Mixed-Norm Amalgam Spaces and Their Predual

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Abstract: In this paper, we introduce mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ and prove the boundedness of maximal function. Then, the dilation argument obtains the necessary and sufficient conditions of fractional integral operators' boundedness. Furthermore, the strong estimates of linear commutators $[b, I_{\gamma}]$ generated by $b \in BMO(\mathbb{R}^n)$ and I_{γ} on mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ are established as well. In order to obtain the necessary conditions of fractional integral commutators' boundedness, we introduce mixed-norm Wiener amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$. We obtain the necessary and sufficient conditions of fractional integral commutators' boundedness by the duality theory. The necessary conditions of fractional integral commutators' boundedness are a new result even for the classical amalgam spaces. By the equivalent norm and the operators $St_r^{(p)}(f)(x)$, we study the duality theory of mixed-norm amalgam spaces, which makes our proof easier. In particular, note that predual of the primal space is not obtained and the predual of the equivalent space does not mean the predual of the primal space.

Keywords: mixed norm; amalgam spaces; predual; fractional integral operators; commutators

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1. Introduction

The fractional power of the Laplacian operators Δ are defined by

$$\left((- \Delta)^{\gamma/2}(f)\right)^{\wedge}(\xi) = (2\pi|\xi|)^{\gamma} \hat{f}(\xi). \quad (1)$$

Comparing (1) to the Fourier transform of $|x|^{-\gamma}$, $0 < \gamma < n$, we are led to define the so-called fractional integral operators I_{γ} by

$$I_{\gamma}f(x) = (- \Delta)^{\gamma/2}(f)(x) = C_{\gamma} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy,$$

where

$$C_{\gamma}^{-1} = \frac{\pi^{n/2} 2^{\gamma} \Gamma(\gamma/2)}{\Gamma((n-\gamma)/2)}.$$

By simple calculation,

$$\int_{\mathbb{R}^n} f(x) I_{\alpha} g(x) dx = \int_{\mathbb{R}^n} g(x) I_{\alpha} f(x) dx$$

can be obtained. According to the symmetry, we can get the following result: if I_{α} are bounded from X to Y , then I_{α} are also bounded from Y' to X' , where Y' and X' are predual of Y and X . An essential application of fractional integral operators' boundedness, via the well-known Hardy–Littlewood–Sobolev theorem, is proving the Sobolev embedding theorem. This paper investigates the generalization of the Hardy–Littlewood–Sobolev theorem on mixed-norm amalgam spaces.

Mixed-norm Lebesgue spaces, as natural generalizations of the classical Lebesgue spaces $L^p(\mathbb{R}^n)$ ($0 < p < \infty$), were first introduced by Benedek and Panzone [1]. Due to

the more precise structure of mixed-norm function spaces than the corresponding classical function spaces, mixed-norm function spaces have extensive applications in the partial differential equations [2–4]. So, the mixed-norm function spaces are widely introduced and studied, such as mixed-norm Lorentz spaces [5], mixed-norm Lorentz–Marcinkiewicz spaces [6], mixed-norm Orlicz spaces [7], anisotropic mixed-norm Hardy spaces [8], mixed-norm Triebel–Lizorkin spaces [9], mixed Morrey spaces [10,11], and weak mixed-norm Lebesgue spaces [12]. More information can be found in [13].

The mixed-norm Lebesgue spaces are stated as follows. Let f be a measurable function on $\mathbb{R}^n$ and $0 < \vec{p} \leq \infty$. We say that f belongs to the mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$, if the norm

$$\|f\|_{L^{\vec{p}}} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \cdots dx_n \right)^{\frac{1}{p_n}} < \infty$$

with the usual modification when $p_i = \infty$. Note that if $p_1 = p_2 = \cdots = p_n = p$, then $L^{\vec{p}}(\mathbb{R}^n)$ are reduced to classical Lebesgue spaces L^p and

$$\|f\|_{L^{\vec{p}}} = \|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}$$

with the usual modification when $p_i = \infty$. Furthermore, let $\{q_1, \dots, q_n\}$ and $\{s_1, \dots, s_n\}$ be rearrangements of the set $\{p_1, \dots, p_n\}$, and satisfy

$$q_1 \leq q_2 \leq \cdots \leq q_n \text{ and } s_1 \geq s_2 \geq \cdots \geq s_n.$$

Then, by Minkowski's inequality,

$$\|f\|_{L^{\vec{q}}} \leq \|f\|_{L^{\vec{p}}} \leq \|f\|_{L^{\vec{s}}},$$

where $\vec{q} = (q_1, \dots, q_n)$, $\vec{s} = (s_1, \dots, s_n)$, and $\vec{p} = (p_1, \dots, p_n)$.

In partial differential equations, Morrey spaces $\mathcal{M}_{p,\lambda}(\mathbb{R}^n)$, introduced by Morrey in 1938 [14], are widely used to investigate the local behavior of solutions to elliptic and parabolic differential equations. These spaces were defined as follows. For $0 \leq \lambda \leq n$, $1 \leq p \leq \infty$, we say that $f \in \mathcal{M}_{p,\lambda}(\mathbb{R}^n)$ if $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and

$$\begin{aligned} \|f\|_{\mathcal{M}_{p,\lambda}} &= \sup_{x \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(Q(x,r))} \\ &= \sup_{r>0} \sup_{x \in \mathbb{R}^n} r^{-\frac{\lambda}{p}} \|f\|_{L_p(Q(x,r))} < \infty. \end{aligned}$$

It is obvious that if $\lambda = 0$, then $\mathcal{M}_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$; if $\lambda = n$, then $\mathcal{M}_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$; if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions almost everywhere equivalent to 0 on $\mathbb{R}^n$.

Moreover, combining mixed Lebesgue spaces and Morrey spaces, Nogayama, in 2019, introduced mixed Morrey spaces [10,11]. Mixed Morrey spaces were stated as follows. Let $\vec{p} = (p_1, p_2, \dots, p_n) \in (0, \infty]^n$ and $p_0 \in (0, \infty]$ satisfy

$$\sum_{j=1}^n \frac{1}{p_j} \geq \frac{n}{p_0}.$$

The mixed Morrey spaces $\mathcal{M}_{\vec{p}}^{p_0}(\mathbb{R}^n)$ were defined to be the set of all measurable functions f such that their quasi-norms

$$\begin{aligned}\|f\|_{\mathcal{M}_{\vec{p}}^{p_0}} &= \sup_{x \in \mathbb{R}^n} \sup_{r > 0} |Q(x, r)|^{\frac{1}{p_0} - \frac{1}{n} \sum_{j=1}^n \frac{1}{p_j}} \|f \chi_{Q(x, r)}\|_{L_{\vec{p}}} \\ &= \sup_{r > 0} \sup_{x \in \mathbb{R}^n} |Q(x, r)|^{\frac{1}{p_0} - \frac{1}{n} \sum_{j=1}^n \frac{1}{p_j}} \|f \chi_{Q(x, r)}\|_{L_{\vec{p}}} \\ &< \infty\end{aligned}$$

are finite. It is obvious that $\mathcal{M}_{\vec{p}}^{p_0}(\mathbb{R}^n) = \mathcal{M}_{\vec{p}}^{p_0}(\mathbb{R}^n) = \mathcal{M}_{p, \lambda}(\mathbb{R}^n)$ if $p_1 = p_2 = \dots = p_n = p$ and $\mathcal{M}_{\vec{p}}^{p_0}(\mathbb{R}^n) = L_{\vec{p}}(\mathbb{R}^n)$ if $\frac{1}{p_0} = \frac{1}{n} \sum_{j=1}^n \frac{1}{p_j}$.

Inspired by global Morrey-type spaces $GM_{p\theta, \omega}(\mathbb{R}^n)$ [15] and mixed Morrey spaces, the global mixed Morrey-type spaces $GM_{\vec{p}\theta, \omega}(\mathbb{R}^n)$ were defined [16] as follows. For any functions $f \in L_1^{\text{loc}}(\mathbb{R}^n)$, we say $f \in LM_{\vec{p}\theta, \omega}(\mathbb{R}^n)$ when the quasi-norms

$$\begin{aligned}\|f\|_{GM_{\vec{p}\theta, \omega}} &= \sup_{x \in \mathbb{R}^n} \|\omega(r)\|_{L_{\vec{p}}} \|f \chi_{Q(x, r)}\|_{L_{\vec{p}}} \|_{L_{\theta}(0, \infty)} \\ &= \sup_{x \in \mathbb{R}^n} \left(\int_0^\infty \left| \omega(r) \|f \chi_{Q(x, r)}\|_{L_{\vec{p}}} \right|^\theta dr \right)^{\frac{1}{\theta}} < \infty.\end{aligned}$$

The global mixed Morrey-type spaces can be regarded as mixed Morrey spaces by replacing the L^∞ -norm for r by the L^θ -norm. Due to symmetry, the interesting mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ are introduced, which can be regarded as mixed Morrey spaces by replacing the L^∞ -norm for x by the $L^{\vec{s}}$ -norm. In order to study $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$, a kind of mixed-norm Wiener amalgam space $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ was also introduced. In particular, Zhao et al. first introduced mixed-norm Wiener amalgam spaces [17] which are different from mixed-norm Wiener amalgam spaces in this paper. Furthermore, according to Proposition 5, the mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ and $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ can also be seen as the generalizations of the classical amalgam spaces $(L^p, \ell^s)^\alpha(\mathbb{R}^n)$ and $(L^p, \ell^s)(\mathbb{R}^n)$. Let us recall some information on classical amalgam spaces.

