$$

From Equation (26) and letting $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, we obtain

$$\begin{aligned} \tfrac{\theta}{4}(2\lambda+1) &= \bar{x} \\ 2\lambda\theta+\theta &= 4\bar{x} \\ 2\lambda\theta &= 4\bar{x}-\theta \\ \hat{\lambda} &= \tfrac{4\bar{x}-\theta}{2\theta}. \end{aligned} \tag{28}$$

Substituting Equation (28) into Equation (27) and letting $T = \frac{1}{n}\sum_{i=1}^{n} x_i^2$, we obtain

$$\begin{aligned} \tfrac{4\left(\frac{4\bar{x}-\theta}{2\theta}\right)^2\theta^2+6\left(\frac{4\bar{x}-\theta}{2\theta}\right)^2\theta^2+3\theta^2}{16} &= T \\ \tfrac{(4\bar{x}-\theta)^2+3\theta(4\bar{x}-\theta)+3\theta^2}{16} &= T \\ \theta^2+16\bar{x}^2+4\theta\bar{x} &= 16T \\ \theta^2+4\bar{x}\theta+16\bar{x}^2-16T &= 0. \end{aligned} \tag{29}$$

From $ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \tag{30}$$

where $a=1$, $b=4\bar{x}$, $c = 16\bar{x}^2 - 16T = 16(\bar{x}^2 - T)$.

$$\hat{\theta} = \frac{-4\bar{x} \pm \sqrt{16\bar{x}^2 - 64(\bar{x}^2-T)}}{2}. \tag{31}$$

Since $\hat{\theta}$ is positive parameter, hence

$$\begin{aligned} \hat{\theta} &= \frac{-4\bar{x}+\sqrt{16\bar{x}^2-64(\bar{x}^2-T)}}{2} \\ &= \frac{-4\bar{x}+4\sqrt{\bar{x}^2-4\bar{x}^2+4T}}{2} \\ &= -2\bar{x}+2\sqrt{-3\bar{x}^2+4T} \\ &= 2\left(\sqrt{4T-3\bar{x}^2}-\bar{x}\right). \end{aligned} \tag{32}$$

Next, substituting Equation (32) into Equation (28)

$$\hat{\lambda} = \frac{4\bar{x} - 2\left(\sqrt{4T-3\bar{x}^2}-\bar{x}\right)}{4\left(\sqrt{4T-3\bar{x}^2}-\bar{x}\right)}. \tag{33}$$

13. Numerical Results

13.1. Simulation Study

In this section, the Monte Carlo simulation to test the performance of the suggested estimators of the TS-LBIG distribution parameters is presented. Different values of the true parameters are considered. All 60 scenarios are the combination of sample size $(n) = 10, 50$ and 100, $\lambda = 1, 3, 5$ and 10, and $\theta = 0.5, 1, 3, 5$ and 10. The proposed estimators, $\hat{\lambda}$ and $\hat{\theta}$, are compared to the estimators presented by Simmachan et al. [20], $\tilde{\lambda}$ and $\tilde{\theta}$, via bias, MSE and variance. The random numbers of the TS-LBIG distribution are generated via the composition method using the "twoCrack" package [23] in R [24], and the replications are repeated 1000 times in each scenario. The parameter estimates with their bias, MSE and variance are reported in Tables 1–6. For easier consideration, bar charts are created and presented in Figures 3 and 4. The blue and yellow bars represent the proposed method and the method of Simmachan et al. [20], respectively. It reveals that the bias, MSE and variance become smaller as the sample size increases and the estimates become closer to the true value of parameters. For bias consideration, the estimators of λ give over-estimates for both methods. The bias of the proposed estimator is slightly smaller than that of the original estimator. On the other hand, the estimators of θ provide mostly under-estimates for both methods. The bias of the proposed estimator is much smaller than that of the original estimator. MSE and variance indicators have similar behavior. For parameter λ, the MSE and variance of the proposed estimator are slightly smaller than those of the original estimator. For parameter θ, however, the MSE and variance of the proposed estimator are much smaller than those of the original estimator. Interestingly, the bias, MSE and variance of the proposed estimators are superior to those of the estimators from the previous study.

