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Fractional Leindler's Inequalities via Conformable Calculus

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Abstract: In this paper, some fractional Leindler and Hardy-type inequalities and their reversed will be proved by using integration by parts and Hölder inequality on conformable fractional calculus. As a special case, some classical integral inequalities will be obtained. Symmetrical properties play an essential role in determining the correct methods to solve inequalities. The new fractional inequalities in special cases yield some recent relevance, which also provide new estimates on inequalities of these type.

Keywords: fractional Hardy's inequality; conformable fractional integral; conformable fractional derivative; Hölder inequality

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1. Introduction

The Hardy discrete inequality is known as

$$\sum_{s=1}^{\infty}\left(\frac{1}{s}\sum_{i=1}^{s}f(i)\right)^{n} \leq \left(\frac{n}{n-1}\right)^{n}\sum_{s=1}^{\infty}f^{n}(s), \quad n>1. \tag{1}$$

where f_s is a nonnegative sequence. Leindler, in [1,2], obtains some generalizations of the inequality (1) by using a new weighted function. Specifically, Leindler, in [1], proved the inequalities:

$$\sum_{r=1}^{\infty}\mu(r)\left(\sum_{k=1}^{r}f(k)\right)^{n} \leq n^{n}\sum_{r=1}^{\infty}\mu^{1-n}(r)\left(\sum_{k=r}^{\infty}\mu(k)\right)^{n}f^{n}(r), \tag{2}$$

and

$$\sum_{s=1}^{\infty}\mu(s)\left(\sum_{k=s}^{\infty}f(k)\right)^{n} \leq n^{n}\sum_{s=1}^{\infty}\mu^{1-n}(s)\left(\sum_{k=1}^{s}\mu(k)\right)^{n}f^{n}(s), \tag{3}$$

where $n>1$ and $\mu(s)>0$.

Copson in [3], established discrete new inequalities (see [4]). Particularly, one of them is presented as

$$\sum_{s=1}^{\infty}\left(\sum_{k=s}^{\infty}f(k)\right)^{n} \geq n^{n}\sum_{s=1}^{\infty}(sf(s))^{n}, \text{ for } 0<n<1, \tag{4}$$

where $\{f_s\}$ is a nonnegative sequence. Leindler [2] proved the reverse of inequalities (2) and (3). Particularly, he proved that

$$\sum_{s=1}^{\infty} \mu(s) \left(\sum_{k=1}^{s} f(k) \right)^n \geq n^n \sum_{s=1}^{\infty} \mu^{1-n}(s) \left(\sum_{k=s}^{\infty} \mu(k) \right)^n f^n(s), \tag{5}$$

and

$$\sum_{s=1}^{\infty} \mu(s) \left(\sum_{k=s}^{\infty} f(k) \right)^n \geq n^n \sum_{s=1}^{\infty} \mu^{1-n}(s) \left(\sum_{k=1}^{s} \mu(n) \right)^n f^n(s), \tag{6}$$

where $0 < n \leq 1$.

A fascinating variation of the inequalities of Hardy–Copson, was presented via Leindler [5]. Indeed, Leindler [5] extended the above-mentioned inequalities and demonstrated that if $\sum_{i=s}^{\infty} \mu(i) < \infty$, $n > 1$ and $0 \leq k < 1$, then

$$\sum_{s=1}^{\infty} \frac{\mu(s)}{(\Psi(s))^k} \left(\sum_{i=1}^{s} \mu(i)\, f(i) \right)^n \leq \left(\frac{n}{1-k} \right)^n \sum_{s=1}^{\infty} \mu(s)(\Psi(s))^{n-k} f^n(s), \tag{7}$$

where $\Psi_s = \sum_{i=s}^{\infty} \mu(i)$, if $1 < k \leq n$, we find

$$\sum_{s=1}^{\infty} \frac{\mu(s)}{(\Psi(s))^k} \left(\sum_{i=s}^{\infty} \mu(i) f(i) \right)^n \leq \left(\frac{n}{k-1} \right)^n \sum_{s=1}^{\infty} \mu(s)(\Psi(s))^{n-k} f^n(s). \tag{8}$$

In recent years, a lot of work has been published for fractional inequalities, the subject has become an active field of research, and several authors were interested in proving inequalities of fractional type by using the Riemann-Liouville and Caputo derivative, see [6–8], for more details about fractional-type inequalities.

In [9,10], the authors expanded fractional calculus to conformable calculus and gave a new definition of the derivative with the base properties of the calculus based on the new definition of derivatives and integrals. During the last few years, by using conformable fractional calculus, authors proved some integral inequalities, such as Hardy's inequality [11], Hermite–Hadamard's inequality [12–17], Opial's inequality [18,19], Steffensen's inequality [20], and Chebyshev's inequality [21]. Additionally, over several decads, many generalizations, extensions and refinements of other types of integral inequalities have been studied we refer the reader to the papers [22–26].

The main question that arises now is: is it possible to prove new α conformable fractional calculus. Therefore, it is natural to look at new fractional inequalities and give an affirmative answer to the above question. In particular, in this paper, we will prove the fractional forms of the Leindler and classical Hardy-type inequalities. The paper is coordinated as: In Section 2, we discuss the preliminaries and basic concepts of conformable fractional calculus, which will be required in proving our main Results. In Section 3, we introduce some fractional Leindler inequalities with their extensions. In Section 4, we demonstrate some reversed fractional Leindler inequalities with their extensions, in addition to sections of Results and Discussion and Conclusions and Future Work.

2. Preliminaries and Basic Concepts

In this part, we show the basics of conformable fractional integral and derivative of order $\alpha \in (0,1]$, that will be used in this paper (see [9,10]).

Definition 1. *The conformable fractional derivative of order α of $g : [0,\infty) \to \mathbb{R}$, is defined by*

$$D_\alpha g(x) = \lim_{\epsilon \to 0} \frac{g(x + \epsilon x^{1-\alpha}) - g(x)}{\epsilon},$$

$\forall x > 0,\ 0 < \alpha \leq 1$, *and* $D_\alpha g(0) = \lim_{x \to 0^+} D_\alpha g(x)$.

