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Some Hermite–Hadamard and Hermite–Hadamard–Fejér Type Fractional Inclusions Pertaining to Different Kinds of Generalized Preinvexities

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Abstract: Fractional derivative and integral operators are often employed to present new generalizations of mathematical inequalities. The introduction of new fractional operators has prompted another direction in different branches of mathematics and applied sciences. First, we investigate and prove new fractional equality. Considering this equality as the auxiliary result, we attain some estimations of a Hermite–Hadamard type inequality involving s -preinvex, s -Godunova–Levin preinvex, and prequasi invex functions. In addition, we investigate a fractional order Hadamard–Fejér inequality and some of its refinements pertaining to h -preinvexity via a non-conformable fractional integral operator. Finally, we present a Pachpatte type inequality for the product of two preinvex functions. The findings as well as the special cases presented in this research are new and applications of our main results.

Keywords: fractional integrals; preinvex functions; Godunova–Levin function; quasi-convex; Godunova–Levin preinvex function; prequasi–invex function

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

1. Introduction

Convexity theory has had a substantial and crucial influence on the development of numerous disciplines such as economics [1], financial mathematics [2], engineering [3], and optimization [4] in modern mathematics. This theory gives a fantastic framework for initiating and developing numerical tools for tackling and studying complex mathematical problems.

In the current decade, many mathematicians always merge new ideas into fractional analysis to bring a new dimension with different features in the field of mathematical analysis and applied mathematics. Fractional analysis has a lot of applications in modeling [5,6], epidemiology [7], fluid flow [8], nano-technology [9], mathematical biology [10], and control systems [11]. Due to these widespread views and their applications, fractional analysis has become an attractive field for scholars, and readers can refer to [12–15].

The theory of inequalities is a subject of many mathematicians' work in the last century. This theory has a lot of applications in numerical quadrature formulas, probability, and statistical problems. Interested readers can refer to [16–21].

Nowadays, inequality theory and fractional analysis have shown synchronous development. Fractional calculus is a fundamental building block in applied sciences and mathematics. Many scholars are encouraged to consider using fractional calculus to find solutions to real-world problems by academics. A number of scholars used the Riemann–Liouville fractional integral operators to study the Hermite–Hadamard type integral inequalities [22], Hermite–Hadamard–Mercer inequalities [23], Simpson type inequality [24], and the Ostrowski inequality [25]. The Hermite–Hadamard inequality and the Fejér type integral inequalities were given via Katugampola type fractional integral operators in [26], while the Simpson–Mercer integral inequality was investigated using the Atangana–Baleanu fractional operator in [27]. Additionally, the Caputo–Fabrizio fractional integrals were used to examine the Hermite–Hadamard inequality and its Mercer counterpart [28,29]. The aforementioned study shows the strong relationship between fractional integral operators and integral inequalities.

Invex functions were first introduced by Hanson [30]. Weir and Mond [31] noted that the generalization of convex functions is the introduction of preinvex functions. The idea of invex sets and preinvex functions are investigated and discussed by Ben-Israel, and Mond [32] involving the bifunction, which can be viewed as a significant contribution to the field of optimization. Mohan and Neogy [33] concluded that the differentiable preinvex and invex functions are equivalent under suitable conditions. Many researchers proved that the properties of the preinvex functions have meaningful uses in the theory of optimization and mathematical programming. For the attention and attraction of the readers, see the references [34,35]. In 1985, the class of Godunova–Levin function was introduced by Godunova and Levin [36]. The idea of quasi-convexity is more general than classical convexity. It means every convex function is quasi-convex, but the converse is not true. Quasi-convex functions have a lot of applications in game theory, mathematical optimization, economics, and mathematical analysis. In the published articles [37,38], the authors examined and celebrated conformable and non-conformable derivatives, respectively. Both terminologies have a lot of meaningful and useful applications, see the references [39,40].

The concept and recent works of inequality, fractional calculus, and preinvexity as discussed above motivated us to work in this direction. Working with different types of preinvexities and fractional operators will attract many authors to generalize the theory of convexity and inequality in a more innovative way in the near future. Throughout the whole paper, the notation $L[\sigma_1, \sigma_2]$ represents an integrable function on the closed interval $[\sigma_1, \sigma_2]$.

We organize and construct the current paper in the following way. First, in Section 2, we explore some known concepts, definitions, and theorems. In Section 3, we prove new fractional equality and related estimations of the Hermite–Hadamard type inequality. In Section 4, we investigate the Hermite–Hadamard–Fejér type inequality for preinvex functions via a non-conformable fractional integral operator. Furthermore, in Section 5, we establish an integral identity and present some refinements of Hermite–Hadamard–Fejér type inequalities. In Section 6, we present a Pachpatte type fractional inequality for the product of two preinvex functions. Section 7 deals with the future directions of these results. In the last Section 8, we present a brief conclusion and discuss some future research ideas.

Let us start with the definition of the non-conformable integral operator, which has an important place among the new operators.

2. Preliminaries

For the sake of completeness, quality, and readers' interest, it will be better to examine and elaborate on several definitions, theorems, and remarks in the preliminary section. The objective of this section is to discuss and study some known concepts and definitions, which we need in our investigation in further sections. We start by introducing the non-conformable derivative, non-conformable fractional integral operator, invex, and preinvex function. In addition, tgs -type s -preinvex function, Godunova–Levin preinvex, s -Godunova–Levin preinvex of 1st type, and s -Godunova–Levin preinvex of 2nd type are

added. We conclude this section by recalling the prequasi-invex function, which will be required in our studies.

Definition 1 ([41]). Let $\mathcal{P} : \mathcal{H} \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a real valued function, then the non-conformable derivative of $\mathcal{P}$ of order α at ϑ is defined by

$${}_{N_3} \mathcal{P}(\vartheta) = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{P}(\vartheta + \epsilon \vartheta^\alpha) - \mathcal{P}(\vartheta)}{\epsilon},$$

where $\alpha \in (0, 1)$ and $\vartheta \in \mathcal{H}$.

If exists ${}_{N_3} \mathcal{P}(\vartheta)$ and is finite, then $\mathcal{P}$ is a α -differentiable at ϑ .

If $\mathcal{P}$ at ϑ is a differentiable, then

$${}_{N_3} \mathcal{P}(\vartheta) = \vartheta^\alpha \mathcal{P}'(\vartheta).$$

It is not difficult to verify the validity of the following properties involving a non-conformable fractional operator:

Property 1 ([38]). Let $\mathcal{P}$ and Ψ be two α -differentiable functions at ϑ and $\alpha \in (0, 1)$, $\vartheta > 0$, then

- (1) ${}_{N_3}(\nu \mathcal{P} + \mu \Psi)(\vartheta) = \nu {}_{N_3}(\mathcal{P})(\vartheta) + \mu {}_{N_3}(\Psi)(\vartheta), \forall \nu, \mu \in \mathbb{R},$
- (2) ${}_{N_3}(\mathcal{P}\Psi)(\vartheta) = \Psi(\vartheta) {}_{N_3}(\mathcal{P})(\vartheta) + \mathcal{P}(\vartheta) {}_{N_3}(\Psi)(\vartheta),$
- (3) ${}_{N_3}(\frac{\mathcal{P}}{\Psi})(\vartheta) = \frac{\Psi(\vartheta) {}_{N_3}(\mathcal{P})(\vartheta) - \mathcal{P}(\vartheta) {}_{N_3}(\Psi)(\vartheta)}{\Psi(\vartheta)^2}, \Psi \neq 0$
- (4) ${}_{N_3}(c) = 0, \forall c \in \mathbb{R},$
- (5) ${}_{N_3}(\frac{1}{1-\alpha} \vartheta^{1-\alpha}) = 1.$

Definition 2 ([42]). Let $\alpha, \sigma_1, \sigma_2 \in \mathbb{R}$ and $\sigma_1 < \sigma_2$. We define the following linear spaces:

$$L_{\alpha,0}[\sigma_1, \sigma_2] = \left\{ \mathcal{P} : [\sigma_1, \sigma_2] \rightarrow \mathbb{R} \mid |\vartheta - u|^{-\alpha} \mathcal{P}(\vartheta) \in L^1[\sigma_1, \sigma_2] \text{ for every } u \in [\sigma_1, \sigma_2] \right\},$$

$$L_\alpha[\sigma_1, \sigma_2] = \left\{ \mathcal{P} : [\sigma_1, \sigma_2] \rightarrow \mathbb{R} \mid (\vartheta - \sigma_1)^{-\alpha} \mathcal{P}(\vartheta), (\sigma_2 - \vartheta)^{-\alpha} \mathcal{P}(\vartheta) \in L[\sigma_1, \sigma_2] \right\}.$$

Note that, if $\alpha \leq 0$, then $L_\alpha[\sigma_1, \sigma_2] = L^1[\sigma_1, \sigma_2]$.