Table 1. The average estimates, the bias, the mean squared errors, and the simulated variance of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$ for $n = 10$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
1	0.5	1.722	0.430	0.722	−0.070	2.355	0.057	1.834	0.052
	1	1.784	0.837	0.784	−0.163	2.120	0.251	1.505	0.224
	3	1.716	2.534	0.716	−0.466	1.696	2.094	1.183	1.877
	5	1.675	4.280	0.675	−0.720	1.649	5.289	1.193	4.770
	10	1.682	8.615	0.682	−1.385	1.893	23.492	1.428	21.575
3	0.5	4.526	0.442	1.526	−0.058	12.504	0.052	10.174	0.049
	1	4.881	0.838	1.881	−0.162	14.427	0.218	10.889	0.192
	3	4.536	2.604	1.536	−0.396	9.135	1.940	6.775	1.783
	5	4.587	4.361	1.587	−0.639	11.243	5.205	8.723	4.797
	10	4.586	8.640	1.586	−1.360	9.841	22.347	7.325	20.498

Table 1. *Cont.*

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
5	0.5	7.504	0.451	2.504	−0.049	28.831	0.063	22.562	0.060
	1	7.746	0.875	2.746	−0.125	40.752	0.220	33.211	0.205
	3	7.519	2.657	2.519	−0.343	32.710	1.823	26.367	1.705
	5	7.239	4.482	2.239	−0.518	22.589	4.973	17.575	4.705
	10	7.498	8.666	2.498	−1.334	25.350	19.060	19.109	17.281
10	0.5	14.842	0.436	4.842	−0.064	100.511	0.051	77.067	0.047
	1	14.325	0.887	4.325	−0.113	85.940	0.191	67.235	0.178
	3	14.980	2.656	4.980	−0.344	154.657	1.975	129.858	1.857
	5	15.466	4.376	5.466	−0.624	163.500	5.670	133.620	5.280
	10	14.174	9.105	4.174	−0.895	106.910	18.675	89.488	17.873

Table 2. The average estimates, the bias, the mean squared errors, and the simulated variance of the original estimators of $\tilde{\lambda}$ and $\tilde{\theta}$ for $n = 10$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
1	0.5	1.827	4.000	0.827	3.500	2.444	17.495	1.760	5.244
	1	1.887	1.928	0.887	0.928	2.220	1.962	1.433	1.102
	3	1.821	0.657	0.821	−2.343	1.790	5.616	1.117	0.125
	5	1.781	0.397	0.781	−4.603	1.739	21.231	1.129	0.045
	10	1.788	0.203	0.788	−9.797	1.980	95.986	1.359	0.012
3	0.5	4.581	0.654	1.581	0.154	12.573	0.130	10.074	0.107
	1	4.933	0.310	1.933	−0.690	14.515	0.500	10.779	0.024
	3	4.590	0.108	1.590	−2.892	9.210	8.368	6.682	0.003
	5	4.641	0.065	1.641	−4.935	11.318	24.359	8.624	0.001
	10	4.640	0.032	1.640	−9.968	9.917	99.354	7.229	0.000
5	0.5	7.541	0.262	2.541	−0.238	28.888	0.077	22.432	0.021
	1	7.782	0.127	2.782	−0.873	40.821	0.767	33.081	0.004
	3	7.555	0.043	2.555	−2.957	32.773	8.746	26.245	0.000
	5	7.276	0.026	2.276	−4.974	22.650	24.745	17.470	0.000
	10	7.534	0.012	2.534	−9.988	25.422	99.751	19.001	0.000
10	0.5	14.862	0.066	4.862	−0.434	100.579	0.189	76.944	0.001
	1	14.345	0.034	4.345	−0.966	86.000	0.934	67.121	0.000
	3	15.000	0.011	5.000	−2.989	154.718	8.933	129.720	0.000
	5	15.486	0.007	5.486	−4.993	163.566	24.934	133.471	0.000
	10	14.195	0.003	4.195	−9.997	106.960	99.931	89.366	0.000