The amalgam spaces $(L^p, \ell^s)(\mathbb{R}^n)$ were first introduced by Wiener [18] in 1926. However, their systematic study goes back to the works of Holland [19], who studied the Fourier transform on $\mathbb{R}^n$. Besides that, the spaces have been widely studied [20–23]. It is obvious that Lebesgue space $L^p(\mathbb{R}^n)$ coincides with the amalgam space $(L^p, \ell^p)(\mathbb{R}^n)$. For any $r > 0$, the dilation operator $St_r^{(p)} : f(x) \mapsto r^{-\frac{n}{p}} f(r^{-1}x)$ is isometric on $L^p(\mathbb{R}^n)$. However, amalgam spaces do not have this property. If $p \neq s$, there does not exist α such that $\sup_{r>0} \|St_r^{(\alpha)}(f)\|_{(L^p, \ell^s)} < \infty$, although $St_r^{(\alpha)}(f) \in (L^p, \ell^s)(\mathbb{R}^n)$ for all $f \in (L^p, \ell^s)(\mathbb{R}^n)$, $\rho > 0$ and $\alpha > 0$ [24]. The amalgam spaces $(L^p, \ell^s)^\alpha(\mathbb{R}^n)$ compensate this shortcoming. The functions spaces $(L^p, \ell^s)^\alpha(\mathbb{R}^n)$ were introduced by Fofana in 1988 [25], which consist of $f \in (L^p, \ell^s)(\mathbb{R}^n)$ and satisfy $\sup_{r>0} \|St_r^{(\alpha)}(f)\|_{(L^p, \ell^s)} < \infty$. Finally, we point out that many new amalgam spaces have been introduced, such as variable exponent amalgam spaces $(L^{p(x)}, L^{s(x)})(\mathbb{R}^n)$ [26] and Orlicz amalgam spaces $(L_t^\Phi, \ell^s)(\mathbb{R}^n)$ [27].

Various mixed-norm function spaces have shown the boundedness properties of I_γ extensively. In 1960, Benedek and Panzone first studied the boundedness of I_γ from mixed-norm Lebesgue spaces $L^{\vec{p}}(\mathbb{R}^n)$ to mixed-norm Lebesgue spaces $L^{\vec{q}}(\mathbb{R}^n)$ [1], which is a generalization of the classical Hardy–Littlewood–Sobolev theorem (see [28]). In 2021, Zhang and Zhou improved the theorem on mixed-norm Lebesgue spaces, which is stated as follows.

Lemma 1 (see [16]). *Let $0 < \gamma < n$ and $1 < \vec{p}, \vec{q} < \infty$. Then,*

$$1 < \vec{p} \leq \vec{q} < \infty, \gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i}$$

if and only if

$$\|I_\gamma f\|_{L_{\vec{q}}} \lesssim \|f\|_{L_{\vec{p}}}.$$

For a locally integrable function b , the commutators of fractional integral operators I_γ are defined by

$$[b, I_\gamma]f(x) := b(x)I_\gamma f(x) - I_\gamma(bf)(x) = C_\gamma \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x - y|^{n-\gamma}} dy,$$

which were introduced by Chanillo in [29]. These commutators can also be used to study the theory of Hardy spaces $H^p(\mathbb{R}^n)$ [30]. In 2019, Nogayama gave an characterization of $BMO(\mathbb{R}^n)$ spaces via the $(\mathcal{M}_{\vec{p}}^{p_0}, \mathcal{M}_{\vec{q}}^{q_0})$ -boundedness of $[b, I_\gamma]$ [11]. In 2021, the result was improved on mixed-norm Lebesgue in [31], which is stated as follows.

Lemma 2 (see [31]). *Let $0 < \gamma < n$, $1 < \vec{p} \leq \vec{q} < \infty$ and*

$$\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i}.$$

Then, the following conditions are equivalent:

- (i) $b \in BMO(\mathbb{R}^n)$.
- (ii) $[b, I_\gamma]$ is bounded from $L^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{q}}(\mathbb{R}^n)$.

Similar to symmetry of I_α , we have

$$\int_{\mathbb{R}^n} f(x)[b, I_\alpha]g(x)dx = - \int_{\mathbb{R}^n} g(x)[b, I_\alpha]f(x)dx,$$

and if I_α are bounded from X to Y , then $[b, I_\alpha]$ are also bounded from Y' to X' , where Y' and X' are predual of Y and X .

In addition, we point out that the boundedness of fractional integral operators and their commutators have been studied in classical amalgam spaces. In 2020, Wang showed the boundedness of fractional integral operators I_γ and their commutators from $(L^p, L^s)^\alpha(\mathbb{R}^n)$ to $(L^q, L^s)^\alpha(\mathbb{R}^n)$ with $1/p - 1/q = 1/\alpha - 1/\beta = \gamma/n$ [32]. Nevertheless, Wang does not prove the necessary conditions of fractional integral operators and their commutators. Using the dilation argument, we obtain the necessary and sufficient conditions of fractional integral operators in this paper. We obtain the necessary and sufficient conditions of fractional integral commutators' boundedness by the duality theory.

Now, let us recall that the definition of $BMO(\mathbb{R}^n)$. $BMO(\mathbb{R}^n)$ is the Banach function space modulo constants with the norm $\|\cdot\|_{BMO}$ defined by

$$\|b\|_{BMO} = \sup_{B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |b(y) - b_B| dy < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and b_B stands for the mean value of b over B ; that is, $b_B := (1/|B|) \int_B b(y) dy$. By John–Nirenberg inequality,

$$\|b\|_{BMO} \sim \sup_{B \subset \mathbb{R}^n} \frac{\|b - b_B\|_{L^p}}{\|\chi_B\|_{L^p}}, \quad 1 < p < \infty.$$

It is also right if we replace the L^p -norm by mixed-norm $L^{\vec{p}}$ -norm (see Lemma 5).

We first define mixed-norm amalgam spaces, which extend classical amalgam spaces and mixed Morrey spaces. Studying the boundedness of I_γ and $[b, I_\gamma]$ in these new spaces is natural and important. Before that, we also studied some properties of the new spaces. This paper is organized as follows. In Section 2, we state definitions of mixed-norm amalgam spaces, some properties of mixed-norm amalgam spaces, and the main results of the present

paper. We give the proof of some properties of mixed-norm amalgam spaces in Section 3. We investigate the predual of mixed-norm amalgam spaces in Section 4. In Section 5, the boundedness of maximal function on mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ is investigated as well as the rationality of fractional integral operators. In Sections 6 and 7, we prove the boundedness of I_{γ} and their commutators generated by $b \in BMO(\mathbb{R}^n)$. In the final section, we study the necessary condition of the boundedness of $[b, I_{\gamma}]$ from $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^{\beta}(\mathbb{R}^n)$, which is a new result even for the classical amalgam spaces.

Next, we make some conventions and recall some notions. Let $\vec{p} = (p_1, p_2, \dots, p_n)$, $\vec{q} = (q_1, q_2, \dots, q_n)$, $\vec{s} = (s_1, s_2, \dots, s_n)$ be n -tuples and $1 < p_i, q_i, s_i < \infty$, $i = 1, 2, \dots, n$. We define that if $\varphi(a, b)$ is a relation or equation among numbers, $\varphi(\vec{p}, \vec{q})$ will mean that $\varphi(p_i, q_i)$ holds for each i . For example, $\vec{p} < \vec{q}$ means that $p_i < q_i$ holds for each i and $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = 1$ means $\frac{1}{p_i} + \frac{1}{p'_i} = 1$ holds for each i . The symbol B denotes the open ball and $B(x, r)$ denotes the open ball centered at x of radius r . Let $\rho B(x, r) = B(x, \rho r)$, where $\rho > 0$. $A \sim B$ means that A is equivalent to B , that is, $A \lesssim B (A \leq CB)$ and $B \lesssim A (B \leq CA)$, where C is a positive constant. Throughout the paper, each positive constant C is not necessarily equal.

2. Mixed-Norm Amalgam Spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ and $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$

The definitions of mixed-norm amalgam spaces and some properties of mixed-norm amalgam spaces are presented in Section 2.1. Then, the main theorems are shown in Section 2.2.

2.1. Definitions and Properties

In this section, we present the definitions of mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ and $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ and their properties. Firstly, the definitions of mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ and $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ are given as follows.

Definition 1. Let $1 \leq \vec{p}, \vec{s}, \alpha \leq \infty$. We define two types of amalgam spaces of $L^{\vec{p}}(\mathbb{R}^n)$ and $L^{\vec{s}}(\mathbb{R}^n)$. If measurable functions f satisfy $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n) := \left\{ f : \|f\|_{(L^{\vec{p}}, L^{\vec{s}})} < \infty \right\}$$

and

$$(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) := \left\{ f : \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} < \infty \right\},$$

where

$$\|f\|_{(L^{\vec{p}}, L^{\vec{s}})} = \left\| \|f\chi_{B(\cdot, 1)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \|f\chi_{B(y, 1)}\|_{L^{\vec{p}}}^{s_1} dy_1 \right)^{\frac{s_2}{s_1}} \cdots dy_n \right)^{\frac{1}{s_n}}$$

and

$$\begin{aligned} \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} &= \sup_{r>0} \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(\cdot, r)}\|_{L^{\vec{p}}(\mathbb{R}^n)} \right\|_{L^{\vec{s}}(\mathbb{R}^n)} \\ &= \sup_{r>0} \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(|B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(y, r)}\|_{L^{\vec{p}}(\mathbb{R}^n)} \right)^{s_1} dy_1 \right)^{\frac{s_2}{s_1}} \cdots dy_n \right)^{\frac{1}{s_n}} \end{aligned}$$

with the usual modification for $p_i = \infty$ or $s_i = \infty$.