Assume $\alpha \in (0,1]$, and g, h be α-differentiable at x, then

$$D_\alpha(gh) = gD_\alpha h + hD_\alpha g, \tag{9}$$

further if $h(x) \neq 0$, then

$$D_\alpha\left(\frac{g}{h}\right) = \frac{hD_\alpha g - gD_\alpha h}{h^2}. \tag{10}$$

Remark 1. *For a differentiable function g, then*

$$D_\alpha g(x) = x^{1-\alpha}\frac{dg(x)}{dx}.$$

Definition 2. *The conformable fractional integral of order α of $g : [0,\infty) \to \mathbb{R}$, is defined by*

$$I_\alpha g(x) = \int_0^x g(s)d_\alpha s = \int_0^x s^{\alpha-1}g(s)ds, \tag{11}$$

$\forall x > 0$ *and* $\alpha \in (0,1]$.

Lemma 1 (Integration by parts formula)**.** *Suppose the two functions $u, v : [0,\infty) \to \mathbb{R}$ are α-differentiable and $\alpha \in (0,1]$, then for any $d > 0$,*

$$\int_0^d u(t)D_\alpha v(t)d_\alpha t = u(t)v(t)|_0^d - \int_0^d v(t)D_\alpha u(t)d_\alpha t. \tag{12}$$

Lemma 2 (Hölder inequality)**.** *Let $g, h : [0,\infty) \to \mathbb{R}$ and $0 < \alpha \leq 1$.Then for any $d > 0$,*

$$\int_0^d |g(t)h(t)|d_\alpha t \leq \left(\int_0^d |g(t)|^n d_\alpha t\right)^{\frac{1}{n}} \left(\int_0^d |h(t)|^m d_\alpha t\right)^{\frac{1}{m}}, \tag{13}$$

at $1/n + 1/m = 1$ (where existing the integrals).

The Hardy conformable fractional operator is defined as

$$Hf(x) = \int_0^x f(t)d_\alpha t, \tag{14}$$

and its dual

$$H^* f(x) = \int_x^\infty f(t)d_\alpha t. \tag{15}$$

Through our paper, we consider that the given integrals exist (are finite, i.e., convergent).

3. Fractional Leindler-Type Inequalities

Here, we will prove some fractional Leindler-type inequalities and their extensions for α-differentiable functions and obtain the classical ones at $\alpha = 1$.

Theorem 1. *If $n > 1$ and $\Omega(y) := \int_y^\infty \theta(t)d_\alpha t$, then*

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y \leq n^n \int_0^\infty \theta^{1-n}(y)\Psi^n(y)f^n(y)d_\alpha y. \tag{16}$$

Proof. Using the integration by parts Formula (12) on $\int_0^\infty \theta(y)H^n f(y)d_\alpha y$, with

$$u(y) = H^n f(y), \text{ and } v(y) = -\Psi(y),$$

we find that

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y = -H^n f(y)\Psi(y)|_0^\infty + n\int_0^\infty y^{1-\alpha}H^{n-1}f(y)(y)H^{'}f(y)\Psi(y)d_\alpha y,$$
$$= n\int_0^\infty y^{1-\alpha}H^{n-1}f(y)(y)H^{'}f(y)\Psi(y)d_\alpha y, \tag{17}$$

where

$$H\,f(\infty) < \infty,\ H\,f(0) = 0,\ \Psi\ (\infty) = 0 \text{ and } \Psi(0) < \infty\,.$$

From (14), we obtain:

$$H^{'}f(y) = y^{\alpha-1}f(y).$$

Substituting into (17), we have

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y = n\int_0^\infty \frac{f(y)\Psi(y)}{(\theta(y))^{\frac{n-1}{n}}}(\theta(y))^{\frac{n-1}{n}}H^{n-1}f(y)d_\alpha y. \tag{18}$$

Using Hölder inequality (13) over the right part of (18) by indices $n/(n-1)$ and n, then

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y \leq$$
$$n\left(\int_0^\infty\left(\frac{f(y)\Psi(y)}{(\theta(y))^{\frac{n-1}{n}}}\right)^n d_\alpha y\right)^{1/n}\left(\int_0^\infty\left((\theta(y))^{\frac{n-1}{n}}H^{n-1}f(y)\right)^{\frac{n}{n-1}}d_\alpha y\right)^{\frac{n-1}{n}},$$

thus

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y \leq n\left(\int_0^\infty \frac{(f(y)\Psi(y))^n}{\theta^{n-1}(y)}d_\alpha y\right)^{1/n}\left(\int_0^\infty \theta(y)H^n f(y)d_\alpha y\right)^{1-1/n},$$

as

$$\left(\int_0^\infty \theta(y)H^n f(y)d_\alpha y\right)^{1-1/n} > 0,$$

then

$$\left(\int_0^\infty \theta(y)y^n f(y)d_\alpha y\right)^{\frac{1}{n}} \leq n\left(\int_0^\infty \frac{(f(y)\Psi(y))^n}{\theta^{n-1}(y)}d_\alpha y\right)^{1/n}.$$

This leads to

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y \leq n^n\int_0^\infty \theta^{1-n}(y)\Psi^n(y)f^n(y)d_\alpha y,$$

which is the wanted inequality (16). □

Remark 2. *If* $\alpha = 1$, *in Theorem 1, we obtain the inequality:*

$$\int_0^\infty \theta(y)\left(\int_0^y f(t)dt\right)^n dy \leq n^n\int_0^\infty \theta^{1-n}(y)\left(\int_y^\infty \theta(t)dt\right)^n f^n(y)dy. \tag{19}$$