Definition 3 ([42]). For each $\mathcal{P} \in L[\sigma_1, \sigma_2]$ and $0 < \sigma_1 < \sigma_2$, we define

$${}_{N_3} J_u^\alpha \mathcal{P}(x) = \int_u^x \vartheta^{-\alpha} \mathcal{P}(\vartheta) d\vartheta,$$

for every $x, u \in [\sigma_1, \sigma_2]$ and $\alpha \in \mathbb{R}$.

Definition 4 ([42]). For each function $\mathcal{P} \in L[\sigma_1, \sigma_2]$, we define the fractional integrals

$${}_{N_3} J_{\sigma_1^+}^\alpha \mathcal{P}(x) = \int_{\sigma_1}^x (x - \vartheta)^{-\alpha} \mathcal{P}(\vartheta) d\vartheta,$$

$${}_{N_3} J_{\sigma_2^-}^\alpha \mathcal{P}(x) = \int_x^{\sigma_2} (\vartheta - x)^{-\alpha} \mathcal{P}(\vartheta) d\vartheta,$$

for every $x \in [\sigma_1, \sigma_2]$ and $\alpha \in \mathbb{R}$.

Remark 1. In the above Definitions 3 and 4, if we put $\alpha = 0$, then we obtain the classical integrals which are represented by

$${}_{N_3} J_{\sigma_1^+}^\alpha \mathcal{P}(x) = {}_{N_3} J_{\sigma_2^-}^\alpha \mathcal{P}(x) = \int_{\sigma_1}^{\sigma_2} \mathcal{P}(\vartheta) d\vartheta.$$

Definition 5 ([43]). The set $\mathcal{H} \subset \mathbb{R}^n$ is called invex with respect to arbitrary bifunction $\Phi(\cdot, \cdot)$, if

$$\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1) \in \mathcal{H},$$

for every $\sigma_1, \sigma_2 \in \mathcal{H}$ and $\vartheta \in [0, 1]$.

It is well-known that there are many applications of invexity in nonlinear optimization, variational inequalities, and in the other branches of pure applied sciences.

Definition 6 ([44]). The function $\mathcal{P}$ is called preinvex with respect to Φ on an invex set $\mathcal{H}$, if

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq (1 - \vartheta)\mathcal{P}(\sigma_1) + \vartheta\mathcal{P}(\sigma_2),$$

holds for all $\sigma_1, \sigma_2 \in \mathcal{H}$ and $\vartheta \in [0, 1]$.

If the above inequality is reversed, then $\mathcal{P}$ is called pre-concave.

Condition-C :

Let $\mathcal{H} \subset \mathbb{R}^n$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$. For any $\Phi, \vartheta \in \mathcal{H}$ and $\vartheta \in [0, 1]$,

$$\begin{aligned}\Phi(\sigma_1, \sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) &= -\vartheta \Phi(\sigma_2, \sigma_1) \\ \Phi(\sigma_2, \sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) &= (1 - \vartheta) \Phi(\sigma_2, \sigma_1).\end{aligned}$$

For any $\sigma_1, \sigma_2 \in \mathcal{H}$, $\vartheta_1, \vartheta_2 \in [0, 1]$ and from condition C, we have

$$\Phi(\sigma_1 + \vartheta_2 \Phi(\sigma_2, \sigma_1), \sigma_1 + \vartheta_1 \Phi(\sigma_2, \sigma_1)) = (\vartheta_2 - \vartheta_1) \Phi(\sigma_2, \sigma_1).$$

In the development of the theory of optimization and inequalities, the decisive role was played by the above Condition-C, see [45,46] and references therein.

Definition 7 ([47]). Let $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ be a function, then $\mathcal{P}$ is called tgs-type s -preinvex function, if

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq \vartheta^s (1 - \vartheta)^s [\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)], \quad (1)$$

holds for all $\vartheta \in [0, 1]$ and $\sigma_1, \sigma_2 \in \mathcal{H}$.

Definition 8 ([48]). Let $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ be a function, then $\mathcal{P}$ is called Godunova–Levin preinvex or Q -preinvex, if

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq \frac{\mathcal{P}(\sigma_1)}{1 - \vartheta} + \frac{\mathcal{P}(\sigma_2)}{\vartheta}, \quad (2)$$

holds for all $\vartheta \in (0, 1)$ and $\sigma_1, \sigma_2 \in \mathcal{H}$.

In 2014, Noor [44] first investigated the classes of s -Godunova–Levin preinvex functions of first and second type.

Definition 9. A function $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is called s -Godunova–Levin preinvex of first type with $s \in (0, 1]$, if

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq \frac{\mathcal{P}(\sigma_1)}{1 - \vartheta^s} + \frac{\mathcal{P}(\sigma_2)}{\vartheta^s}, \quad (3)$$

holds for all $\vartheta \in (0, 1)$ and $\sigma_1, \sigma_2 \in \mathcal{H}$.

Definition 10. A function $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is called s -Godunova–Levin preinvex of second type with $s \in [0, 1]$, if

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq \frac{\mathcal{P}(\sigma_1)}{(1 - \vartheta)^s} + \frac{\mathcal{P}(\sigma_2)}{\vartheta^s}, \quad (4)$$

holds for all $\vartheta \in (0, 1)$ and $\sigma_1, \sigma_2 \in \mathcal{H}$.

Prequasi-invex functions were introduced by Pini [49]. Prequasi-invex function satisfying condition C is not necessarily a quasi-convex function, see [49] (Example 1.1).

Definition 11. Let $\mathcal{H} \subseteq \mathbb{R}^n$ be an invex set with respect to a bifunction $\Phi(\cdot, \cdot)$, then the function $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is said to be prequasi-invex on $\mathcal{H}$, if the inequality

$$\mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \leq \max\{\mathcal{P}(\sigma_1), \mathcal{P}(\sigma_2)\}, \quad (5)$$

holds for all $\sigma_1, \sigma_2 \in \mathcal{H}$ and $\vartheta \in [0, 1]$.

Remark 2. In the above Definition 11, if we take $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then prequasi-invex function reduces to quasi-convex function.

3. Estimations of Hermite–Hadamard Type Inequality via Generalized Fractional Integral Operator

Since the concept of convexity was first proposed more than a century ago, numerous significant inequalities have been presented for the class of convex functions. The alleged Hamard inequality, also known as the Hermite–Hadamard inequality, is the most notable. Hermite and Hadamard introduced this inequality in their ways. It has a variety of applications and an intriguing geometric interpretation. Jensen's inequality leads to the Hermite–Hadamard inequalities, which are a development of the idea of convexity. It is also quite interesting to note that with the aid of peculiar convex functions, some of the classical inequalities for means can be derived from Hadamard's inequality. Hermite–Hadamard inequalities for convex functions have attracted a lot of attention lately, leading to an impressive array of improvements and generalizations.