Remark 1. By Definition 1, we have

(i) If $p_i = p$ and $s_i = s$ for each i , then

$$(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n) = (L^p, L^s)(\mathbb{R}^n), \quad (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) = (L^p, L^s)^{\alpha}(\mathbb{R}^n);$$

(ii) If $s_i = \infty$ for each i and $\frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, then

$$(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) = \mathcal{M}_{\vec{p}}^{\alpha}(\mathbb{R}^n),$$

where $\mathcal{M}_{\vec{p}}^{\alpha}(\mathbb{R}^n)$ is mixed Morrey spaces defined as [10,11]. In particular, the condition $\frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$ inspired us to study Proposition 2.

Next, we claim that the mixed-norm amalgam spaces defined in Definition 1 are Banach spaces.

Proposition 1. Let $1 \leq \vec{p}, \vec{s}, \alpha \leq \infty$. Mixed norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ and $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ are also Banach spaces.

The following proposition shows the necessary relationship of the index $\vec{p}, \vec{s}$ and α .

Proposition 2. The spaces $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ are nontrivial if and only if $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$.

Some embedding results are shown as follows.

Proposition 3. Let $1 \leq \vec{p}, \vec{q}, \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Then,

- (i) $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) \subset (L^{\vec{q}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ with $\|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \leq \|f\|_{(L^{\vec{q}}, L^{\vec{s}})^{\alpha}}$;
- (ii) If $\vec{p} \leq \vec{q}$, $(L^{\vec{q}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) \subseteq (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ with $\|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \leq \|f\|_{(L^{\vec{q}}, L^{\vec{s}})^{\alpha}}$.

We give an estimate of characteristic function on $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ and $\mathcal{H}(\vec{p}', \vec{s}', \alpha')(\mathbb{R}^n)$ as follows.

Proposition 4. Let $0 \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} < 1$ and $\chi_{B(x_0, r_0)}$ is a characteristic function on $B(x_0, r_0)$. Then, we have

$$\|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \lesssim r_0^{n/\alpha} \text{ and } \|\chi_{B(x_0, r_0)}\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \lesssim r_0^{n/\alpha'}.$$

2.2. Main Theorems

In this section, we show the main theorems in this paper. First, we define two types “discrete” mixed-norm amalgam spaces which are equivalent to mixed-norm amalgam spaces in Definition 1. Let $Q_{r,k} = r[k + [0, 1)^n]$ and

$$\|\{a_k\}_{k \in \mathbb{Z}^n}\|_{\ell^{\vec{s}}} := \left(\sum_{k_n \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}} |a_k|^{s_1} \right)^{\frac{s_2}{s_1}} \cdots \right)^{\frac{1}{s_n}}$$

with the usual modification for $s_i = \infty$.

Proposition 5. Let $1 \leq \vec{p}, \vec{s}, \alpha \leq \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. We define two types “discrete” mixed-norm amalgam spaces.

$$(L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n) := \left\{ f \in L_{loc}^1 : \|f\|_{\vec{p}, \vec{s}} := \left\| \left\{ \|f \chi_{Q_{1,k}}\|_{\vec{p}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} < \infty \right\}$$

and

$$(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n) := \left\{ f \in L_{loc}^1 : \|f\|_{\vec{p}, \vec{s}, \alpha} := \sup_{r>0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{p_i}} \|f\|_{\vec{p}, \vec{s}} < \infty \right\},$$

where

$$r\|f\|_{\vec{p},\vec{s}} := \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{\vec{p}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\vec{s}}.$$

In fact, we have

$$(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n) = (L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n) \text{ and } (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) = (L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n).$$

According to Proposition 5, we give the definition of the predual of mixed-norm amalgam spaces $(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n)$.

Definition 2. Let $1 \leq \vec{p}, \vec{s}, \alpha \leq \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. The space $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$ is defined as the set of all elements of $L_{loc}^1(\mathbb{R}^n)$ for which there exists a sequence $\{(c_j, r_j, f_j)\}_{j \geq 1}$ of elements of $\mathbb{C} \times (0, \infty) \times (L^{\vec{p}'}, \ell^{\vec{s}'})^{\alpha'}(\mathbb{R}^n)$ such that

$$f := \sum_{j \geq 1} c_j St_{r_j}^{(\alpha')}(f_j) \text{ in the sense of } L_{loc}^1(\mathbb{R}^n); \quad (2)$$

$$\|f_j\|_{\vec{p}', \vec{s}'} \leq 1, j \geq 1; \quad (3)$$

$$\sum_{j \geq 1} |c_j| < \infty. \quad (4)$$

We will always refer to any sequence $\{(c_j, r_j, f_j)\}_{j \geq 1}$ of elements of $\mathbb{C} \times (0, \infty) \times (L^{\vec{p}'}, \ell^{\vec{s}'})^{\alpha'}(\mathbb{R}^n)$ satisfying (2)–(4) as a block decomposition of f . For any element f of $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$, we set

$$\|f\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} := \inf \left\{ \sum_{j \geq 1} |c_j| : f := \sum_{j \geq 1} c_j St_{r_j}^{(\alpha')}(f_j) \right\},$$

where the infimum is taken over all block decompositions of f .

Theorem 1. (i) Let $1 \leq \vec{p}, \vec{s} \leq \infty$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. If $g \in (L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}$ and $f \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')$, we obtain $fg \in L^1(\mathbb{R}^n)$ and

$$\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| \leq \|g\|_{(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}} \|f\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')}. \quad (5)$$

(ii) Let $1 < \vec{p}, \vec{s} \leq \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. The operator $T : g \mapsto T_g$ defined as

$$\langle T_g, f \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx, \quad g \in (L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n) \text{ and } f \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')$$

is an isometric isomorphism of $(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ into $\mathcal{H}(\vec{p}', \vec{s}', \alpha')^*$.

(iii) Symmetric with

$$\|f\|_{(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in \mathcal{H}(\vec{p}', \vec{s}', \alpha'), \|g\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \leq 1 \right\},$$

we have

$$\|f\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} = \sup \left\{ \int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in (L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}(\mathbb{R}^n), \|g\|_{(L^{\vec{p}}, \ell^{\vec{s}})^{\alpha}} \leq 1 \right\}. \quad (6)$$

Before studying fractional integrals, we give the boundedness of maximal function, which shows the rationality of fractional integral operators on mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$.

Theorem 2. Let $0 < \gamma < n$, $1 < \vec{p}, \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. Then, the maximal function M is bounded on $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$. The maximal function M is defined as

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x .

Now, we show the boundedness of fractional integral operators on mixed-norm amalgam spaces.

Theorem 3. Let $0 < \gamma < n$, $1 < \vec{p} \leq \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Assume that $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i}$. Then, the fractional integral operators I_γ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$ if and only if

$$\gamma = \frac{n}{\alpha} - \frac{n}{\beta}.$$

Remark 2. In fact, the condition $\gamma = \frac{n}{\alpha} - \frac{n}{\beta}$ is necessary for the boundedness of fractional integral operators I_γ . Let $\delta_t f(x) = f(tx)$, where $(t > 0)$. Then,

$$I_\gamma(\delta_t f) = t^{-\gamma} \delta_t I_\gamma(f)$$

$$\|\delta_{t^{-1}} f\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta} = t^{\frac{n}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta}.$$

$$\|\delta_t f\|_{(L^{\vec{p}}, L^{\vec{r}})^\alpha} = t^{-\frac{n}{\alpha} + \sum_{i=1}^n \frac{1}{r_i}} \|f\|_{(L^{\vec{p}}, L^{\vec{r}})^\alpha}.$$

Thus, by the boundedness of I_γ from $(L^{\vec{p}}, L^{\vec{r}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$,

$$\begin{aligned} \|I_\gamma f\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta} &= t^\gamma \|\delta_{t^{-1}} I_\gamma(\delta_t f)\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta} \\ &= t^{\gamma + \frac{n}{\beta} - \sum_{i=1}^n \frac{1}{s_i}} \|I_\gamma(\delta_t f)\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta} \\ &\lesssim t^{\gamma + \frac{n}{\beta} - \sum_{i=1}^n \frac{1}{s_i}} \|\delta_t f\|_{(L^{\vec{p}}, L^{\vec{r}})^\alpha} \\ &= t^{\gamma + \frac{n}{\beta} - \sum_{i=1}^n \frac{1}{s_i} - \frac{n}{\alpha} + \sum_{i=1}^n \frac{1}{r_i}} \|f\|_{(L^{\vec{p}}, L^{\vec{r}})^\alpha}. \end{aligned}$$

Thus, $\gamma = \frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{r_i} - \frac{n}{\beta} + \sum_{i=1}^n \frac{1}{s_i}$ and $\gamma = \frac{n}{\alpha} - \frac{n}{\beta}$ when $\vec{s} = \vec{r}$.

Let $[b, I_\gamma]$ be the linear commutators generated by I_γ and BMO function b . We have the following result for the strong-type estimates of $[b, I_\gamma]$ on the mixed-norm amalgam spaces.