Remark 3. *If* $\theta(y) = 1/y^n$, *in Theorem 1, we obtain the inequality:*

$$\int_0^\infty\left(\frac{1}{y}\int_0^y f(t)d_\alpha t\right)^n d_\alpha y \leq \left(\frac{n}{n-\alpha}\right)^n\int_0^\infty\left(y^{\alpha-1}f(y)\right)^n d_\alpha y. \tag{20}$$

which is the α*-fractional Hardy inequality.*

Remark 4. *As a result, if $\alpha = 1$ in (20), we obtain the classical Hardy inequality:*

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(t)dt\right)^n dy \le \left(\frac{n}{n-1}\right)^n \int_0^\infty f^n(y)dy. \tag{21}$$

Theorem 2. *If $\Phi(y) := \int_0^y \theta(t)d_\alpha t$ and $n > 1$, we find that*

$$\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y \le n^n \int_0^\infty \theta^{1-n}(y)\Phi^n(y)f^n(y)d_\alpha y. \tag{22}$$

Proof. Using the integration by parts Formula (12) on $\int_0^\infty \theta(y)y^{*^n} f(y)d_\alpha y$, with

$$u(y) = H^{*^n} f(y) \text{ and } v(y) = \Phi(y),$$

we have

$$\begin{aligned}\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y &= H^{*^n} f(y)\Phi(y)\Big|_0^\infty - n\int_0^\infty y^{1-\alpha}H^{*^{n-1}} f(y)H^{*'}f(y)\Phi(y)d_\alpha y,\\ &= -n\int_0^\infty y^{1-\alpha}H^{*^{n-1}} f(y)H^{*'}f(y)\Phi(y)d_\alpha y,\end{aligned} \tag{23}$$

where

$$H^* f(\infty) = 0, H^* f(0) < \infty, \Phi(\infty) < \infty \text{ and } \Phi(0) = 0.$$

From (15), we obtain

$$H^{*'} f(y) = -y^{\alpha-1} f(y).$$

Substituting into (23), we have

$$\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y = n\int_0^\infty \frac{f(y)\Phi(y)}{(\theta(y))^{\frac{n-1}{n}}}(\theta(y))^{\frac{n-1}{n}}H^{*^{n-1}} f(y)d_\alpha y. \tag{24}$$

Using the Hölder inequality (13) over the right part of (24) by indices $n/(n-1)$ and n, we find that

$$\begin{aligned}\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y \le{}& n\left(\int_0^\infty \left(\frac{f(y)\Phi(y)}{(\theta(y))^{\frac{n-1}{n}}}\right)^n d_\alpha y\right)^{1/n}\\ &\times\left(\int_0^\infty \left((\theta(y))^{\frac{n-1}{n}}H^{*^{n-1}} f(y)\right)^{\frac{n}{n-1}} d_\alpha y\right)^{\frac{n-1}{n}}.\end{aligned}$$

Then,

$$\left(\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y\right)^{\frac{1}{n}} \le n\left(\int_0^\infty \frac{(f(y)\Phi(y))^n}{\theta^{n-1}(y)}d_\alpha y\right)^{1/n},$$

thus

$$\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y \le n^n\int_0^\infty \theta^{1-n}(y)\Phi^n(y)f^n(y)d_\alpha y,$$

which is the wanted inequality (22). □

Remark 5. *If $\alpha = 1$, in Theorem 2, we achieve the inequality:*

$$\int_0^\infty \theta(y)\left(\int_y^\infty f(t)dt\right)^n dy \le n^n\int_0^\infty \theta^{1-n}(y)\left(\int_0^y \theta(t)dt\right)^n f^n(y)dy. \tag{25}$$

Remark 6. *If* $\theta(y) = 1$, *in Theorem 2, we arrive at the inequality:*

$$\int_0^\infty \left(\int_y^\infty f(t) d_\alpha t \right)^n d_\alpha y \leq \left(\frac{n}{\alpha} \right)^n \int_0^\infty (y^\alpha f(y))^n d_\alpha y, \tag{26}$$

which is the α*-fractional Hardy inequality.*

Remark 7. *As a result, if* $\alpha = 1$, *in (26), we obtain the Hardy inequality:*

$$\int_0^\infty \left(\int_y^\infty f(t) dt \right)^n dy \leq n^n \int_0^\infty (y f(y))^n dy. \tag{27}$$

Theorem 3. *If* $n \geq 1$ *and* $0 \leq k < 1$, *then*

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Lambda^n(y) d_\alpha y \leq \left(\frac{n}{1-k} \right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)} f^n(y) d_\alpha y, \tag{28}$$

where

$$\Psi(y) := \int_y^\infty \theta(t) d_\alpha t \text{ and } \Lambda(y) := \int_0^y \theta(t) f(t) d_\alpha t.$$

Proof. Employing the formula of integration by parts (12) on $\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Lambda^n(y) d_\alpha y$, with

$$u(y) = \Lambda^n(y) \text{ and } v(y) = -\frac{\Psi^{1-k}(y)}{1-k},$$

also

$$D_\alpha u(y) = n y^{1-\alpha} \Lambda^{n-1}(y) \Lambda'(y) \text{ and } D_\alpha v(y) = \theta(y) \Psi^{-k}(y),$$

thus

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Lambda^n(y) d_\alpha y = -\frac{\Lambda^n(y) \Psi^{1-k}(y)}{1-k} \bigg|_0^\infty + \int_0^\infty \frac{n y^{1-\alpha} \Lambda^{n-1}(y) \Lambda'(y) \Psi^{1-k}(y)}{1-k} d_\alpha y.$$

From

$$\Lambda'(y) = y^{\alpha-1} \theta(y) f(y), \Lambda(\infty) < \infty, \Lambda(0) = 0, \Psi(\infty) = 0 \text{ and } \Psi(0) < \infty,$$

we get

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Lambda^n(y) d_\alpha y = -\frac{n}{1-k} \int_0^\infty y^{1-\alpha} \Lambda^{n-1}(y) y^{\alpha-1} \theta(y) f(y) \Psi^{1-k}(y) d_\alpha y,$$

hence

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Lambda^n(y) d_\alpha y = \frac{n}{1-k} \int_0^\infty \frac{\theta(y) \Psi^{1-k}(y) f(y)}{\left(\frac{\theta(y)}{\Psi^k(y)} \right)^{\frac{n-1}{n}}} \left(\frac{\theta(y)}{\Psi^k(y)} \right)^{\frac{n-1}{n}} \Lambda^{n-1}(y) d_\alpha y. \tag{29}$$