This section aims to examine and prove a new lemma. Based on this newly introduced lemma, we attain some refinements of the Hermite–Hadamard type inequality using the non-conformable fractional integral operator. For the quality and interest of readers, we add some remarks. So now, we proceed by giving an important identity involving non-conformable fractional integral operators.

Lemma 1. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\sigma_1 \leq \sigma_1 + \Phi(\sigma_2, \sigma_1)$. If $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable function such that $\mathcal{P}'' \in L_{\alpha,0}[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$, then for $\alpha \leq -1$ the following equality for non-conformable fractional integral operator holds:

$$\begin{aligned} & \left[\frac{\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{2} \right] \\ & - \frac{1 - \alpha}{2(\Phi(\sigma_2, \sigma_1))^{-\alpha}} \left\{ \alpha \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)}^{\alpha+1} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha+1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right. \\ & \left. + \frac{(2 - \alpha)}{\Phi(\sigma_2, \sigma_1)} \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)}^{\alpha} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right\} \\ & = \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} (I_1 + I_2), \end{aligned} \quad (6)$$

where

$$\begin{aligned} I_1 &= \int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta, \\ I_2 &= \int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

Proof. It is obvious that

$$\begin{aligned} I_1 &= \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + (1-\vartheta)(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 \vartheta^{1-\alpha} (1-\vartheta) \mathcal{P}''(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 (\vartheta^{1-\alpha} - \vartheta^{2-\alpha}) \mathcal{P}''(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

Now, integrating by parts, we obtain

$$\begin{aligned} I_1 &= \frac{\mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{(\Phi(\sigma_2, \sigma_1))^2} + \frac{1-\alpha}{(\Phi(\sigma_2, \sigma_1))^2} \left[-\alpha \int_0^1 \vartheta^{-\alpha-1} \mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \right. \\ &\quad \left. - (2-\alpha) \int_0^1 \vartheta^{-\alpha} \mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \right] \\ &= \frac{\mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{(\Phi(\sigma_2, \sigma_1))^2} + \frac{1-\alpha}{(\Phi(\sigma_2, \sigma_1))^{2-\alpha}} \left[\alpha \int_{\sigma_1}^{\sigma_1 + \Phi(\sigma_2, \sigma_1)} (x - \sigma_1)^{-\alpha-1} \mathcal{P}(x) dx \right. \\ &\quad \left. - \frac{(2-\alpha)}{(\Phi(\sigma_2, \sigma_1))} \int_{\sigma_1}^{\sigma_1 + \Phi(\sigma_2, \sigma_1)} (x - \sigma_1)^{-\alpha} \mathcal{P}(x) dx \right] \\ &= \frac{\mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{(\Phi(\sigma_2, \sigma_1))^2} \\ &\quad - \frac{1-\alpha}{(\Phi(\sigma_2, \sigma_1))^{2-\alpha}} \left[\alpha N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)}^{\alpha+1} \mathcal{P}(\sigma_1) + \frac{2-\alpha}{\Phi(\sigma_2, \sigma_1)} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)}^{\alpha} \mathcal{P}(\sigma_1) \right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} I_2 &= \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 \vartheta^{1-\alpha} (1-\vartheta) \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 (\vartheta^{1-\alpha} - \vartheta^{2-\alpha}) \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

Again, applying integration by parts, we find

$$\begin{aligned} I_2 &= \frac{\vartheta^{1-\alpha} - \vartheta^{2-\alpha}}{\Phi(\sigma_2, \sigma_1)} \mathcal{P}'(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) \Big|_0^1 \\ &\quad - \frac{1}{\Phi(\sigma_2, \sigma_1)} \int_0^1 ((1-\alpha)\vartheta^{-\alpha} - (2-\alpha)\vartheta^{1-\alpha}) \mathcal{P}'(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= -\frac{1}{\Phi(\sigma_2, \sigma_1)} \left[\frac{(1-\alpha)\vartheta^{-\alpha} - (2-\alpha)\vartheta^{1-\alpha}}{(\Phi(\sigma_2, \sigma_1))} \mathcal{P}(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) \Big|_0^1 \right. \\ &\quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} \int_0^1 (\alpha(1-\alpha)\vartheta^{-\alpha-1} - (1-\alpha)(2-\alpha)\vartheta^{-\alpha}) \mathcal{P}(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{\mathcal{P}(\sigma_1)}{(\Phi(\sigma_2, \sigma_1))^2} + \frac{(1-\alpha)}{(\Phi(\sigma_2, \sigma_1))^2} \left[-\alpha \int_0^1 \vartheta^{-\alpha-1} \mathcal{P}(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \right. \\
&\quad \left. - (2-\alpha) \int_0^1 \vartheta^{-\alpha} \mathcal{P}(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \right]. \\
&= \frac{\mathcal{P}(\sigma_1)}{(\Phi(\sigma_2, \sigma_1))^2} + \frac{(1-\alpha)}{(\Phi(\sigma_2, \sigma_1))^2} \\
&\quad \times \left[\frac{-\alpha}{(\Phi(\sigma_2, \sigma_1))^{-\alpha}} \int_{\sigma_1}^{\sigma_1 + \Phi(\sigma_2, \sigma_1)} (\sigma_1 + \Phi(\sigma_2, \sigma_1) - x)^{-\alpha-1} \mathcal{P}(x) dx \right. \\
&\quad \left. - \frac{2-\alpha}{(\Phi(\sigma_2, \sigma_1))^{-\alpha+1}} \int_{\sigma_1}^{\sigma_1 + \Phi(\sigma_2, \sigma_1)} (\sigma_1 + \Phi(\sigma_2, \sigma_1) - x)^{-\alpha} \mathcal{P}(x) dx \right] \\
&= \frac{\mathcal{P}(\sigma_1)}{(\Phi(\sigma_2, \sigma_1))^2} - \frac{(1-\alpha)}{(\Phi(\sigma_2, \sigma_1))^{2-\alpha}} \\
&\quad \times \left[\alpha \left({}_{N_3}J_{\sigma_1^+}^{\alpha+1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right) + \frac{2-\alpha}{\Phi(\sigma_2, \sigma_1)} \left({}_{N_3}J_{\sigma_1^+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right) \right].
\end{aligned}$$

From the above developments, we obtain

$$\begin{aligned}
I_1 + I_2 &= \frac{\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{(\Phi(\sigma_2, \sigma_1))^2} - \frac{1-\alpha}{(\Phi(\sigma_2, \sigma_1))^{2-\alpha}} \\
&\quad \times \left\{ \alpha \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha+1} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha+1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right. \\
&\quad \left. + \frac{(2-\alpha)}{\Phi(\sigma_2, \sigma_1)} \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right\}.
\end{aligned}$$

Multiplying the above equality by $\frac{\Phi^2(\sigma_2, \sigma_1)}{2}$, we complete the proof of the desired Lemma 1. $\square$

Theorem 1. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\sigma_1 \leq \sigma_1 + \Phi(\sigma_2, \sigma_1)$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable function such that $\mathcal{P}'' \in L[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. If $|\mathcal{P}''|$ is tgs -type s -preinvex function on $[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. Then, for all $\alpha \leq -1$ and $s \in [0, 1)$, the following inequality for non-conformable fractional integral operator holds:

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} B(2+s, 2-\alpha+s) \left(|\mathcal{P}''(\sigma_1)| + |\mathcal{P}''(\sigma_2)| \right), \quad (7)$$

where

$$\begin{aligned}
U &= \left[\frac{\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))}{2} \right] \\
&\quad - \frac{1-\alpha}{2(\Phi(\sigma_2, \sigma_1))^{-\alpha}} \left\{ \alpha \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha+1} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha+1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right. \\
&\quad \left. + \frac{(2-\alpha)}{\Phi(\sigma_2, \sigma_1)} \left[{}_{N_3}J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \right\}
\end{aligned}$$

and

$$B(x, y) = \int_0^1 \vartheta^{x-1} (1-\vartheta)^{y-1} d\vartheta,$$

for $x > 0$ and $y > 0$.