Theorem 4. Let $0 < \gamma < n$, $1 < \vec{p} \leq \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Assume that $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \frac{n}{\alpha} - \frac{n}{\beta}$. If $b \in \text{BMO}(\mathbb{R}^n)$, then the linear commutators $[b, I_\gamma]$ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$.

In fact, if the linear commutators $[b, I_\gamma]$ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$, then $b \in \text{BMO}(\mathbb{R}^n)$. This result can be stated as follows.

Theorem 5. Let $0 < \gamma < n$, $1 < \vec{p} \leq \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Assume that $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \frac{n}{\alpha} - \frac{n}{\beta}$. If the linear commutators $[b, I_\gamma]$ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$, then $b \in \text{BMO}(\mathbb{R}^n)$.

Theorem 5 is proved by Proposition 5 and Theorem 1. By this new result, we can obtain the following result.

Corollary 1. Let $0 < \gamma < n$, $1 < \vec{p} \leq \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. If $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \frac{n}{\alpha} - \frac{n}{\beta}$, then the following statements are equivalent:

- (i) The linear commutators $[b, I_\gamma]$ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$;
- (ii) $b \in \text{BMO}(\mathbb{R}^n)$.

3. Some Basic Properties

In this section, we give proofs of the properties of mixed-norm amalgam spaces.

Proof of Proposition 1. First, we will check the triangle inequality. For $f, g \in (L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$,

$$\begin{aligned} \|f + g\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} &= \sup_{r>0} \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|(f + g)\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &\leq \sup_{r>0} \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &\quad + \sup_{r>0} \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|g\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &= \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} + \|g\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha}. \end{aligned}$$

The positivity and the homogeneity are both clear. Thus, we prove that $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ are spaces with norm $\|\cdot\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha}$. It remains to check the completeness. Without losing the generality, let a Cauchy sequence $\{f_j\}_{j=1}^\infty \subset (L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ satisfy

$$\|f_{j+1} - f_j\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} < 2^{-j}.$$

We write $f = f_1 + \sum_{j=1}^\infty (f_{j+1} - f_j) = \lim_{j \rightarrow \infty} f_j$. Then,

$$\|f\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} \leq \|f_1\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} + \sum_{j=1}^\infty \|f_{j+1} - f_j\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} < \infty.$$

Thus, for almost everywhere $x \in \mathbb{R}^n$,

$$f(x) = f_1(x) + \sum_{j=1}^\infty (f_{j+1}(x) - f_j(x)) \leq |f_1(x)| + \sum_{j=1}^\infty |f_{j+1}(x) - f_j(x)| < \infty$$

and $f \in (L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$. Furthermore,

$$\begin{aligned} \|f - f_J\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} &= \left\| \sum_{j=1}^\infty (f_{j+1} - f_j) - \sum_{j=1}^{J-1} (f_{j+1} - f_j) \right\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} \\ &\leq \sum_{j=J}^\infty \|f_{j+1} - f_j\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} \\ &\leq 2 \cdot 2^{-J} \end{aligned}$$

and

$$\lim_{J \rightarrow \infty} \|f - f_J\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} = 0.$$

So, we prove that $(L^{\vec{p}}, L^{\vec{s}})^\alpha$ are Banach spaces. By the same discussion, we can prove $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ are also Banach spaces. $\square$

Proof of Proposition 2. We prove these by contradiction. In fact, by the Lebesgue differential theorem in the mixed-norm Lebesgue spaces [16], we know

$$\lim_{r \rightarrow 0} \frac{\|f\chi_{B(x,r)}\|_{L^{\vec{p}}}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}}} = f(x) \text{ a.e. } x \in \mathbb{R}^n.$$

Thus, if $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{\alpha} > 0$ and $f \neq 0$,

$$\lim_{r \rightarrow 0} |B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \frac{\|f\chi_{B(x,r)}\|_{L^{\vec{p}}}}{\|\chi_{B(x,r)}\|_{L^{\vec{p}}}} = \infty.$$

Therefore, we prove $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} < \frac{1}{\alpha}$.

If $\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} > 0$, then we claim $\|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} = \infty$ for any ball $B(x_0, r_0)$, which shows that $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ are trivial spaces. Hence, we acquire $\frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. Indeed, if $x \in B(x_0, \frac{r}{2})$ and $2r_0 < r$, then for any $y \in B(x_0, r_0)$, we have

$$|x - y| \leq |x_0 - x| + |x_0 - y| \leq \frac{r}{2} + r_0 < r,$$

that is $B(x_0, r_0) \subset B(x, r)$. Therefore,

$$\begin{aligned} \|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} &\sim \sup_{r > 0} r^{\frac{n}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, r_0)} \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &\geq \sup_{r > 2r_0} r^{\frac{n}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, \frac{r}{2})} \chi_{B(x_0, r_0)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &\gtrsim \sup_{r > 2r_0} r^{\frac{n}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \cdot r^{\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \\ &\geq \lim_{r \rightarrow +\infty} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{p_i}} \\ &= +\infty. \end{aligned}$$

For the opposite side, by calculation, we can prove $\chi_{B(0,1)} \in (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ if $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. $\square$

Proof of Proposition 3. By direct calculation, we have

$$\begin{aligned} \|f\|_{(L^{\vec{p}}, L^{\vec{s}})} &\sim \left\| |B(\cdot, 1)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(\cdot, 1)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &\leq \sup_{r > 0} \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ &= \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \end{aligned}$$

Therefore, $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) \subset (L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ with $\|f\|_{(L^{\vec{p}}, L^{\vec{s}})} \leq \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}$. Particularly, if $\vec{p} \leq \vec{q}$, by Hölder's inequality,

$$|B(x, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(x, r)}\|_{L^{\vec{p}}} \leq |B(x, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(x, r)}\|_{L^{\vec{q}}}$$

Thus, $\|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \leq \|f\|_{(L^{\vec{q}}, L^{\vec{s}})^{\alpha}}$ and $(L^{\vec{q}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n) \subseteq (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$. $\square$

Now, we show the proof of Proposition 4.

Proof of Proposition 4. It is obvious that

$$\|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \sim \sup_{r > 0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, r_0)} \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

If $r > r_0$, then by $\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \leq 0$

$$\begin{aligned} & \sup_{r > r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, r_0)} \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ & \leq \sup_{r > r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, r_0)}\|_{L^{\vec{p}}} \cdot \chi_{B(x_0, r+r_0)} \right\|_{L^{\vec{s}}} \\ & \lesssim r_0^{\sum_{j=1}^n \frac{1}{p_j}} \sup_{r > r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} (r + r_0)^{\sum_{i=1}^n \frac{n}{s_i}} \\ & = r_0^{\sum_{j=1}^n \frac{1}{p_j}} \sup_{r > r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{p_i}} \left(1 + \frac{r_0}{r}\right)^{\sum_{i=1}^n \frac{1}{s_i}} \\ & \lesssim r_0^{\frac{n}{\alpha}}. \end{aligned}$$

For $r \leq r_0$, by $\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \geq 0$, we have

$$\begin{aligned} & \sup_{r \leq r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(x_0, r_0)} \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \\ & \leq \sup_{r > r_0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \|\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \cdot \chi_{B(x_0, r+r_0)} \right\|_{L^{\vec{s}}} \\ & \lesssim \sup_{r > 0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{s_i} - \sum_{i=1}^n \frac{1}{p_i}} r^{\sum_{j=1}^n \frac{1}{p_j}} (r + r_0)^{\sum_{i=1}^n \frac{1}{s_i}} \\ & = \sup_{r > 0} r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{p_i}} (r + r_0)^{\sum_{i=1}^n \frac{1}{s_i}} \\ & \lesssim r_0^{\frac{n}{\alpha}}. \end{aligned}$$

Thus, $\|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \lesssim r_0^{n/\alpha}$.

Next, we show that $\|\chi_{B(x_0, r_0)}\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \lesssim r_0^{n/\alpha'}$. First, by the similar argument dilation operator of (9), let

$$\chi_{B(x_0, r_0)} = r^{\frac{n}{\alpha'}} \|\chi_{B(x_0/r, r_0/r)}\|_{\vec{p}', \vec{s}'} \cdot St_r^{\alpha'}(\|\chi_{B(x_0/r, r_0/r)}\|_{\vec{p}', \vec{s}'}^{-1} \chi_{B(x_0/r, r_0/r)}).$$

It is obvious that

$$\left\| \|\chi_{B(x_0/r, r_0/r)}\|_{\vec{p}', \vec{s}'}^{-1} \chi_{B(x_0/r, r_0/r)} \right\|_{\vec{p}', \vec{s}'} \leq 1.$$

From Definition 2 and Proposition 5,

$$\begin{aligned} \|\chi_{B(x_0, r_0)}\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} & \leq \sup_{r > 0} r^{\frac{n}{\alpha'}} \|\chi_{B(x_0/r, r_0/r)}\|_{\vec{p}', \vec{s}'} \\ & \lesssim \sup_{r > 0} r^{\frac{n}{\alpha'}} \left\| \|\chi_{B(x_0/r, r_0/r)} \chi_{B(\cdot, 1)}\|_{L^{\vec{p}'}} \right\|_{L^{\vec{s}'}}. \end{aligned}$$

Using the same argument of the proof of $\|\chi_{B(x_0, r_0)}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \lesssim r_0^{n/\alpha}$ with $r_0/r > 1$ and $r_0/r \leq 1$, we have

$$\|\chi_{B(x_0, r_0)}\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \lesssim \sup_{r > 0} r^{\frac{n}{\alpha'}} \left\| \|\chi_{B(x_0/r, r_0/r)} \chi_{B(\cdot, 1)}\|_{L^{\vec{p}'}} \right\|_{L^{\vec{s}'}} \lesssim r_0^{n/\alpha'}.$$

The proof is completed. $\square$

Before the proof of Proposition 5, the following two lemmas are necessary.