Using the Hölder inequality (13) over the right part of (29) by indices $n/n-1$ and n, we find that

$$
\begin{aligned}
&\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Lambda^n(y)d_\alpha y,\\
&\leq \frac{n}{1-k}\left(\int_0^\infty \left(\theta(y)\Psi^{1-k}(y)f(y)\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{-\frac{n-1}{n}}\right)^n d_\alpha y\right)^{\frac{1}{n}}\\
&\times \left(\int_0^\infty \left(\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{\frac{n-1}{n}}\Lambda^{n-1}(y)\right)^{\frac{n}{n-1}} d_\alpha y\right)^{\frac{n-1}{n}},\\
&= \frac{n}{1-k}\left(\int_0^\infty \theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y\right)^{\frac{1}{n}}\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Lambda^n(y)d_\alpha y\right)^{\frac{n-1}{n}}.
\end{aligned}
$$

So, we have

$$
\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Lambda^n(y)d_\alpha y\right)^{\frac{1}{n}} \leq \frac{n}{1-k}\left(\int_0^\infty \theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y\right)^{\frac{1}{n}},
$$

hence

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Lambda^n(y)d_\alpha y \leq \left(\frac{n}{1-k}\right)^n \int_0^\infty \theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y,
$$

which is the wanted inequality (28). □

Remark 8. *If $\alpha = 1$, in Theorem 3, we obtain:*

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\left(\int_0^y \theta(t)f(t)dt\right)^n dy \leq \left(\frac{n}{1-k}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)}f^n(y)dy, \tag{30}
$$

where $\Psi(y) = \int_y^\infty \theta(t)dt$, $0 \leq k < 1$ and $n > 1$.

Theorem 4. *If $n > k-1$ and $1 < k \leq n$, then*

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y \leq \left(\frac{n}{k-1}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)}f^n(y)d_\alpha y, \tag{31}
$$

where

$$
\Psi(y) := \int_y^\infty \theta(t)d_\alpha t \text{ and } \Phi(y) := \int_y^\infty \theta(t)f(t)d_\alpha t.
$$

Proof. Employing the formula of integration by parts (12) on $\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y$, with

$$
u(y) = \Phi^n(y) \text{ and } v(y) = -\frac{\Psi^{1-k}(y)}{1-k},
$$

also

$$
D_\alpha u(y) = ny^{1-\alpha}\Phi^{n-1}(y)\Phi'(y) \text{ and } D_\alpha v(y) = \theta(y)\Psi^{-k}(y),
$$

we have

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y = -\frac{\Phi^n(y)\Psi^{1-k}(y)}{1-k}\Bigg|_0^\infty + \int_0^\infty \frac{ny^{1-\alpha}\Phi^{n-1}(y)\Phi'(y)\Psi^{1-k}(y)}{1-k}d_\alpha y.
$$

By using

$\Phi(\infty) = 0, \Phi(0) < \infty, \Psi(\infty) = 0, \Psi(0) < \infty, \Phi'(y) = -y^{\alpha-1}\theta(y)f(y)$ and $k > 1$, and noting

$$\lim_{y\to\infty}\Psi^{\frac{1-k}{n}}(y)\Phi(y)=\lim_{y\to\infty}\frac{\int_y^\infty t^{\alpha-1}\theta(t)f(t)\,dt}{(\Psi(y))^{\frac{k-1}{n}}},$$
$$=\lim_{y\to\infty}\frac{-y^{\alpha-1}\theta(y)f(y)}{-\frac{k-1}{n}(\Psi(y))^{\frac{k-1}{n}-1}y^{\alpha-1}\theta(y)},$$
$$=\lim_{y\to\infty}\frac{n(\Psi(y))^{1-\frac{k-1}{n}}f(y)}{k-1}=0,$$

we find,

$$\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\le\frac{n}{k-1}\int_0^\infty y^{1-\alpha}\Phi^{n-1}(y)y^{\alpha-1}\theta(y)f(y)\Psi^{1-k}(y)d_\alpha y,$$

then

$$\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\le\frac{n}{k-1}\int_0^\infty\frac{\theta(y)\Psi^{1-k}(y)f(y)}{\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{\frac{n-1}{n}}}\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{\frac{n-1}{n}}\Phi^{n-1}(y)d_\alpha y. \tag{32}$$

Using Hölder inequality (13) over the right part of (32) by indices $n/(n-1)$ and n, then

$$\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y,$$
$$\le\frac{n}{k-1}\left(\int_0^\infty\left(\theta(y)\Psi^{1-k}(y)f(y)\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{-\frac{n-1}{n}}\right)^n d_\alpha y\right)^{\frac{1}{n}}$$
$$\times\left(\int_0^\infty\left(\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{\frac{n-1}{n}}\Phi^{n-1}(y)\right)^{\frac{n}{n-1}}d_\alpha y\right)^{\frac{n-1}{n}},$$
$$=\frac{n}{k-1}\left(\int_0^\infty\theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y\right)^{\frac{1}{n}}\left(\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\right)^{\frac{n-1}{n}}.$$

Therefore, we have

$$\left(\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\right)^{\frac{1}{n}}\le\frac{n}{k-1}\left(\int_0^\infty\theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y\right)^{\frac{1}{n}},$$

and hence

$$\int_0^\infty\frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\le\left(\frac{n}{k-1}\right)^n\int_0^\infty\theta(y)\Psi^{n-k}(y)f^n(y)d_\alpha y,$$

which is the wanted inequality (31). □

Remark 9. *From Theorem 4, we find that if $y^{\alpha-1}\theta(y)f(y)$ and $y^{\alpha-1}\theta(y)$ are continuous on $[0,\infty)$ exchanged either by:*

(i) *$y^{\alpha-1}\theta(y)f(y)$, $y^{\alpha-1}\theta(y)$ is continuous on $(0,\infty)$ and $\lim_{y\to\infty}(\Psi(y))^{1-\frac{k-1}{n}}f(y)=0$;*
(ii) *$\lim_{y\to\infty}\Psi^{1-k}(y)\Phi^n(y)=0$;*

then (31) is also true.