Proof. From Lemma 1, we have

$$\left| \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta + \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta \right| \\ \leq \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) \right| d\vartheta + \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \right| d\vartheta.$$

Since $|\mathcal{P}''|$ is tgs-type s -preinvex function, applying integration to every integral, we obtain

$$\int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) \right| d\vartheta \\ \leq \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \vartheta^s (1-\vartheta)^s \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right) d\vartheta \\ = B(2+s, 2-\alpha+s) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right)$$

and

$$\int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \right| d\vartheta \\ = B(2+s, 2-\alpha+s) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right).$$

Finally, we have

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} B(2+s, 2-\alpha+s) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right).$$

□

Remark 3. If we consider $s = 1$, then we obtain the following inequality

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} B(3, 3-\alpha) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right),$$

where U is defined in Theorem 1.

Remark 4. In the above Theorem 1, if we put $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then we obtain the following inequality

$$|U_x| \leq \frac{(\sigma_2 - \sigma_1)^2}{2} B(2+s, 2-\alpha+s) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right),$$

where

$$U_x = \left[\frac{\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)}{2} \right] - \frac{1-\alpha}{2(\sigma_2 - \sigma_1)^{-\alpha}} \left\{ \alpha \left[{}_{N_3}J_{\sigma_2^-}^{\alpha+1} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha+1} \mathcal{P}(\sigma_2) \right] \right. \\ \left. + \frac{(2-\alpha)}{(\sigma_2 - \sigma_1)} \left[{}_{N_3}J_{\sigma_2^-}^{\alpha} \mathcal{P}(\sigma_1) + {}_{N_3}J_{\sigma_1^+}^{\alpha} \mathcal{P}(\sigma_2) \right] \right\}.$$

Remark 5. In the above Theorem 1, assume that Φ satisfies condition C, and using inequality we obtain

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} B(2+s, 2-\alpha+s) \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right| \right),$$

where U is defined in Theorem 1.

Theorem 2. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\sigma_1 \leq \sigma_1 + \Phi(\sigma_2, \sigma_1)$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable function such that $\mathcal{P}'' \in L[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. If $|\mathcal{P}''|$ is s -Godunova-Levin preinvex function on $[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. Then, for all $\alpha \leq -1$ and $s \in [0, 1)$, the following inequality for non-conformable fractional integral operator holds:

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} [B(2-s, 2-\alpha) + B(2, 2-\alpha-s)] \left(|\mathcal{P}''(\sigma_1)| + |\mathcal{P}''(\sigma_2)| \right), \quad (8)$$

where U is defined in Theorem 1 and

$$B(x, y) = \int_0^1 \vartheta^{x-1} (1-\vartheta)^{y-1} d\vartheta, \quad x > 0, y > 0.$$

Proof. From Lemma 1, we have

$$\begin{aligned} & \left| \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta + \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta \right| \\ & \leq \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} |\mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1))| d\vartheta + \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} |\mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))| d\vartheta. \end{aligned}$$

Since $|\mathcal{P}''|$ is an s -Godunova-Levin preinvex function, applying integration to every integral, respectively, we obtain

$$\begin{aligned} & \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} |\mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1))| d\vartheta \\ & \leq |\mathcal{P}''(\sigma_1)| \int_0^1 \vartheta^{1-s} (1-\vartheta)^{1-\alpha} d\vartheta + |\mathcal{P}''(\sigma_2)| \int_0^1 \vartheta(1-\vartheta)^{1-\alpha-s} d\vartheta \\ & = \left[|\mathcal{P}''(\sigma_1)| B(2-s, 2-\alpha) + |\mathcal{P}''(\sigma_2)| B(2, 2-\alpha-s) \right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \vartheta(1-\vartheta)^{1-\alpha} |\mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))| d\vartheta \\ & = \left[|\mathcal{P}''(\sigma_1)| B(2, 2-\alpha-s) + |\mathcal{P}''(\sigma_2)| B(2-s, 2-\alpha) \right]. \end{aligned}$$

Finally, we have

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} [B(2-s, 2-\alpha) + B(2, 2-\alpha-s)] \left(|\mathcal{P}''(\sigma_1)| + |\mathcal{P}''(\sigma_2)| \right).$$

□

Remark 6. If we consider $s = 1$, then we obtain the following inequality

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} [B(1, 2-\alpha) + B(2, 1-\alpha)] \left(|\mathcal{P}''(\sigma_1)| + |\mathcal{P}''(\sigma_2)| \right),$$

where U is defined in Theorem 1.

Remark 7. If we consider $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then we obtain the following inequality

$$|U_x| \leq \frac{(\Phi(\sigma_2 - \sigma_1))^2}{2} [B(2-s, 2-\alpha) + B(2, 2-\alpha-s)] \left(|\mathcal{P}''(\sigma_1)| + |\mathcal{P}''(\sigma_2)| \right).$$

where U_x is defined in Remark 4.

Remark 8. If we consider $s = 1$ and $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then we obtain the following inequality

$$|U_x| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} [B(1, 2 - \alpha) + B(2, 1 - \alpha)] \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_2) \right| \right),$$

where U_x is defined in Remark 4.

Remark 9. In the above Theorem 2, assume that Φ satisfies condition C, and using inequality we obtain

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} [B(2 - s, 2 - \alpha) + B(2, 2 - \alpha - s)] \left(\left| \mathcal{P}''(\sigma_1) \right| + \left| \mathcal{P}''(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right| \right),$$

where U is defined in Theorem 1.

Theorem 3. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\sigma_1 \leq \sigma_1 + \Phi(\sigma_2, \sigma_1)$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable function such that $\mathcal{P}'' \in L[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. If $|\mathcal{P}''|$ is prequasi-invex on $[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. Then, for all $q \geq 1, \vartheta \in [0, 1]$ and $\alpha \leq -1$, the following inequality for a non-conformable fractional integral operator holds:

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{(2 - \alpha)(3 - \alpha)} \left[\max \left(\left| \mathcal{P}''(\sigma_1) \right|^q, \left| \mathcal{P}''(\sigma_2) \right|^q \right) \right]^{\frac{1}{q}}, \quad (9)$$

where U is defined in Theorem 1.