Lemma 3. Let $1 \leq \vec{p}, \vec{s} \leq \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. For any constant $\rho \in (0, \infty)$, we have

$$\left\| \|f \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \sim \left\| \|f \chi_{B(\cdot, \rho r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}$$

where the positive equivalence constants are independent of f and t .

Proof. Firstly, we prove the lemma holds when $\rho > 1$. It is obvious that

$$\left\| \|f \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \leq \left\| \|f \chi_{B(\cdot, \rho r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

Next, we prove the reverse inequality. We can find $N \in \mathbb{N}$ and $\{x_1, x_2, \dots, x_N\}$, such that

$$B(0, \rho r) \subset \bigcup_{j=1}^N B(x_j, r),$$

where N is independent of r and $N \lesssim 1$. Therefore, we have

$$\|f \chi_{B(x, \rho r)}\|_{L^{\vec{p}}} \leq \left\| f \sum_{j=1}^N \chi_{B(x+x_j, r)} \right\|_{L^{\vec{p}}} \leq \sum_{j=1}^N \|f \chi_{B(x+x_j, r)}\|_{L^{\vec{p}}}$$

for any $x \in \mathbb{R}^n$. According to the translation invariance of the Lebesgue measure and $N \lesssim 1$, it follows that

$$\left\| \|f \chi_{B(\cdot, \rho r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \leq \sum_{j=1}^N \left\| \|f \chi_{B(\cdot+x_j, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \lesssim \left\| \|f \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

For the $\rho \in (0, 1)$, we only need replace r by r/ρ . The proof is completed. $\square$

Remark 3. If taking $r = 1$, we have

$$\|f\|_{(L^{\vec{p}}, L^{\vec{s}})} \sim \left\| \|f \chi_{B(\cdot, \rho)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}, \quad \rho \in (0, \infty),$$

where the positive equivalence constants are independent of f .

The following result plays an indispensable role in the proof of Proposition 5.

Lemma 4. Let $1 \leq \vec{p}, \vec{s} \leq \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. Then, we have

$$\left\| \left\{ \|f \chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} \sim r^{-\sum_{i=1}^n \frac{1}{s_i}} \left\| \|f \chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}},$$

where the positive equivalence constants are independent of f and t .

Proof. By Lemma 3, we only need to show that

$$\left\| \left\{ \|f \chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} \sim r^{-\sum_{i=1}^n \frac{1}{s_i}} \left\| \|f \chi_{B(\cdot, 2\sqrt{nr})}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

For any given $x \in \mathbb{R}^n$, we let

$$A_x := \{k \in \mathbb{Z} : Q_{r,k} \cap B(x, 2\sqrt{nr}) \neq \emptyset\}.$$

Then the cardinality of A_x is finite and $x \in B(rk, 4\sqrt{nr})$ for any $k \in A_x$. Thus,

$$\begin{aligned} \|f\chi_{B(x, 2\sqrt{nr})}\|_{L^{\vec{p}}} &\leq \left\| \sum_{k \in A_x} f\chi_{Q_{r,k}} \right\|_{L^{\vec{p}}} \leq \sum_{k \in A_x} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \\ &\leq \sum_{k \in \mathbb{Z}^n} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \chi_{B(rk, 4\sqrt{nr})}(x). \end{aligned}$$

Taking the $L^{\vec{s}}$ -norm on x , we have

$$\left\| \|f\chi_{B(\cdot, 2\sqrt{nr})}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \leq \left\| \sum_{k \in \mathbb{Z}^n} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \chi_{B(rk, 4\sqrt{nr})} \right\|_{L^{\vec{s}}}.$$

By the similar argument of Lemma 3, there exist $N \in \mathbb{N}$ and $\{k_1, k_2, \dots, k_N\}$, such that

$$B(0, 4\sqrt{nr}) \subset \bigcup_{j=1}^N Q_{k_j, r}$$

where N is independent of r and $N \sim 1$. According to the translation invariance of the Lebesgue measure and $N \sim 1$, it follows that

$$\begin{aligned} \left\| \|f\chi_{B(\cdot, 2\sqrt{nr})}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} &\leq \left\| \sum_{k \in \mathbb{Z}^n} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \chi_{B(rk, 4\sqrt{nr})} \right\|_{L^{\vec{s}}} \\ &\leq \left\| \sum_{j=1}^N \sum_{k \in \mathbb{Z}^n} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \chi_{Q_{r,k_j+k}} \right\|_{L^{\vec{s}}} \\ &\lesssim r^{\sum_{i=1}^n \frac{1}{s_i}} \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}}. \end{aligned}$$

Indeed, the last inequality is obtained by the following fact that

$$\begin{aligned} &\left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left| \sum_{k \in \mathbb{Z}^n} C_k \chi_{kr+(0,r]^n}(x) \right|^{s_1} dx_1 \right)^{\frac{s_2}{s_1}} \cdots dx_n \right)^{\frac{1}{s_n}} \\ &= \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left| \sum_{k \in \mathbb{Z}^n} C_k \prod_{i=1}^n \chi_{I_{k_i}}(x_i) \right|^{s_1} dx_1 \right)^{\frac{s_2}{s_1}} \cdots dx_n \right)^{\frac{1}{s_n}} \\ &= \left(\sum_{k_n \in \mathbb{Z}} \int_{I_{k_n}} \cdots \left(\sum_{k_1 \in \mathbb{Z}^n} \int_{I_{k_1}} |C_k|^{s_1} dx_1 \right)^{\frac{s_2}{s_1}} \cdots dx_n \right)^{\frac{1}{s_n}} \\ &= r^{\sum_{i=1}^n \frac{1}{s_i}} \cdot \left(\sum_{k_n \in \mathbb{Z}} \cdots \left(\sum_{k_1 \in \mathbb{Z}^n} |C_k|^{s_1} dx_1 \right)^{\frac{s_2}{s_1}} \cdots dx_n \right)^{\frac{1}{s_n}}, \end{aligned}$$

where $C_k = \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}}$ and $I_{k_i} = rk_i + (0, r]$. Thus, we prove that

$$r^{-\sum_{i=1}^n \frac{1}{s_i}} \left\| \|f\chi_{B(\cdot, 2\sqrt{nr})}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}} \lesssim \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}}.$$

For the opposite inequality, it is obvious that

$$r^{\sum_{i=1}^n \frac{1}{s_i}} \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} = \left\| \sum_{k \in \mathbb{Z}^n} \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \chi_{Q_{r,k}} \right\|_{L^{\vec{s}}}.$$

By $Q_{r,k} \subset B(x, 2\sqrt{nr})$ for $x \in Q_{r,k}$, we have

$$r^{\sum_{i=1}^n \frac{1}{s_i}} \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} \leq \left\| \|f\chi_{B(\cdot, 2\sqrt{nr})}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

The proof is completed. $\square$

Lemma 4 can prove the proof of Proposition 5.

Proof of Proposition 5. According to Lemma 4, we obtain that

$$\left\| \left\{ \|f\chi_{Q_{1,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{\ell^{\vec{s}}} \sim \left\| \|f\chi_{B(\cdot, 1)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}$$

and

$$r^{\frac{n}{\alpha} - \sum_{i=1}^n \frac{1}{p_i}} \left\| \left\{ \|f\chi_{Q_{r,k}}\|_{L^{\vec{p}}} \right\}_{k \in \mathbb{Z}^n} \right\|_{L^{\vec{s}}} \sim \left\| |B(\cdot, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{B(\cdot, r)}\|_{L^{\vec{p}}} \right\|_{L^{\vec{s}}}.$$

Thus, we prove Proposition 5. $\square$

4. The Predual of Amalgam Spaces

In this section, we will prove Theorem 1, whose ideal comes from [33]. Before that, the dual of mixed-norm amalgam spaces $(L^{\vec{p}}, L^{\vec{s}})(\mathbb{R}^n)$ is given as follows.

Lemma 5. (i) Let $1 \leq \vec{p}, \vec{s} \leq \infty$. For $r \in (0, \infty)$, we have

$$\|fg\|_1 \leq r \|f\|_{\vec{p}, \vec{s}} \cdot r \|g\|_{\vec{p}', \vec{s}'}, \quad f, g \in L_{loc}^1(\mathbb{R}^n) \quad (7)$$

(ii) Let $1 \leq \vec{p}, \vec{s} < \infty$. The dual of mixed-norm amalgam spaces $(L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$ is $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$.