Remark 10. *If $\alpha = 1$, in Theorem 4, we find:*

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \left(\int_y^\infty \theta(t) f(t) dt \right)^n dy \leq \left(\frac{n}{k-1} \right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)} f^n(y) dy, \tag{33}$$

where $1 < k \leq n$ and $\Psi(y) = \int_y^\infty \theta(t) dt$.

4. Reversed Fractional Leindler Inequalities

In this section, we will deduce some reversed fractional inequalities and some fractional extensions.

Theorem 5. *If $\Psi(y) := \int_y^\infty \theta(t) d_\alpha t$ and $0 < n \leq 1$, then*

$$\int_0^\infty \theta(y) H^n f(y) d_\alpha y \geq n^n \int_0^\infty \theta^{1-n}(y) \Psi^n(y) f^n(y) d_\alpha y. \tag{34}$$

Proof. Using the integration by parts Formula (12) on $\int_0^\infty \theta(y) y^n f(y) d_\alpha y$, with

$$u(y) = H^n f(y) \text{ and } v(y) = -\Psi(y),$$

and we obtain

$$\int_0^\infty \theta(y) H^n f(y) d_\alpha y = -H^n f(y) \Psi(y)|_0^\infty + \int_0^\infty n y^{1-\alpha} H^{n-1} f(y) H' f(y) \Psi(y) d_\alpha y.$$

By using

$$H\, f(\infty) < \infty, H\, f(0) = 0, \Psi(\infty) = 0 \text{ and } \Psi(0) < \infty,$$

and from (14), we get

$$H' f(y) = y^{\alpha-1} f(y),$$

so, we obtain

$$\int_0^\infty \theta(y) H^n f(y) d_\alpha y = n \int_0^\infty y^{1-\alpha} H^{n-1} f(y) y^{\alpha-1} f(y) \Psi(y) d_\alpha y,$$

then

$$\int_0^\infty \theta(y) H^n f(y) d_\alpha y = n \int_0^\infty \left(\frac{\Psi^n(y) f^n(y)}{H^{n(1-n)} f(y)} \right)^{\frac{1}{n}} d_\alpha y,$$

which can be reformed in the shape

$$\left(\int_0^\infty \theta(y) H^n f(y) d_\alpha y \right)^n = n^n \left(\int_0^\infty \left(\frac{\Psi^n(y) f^n(y)}{H^{n(1-n)} f(y)} \right)^{\frac{1}{n}} d_\alpha y \right)^n. \tag{35}$$

Using Hölder's inequality

$$\int_0^\infty g(y) h(y) d_\alpha y \leq \left(\int_0^\infty g^a(y) d_\alpha y \right)^{\frac{1}{a}} \left(\int_0^\infty h^b(y) d_\alpha y \right)^{\frac{1}{b}},$$

with $a = 1/n$, $b = 1/(1-n)$,

$$g(y) = \frac{\Psi^n(y) f^n(y)}{H^{n(1-n)} f(y)} \text{ and } h(y) = \theta^{1-n}(y) H^{n(1-n)} f(y),$$

we get

$$\begin{aligned}
\left(\int_0^\infty g^{\frac{1}{n}}(y)d_\alpha y\right)^n &= \left(\int_0^\infty \left(\frac{\Psi^n(y)f^n(y)}{H^{n(1-n)}f(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n, \\
&\geq \frac{\int_0^\infty |g(y)h(y)|d_\alpha y}{\left(\int_0^\infty h^{\frac{1}{1-n}}(y)d_\alpha y\right)^{1-n}} = \int_0^\infty \frac{\Psi^n(y)f^n(y)}{H^{n(1-n)}f(y)}\theta^{1-n}(y)H^{n(1-n)}f(y)d_\alpha y \\
&\times \left(\int_0^\infty \left(\theta^{1-n}(y)H^{n(1-n)}f(y)\right)^{\frac{1}{1-n}} d_\alpha y\right)^{n-1},
\end{aligned}$$

then

$$\left(\int_0^\infty \left(\frac{\Psi^n(y)f^n(y)}{H^{n(1-n)}f(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n \geq \left(\int_0^\infty \Psi^n(y)f^n(y)\theta^{1-n}(y)d_\alpha y\right) \tag{36}$$

$$\times\left(\int_0^\infty (\theta(y)H^n f(y))d_\alpha y\right)^{n-1}. \tag{37}$$

Substituting (37) into (35), we obtain

$$\left(\int_0^\infty \theta(y)H^n f(y)d_\alpha y\right)^n \geq n^n \frac{\int_0^\infty \Psi^n(y)f^n(y)\theta^{1-n}(y)d_\alpha y}{\left(\int_0^\infty (\theta(y)H^n f(y))d_\alpha y\right)^{1-n}}.$$

Thus

$$\int_0^\infty \theta(y)H^n f(y)d_\alpha y \geq n^n \int_0^\infty \theta^{1-n}(y)\Psi^n(y)f^n(y)d_\alpha y,$$

which is the wanted inequality (34). □

Remark 11. *If $\alpha = 1$, in Theorem 5, we get the inequality:*

$$\int_0^\infty \theta(y)\left(\int_0^y f(t)dt\right)^n dy \geq n^n \int_0^\infty \theta^{1-n}(y)\left(\int_y^\infty \theta(t)dt\right)^n f^n(y)dy. \tag{38}$$

Remark 12. *If $\theta(y) = 1/y^n$ and $n > \alpha$, in Theorem 5, we get the inequality:*

$$\int_0^\infty \left(\frac{\int_0^y f(t)d_\alpha t}{y}\right)^n d_\alpha y \geq \left(\frac{n}{n-\alpha}\right)^n \int_0^\infty \left(y^{\alpha-1}f(y)\right)^n d_\alpha y, \tag{39}$$

which is the fractional reversed Hardy inequality.