Proof. By using Lemma 1 and the power-mean inequality with properties of modulus, we can write

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} \left[\int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) \right| d\vartheta + \int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \right| d\vartheta \right]$$

It is obvious that

$$\begin{aligned} I_1 &= \int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 \vartheta^{1-\alpha} (1 - \vartheta) \mathcal{P}''(\sigma_2 + \vartheta\Phi(\sigma_1, \sigma_2)) d\vartheta \\ &= \int_0^1 (\vartheta^{1-\alpha} - \vartheta^{2-\alpha}) \mathcal{P}''(\sigma_2 + \vartheta\Phi(\sigma_1, \sigma_2)) d\vartheta \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^1 \vartheta(1 - \vartheta)^{1-\alpha} \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 \vartheta^{1-\alpha} (1 - \vartheta) \mathcal{P}''(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \int_0^1 (\vartheta^{1-\alpha} - \vartheta^{2-\alpha}) \mathcal{P}''(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

Thus, we have

$$\begin{aligned} |U| &\leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{2} \left[\left(\int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + (1-\vartheta)\Phi(\sigma_2, \sigma_1)) \right|^q d\vartheta \right)^{\frac{1}{q}} \right. \\ &\quad \left. + \left(\int_0^1 \vartheta(1-\vartheta)^{1-\alpha} \left| \mathcal{P}''(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \right|^q d\vartheta \right)^{\frac{1}{q}} \right] \\ &= \frac{(\Phi(\sigma_2, \sigma_1))^2}{(2-\alpha)(3-\alpha)} \left[\max \left(\left| \mathcal{P}''(\sigma_1) \right|^q, \left| \mathcal{P}''(\sigma_2) \right|^q \right) \right]^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. $\square$

Remark 10. In the above Theorem 3, if we put $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then we obtain

$$|U_x| \leq \frac{(\sigma_2 - \sigma_1)^2}{(2-\alpha)(3-\alpha)} \left[\max \left(\left| \mathcal{P}''(\sigma_1) \right|^q, \left| \mathcal{P}''(\sigma_2) \right|^q \right) \right]^{\frac{1}{q}},$$

where U_x is defined in Remark 4.

Remark 11. In the above Theorem 3, assume that Φ satisfies condition C, then using inequality, we obtain

$$|U| \leq \frac{(\Phi(\sigma_2, \sigma_1))^2}{(2-\alpha)(3-\alpha)} \left[\max \left(\left| \mathcal{P}''(\sigma_1) \right|^q, \left| \mathcal{P}''(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right|^q \right) \right]^{\frac{1}{q}},$$

where U is defined in Theorem 1.

4. Hermite–Hadamard–Fejér Inequality via Generalized Fractional Integral Operator

The subject of integral inequalities has importance and applications in number theory, quantum theory, combinatorics, linear programming, orthogonal polynomials, dynamics, optimization theory, and the theory of relativity. Scientists and mathematicians have given a lot of attention to this problem. The most popular and well-known inequality in the literature associated with the field of convex theory is the Hermite–Hadamard or Hermite–Hadamard–Fejér type inequality. The Fejér type inequality is the weighted generalization of the Hermite–Hadamard type inequality. This inequality was investigated and examined by Fejér [50] in 1906.

This section aims to explore and examine the Hermite–Hadamard–Fejér inequality via a generalized fractional integral operator, namely the non-conformable fractional integral operator. For the comprehensiveness of this section, some corollaries are presented.

Theorem 4. Suppose $\mathcal{P} : [\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)] \rightarrow \mathbb{R}$ is an h -preinvex function, condition-C for Φ holds and $\Phi(\sigma_2, \sigma_1) > 0$, $h(\frac{1}{2}) > 0$ and $\mathcal{S} : [\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)] \rightarrow \mathbb{R}$, $\mathcal{S} \geq 0$ is symmetric with respect to $\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)$ and $\alpha \leq -1$. Then the following inequality holds:

$$\begin{aligned} &\frac{\mathcal{P}\left(\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)\right)}{2h\left(\frac{1}{2}\right)\Phi(\sigma_2, \sigma_1)^{1-\alpha}} \left[N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)-}^{\alpha} \mathcal{S}(\sigma_1) + N_3 J_{\sigma_1+}^{\alpha} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \\ &\leq \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} \left[N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)-}^{\alpha} \mathcal{P}(\sigma_1) \mathcal{S}(\sigma_1) + N_3 J_{\sigma_1+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \\ &\leq [\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)] \int_0^1 \vartheta^{-\alpha} [h(\vartheta) + h(1-\vartheta)] \mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned} \quad (10)$$

Proof. From the definition of an h -preinvex function and from Condition C for Φ , we have

$$\mathcal{P}\left(\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)\right) \leq h\left(\frac{1}{2}\right)[\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) + \mathcal{P}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1))].$$

Multiplying both sides by

$$\vartheta^{-\alpha}\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) = \vartheta^{-\alpha}\mathcal{S}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1))$$

and then integrating the resulting inequality with respect to ϑ over $[0, 1]$, we obtain

$$\begin{aligned} & \int_0^1 \mathcal{P}\left(\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)\right) \vartheta^{-\alpha}\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta \\ & \leq h\left(\frac{1}{2}\right) \left[\int_0^1 \vartheta^{-\alpha}\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta \right. \\ & \quad \left. + \int_0^1 \vartheta^{-\alpha}\mathcal{P}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1))d\vartheta \right]. \end{aligned}$$

Since

$$\begin{aligned} & \int_0^1 \vartheta^{-\alpha}\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta = \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{S}(\sigma_1), \\ & \int_0^1 \vartheta^{-\alpha}\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta \\ & = \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1)\mathcal{S}(\sigma_1) \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \vartheta^{-\alpha}\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1))d\vartheta \\ & = \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)). \end{aligned}$$

It follows from the above developments that

$$\begin{aligned} & \frac{\mathcal{P}\left(\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)\right)}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{S}(\sigma_1) \\ & \leq h\left(\frac{1}{2}\right) \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} \left[N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1)\mathcal{S}(\sigma_1) + N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right]. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} & \frac{\mathcal{P}\left(\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)\right)}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \\ & \leq h\left(\frac{1}{2}\right) \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} \left[N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1)\mathcal{S}(\sigma_1) + N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)^-}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))\mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right]. \end{aligned}$$

If we add the above two inequalities, then we have the proof of the first inequality.

For the second inequality, we use the definition of the h -preinvex function given as

$$\begin{aligned} \mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) & \leq h(1 - \vartheta)\mathcal{P}(\sigma_1) + h(\vartheta)\mathcal{P}(\sigma_2), \\ \mathcal{P}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) & \leq h(\vartheta)\mathcal{P}(\sigma_1) + h(1 - \vartheta)\mathcal{P}(\sigma_2). \end{aligned}$$

By adding these inequalities, we have

$$\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) + \mathcal{P}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) \leq h(1 - \vartheta) + h(\vartheta)[\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)].$$

If we multiply both sides by

$$\vartheta^{-\alpha}\mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) = \vartheta^{-\alpha}\mathcal{S}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)),$$

and then integrating the obtained inequality over $[0, 1]$, we find

$$\begin{aligned} & \int_0^1 \vartheta^{-\alpha} \mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta \\ & + \int_0^1 \vartheta^{-\alpha} \mathcal{P}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ & \leq [\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)] \int_0^1 \vartheta^{-\alpha} [h(\vartheta) + h(1 - \vartheta)] \mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

It readily follows

$$\begin{aligned} & \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-\alpha}} \left[{}_{N_3}J_{\sigma_1+\Phi(\sigma_2, \sigma_1)-}^{\alpha} \mathcal{P}(\sigma_1) \mathcal{S}(\sigma_1) + {}_{N_3}J_{\sigma_1+}^{\alpha} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \\ & \leq [\mathcal{P}(\sigma_1) + \mathcal{P}(\sigma_2)] \int_0^1 \vartheta^{-\alpha} [h(\vartheta) + h(1 - \vartheta)] \mathcal{S}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) d\vartheta. \end{aligned}$$

This completes the proof of the desired Theorem 4. $\square$

Corollary 1. If we choose $\alpha = 0$ in Theorem 4, then we recover the Hermite–Hadamard–Fejér inequality for the h -preinvex function established by Matloka (see [51]).

Corollary 2. If we choose $\alpha = 0$ and $\mathcal{S}(x) = 1$ in Theorem 4, then we recover the Hermite–Hadamard type inequality for the h -preinvex function established by Matloka (see [52]).

Corollary 3. If we choose $\alpha = 0$, $\mathcal{S}(x) = 1$ and $h(\vartheta) = \vartheta$ in Theorem 4, then we recapture the Hermite–Hadamard type inequality for the preinvex function introduced by Noor (see [53]).