Proof. For $0 < r < \infty$, by Hölder's inequality, we have

$$\|fg\|_1 \leq r \|f\|_{\vec{p}, \vec{s}} \cdot r \|g\|_{\vec{p}', \vec{s}'}, \quad f, g \in L_{loc}^1(\mathbb{R}^n).$$

According to ([19] Theorem 2) and ([1] Theorem 1a of Section 3), we deduce that the dual of $(L^{\vec{p}}, \ell^{s_1})(\mathbb{R}^n)$ is $(L^{\vec{p}'}, \ell^{s_1'})(\mathbb{R}^n)$. If the dual of $(L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$ is $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ with $\vec{s} = (s_1, s_2, \dots, s_{n-1})$, using ([19] Theorem 2),

$$\begin{aligned} (L^{\vec{p}}, \ell^{\vec{s}})^* &= \left(\prod_{k_n \in \mathbb{Z}} (L^{\vec{p}}, \ell^{s_n}) \right)^* = \left(\prod_{k_n \in \mathbb{Z}} (L^{\vec{p}}, \ell^{s_n})^*, (\ell^{s_n})^* \right) \\ &= \left(\prod_{k_n \in \mathbb{Z}} (L^{\vec{p}'}, \ell^{s_n'}) \right) = (L^{\vec{p}'}, \ell^{\vec{s}'}). \end{aligned}$$

Hence, $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ is isometrically isomorphic to the dual of $(L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$. There is a unique element $\phi(T)$ of $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ such that

$$T(f) = \int_{\mathbb{R}^n} f(x) \phi(T)(x) dx, \quad f \in (L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$$

and, furthermore,

$$\|\phi(T)\|_{\vec{p}', \vec{s}'} = \|T\| \quad (8)$$

where $\|T\| := \sup\{|T(f)| : f \in L^1_{loc}(\mathbb{R}^n) \text{ and } \|f\|_{\vec{p}, \vec{s}} \leq 1\}$. $\square$

Now, we discuss the properties of the dilation operator $St_r^{(\alpha)} : f(x) \mapsto r^{-\frac{n}{\alpha}} f(r^{-1}x)$ for $0 < \alpha < \infty$ and $0 < r < \infty$. By direct computation, we have the following properties.

Proposition 6. Let $f \in L^1_{loc}(\mathbb{R}^n)$, $0 < \alpha < \infty$, and $0 < r < \infty$.

- (i) $St_r^{(\alpha)}$ maps $L^1_{loc}(\mathbb{R}^n)$ into itself.
- (ii) $f = St_1^{(\alpha)}(f)$.
- (iii) $St_{r_1}^{(\alpha)} \circ St_{r_2}^{(\alpha)} = St_{r_2}^{(\alpha)} St_{r_1}^{(\alpha)} = St_{r_1 r_2}^{(\alpha)}$.
- (iv) $\sup_{r>0} \|St_r^{(\alpha)}(f)\|_{\vec{p}, \vec{s}} = \|f\|_{\vec{p}, \vec{s}, \alpha}$, where $1 \leq \vec{p}, \vec{s} < \infty$ and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$.

Proposition 6 and Definition 2 prove the following result.

Proposition 7. Let $1 \leq \vec{p}, \vec{s} \leq \infty$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$. $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ is a dense subspace of $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$.

Proof. First, we verify that $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ is continuously embedded into $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$. For any $f \in (L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$, we have

$$f = \|f\|_{\vec{p}', \vec{s}'} St_1^{\alpha'} (\|f\|_{\vec{p}', \vec{s}'}^{-1} f) \quad (9)$$

and

$$\left\| \|f\|_{\vec{p}', \vec{s}'}^{-1} f \right\|_{\vec{p}', \vec{s}'} = 1.$$

Thus, $f \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')$ and satisfies

$$\|f\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \leq \|f\|_{\vec{p}', \vec{s}'}.$$

Let us show the denseness of $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ in $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$. It is clear that if $\{(c_j, r_j, f_j)\}_{j \geq 1}$ is a block decomposition of $f \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')$, then

$$\left\{ \sum_{j=1}^J c_j St_{r_j}^{(\alpha')}(f_j) \right\}_{J \geq 1} \subset (L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$$

and

$$\left\| f - \sum_{j=1}^J c_j St_{r_j}^{(\alpha')}(f_j) \right\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} = \left\| \sum_{j=J+1}^{\infty} c_j St_{r_j}^{(\alpha')}(f_j) \right\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \leq \sum_{j=J+1}^{\infty} |c_j| \rightarrow 0$$

with $J \rightarrow \infty$. Thus, $(L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ is a dense subspace of $\mathcal{H}(\vec{p}', \vec{s}', \alpha')$. $\square$

Now, let us prove the main theorem in this section.

Proof of Theorem 1. Let us prove (i). Let $\{(c_j, r_j, f_j)\}_{j \geq 1}$ be a block decomposition of f . By Proposition 5 and (7), we have for any $j \geq 1$

$$\begin{aligned} \left| \int_{\mathbb{R}^n} St_{r_j}^{(\alpha')} (f_j)(x) g(x) dx \right| &= \left| \int_{\mathbb{R}^n} St_{r_j^{-1}}^{(\alpha)} (g)(x) f_j(x) dx \right| \\ &\leq \int_{\mathbb{R}^n} \left| St_{r_j^{-1}}^{(\alpha)} (g)(x) f_j(x) \right| dx \\ &\leq \|f_j\|_{\vec{p}', \vec{s}'} \left\| St_{r_j^{-1}}^{(\alpha)} (g) \right\|_{\vec{p}, \vec{s}} \\ &\leq \left\| St_{r_j^{-1}}^{(\alpha)} (g) \right\|_{\vec{p}, \vec{s}} \leq \|g\|_{\vec{p}, \vec{s}, \alpha}. \end{aligned}$$

Therefore, we have

$$\sum_{j \geq 1} \int_{\mathbb{R}^n} \left| c_j St_{r_j}^{(\alpha')} (f_j)(x) g(x) \right| dx \leq \|g\|_{\vec{p}, \vec{s}, \alpha} \sum_{j \geq 1} |c_j|.$$

Thus, we have $fg \in L^1(\mathbb{R}^n)$ and

$$\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \leq \int_{\mathbb{R}^n} |f(x) g(x)| dx \leq \|g\|_{\vec{p}, \vec{s}, \alpha} \sum_{j \geq 1} |c_j|.$$

Taking the infimum with respect to all block decompositions of f , we obtain

$$\left| \int_{\mathbb{R}^n} f(x) g(x) dx \right| \leq \int_{\mathbb{R}^n} |f(x) g(x)| dx \leq \|g\|_{\vec{p}, \vec{s}, \alpha} \|f\|_{\vec{p}', \vec{s}', \alpha'}.$$

Now, let us prove (ii). By (i), we have

$$T_g \in \mathcal{H}(\vec{p}, \vec{s}, \alpha)^*.$$

For any $a_1, a_2 \in \mathbb{R}$, $g_1, g_2 \in (L^{\vec{p}}, \ell^{\vec{s}})^\alpha(\mathbb{R}^n)$

$$T(a_1 g_1 + a_2 g_2) = a_1 T_{g_1} + a_2 T_{g_2}$$

and

$$\|T_g\| = \sup_{\|f\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \leq 1} |T_g(f)| \leq \|g\|_{\vec{p}, \vec{s}, \alpha},$$

that is, T is linear and bounded mapping from $(L^{\vec{p}}, \ell^{\vec{s}})^\alpha(\mathbb{R}^n)$ into $\mathcal{H}(\vec{p}', \vec{s}', \alpha')^*$ satisfying $\|T\| \leq 1$. For any $g_1, g_2 \in (L^{\vec{p}}, \ell^{\vec{s}})^\alpha(\mathbb{R}^n) \subset (L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$, if $T_{g_1} = T_{g_2}$, then for any $f \in (L^{\vec{p}', \ell^{\vec{s}'}})(\mathbb{R}^n) \subset \mathcal{H}(\vec{p}', \vec{s}', \alpha')$, we have

$$T_{g_1}(f) = T_{g_2}(f).$$

Thus, $g_1 = g_2$, that is, T is injective.

Now, we will prove that T is a surjection and $\|g\|_{\vec{p}, \vec{s}, \alpha} \leq \|T_g\|$ (or $\|T\| \geq 1$). Let T^* be an element of $\mathcal{H}(\vec{p}', \vec{s}', \alpha')^*$. From Proposition 7, it follows that the restriction T_0^* of T^* to $(L^{\vec{p}', \ell^{\vec{s}'}})(\mathbb{R}^n)$ belongs to $\mathcal{H}(\vec{p}', \vec{s}', \alpha')^*$. Furthermore, we have

$$\frac{1}{n} \sum_{i=1}^n \frac{1}{p'_i} \leq \frac{1}{\alpha'} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{s'_i}.$$

There is an element g of $(L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$ such that for any $f \in (L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$

$$T^*(f) = T_0^*(f) = \int_{\mathbb{R}^n} f(x)g(x)dx \quad (10)$$

Hence, for $f \in (L^{\vec{p}'}, \ell^{\vec{s}'}) (\mathbb{R}^n)$ and $\rho > 0$, we have

$$\int_{\mathbb{R}^n} St_r^{(\alpha)}(g)(x)f(x)dx = \int_{\mathbb{R}^n} g(x)St_{r-1}^{(\alpha')}(f)(x)dx = T^*[St_{r-1}^{(\alpha')}(f)].$$

and $St_{r-1}^{(\alpha')}(f) \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')$. By the the assumption $T^* \in \mathcal{H}(\vec{p}', \vec{s}', \alpha')^*$, we have

$$\left| \int_{\mathbb{R}^n} St_r^{(\alpha)}(g)(x)f(x)dx \right| \leq \|T^*\| \cdot \|St_{r-1}^{(\alpha')}(f)\|_{\mathcal{H}(\vec{p}', \vec{s}', \alpha')} \leq \|T^*\| \cdot \|f\|_{\vec{p}', \vec{s}'}.$$

Due to (6), it follows that

$$\|St_r^{(\alpha)}(g)\|_{\vec{p}, \vec{s}} \leq \|T^*\|.$$

Therefore, for any $g \in (L^{\vec{p}}, \ell^{\vec{s}})(\mathbb{R}^n)$, by Proposition 6,

$$\|g\|_{\vec{p}, \vec{s}, \alpha} \leq \|T^*\|.$$

According to (10) and Proposition 7, we obtain

$$T^*(f) = \int_{\mathbb{R}^n} f(x)g(x)dx, \quad f \in \mathcal{H}(\vec{p}', \vec{s}', \alpha').$$

Thus, T is a surjection and $\|g\|_{\vec{p}, \vec{s}, \alpha} \leq \|T\|$.