Table 3. The average estimates, the bias, the mean squared errors, and the simulated variance of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$ for $n = 50$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
1	0.5	1.130	0.485	0.130	−0.015	0.142	0.017	0.125	0.017
	1	1.120	0.964	0.120	−0.036	0.127	0.058	0.113	0.057
	3	1.141	2.886	0.141	−0.114	0.138	0.525	0.119	0.512
	5	1.138	4.832	0.138	−0.168	0.150	1.627	0.131	1.599
	10	1.143	9.580	0.143	−0.420	0.136	5.899	0.116	5.722

Table 3. *Cont.*

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
3	0.5	3.285	0.484	0.285	−0.016	0.705	0.012	0.623	0.012
	1	3.251	0.979	0.251	−0.021	0.725	0.052	0.661	0.052
	3	3.298	2.905	0.298	−0.095	0.743	0.469	0.654	0.460
	5	3.295	4.822	0.295	−0.178	0.792	1.320	0.705	1.289
	10	3.232	9.820	0.232	−0.180	0.677	5.197	0.623	5.165
5	0.5	5.393	0.488	0.393	−0.012	1.694	0.012	1.539	0.011
	1	5.443	0.970	0.443	−0.030	1.873	0.050	1.677	0.049
	3	5.404	2.926	0.404	−0.074	1.842	0.414	1.678	0.408
	5	5.432	4.853	0.432	−0.147	1.758	1.177	1.571	1.156
	10	5.403	9.724	0.403	−0.276	1.654	4.850	1.492	4.774
10	0.5	10.729	0.488	0.729	−0.012	6.058	0.011	5.527	0.010
	1	10.667	0.983	0.667	−0.017	6.356	0.044	5.910	0.044
	3	10.692	2.935	0.692	−0.065	5.969	0.372	5.490	0.368
	5	10.809	4.841	0.809	−0.159	6.479	1.085	5.824	1.059
	10	10.628	9.843	0.628	−0.157	5.947	4.176	5.553	4.151

Table 4. The average estimates, the bias, the mean squared errors, and the simulated variance of the original estimators of $\tilde{\lambda}$ and $\tilde{\theta}$ for $n = 50$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	Bias ($\hat{\lambda}$)	Bias ($\hat{\theta}$)	MSE ($\hat{\lambda}$)	MSE ($\hat{\theta}$)	Var ($\hat{\lambda}$)	Var ($\hat{\theta}$)
1	0.5	1.251	4.600	0.251	4.100	0.175	18.484	0.112	1.670
	1	1.241	2.319	0.241	1.319	0.159	2.121	0.101	0.383
	3	1.260	0.755	0.260	−2.245	0.174	5.082	0.106	0.040
	5	1.258	0.458	0.258	−4.542	0.184	20.645	0.118	0.017
	10	1.263	0.226	0.263	−9.774	0.173	95.541	0.104	0.003
3	0.5	3.346	0.721	0.346	0.221	0.726	0.076	0.607	0.028
	1	3.312	0.365	0.312	−0.635	0.741	0.410	0.644	0.007
	3	3.358	0.120	0.358	−2.880	0.765	8.297	0.637	0.001
	5	3.355	0.073	0.355	−4.927	0.813	24.280	0.687	0.000
	10	3.293	0.037	0.293	−9.963	0.692	99.267	0.606	0.000
5	0.5	5.433	0.281	0.433	−0.219	1.709	0.052	1.521	0.004
	1	5.483	0.140	0.483	−0.860	1.891	0.741	1.658	0.001
	3	5.445	0.047	0.445	−2.953	1.857	8.721	1.659	0.000
	5	5.473	0.028	0.473	−4.972	1.776	24.721	1.553	0.000
	10	5.444	0.014	0.444	−9.986	1.670	99.719	1.473	0.000
10	0.5	10.751	0.074	0.751	−0.426	6.072	0.182	5.508	0.000
	1	10.690	0.037	0.690	−0.963	6.365	0.927	5.889	0.000
	3	10.714	0.012	0.714	−2.988	5.981	8.926	5.471	0.000
	5	10.831	0.007	0.831	−4.993	6.495	24.926	5.804	0.000
	10	10.650	0.004	0.650	−9.996	5.956	99.925	5.534	0.000