Corollary 4. If we choose $\alpha = 0$, and $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$ in Theorem 4, then the Hermite–Hadamard–Fejér inequality for the h -convex function given by Bombardelli and Varosanec (see [54]) is recovered.

Corollary 5. If we choose $\alpha = 0$, $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$ and $\mathcal{S}(x) = 1$ in Theorem 4, then the Hermite–Hadamard type inequality for the h -convex function presented by Sarikaya et al. (see [55]) is recovered.

Corollary 6. If we choose $\alpha = 0$, $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, $\mathcal{S}(x) = 1$ and $h(\vartheta) = \vartheta$ in Theorem 4, then the classical Hermite–Hadamard type inequality for the convex function proved by Hadamard (see [56]) is recaptured.

Corollary 7. If we choose $\alpha = 0$, $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, $\mathcal{S}(x) = 1$ and $h(\vartheta) = \vartheta^s$ in Theorem 4, then the classical Hermite–Hadamard type inequality for the s -convex function proved by Dragomir and Fitzpatrick (see [57]) is recovered.

5. Refinements of the Hermite–Hadamard–Fejér Inequality via Generalized Fractional Integral Operator

Recently, many researchers and mathematicians worked on new concepts associated with this problem with different aspects in the field of convex analysis. Chen [58] investigated Fejér and Hermite–Hadamard type inequalities for harmonic convexity in

2014. Professor Iscan [59] published some work in 2014 in the direction of weighted integral inequality, namely the Hermite–Hadamard–Fejér inequality for convexity via the Riemann–Liouville fractional integral operator. Dragomir et al. [60] worked in this direction in 2015 and explored the Fejér type inequality via the concept of preinvexity. In 2017, Yang [61] investigated this inequality for differentiable functions involving quantum calculus on finite intervals. Sarikaya [62] introduced some new Fejér type inequalities with fractional integrals in 2018. Khan [63] worked in 2018 on the conformable fractional integral operator and investigated some Hermite–Hadamard–Fejér inequalities using the idea of preinvexity. In 2018, Delavar [64] used the concept of (η_1, η_2) convexity and proved some Hermite–Hadamard–Fejér inequality pertaining to fractional integrals. Bilal [65] and Farid [66] worked on this inequality in 2021 in the direction of interval analysis and fractional calculus, respectively.

The main aim of this section is to prove and examine a new integral identity. To this newly introduced identity, we obtain some generalizations, estimations, and extensions of Hermite–Hadamard–Fejér inequality via generalized fractional integral operator, namely a non-conformable fractional integral operator. Here, for obtaining the main results, we will use the concept of h -preinvexity, Hölder and power mean inequality. For the interest of readers, we added some corollaries.

Lemma 2. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\Phi(\sigma_2, \sigma_1) > 0$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is differentiable mapping on $\mathcal{H}$ such that $\mathcal{P}' \in L([\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)])$. If $\mathcal{S} : \mathcal{H} \rightarrow [0, \infty)$ is differentiable and $\alpha \leq -1$, then the following equality holds:

$$\begin{aligned} & \int_0^1 \left[(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right] \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \cdot \mathcal{P}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))]. \end{aligned} \quad (11)$$

Proof. By integration by parts, we have

$$\begin{aligned} & \int_0^1 \left[(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right] \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \cdot \mathcal{P}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= \frac{1}{\Phi(\sigma_2, \sigma_1)} \mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \left[(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right]_0^1 \\ & \quad + \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)} \int_0^1 \left[(1-\vartheta)^{-2\alpha-1} + \vartheta^{-2\alpha-1} \right] \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \cdot \mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ & \quad - \int_0^1 \left[(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right] \mathcal{S}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) \cdot \mathcal{P}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) d\vartheta \\ &= -\frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \\ & \quad + \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right), \end{aligned}$$

which completes the proof. $\square$

Theorem 5. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\Phi(\sigma_2, \sigma_1) > 0$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{S} : \mathcal{H} \rightarrow$

$[0, \infty)$ is differentiable and symmetric to $\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)$. If $|\mathcal{P}'|$ is h -preinvex on $\mathcal{H}$ and $\alpha \leq -1$, we have the following inequality:

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left({}_{N_3}J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + {}_{N_3}J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left({}_{N_3}J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + {}_{N_3}J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq [|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|] \cdot \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) [h(\vartheta) + h(1 - \vartheta)] d\vartheta. \end{aligned} \quad (12)$$

Proof. Using Lemma 2 and the h -preinvexity of $|\mathcal{P}'|$, we have

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left({}_{N_3}J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + {}_{N_3}J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left({}_{N_3}J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + {}_{N_3}J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \int_0^1 \left| (1 - \vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right| \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) |\mathcal{P}'(\sigma_1 + \Phi(\sigma_2, \sigma_1))| d\vartheta \\ & \leq \int_0^1 \left[(1 - \vartheta)^{-2\alpha} + \vartheta^{-2\alpha} \right] \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) [h(1 - \vartheta) |\mathcal{P}'(\sigma_1)| + h(\vartheta) |\mathcal{P}'(\sigma_2)|] d\vartheta \\ & = |\mathcal{P}'(\sigma_1)| \int_0^1 (1 - \vartheta)^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(1 - \vartheta) d\vartheta \\ & \quad + |\mathcal{P}'(\sigma_1)| \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(1 - \vartheta) d\vartheta \\ & \quad + |\mathcal{P}'(\sigma_2)| \int_0^1 (1 - \vartheta)^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(\vartheta) d\vartheta \\ & \quad + |\mathcal{P}'(\sigma_2)| \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(\vartheta) d\vartheta \\ & = [|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|] \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(\vartheta) d\vartheta \\ & \quad + [|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|] \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) h(1 - \vartheta) d\vartheta \\ & = [|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|] \int_0^1 \vartheta^{-2\alpha} \mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1)) [h(\vartheta) + h(1 - \vartheta)] d\vartheta, \end{aligned}$$

which completes the proof. $\square$

Corollary 8. If we take $\mathcal{S}(x) \equiv 1$ and $h(\vartheta) = \vartheta$, then Theorem 5 becomes for the preinvex function:

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left({}_{N_3}J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + {}_{N_3}J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \frac{|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|}{1 - 2\alpha}. \end{aligned} \quad (13)$$

Corollary 9. If we take $\mathcal{S}(x) \equiv 1$ and $h(\vartheta) = \vartheta^s$, then the Theorem 5 become for the s -preinvex function:

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq [|\mathcal{P}'(\sigma_1)| + |\mathcal{P}'(\sigma_2)|] \cdot \left[\frac{1}{1+s-2\alpha} + \frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(s-2\alpha-2)} \right]. \end{aligned} \quad (14)$$

Theorem 6. Let $\mathcal{H} \subseteq \mathcal{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\Phi(\sigma_2, \sigma_1) > 0$. Suppose that $\mathcal{P} : \mathcal{H} \rightarrow \mathcal{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{S} : \mathcal{H} \rightarrow [0, \infty)$ is differentiable and symmetric to $\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)$. If $|\mathcal{P}'|^q$, $q \geq 1$, is h -preinvex on $\mathcal{H}$ and $\alpha \leq -1$, then one has:

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \left(\frac{2}{1-2\alpha} \right)^{1-\frac{1}{q}} \left([|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q] \int_0^1 \vartheta^{-2\alpha} [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q [h(\vartheta) + h(1-\vartheta)] d\vartheta \right)^{\frac{1}{q}}. \end{aligned} \quad (15)$$

Proof. By using Lemma 2, h -preinvexity of $|\mathcal{P}'|^q$ and the well known power mean inequality, we have