For (iii), the Hahn–Banach theorem shows that (6) holds. $\square$

5. The Boundedness of Maximal Function

In this section, we prove the boundedness of maximal function and discuss the rationality of fractional integral operators and their commutators on mixed-norm amalgam spaces.

Proof of Theorem 2. From Theorem 1.2 [10], we obtain that

$$\|Mf\|_{L^{\vec{p}}} \leq C\|f\|_{L^{\vec{p}}}.$$

Thus, let $B = B(y, r)$, for $f = f\chi_{3B} + f\chi_{(3B)^c} =: f_1 + f_2$,

$$\begin{aligned} & |B(y, r)|^{\frac{1}{\vec{p}} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|Mf\chi_{B(y, r)}\|_{L^{\vec{p}}} \\ & \leq |B(y, r)|^{\frac{1}{\vec{p}} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|M(f_1)\chi_{B(y, r)}\|_{L^{\vec{p}}} \\ & + |B(y, r)|^{\frac{1}{\vec{p}} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|M(f_2)\chi_{B(y, r)}\|_{L^{\vec{p}}} \\ & := I(y, r) + II(y, r). \end{aligned}$$

For $I(y, r)$, we have

$$I(y, r) \lesssim |3B(y, r)|^{\frac{1}{\vec{p}} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f\chi_{3B(y, r)}\|_{L^{\vec{p}}}. \quad (11)$$

For $II(y, r)$, from

$$Mf_2(y) \lesssim \sup_{B \subseteq R} \frac{1}{|R|} \int_R |f(x)|dx, \quad y \in B,$$

we have

$$\Pi(y, r) < \sup_{B \subseteq R} \frac{1}{|R|} \int_R |f(x)| dx \cdot |B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}},$$

where R denotes the open ball. According to the definition of $\sup$, there exists $z \in B$ and $r_1 > 2r$, such that

$$\sup_{B \subseteq R} \frac{1}{|R|} \int_R |f(x)| dx < \frac{2}{|B(z, r_1)|} \int_{B(z, r_1)} |f(x)| dx.$$

By $B(z, r_1) \subset B(y, 2r_1)$,

$$\sup_{B \subseteq R} \frac{1}{|R|} \int_R |f(x)| dx \lesssim \frac{1}{|B(y, 2r_1)|} \int_{B(y, 2r_1)} |f(x)| dx.$$

Thus,

$$\Pi(y, r) \lesssim |B(y, 2r_1)|^{\frac{1}{\alpha} - 1 - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \int_{B(y, 2r_1)} |f(x)| dx. \quad (12)$$

According to (11), (12), and Proposition 3(ii), we have

$$\begin{aligned} \|Mf\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} &\leq \sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi(\cdot, r)\|_{L^{\vec{s}}} \\ &\lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} + \|f\|_{(L^1, L^{\vec{s}})^{\alpha}} \\ &\lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \end{aligned}$$

The proof is complete. $\square$

To discuss the rationality of fractional integral operators and their commutators on mixed-norm amalgam spaces, we need the following lemmas about the $BMO(\mathbb{R}^n)$ function.

Lemma 6. Let b be a function in $BMO(\mathbb{R}^n)$.

(i) For any ball B in $\mathbb{R}^n$ and for any positive integer $j \in \mathbb{Z}^+$,

$$|b_{2^{j+1}B} - b_B| \leq Cj \|b\|_{BMO}.$$

(ii) Let $1 < \vec{p} < \infty$. There exist positive constants $C_1 \leq C_2$ such that for all $b \in BMO(\mathbb{R}^n)$,

$$C_1 \|b\|_{BMO} \leq \sup_{B \subseteq \mathbb{R}^n} \frac{\|b - b_B\|_{L^{\vec{p}}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{\vec{p}}(\mathbb{R}^n)}} \leq C_2 \|b\|_{BMO}.$$

Proof. For (i), we have

$$\begin{aligned} |b_{2^{j+1}B} - b_B| &\leq \sum_{i=1}^j |b_{2^{i+1}B} - b_{2^iB}| \\ &\leq C \sum_{i=1}^j \frac{1}{|2^{i+1}B|} \int_{2^{i+1}B} |b(x) - b_{2^{i+1}B}| dx \\ &\leq 2Cj \|b\|_{BMO}. \end{aligned}$$

By Lemma 3.5 [34], the Mf is bounded on $L^{\vec{p}}(\mathbb{R}^n)$ with $1 < \vec{p} = (p_1, p_2, \dots, p_n) < \infty$. According to the dual theorem of Theorem 1.a [1], the associate space of $L^{\vec{p}}(\mathbb{R}^n)$ is $L^{\vec{p}'}(\mathbb{R}^n)$. Finally, by Theorem 1.1 [35], the proof of (ii) can be proved. $\square$

Let $M_t f(x) = M(|f|^t)^{\frac{1}{t}}$. We only discuss the rationality of fractional integral operators' commutators on mixed-norm amalgam spaces.

Let $f \in (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$. By the definition of $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$, we have

$$\|f\chi_{B(y,r)}\|_{L^{\vec{p}}}\|_{L^{\vec{s}}} < \infty.$$

So there exists $B = B(y, r) \ni x$ such that $f\chi_{B(y,r)} \in L^{\vec{p}}(\mathbb{R}^n)$ ($1 < \vec{p} < \infty$). For $[b, I_{\gamma}]f_1(x) = b(x)I_{\gamma}(f\chi_B)(x) - I_{\gamma}(f\chi_B)(x)$, we have

$$|I_{\gamma}(f\chi_B)(x)| \leq \int_{\mathbb{R}^n} \frac{|f(z)\chi_B(z)|}{|x-z|^{n-\gamma}} dz \leq C_n r^{\gamma} Mf(x)$$

and

$$\begin{aligned} |I_{\gamma}(f\chi_B)(x)| &\leq \int_{\mathbb{R}^n} \frac{|b(z)f(z)\chi_B(z)|}{|x-z|^{n-\gamma}} dz \\ &\leq b_B \int_{\mathbb{R}^n} \frac{|f(z)\chi_B(z)|}{|x-z|^{n-\gamma}} dz + \int_{\mathbb{R}^n} \frac{|b(z) - b_B||f(z)\chi_B(z)|}{|x-z|^{n-\gamma}} dz \\ &\leq C_n b_B r^{\gamma} Mf(x) + \|b(\cdot) - b_B\|_{L^{t'}} \left(\int_{\mathbb{R}^n} \frac{|f(z)|^t \chi_B(z)}{|x-z|^{t(n-\gamma)}} dz \right)^{\frac{1}{t}} \\ &\leq C_n b_B r^{\gamma} Mf(x) + C_n r^{n-t(n-\gamma)+\frac{n}{t'}} \|b\|_{BMO} Mf(x), \end{aligned}$$

where $1 < t < \frac{n}{n-\gamma}$ is small enough.

For $[b, I_{\gamma}](f\chi_{B(y,r)^c})$,

$$\begin{aligned} |[b, I_{\gamma}](f\chi_{B(y,r)^c})| &\leq \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(x) - b_B| \frac{|f(z)|}{|x-z|^{n-\gamma}} dz \\ &\quad + \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_B| \frac{|f(z)|}{|x-z|^{n-\gamma}} dz \\ &=: J_1 + J_2 \end{aligned}$$

By Theorem 2,

$$\begin{aligned} J_1 &= |b(x) - b_B| \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f(z)|}{|x-z|^{n-\gamma}} dz \\ &\leq |b(x) - b_B| \sum_{k=0}^{\infty} |2^{k+1}B|^{-1+\gamma/n} \int_{2^{k+1}B} |f(z)| dz \\ &\leq |b(x) - b_B| \sum_{k=0}^{\infty} |2^{k+1}B|^{\gamma/n} \inf_{x \in 2^{k+1}B} Mf(x) \\ &\leq |b(x) - b_B| \sum_{k=0}^{\infty} \frac{\|\chi_{2^{k+1}B}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}}{\|\chi_{2^{k+1}B}\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}} |2^{k+1}B|^{\gamma/n} \inf_{x \in 2^{k+1}B} Mf(x) \\ &\leq |b(x) - b_B| \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \sum_{k=0}^{\infty} |2^{k+1}B|^{\gamma/n-1/\alpha} \\ &\lesssim |B|^{\gamma/n-1/\alpha} \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} |b(x) - b_B|. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} J_2 &\leq \sum_{k=0}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_B| \frac{|f(z)|}{|x - z|^{n-\alpha}} dz \\ &\leq \sum_{k=0}^{\infty} |2^{k+1}B|^{-1+\frac{\alpha}{n}} \int_{2^{k+1}B \setminus 2^k B} |b(z) - b_B| |f(z)| dz \\ &\leq \sum_{k=0}^{\infty} |2^{k+1}B|^{\frac{\alpha}{n}} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |b(z) - b_B|^{t'} dz \right)^{\frac{1}{t'}} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |f(z)|^t dz \right)^{\frac{1}{t}}, \end{aligned}$$

where $1 < t < \min\{p_1, p_2, \dots, p_n\}$. From Lemma 6,

$$J_2 \lesssim \sum_{k=0}^{\infty} |2^{k+1}B|^{\frac{\alpha}{n}} k \|b\|_{BMO} \inf_{x \in 2^{k+1}B} M_t f(x).$$

Similar to J_1 , we have

$$J_2 \lesssim |B|^{\gamma/n-1/\alpha} \|b\|_{BMO} \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}.$$

Now, we can say $[b, I_{\alpha}]f$ are well defined.