Remark 13. *If $\alpha = 1$ in (39), we get the reversed Hardy inequality:*

$$\int_0^\infty \left(\frac{\int_0^y f(t)dt}{y}\right)^n dy \geq \left(\frac{n}{n-1}\right)^n \int_0^\infty f^n(y)dy. \tag{40}$$

Theorem 6. *If $0 < n \leq 1$ and $\Phi(y) := \int_0^y \theta(t)d_\alpha t$, then*

$$\int_0^\infty \theta(y)H^{*^n} f(y)d_\alpha y \geq n^n \int_0^\infty \theta^{1-n}(y)\Phi^n(y)f^n(y)d_\alpha y. \tag{41}$$

Proof. Employing the formula of integration by parts (12) on $\int_0^\infty \theta(y)y^{*^n} f(y)d_\alpha y$, with

$$u(y) = H^{*^n} f(y) \text{ and } v(y) = \Phi(y),$$

we obtain

$$\int_0^\infty \theta(y) y^{*^n} f(y) d_\alpha y = H^{*^n} f(y)\Phi(y)\Big|_0^\infty - \int_0^\infty n y^{1-\alpha} H^{*^{n-1}} f(y) H^{*'} f(y)\Phi(y) d_\alpha y.$$

Since

$$H^* f(\infty) = 0, H^* f(0) < \infty, \Phi(\infty) < \infty \text{ and } \Phi(0) = 0,$$

and from (15), we find that

$$H^{*'} f(y) = -y^{\alpha-1} f(y),$$

so

$$\int_0^\infty \theta(y) H^{*^n} f(y) d_\alpha y = n \int_0^\infty y^{1-\alpha} H^{*^{n-1}} f(y) y^{\alpha-1} f(y)\Phi(y) d_\alpha y,$$

then

$$\int_0^\infty \theta(y) H^{*^n} f(y) d_\alpha y = n \int_0^\infty \left(\frac{\Phi^n(y) f^n(y)}{H^{*^{n(1-n)}} f(y)}\right)^{\frac{1}{n}} d_\alpha y,$$

which can be reformed in the shape

$$\left(\int_0^\infty \theta(y) H^{*^n} f(y) d_\alpha y\right)^n = n^n \left(\int_0^\infty \left(\frac{\Phi^n(y) f^n(y)}{H^{*^{n(1-n)}} f(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n. \tag{42}$$

Similar to the proof of Theorem 5, we obtain

$$\int_0^\infty \theta(y) H^{*^n} f(y) d_\alpha y \geq n^n \int_0^\infty \theta^{1-n}(y)\Phi^n(y) f^n(y) d_\alpha y,$$

which is the wanted inequality (41). □

Remark 14. *If $\alpha = 1$, in Theorem 6, we find:*

$$\int_0^\infty \theta(y)\left(\int_y^\infty f(t)dt\right)^n dy \geq n^n \int_0^\infty \theta^{1-n}(y)\left(\int_0^y \theta(t)dt\right)^n f^n(y)dy. \tag{43}$$

Remark 15. *If $\theta(y) = 1$ and $n \geq \alpha$, in Theorem 6, we find:*

$$\int_0^\infty \left(\int_y^\infty f(t)d_\alpha t\right)^n d_\alpha y \geq \left(\frac{n}{\alpha}\right)^n \int_0^\infty (y^\alpha f(y))^n d_\alpha y,$$

since $(n/\alpha)^n > 1$, then

$$\int_0^\infty \left(\int_y^\infty f(t)d_\alpha t\right)^n d_\alpha y \geq \int_0^\infty (y^\alpha f(y))^n d_\alpha y, \tag{44}$$

which is the fractional reversed Hardy inequality.

Remark 16. *If $\alpha = 1$ in (44), we have the reversed Hardy inequality:*

$$\int_0^\infty \left(\int_y^\infty f(t)dt\right)^n dy \geq \int_0^\infty (yf(y))^n dy. \tag{45}$$

Theorem 7. *If $k \leq 0 < n < 1$, then*

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Omega^n(y) d_\alpha y \geq \left(\frac{n}{1-k}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)} f^n(y) d_\alpha y, \tag{46}$$

where

$$\Psi(y) := \int_y^\infty \theta(t) d_\alpha t \text{ and } \Omega(y) := \int_0^y \theta(t) f(t) d_\alpha t.$$

Proof. Employing the formula of integration by parts (12) on $\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Omega^n(y) d_\alpha y$, with

$$u(y) = \Omega^n(y) \Psi^{-k}(y) \text{ and } v(y) = -\Psi(y),$$

where

$$D_\alpha u(y) = y^{1-\alpha} \Big(n \Omega^{n-1}(y) \Omega'(y) \Psi^{-k}(y) - k \Omega^n(y) \Psi^{-k-1}(y) \Psi'(y) \Big),$$

then

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Omega^n(y) d_\alpha y = -\Psi^{1-k}(y) \Omega^n(y) \Big|_0^\infty + \int_0^\infty y^{1-\alpha} \Big(n \Omega^{n-1}(y) \Omega'(y) \Psi^{-k}(y) - k \Omega^n(y) \Psi^{-k-1}(y) \Psi'(y) \Big) \Psi(y) d_\alpha y.$$

Since

$$\Omega'(y) = y^{\alpha-1} \theta(y) f(y), \Psi'(y) = -y^{\alpha-1} \theta(y), \Omega(\infty) < \infty, \Omega(0) = 0, \Psi(0) < \infty \text{ and } \Psi(\infty) = 0.$$