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\ & \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \left(\int_0^1 [(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha}] d\vartheta \right)^{1-\frac{1}{q}} \\ & \quad \times \left(\int_0^1 [(1-\vartheta)^{-2\alpha} + \vartheta^{-2\alpha}] [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q |\mathcal{P}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))|^q d\vartheta \right)^{\frac{1}{q}} \\ & \leq \left(\frac{2}{1-2\alpha} \right)^{1-\frac{1}{q}} \left([|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q] \int_0^1 \vartheta^{-2\alpha} [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q [h(\vartheta) + h(1-\vartheta)] d\vartheta \right)^{\frac{1}{q}}, \end{aligned}$$

which completes the proof. $\square$

Corollary 10. If we take $w(x) \equiv 1$ and $h(\vartheta) = \vartheta$, then Theorem 6 for the preinvex function

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \left(\frac{2}{1-2\alpha} \right)^{1-\frac{1}{q}} \left(\frac{|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q}{1-2\alpha} \right)^{\frac{1}{q}}. \end{aligned} \quad (16)$$

Corollary 11. If we take $w(x) \equiv 1$ and $h(\vartheta) = \vartheta^s$, then Theorem 6 for the s -preinvex function

$$\begin{aligned}
& \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\
& \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\
& \leq \left(\frac{2}{1-2\alpha} \right)^{1-\frac{1}{q}} \left([|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q] \cdot \left[\frac{1}{1+s-2\alpha} + \frac{\Gamma(1-2\alpha)\Gamma(s+1)}{\Gamma(2+s-2\alpha)} \right] \right)^{\frac{1}{q}}. \quad (17)
\end{aligned}$$

Theorem 7. Let $\mathcal{H} \subseteq \mathbb{R}$ be an open invex subset with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathbb{H}$ with $\Phi(\sigma_2, \sigma_1) > 0$. Suppose that $\mathcal{H} : \mathcal{H} \rightarrow \mathbb{R}$ is a differentiable mapping on $\mathcal{H}$ and $\mathcal{S} : \mathcal{H} \rightarrow [0, \infty)$ is differentiable and symmetric to $\sigma_1 + \frac{1}{2}\Phi(\sigma_2, \sigma_1)$. If $|\mathcal{P}'|^q, q > 1$, is h -preinvex on $\mathcal{H}$ and $\alpha \leq -1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}
& \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\
& \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\
& \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\
& \leq \frac{2}{(1-2\alpha p)^{\frac{1}{p}}} \left([|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q] \cdot \int_0^1 [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q h(\vartheta) d\vartheta \right)^{\frac{1}{q}}, \quad (18)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2 and using the well known Hölder inequality, we have

$$\begin{aligned}
& \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) \right) \right. \\
& \quad - \frac{1}{\Phi(\sigma_2, \sigma_1)^{1-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha} \mathcal{S}'(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha} \mathcal{S}'(\sigma_1) \mathcal{P}(\sigma_1) \right) \\
& \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{S}(\sigma_1) \mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \mathcal{S}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\
& \leq \left(\int_0^1 (1-\vartheta)^{-2\alpha p} d\vartheta \right)^{\frac{1}{p}} \left(\int_0^1 [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q |\mathcal{P}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))|^q d\vartheta \right)^{\frac{1}{q}} \\
& \quad + \left(\int_0^1 \vartheta^{-2\alpha p} d\vartheta \right)^{\frac{1}{p}} \cdot \left(\int_0^1 [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q |\mathcal{P}'(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))|^q d\vartheta \right)^{\frac{1}{q}} \\
& \leq \frac{2}{(1-2\alpha p)^{\frac{1}{p}}} \left([|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q] \cdot \int_0^1 [\mathcal{S}(\sigma_1 + \vartheta \Phi(\sigma_2, \sigma_1))]^q h(\vartheta) d\vartheta \right)^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 12. If we take $w(x) \equiv 1$, and $h(\vartheta) = \vartheta$, then Theorem 7 for the preinvex function

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \frac{2}{(1 - 2\alpha\rho)^{\frac{1}{p}}} \left(\frac{|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned} \quad (19)$$

Corollary 13. If we take $w(x) \equiv 1$ and $h(\vartheta) = \vartheta^s$, then Theorem 7 for the s -preinvex function

$$\begin{aligned} & \left| \frac{2\alpha}{\Phi(\sigma_2, \sigma_1)^{-2\alpha}} \left(N_3 J_{\sigma_1^+}^{2\alpha-1} \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) + N_3 J_{(\sigma_1 + \Phi(\sigma_2, \sigma_1))^-}^{2\alpha-1} \mathcal{P}(\sigma_1) \right) \right. \\ & \quad \left. - \frac{1}{\Phi(\sigma_2, \sigma_1)} [\mathcal{P}(\sigma_1) - \mathcal{P}(\sigma_1 + \Phi(\sigma_2, \sigma_1))] \right| \\ & \leq \frac{2}{(1 - 2\alpha\rho)^{\frac{1}{p}} (s+1)^{\frac{1}{q}}} \left(|\mathcal{P}'(\sigma_1)|^q + |\mathcal{P}'(\sigma_2)|^q \right)^{\frac{1}{q}}. \end{aligned} \quad (20)$$

6. Inequality for the Product of Two Preinvex Functions Involving Generalized Fractional Integral Operators

Convex analysis has received attention in recent years due to its connection with the topic of inequality. In the literature, various inequalities are documented due to applications of convexity theory in both pure and practical sciences. Preinvexity has also been explored by a number of mathematicians, and many articles have been published that provide new estimates, extensions, and generalizations. These research studies greatly improve the well-known Hermite–Hadamard inequality for preinvex functions. Preinvexity is a concept that has been fundamental to the growth of generalized convex programming. The first Hermite–Hadamard type inequalities for the product of two preinvex functions were established by Noor [53] in 2009. The Hermite–Hadamard type inequality for the product of the r -preinvex function and the s -preinvex function was introduced and proven by Iqbal [67] in 2013. Kashuri [68] established various refinements of Hermite–Hadamard type inequalities for products of two $MT(r; g, m, \phi)$ -preinvex functions in the context of Riemann–Liouville fractional integrals. Several extensions of this inequality for products of two generalized beta (r, g) -preinvex functions were examined by Kashuri [69] in 2021. In the direction of interval analysis, Mishra [70] explored this inclusion for the product of two harmonically h -preinvexity. Motivated by the above results and the literature, we are going to examine and investigate inequality for the product of two preinvex functions involving a generalized fractional integral operator, namely a non-conformable fractional integral operator. To enhance the content of this section, some remarks are presented.