Table 5. The average estimates, the bias, the mean squared errors, and the simulated variance of the proposed estimators $\hat{\lambda}$ and $\hat{\theta}$ for $n = 100$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	**Bias** $(\hat{\lambda})$	**Bias** $(\hat{\theta})$	**MSE** $(\hat{\lambda})$	**MSE** $(\hat{\theta})$	**Var** $(\hat{\lambda})$	**Var** $(\hat{\theta})$
1	0.5	1.064	0.495	0.064	−0.005	0.065	0.009	0.061	0.009
	1	1.068	0.982	0.068	−0.018	0.064	0.032	0.059	0.032
	3	1.064	2.951	0.064	−0.049	0.063	0.297	0.059	0.295
	5	1.072	4.923	0.072	−0.077	0.063	0.875	0.058	0.869
	10	1.072	9.843	0.072	−0.157	0.068	3.525	0.062	3.501
3	0.5	3.134	0.492	0.134	−0.008	0.308	0.006	0.290	0.006
	1	3.145	0.983	0.145	−0.017	0.321	0.027	0.300	0.027
	3	3.148	2.941	0.148	−0.059	0.301	0.218	0.279	0.214
	5	3.113	4.969	0.113	−0.031	0.320	0.726	0.307	0.725
	10	3.115	9.933	0.115	−0.067	0.329	2.863	0.316	2.858
5	0.5	5.200	0.493	0.200	−0.007	0.746	0.006	0.706	0.006
	1	5.152	0.996	0.152	−0.004	0.747	0.024	0.724	0.024
	3	5.233	2.950	0.233	−0.050	0.861	0.226	0.806	0.223
	5	5.226	4.912	0.226	−0.088	0.760	0.613	0.709	0.606
	10	5.238	9.807	0.238	−0.193	0.817	2.506	0.760	2.469
10	0.5	10.310	0.496	0.310	−0.004	2.513	0.005	2.416	0.005
	1	10.402	0.984	0.402	−0.016	2.662	0.022	2.500	0.021
	3	10.407	2.954	0.407	−0.046	2.832	0.205	2.666	0.203
	5	10.336	4.948	0.336	−0.052	2.688	0.558	2.575	0.556
	10	10.393	9.832	0.393	−0.168	2.532	2.165	2.377	2.136

Table 6. The average estimates, the bias, the mean squared errors, and the simulated variance of the original estimators of $\tilde{\lambda}$ and $\tilde{\theta}$ for $n = 100$.

λ	θ	$\hat{\lambda}$	$\hat{\theta}$	**Bias** $(\hat{\lambda})$	**Bias** $(\hat{\theta})$	**MSE** $(\hat{\lambda})$	**MSE** $(\hat{\theta})$	**Var** $(\hat{\lambda})$	**Var** $(\hat{\theta})$
1	0.5	1.187	4.656	0.187	4.156	0.089	18.094	0.054	0.820
	1	1.191	2.326	0.191	1.326	0.089	1.954	0.053	0.196
	3	1.187	0.780	0.187	−2.220	0.087	4.953	0.052	0.024
	5	1.194	0.462	0.194	−4.538	0.089	20.603	0.051	0.008
	10	1.195	0.231	0.195	−9.769	0.093	95.427	0.055	0.002
3	0.5	3.195	0.736	0.195	0.236	0.319	0.071	0.281	0.015
	1	3.206	0.367	0.206	−0.633	0.333	0.404	0.291	0.004
	3	3.209	0.122	0.209	−2.878	0.315	8.284	0.271	0.000
	5	3.175	0.074	0.175	−4.926	0.329	24.264	0.298	0.000
	10	3.176	0.037	0.176	−9.963	0.338	99.260	0.307	0.000
5	0.5	5.241	0.284	0.241	−0.216	0.755	0.048	0.697	0.002
	1	5.193	0.144	0.193	−0.856	0.752	0.734	0.714	0.001
	3	5.274	0.047	0.274	−2.953	0.871	8.719	0.796	0.000
	5	5.267	0.028	0.267	−4.972	0.771	24.718	0.700	0.000
	10	5.278	0.014	0.278	−9.986	0.828	99.717	0.751	0.000
10	0.5	10.333	0.075	0.333	−0.425	2.518	0.180	2.407	0.000
	1	10.424	0.037	0.424	−0.963	2.671	0.927	2.491	0.000
	3	10.430	0.012	0.430	−2.988	2.841	8.925	2.656	0.000
	5	10.359	0.008	0.359	−4.992	2.694	24.925	2.565	0.000
	10	10.416	0.004	0.416	−9.996	2.541	99.925	2.368	0.000