We find that

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Omega^n(y) d_\alpha y = n \int_0^\infty \theta(y) f(y) \Psi^{1-k}(y) \Omega^{n-1}(y) d_\alpha y + k \int_0^\infty \frac{\Omega^n(y) \theta(y)}{\Psi^k(y)} d_\alpha y.$$

Then

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Omega^n(y) d_\alpha y = \frac{n}{1-k} \int_0^\infty \theta(y) f(y) \Omega^{1-k}(y) \Omega^{n-1}(y) d_\alpha y,$$

which can be reformed in the shape

$$\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)} \Omega^n(y) d_\alpha y \right)^n = \left(\frac{n}{1-k} \right)^n \left(\int_0^\infty \left(\frac{(\theta(y) f(y))^n}{\Psi^{n(k-1)}(y) \Omega^{n(1-n)}(y)} \right)^{\frac{1}{n}} d_\alpha y \right)^n.$$

Using Hölder's inequality

$$\int_0^\infty g(y) h(y) d_\alpha y \leq \left(\int_0^\infty g^a(y) d_\alpha y \right)^{\frac{1}{a}} \left(\int_0^\infty h^b(y) d_\alpha y \right)^{\frac{1}{b}},$$

with $a = 1/n$ and $b = 1/(1-n)$ where

$$g(y) = \frac{(\theta(y) f(y))^n}{\Psi^{n(k-1)}(y) \Omega^{n(1-n)}(y)} \text{ and } h(y) = \left(\frac{\theta(y)}{\Psi^k(y)} \right)^{1-n} \Omega^{n(1-n)}(y),$$

we have

$$\left(\int_0^\infty g^{\frac{1}{n}}(y)d_\alpha y\right)^n = \left(\int_0^\infty \left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Omega^{n(1-n)}(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n,$$
$$\geq \frac{\int_0^\infty g(y)h(y)d_\alpha y}{\left(\int_0^\infty h^{\frac{1}{1-n}}(y)d_\alpha y\right)^{1-n}}$$
$$= \int_0^\infty \frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Omega^{n(1-n)}(y)}\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{1-n}\Omega^{n(1-n)}(y)d_\alpha y$$
$$\times\left(\int_0^\infty \left(\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{1-n}\Omega^{n(1-n)}(y)\right)^{\frac{1}{1-n}} d_\alpha y\right)^{n-1},$$

thus

$$\left(\int_0^\infty \left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Omega^{n(1-n)}(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n \geq \left(\int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y\right)$$
$$\times\left(\int_0^\infty \left(\frac{\theta(y)}{\Psi^k(y)}\right)\Omega^n(y)d_\alpha y\right)^{n-1},$$

then

$$\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Omega^n(y)d_\alpha y\right)^n \geq \left(\frac{n}{1-k}\right)^n\left(\int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y\right)$$
$$\times\left(\int_0^\infty \left(\frac{\theta(y)}{\Psi^k(y)}\right)\Omega^n(y)d_\alpha y\right)^{n-1}.$$

Hence,

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Omega^n(y)d_\alpha y \geq \left(\frac{n}{1-k}\right)^n \int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y,$$

which the wanted inequality (46). □

Remark 17. *If $\alpha = 1$, in Theorem 7, we obtain:*

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\left(\int_0^y \theta(t)f(t)dt\right)^n dy \geq \left(\frac{n}{1-k}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)}f^n(y)dy, \tag{47}$$

where $\Psi(y) = \int_y^\infty \theta(t)dt$ and $k \leq 0 < n < 1$.

Theorem 8. *If $n < k-1$ and $0 < n < 1 < k$, then*

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y \geq \left(\frac{n}{k-1}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)}f^n(y)d_\alpha y, \tag{48}$$

where

$$\Psi(y) := \int_y^\infty \theta(t)d_\alpha t \text{ and } \Phi(y) := \int_y^\infty \theta(t)f(t)d_\alpha t.$$

Proof. Employing the formula of integration by parts (12) on $\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y$, with

$$u(y) = \Phi^n(y) \text{ and } v(y) = -\frac{\Psi^{1-k}(y)}{1-k},$$

then

$$D_\alpha u(y) = ny^{1-\alpha}\Phi^{n-1}(y)\Phi^{'}(y) \text{ and } D_\alpha v(y) = \theta(y)\Psi^{-k}(y).$$

We find that

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y = -\frac{\Phi^n(y)\Psi^{1-k}(y)}{1-k}\bigg|_0^\infty$$
$$+\int_0^\infty \frac{ny^{1-\alpha}\Phi^{n-1}(y)\Phi'(y)\Psi^{1-k}(y)}{1-k}d_\alpha y.$$

Since

$$\Phi^{'}(y) = -y^{\alpha-1}\theta(y)f(y), \Phi(\infty) = 0, \Phi(0) < \infty, \Psi(\infty) = 0, \Psi(0) < \infty \text{ and } k > 1,$$

and noting that

$$\begin{aligned}\lim_{y\to\infty}\Psi(y)\Phi^{\frac{n}{1-k}}(y) &= \lim_{y\to\infty}\frac{\int_y^\infty t^{\alpha-1}\,\theta(t)\,dt}{(\Phi(y))^{\frac{n}{k-1}}} \\ &= \lim_{y\to\infty}\frac{-y^{\alpha-1}\theta(y)}{-\frac{n}{k-1}(\Phi(y))^{\frac{n}{k-1}-1}y^{\alpha-1}\theta(y)f(y)} \\ &= \lim_{y\to\infty}\frac{(k-1)(\Phi(y))^{1-\frac{n}{k-1}}}{nf(y)} = 0.\end{aligned}$$

We find that

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y \le \frac{n}{k-1}\int_0^\infty y^{1-\alpha}\Phi^{n-1}(y)y^{\alpha-1}\theta(y)f(y)\Psi^{1-k}(y)d_\alpha y.$$