Theorem 8. Let $\mathcal{H} \subseteq \mathbb{R}$ be an invex set with respect to $\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$ and $\sigma_1, \sigma_2 \in \mathcal{H}$ with $\sigma_1 < \sigma_1 + \Phi(\sigma_2, \sigma_1)$. Suppose that $\mathcal{P}, \mathcal{O} : \mathcal{H} \rightarrow \mathbb{R}$ are differentiable functions such that $\mathcal{P}, \mathcal{O} \in L_{\alpha,0}[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$. If $\mathcal{P}, \mathcal{O}$ are preinvex functions on $[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$, then the following inequality for non-conformable fractional integral operators with $\alpha \leq 0$ holds:

$$\begin{aligned}
& \frac{1}{(\Phi(\sigma_2, \sigma_1))^{1-\alpha}} \left[N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)-}^{\alpha} \mathcal{P}\mathcal{O}(\sigma_1) + N_3 J_{\sigma_1+}^{\alpha} \mathcal{P}\mathcal{O}(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \right] \\
& \leq \left(\frac{1}{3-\alpha} + \frac{-2}{\alpha^3 - 6\alpha^2 + 11\alpha - 6} \right) (\mathcal{P}\mathcal{O})(\sigma_1) \\
& \quad + \left(\frac{1}{3-\alpha} + \frac{-2}{\alpha^3 - 6\alpha^2 + 11\alpha - 6} \right) (\mathcal{P}\mathcal{O})(\sigma_2) \\
& \quad + \frac{2}{\alpha^2 - 5\alpha + 6} (\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)). \tag{21}
\end{aligned}$$

Proof. Since $\mathcal{P}, \mathcal{O}$ are non-negative preinvex functions on $[\sigma_1, \sigma_1 + \Phi(\sigma_2, \sigma_1)]$, then by the definitions of preinvexity, we have

$$\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \leq (1 - \vartheta)\mathcal{P}(\sigma_1) + \vartheta\mathcal{P}(\sigma_2)$$

and

$$\mathcal{O}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) \leq (1 - \vartheta)\mathcal{O}(\sigma_1) + \vartheta\mathcal{O}(\sigma_2).$$

If we multiply these inequalities, we obtain

$$\begin{aligned}
\mathcal{P}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))\mathcal{O}(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) & \leq (1 - \vartheta)^2(\mathcal{P}\mathcal{O})(\sigma_1) + \vartheta^2(\mathcal{P}\mathcal{O})(\sigma_2) \\
& \quad + \vartheta(1 - \vartheta)[\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)].
\end{aligned}$$

Multiplying the above inequality by $\vartheta^{-\alpha}$, we have the following inequality

$$\begin{aligned}
\vartheta^{-\alpha}(\mathcal{P}\mathcal{O})(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1)) & \leq \vartheta^{-\alpha}(1 - \vartheta)^2(\mathcal{P}\mathcal{O})(\sigma_1) + \vartheta^{2-\alpha}(\mathcal{P}\mathcal{O})(\sigma_2) \\
& \quad + \vartheta^{1-\alpha}(1 - \vartheta)[\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)].
\end{aligned}$$

Now, by integrating the resulting inequality with respect to ϑ over $[0, 1]$, we have

$$\begin{aligned}
\int_0^1 \vartheta^{-\alpha}(\mathcal{P}\mathcal{O})(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta & \leq \int_0^1 \left(\vartheta^{-\alpha}(1 - \vartheta)^2(\mathcal{P}\mathcal{O})(\sigma_1) + \vartheta^{2-\alpha}(\mathcal{P}\mathcal{O})(\sigma_2) \right) d\vartheta \\
& \quad + \int_0^1 \left(\vartheta^{1-\alpha}(1 - \vartheta)[\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)] \right) d\vartheta.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^1 \vartheta^{-\alpha}(\mathcal{P}\mathcal{O})(\sigma_1 + \vartheta\Phi(\sigma_2, \sigma_1))d\vartheta & \leq (\mathcal{P}\mathcal{O})(\sigma_1) \int_0^1 \vartheta^{-\alpha}(1 - \vartheta)^2d\vartheta + (\mathcal{P}\mathcal{O})(\sigma_2) \int_0^1 \vartheta^{2-\alpha}d\vartheta \\
& \quad + [\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)] \int_0^1 \vartheta^{1-\alpha}(1 - \vartheta)d\vartheta.
\end{aligned}$$

By computing the above integrals and changing the variables, we deduce

$$\begin{aligned}
& \frac{1}{(\Phi(\sigma_2, \sigma_1))^{1-\alpha}} N_3 J_{\sigma_1 + \Phi(\sigma_2, \sigma_1)-}^{\alpha} (\mathcal{P}\mathcal{O})(\sigma_1) \\
& \leq \frac{-2}{\alpha^3 - 6\alpha^2 + 11\alpha - 6} (\mathcal{P}\mathcal{O})(\sigma_1) + \frac{1}{3-\alpha} (\mathcal{P}\mathcal{O})(\sigma_2) \\
& \quad + \frac{1}{\alpha^2 - 5\alpha + 6} (\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)). \tag{22}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned} & \int_0^1 \vartheta^{-\alpha} (\mathcal{PO})(\sigma_1 + (1 - \vartheta)\Phi(\sigma_2, \sigma_1)) d\vartheta \\ & \leq (\mathcal{PO})(\sigma_1) \int_0^1 \vartheta^{2-\alpha} d\vartheta + (\mathcal{PO})(\sigma_2) \int_0^1 \vartheta^{-\alpha} (1 - \vartheta)^2 d\vartheta \\ & \quad + [\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)] \int_0^1 \vartheta^{1-\alpha} (1 - \vartheta) d\vartheta. \end{aligned}$$

So, by computing the above integrals and changing the variables, we deduce

$$\begin{aligned} & \frac{1}{(\mathcal{P}(\sigma_2, \sigma_1))^{1-\alpha}} {}_{N_3}J_{\sigma_1^+}^{\alpha} (\mathcal{PO})(\sigma_1 + \Phi(\sigma_2, \sigma_1)) \\ & \leq \frac{1}{3-\alpha} (\mathcal{PO})(\sigma_1) + \frac{-2}{\alpha^3 - 6\alpha^2 + 11\alpha - 6} (\mathcal{PO})(\sigma_2) \\ & \quad + \frac{1}{\alpha^2 - 5\alpha + 6} [\mathcal{P}(\sigma_1)\mathcal{O}(\sigma_2) + \mathcal{P}(\sigma_2)\mathcal{O}(\sigma_1)]. \end{aligned} \quad (23)$$

By adding the inequalities (22) and (23), we obtain the desired result. $\square$

Remark 12.

- (i) In case of $\alpha = 0$, then we obtain a Theorem 3.4, which is proved by A. G. Ghazanfari in a published article [71];
- (ii) In case of $\alpha = 0$ and $\Phi(\sigma_2, \sigma_1) = \sigma_2 - \sigma_1$, then we obtain a Theorem 1 (part1), which is proved by B. G. Pachpatte in a published article [72].

7. Future Scopes

This innovative concept could be applied to future presentations of various inequalities, such as those of the Hermite–Hadamard, Ostrowski, Hadamard–Mercer, Simpson, Fejér, and Bullen types. Numerous interval-valued, L-R interval-valued convexities, fuzzy interval convexities, and C-R order interval-valued convexities can be used to illustrate the inequalities above. Additionally, these results will be used for quantum calculus and coordinated interval-valued functions using a variety of innovative fractional operators. Any mathematicians will be interested in learning how various forms of interval-valued analysis and quantum calculus might be applied to integral inequalities, as these are the most active areas of research in the field of integral inequalities.

8. Conclusions

Fractional calculus becomes more important and produces more accurate results when evaluating the effect of memory on computer models. That is to say, due to hereditary characteristics and the notion of memory, fractional calculus is more adaptable than classical calculus. The study of fractional calculus has drawn the attention of numerous writers and researchers across scientific fields. The present applications of fractional integrodifferential equations in various domains, particularly physics, have served as the inspiration for recent developments in fractional calculus. We can use convexity theory to create unified frameworks for effective, captivating, and potent numerical techniques that can be used to attack and resolve a wide range of problems in both pure and applied sciences. In this paper,

- (1) We proved and examined a new lemma, and based on this new lemma, some estimations of Hermite–Hadamard type inequality are presented.
- (2) We investigated Hadamard–Fejér inequality pertaining to non-conformable fractional integral operators with some interesting remarks.

- (3) We investigated and validated a new integral identity. Some refinements of Hadamard–Fejér type inequality about h -preinvexity via non-conformable fractional integral operator were investigated for this new integral identity.
- (4) Finally, the product of the two preinvex functions via non-conformable fractional integral operators was examined and proved.

Our approach, as well as this broad and intriguing new concept, could lead to extensive research on this fascinating topic.

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