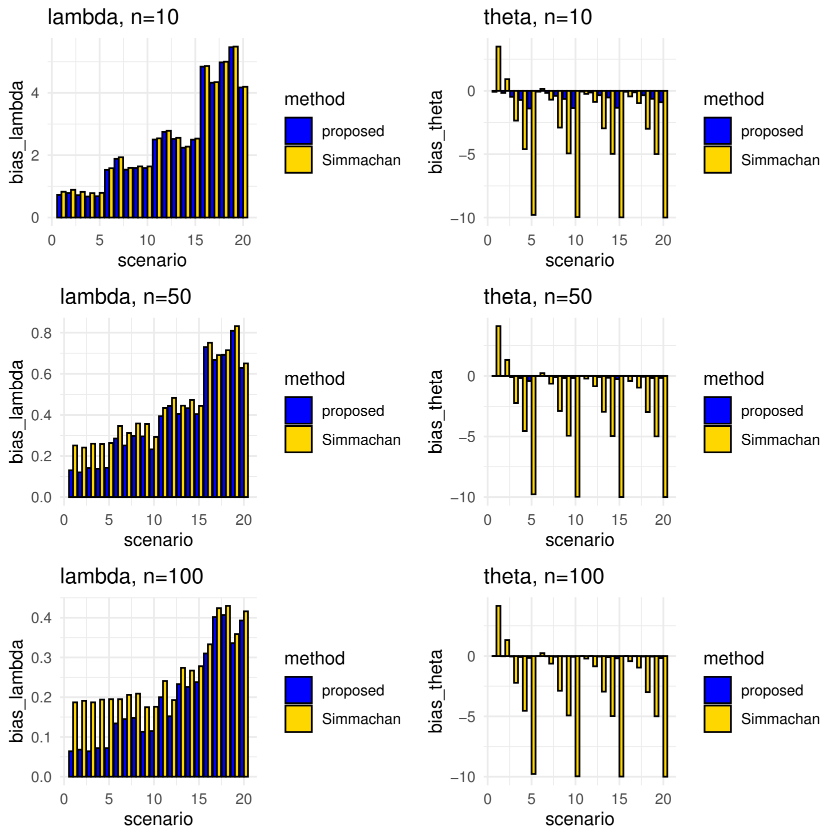

Figure 3. Bar chart for the bias of estimator λ(*lambda*) and θ(*theta*).