Then

$$\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y \le \frac{n}{k-1}\int_0^\infty \theta(y)f(y)\Psi^{1-k}(y)\Phi^{n-1}(y)d_\alpha y,$$

which can be reformed in the shape

$$\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\right)^n = \left(\frac{n}{k-1}\right)^n\left(\int_0^\infty\left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)}\right)^{\frac{1}{n}}d_\alpha y\right)^n.$$

Using Hölder's inequality

$$\int_0^\infty g(y)h(y)d_\alpha y \le \left(\int_0^\infty g^a(y)d_\alpha y\right)^{\frac{1}{a}}\left(\int_0^\infty h^b(y)d_\alpha y\right)^{\frac{1}{b}}$$

with $a = 1/n$, $b = 1/(1-n)$,

$$g(y) = \frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)} \text{ and } h(y) = \left(\frac{\theta(y)}{\Psi^k(y)}\right)^{1-n}\Phi^{n(1-n)}(y),$$

then

$$
\begin{aligned}
\left(\int_0^\infty g^{\frac{1}{n}}(y)d_\alpha y\right)^n &= \left(\int_0^\infty \left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n, \\
&\geq \frac{\int_0^\infty g(y)h(y)d_\alpha y}{\left(\int_0^\infty h^{\frac{1}{1-n}}(y)d_\alpha y\right)^{1-n}}, \\
&= \int_0^\infty \frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)}\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{1-n}\Phi^{n(1-n)}(y)d_\alpha y \\
&\times \left(\int_0^\infty \left(\left(\frac{\theta(y)}{\Psi^k(y)}\right)^{1-n}\Phi^{n(1-n)}(y)\right)^{\frac{1}{1-n}} d_\alpha y\right)^{n-1}.
\end{aligned}
$$

This lead to

$$
\begin{aligned}
\left(\int_0^\infty \left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n \geq \left(\int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y\right)\times \\
\left(\int_0^\infty \left(\frac{\theta(y)}{\Psi^k(y)}\right)\Phi^n(y)d_\alpha y\right)^{n-1},
\end{aligned}
$$

since, we have

$$
\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\right)^n = \left(\frac{n}{k-1}\right)^n \left(\int_0^\infty \left(\frac{(\theta(y)f(y))^n}{\Psi^{n(k-1)}(y)\Phi^{n(1-n)}(y)}\right)^{\frac{1}{n}} d_\alpha y\right)^n.
$$

Since

$$
\begin{aligned}
\left(\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y\right)^n \geq \left(\frac{n}{k-1}\right)^n \left(\int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y\right) \\
\times \left(\int_0^\infty \left(\frac{\theta(y)}{\Psi^k(y)}\right)\Phi^n(y)d_\alpha y\right)^{n-1}.
\end{aligned}
$$

Hence

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\Phi^n(y)d_\alpha y \geq \left(\frac{n}{k-1}\right)^n \int_0^\infty \frac{\theta(y)f^n(y)}{\Psi^{k-n}(y)}d_\alpha y,
$$

which is the desired inequality (48). □

Remark 18. *In Theorem 8, we get that if $y^{\alpha-1}\theta(y)f(y)$ and $y^{\alpha-1}\theta(y)$ is continuous on $[0,\infty)$ replaced either by:*

(i) *$y^{\alpha-1}\theta(y)f(y)$, $y^{\alpha-1}\theta(y)$ is continuous on $(0,\infty)$ and $\lim_{y\to\infty} \frac{(\Phi(y))^{1-\frac{n}{k-1}}}{f(y)} = 0$; or*

(ii) *$\lim_{y\to\infty} \Psi^{1-k}(y)\Phi^n(y) = 0$;*

then (48) is also true.

Remark 19. *If $\alpha = 1$, in Theorem 8, we obtain the inequality:*

$$
\int_0^\infty \frac{\theta(y)}{\Psi^k(y)}\left(\int_y^\infty \theta(t)f(t)dt\right)^n dy \geq \left(\frac{n}{k-1}\right)^n \int_0^\infty \frac{\theta(y)}{\Psi^{k-n}(y)}f^n(y)dy, \tag{49}
$$

where $\Psi(y) = \int_y^\infty \theta(t)dt$ and $0 < n < 1 < k$.

5. Results and Discussion

It is great to take a look at the obtained number of new Leindler and Hardy-type inequalities by the utilization of the conformable factional calculus. We generalize a number of those inequalities to a general fractional form, and also get the α-fractional Hardy inequality:

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(t)d_\alpha t\right)^n d_\alpha y \le \left(\frac{n}{n-\alpha}\right)^n \int_0^\infty \left(y^{\alpha-1}f(y)\right)^n d_\alpha y,$$

as a result, if $\alpha = 1$ we obtain the classical Hardy inequality:

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(t)dt\right)^n dy \le \left(\frac{n}{n-1}\right)^n \int_0^\infty f^n(y)dy.$$

Furthermore, we also get the inequality:

$$\int_0^\infty \left(\int_y^\infty f(t)d_\alpha t\right)^n d_\alpha y \le \left(\frac{n}{\alpha}\right)^n \int_0^\infty (y^\alpha f(y))^n d_\alpha y,$$

which is the α-fractional Hardy inequality, and as a result, if $\alpha = 1$, we get the Hardy inequality:

$$\int_0^\infty \left(\int_y^\infty f(t)dt\right)^n dy \le n^n \int_0^\infty (yf(y))^n dy.$$

In addition to this, we also extend our reversed inequalities to the fractional shape.

6. Conclusions and Future Work

In this study, we established certain fractional inequalities of Leindler's type by employing the conformable fractional calculus. The technique is based on the applications of well-known inequalities and new tools from conformable fractional calculus. The results established in this paper give some contribution in the field of fractional calculus and fractional inequalities of Leindler' type. From these results, some work directions remain open, for example:

- Extending these results to other types of integral fractional operators, which contains as particular cases many of those reported in the literature.
- Obtain new results for other well-known inequalities, such as Opial, Hilbert, Copson, among others.

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