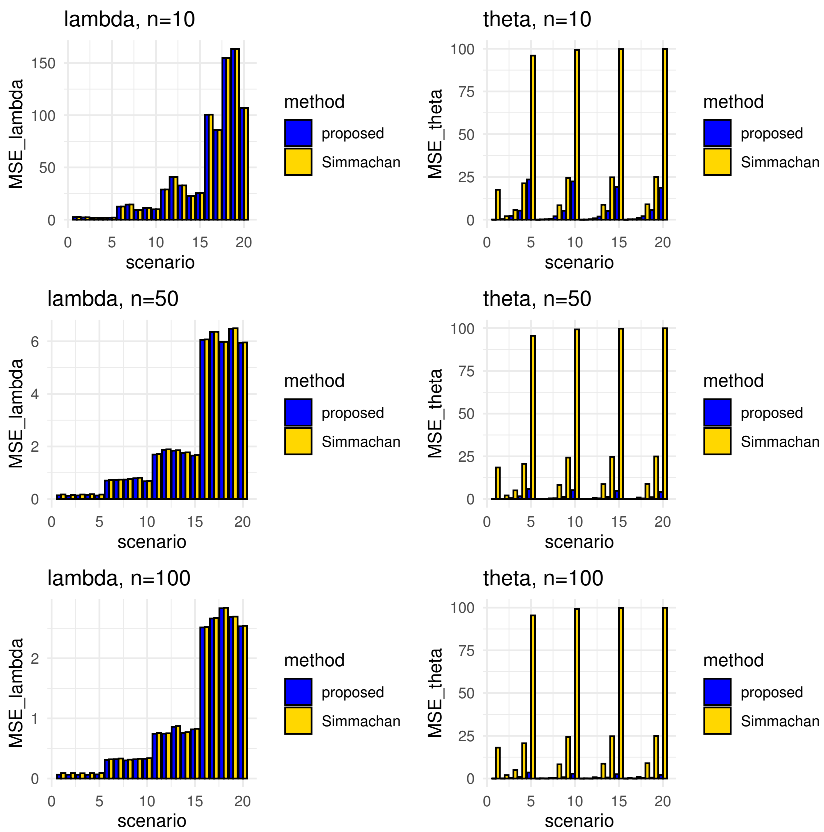

Figure 4. Bar chart for the MSE of estimator λ(*lambda*) and θ(*theta*).

13.2. Illustrative Examples

In this section, the suggested distribution is implemented via a real dataset. The following data are collected from the BackBlaze data center [25], and they present the lifetime of the hard drives (in days) containing only the model ST8000DM002 in December 2017: 490, 497, 521, 394, 489, 323, 376, 319, 431, 484, 547, 383, 534 and 316. This dataset was analyzed by Chananet and Phaphan [26], and the result indicated that the lifetime of the hard drives follow the right skewed distribution. Consequently, five right skewed distributions—two-parameter crack [18], Birnbaum–Saunders [16], inverse Gaussian [17], length-biased inverse Gaussian [17], and the proposed two-sided length-biased inverse Gaussian—are selected for goodness of fit comparison. The parameters of the TS-LBIG distribution are estimated by the suggested estimators. The parameters of other candidate distributions are estimated via the maximum likelihood estimation. The "nlminb" function in R [24] is employed for maximizing their likelihood functions. The Akaike information criterion (AIC) is used as an assessment criterion; hence, the best model is the one that provides the minimum AIC. As the result in Table 7 indicates, the TS-LBIG distribution gives the minimum AIC. This indicates that the proposed distribution is the best of the candidate distributions by considering at the value of AIC. Hence, by Equations (24) and (25), the average and standard deviation of the lifetime of the hard drives are 436 and 80.15877 days, respectively.

Table 7. The MLE of the model parameters for the hard drive failure dataset, and AIC measure.

Fitting Dist.	Estimate Parameters		AIC
	λ	θ	
TCR	110.9891	0.1000	13.39908
BS	0.001	0.100	67.6742
IG	0.001	0.001	93.44156
LBIG	26.79801	15.68458	9.106067
TS-LBIG	14.77663	57.08065	7.748027

14. Conclusions and Discussion

In this article, a new form of the TS-LBIG distribution is introduced, since the original version offered by Simmachan et al. [20] does not present a closed-form PDF. This distribution is a right-skewed distribution. Some distributional properties of this distribution were studied, and its two parameters were estimated using the method of moment. Sixty combination scenarios are used to construct the simulation study in assessing the performance of the proposed method. An application of the TS-LBIG distribution was implemented in the lifetime of the hard drives. Results show that the proposed estimators are more efficient than the Simmachan et al. [20] estimators. The original study dealing with the indirect method of parameter estimation affects the parameter estimates far from the true values, especially the parameter θ. This is different from the proposed estimators that dealt with the direct method. The TS-LBIG distribution gives a better fit than the other candidate distributions in terms of AIC. Our contribution provides an alternative right-skewed distribution that can be applied in other aspects such as survival analysis and forestry.

For the future directions of this work, other methods of parameter estimation could be considered. Confidence intervals of the parameters could be also examined. Additionally, the concept of the two-sided model could be extended to generate a new distribution. Moreover, other applications of the TS-LBIG distribution should be applied.

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