6. The Boundedness of I_{γ}

In this section, we will prove the conclusions of Theorem 3.

Proof of Theorem 3. By Remark 5, we only need to prove the boundedness of I_{γ} on mixed-norm amalgam spaces if $\gamma = \frac{n}{\alpha} - \frac{n}{\beta}$. Let $f \in (L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$, $B = B(y, r)$, and

$$f = f_1 + f_2 = f\chi_{2B} + f\chi_{(2B)^c}.$$

where χ_{2B} is the characteristic function of $2B$. By the linearity of the fractional integral operator I_{γ} , one can write

$$\begin{aligned} &|B(y, r)|^{\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^n \frac{1}{q_i}-\frac{1}{n}\sum_{i=1}^n \frac{1}{s_i}} \|I_{\alpha}(f)\chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &= |B(y, r)|^{\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^n \frac{1}{q_i}-\frac{1}{n}\sum_{i=1}^n \frac{1}{s_i}} \|I_{\alpha}(f_1)\chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &+ |B(y, r)|^{\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^n \frac{1}{q_i}-\frac{1}{n}\sum_{i=1}^n \frac{1}{s_i}} \|I_{\alpha}(f_2)\chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &:= I(y, r) + II(y, r) \end{aligned}$$

Below, we will give the estimates of $I(y, r)$ and $II(y, r)$, respectively. By the $(L^{\vec{p}}, L^{\vec{q}})$ -boundedness of I_{γ} (see Lemma 1),

$$\begin{aligned} I(y, r) &\leq |B(y, r)|^{\frac{1}{\beta}-\frac{1}{n}\sum_{i=1}^n \frac{1}{q_i}-\frac{1}{n}\sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{2B(y, r)}\|_{L^{\vec{p}}} \\ &\sim |2B(y, r)|^{\frac{1}{\alpha}-\frac{1}{n}\sum_{i=1}^n \frac{1}{p_i}-\frac{1}{n}\sum_{i=1}^n \frac{1}{s_i}} \|f\chi_{2B(y, r)}\|_{L^{\vec{p}}}. \end{aligned}$$

Thus,

$$\sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} \lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \quad (13)$$

Let us now turn to the estimate of $\Pi(y, r)$. First, it is clear that when $x \in B(y, r)$ and $z \in (2B)^c$, we obtain $|x - z| \sim |y - z|$. Then, we decompose $\mathbb{R}^n$ into a geometrically increasing sequence of concentric balls and obtain the following pointwise estimate:

$$\begin{aligned} I_\gamma(f_2)(x) &= \int_{(2B)^c} \frac{|f(z)|}{|x - z|^{n-\gamma}} dz \\ &\sim \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^jB} \frac{|f(z)|}{|y - z|^{n-\gamma}} dz \\ &\lesssim \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B|^{1-\frac{\gamma}{n}}} \int_{2^{j+1}B} |f(z)| dz. \end{aligned} \quad (14)$$

Combining (14) and Hölder's inequality, we obtain

$$\begin{aligned} \Pi &= |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \left\| \chi_{B(y, r)} \int_{\mathbb{R}^n} \frac{|f_2(z)|}{|\cdot - z|^{n-\gamma}} dz \right\|_{L^{\vec{q}}} \\ &\lesssim |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}B(y, r)|^{1-\frac{\gamma}{n}}} \int_{2^{j+1}B(y, r)} |f(z)| dz \\ &\leq \sum_{j=1}^{\infty} 2^{-j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})} |2^{j+1}B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f \chi_{2^{j+1}B(y, r)}\|_{L^{\vec{p}}}, \end{aligned}$$

where $\frac{1}{\alpha} = \frac{\gamma}{n} + \frac{1}{\beta}$. By $\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} > 0$,

$$\sum_{j=1}^{\infty} 2^{-j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})} \sim 1.$$

Thus,

$$\sup_{r>0} \|\Pi(\cdot, r)\|_{L^{\vec{s}}} \lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \quad (15)$$

Therefore, using (13) and (15),

$$\begin{aligned} \|I_\alpha f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\beta}} &= \sup_{r>0} \left\| |B(\cdot, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \|I_\alpha(f) \chi_{B(\cdot, r)}\|_{L^{\vec{q}}} \right\|_{L^{\vec{s}}} \\ &\leq \sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi(\cdot, r)\|_{L^{\vec{s}}} \\ &\lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \end{aligned}$$

The proof is completed. $\square$

Let $0 < \gamma < n$. The related fractional maximal function is defined as

$$M_\gamma f(x) := \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\gamma}{n}}} \int_B |f(y)| dy,$$

where the supremum is taken over all cubes $B \subset \mathbb{R}^n$ containing x . It is well known that

$$|M_\gamma f(x)| \lesssim I_\gamma(|f|)(x). \quad (16)$$

An immediate application of the above inequality (16) is the following strong-type for the operators M_γ .

Corollary 2. Let $0 < \gamma < n$, $1 < \vec{p}, \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Assume that $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \frac{n}{\alpha} - \frac{n}{\beta}$. Then, the fractional integral operators M_γ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^{\beta}(\mathbb{R}^n)$.

Before the following corollary, let us recall generalized fractional integral operators.

Suppose that $\mathcal{L}$ are linear operators which generate an analytic semigroup $\{e^{-t\mathcal{L}}\}_{t>0}$ on $L^2(\mathbb{R}^n)$ with a kernel $p_t(x, y)$ satisfying

$$|p_t(x, y)| \leq \frac{C_1}{t^{n/2}} e^{-C_2 \frac{|x-y|^2}{t}} \quad x, y \in \mathbb{R}^n,$$

where $C_1, C_2 > 0$ are independent of x, y and t .

For any $0 < \gamma < n$, the generalized fractional integral operators $\mathcal{L}^{-\gamma/2}$ associated with the operator $\mathcal{L}$ are defined by

$$\mathcal{L}^{-\gamma/2} f(x) = \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t\mathcal{L}}(f)(x) \frac{dt}{t^{-\gamma/2+1}}.$$

Note that if $\mathcal{L} = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $\mathcal{L}^{-\gamma/2}$ is the classical fractional integral operators I_γ . See, for example, [28] (Chapter 5). By the Gaussian upper bound of kernel $p_t(x, y)$, for all $x \in \mathbb{R}^n$,

$$|\mathcal{L}^{-\gamma/2} f(x)| \leq C I_\gamma(|f|)(x). \quad (17)$$

(see [36]). In fact, if we denote the kernel of $\mathcal{L}^{-\gamma/2}$ by $K_\gamma(x, y)$, then

$$\begin{aligned} \mathcal{L}^{-\gamma/2} f(x) &= \frac{1}{\Gamma(\gamma/2)} \int_0^\infty e^{-t\mathcal{L}}(f)(x) \frac{dt}{t^{-\gamma/2+1}} \\ &= \frac{1}{\Gamma(\gamma/2)} \int_0^\infty \int_{\mathbb{R}^n} p_t(x, y) f(y) dy \frac{dt}{t^{-\gamma/2+1}} \\ &= \int_{\mathbb{R}^n} \frac{1}{\Gamma(\gamma/2)} \int_0^\infty p_t(x, y) \frac{dt}{t^{-\gamma/2+1}} \cdot f(y) dy \\ &= \int_{\mathbb{R}^n} K_\gamma(x, y) \cdot f(y) dy. \end{aligned}$$

Hence, by the Gaussian upper bound,

$$\begin{aligned} |K_\gamma(x, y)| &= \left| \frac{1}{\Gamma(\gamma/2)} \int_0^\infty p_t(x, y) \frac{dt}{t^{-\gamma/2+1}} \right| \\ &\leq \frac{1}{\Gamma(\gamma/2)} \int_0^\infty |p_t(x, y)| \frac{dt}{t^{-\gamma/2+1}} \\ &\leq C \int_0^\infty e^{-C_2 \frac{|x-y|^2}{t}} \frac{dt}{t^{n/2-\gamma/2+1}} \\ &\leq C \cdot \frac{1}{|x-y|^{n-\gamma}}. \end{aligned}$$

Considering the pointwise inequality (17), as a consequence of Theorem 2, we have the following corollary.

Corollary 3. Let $0 < \gamma < n$, $1 < \vec{p}, \vec{q} < \infty$, $1 < \vec{s} \leq \infty$, $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\alpha} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$, and $\frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} \leq \frac{1}{\beta} \leq \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}$. Assume that $\gamma = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} = \frac{n}{\alpha} - \frac{n}{\beta}$. Then the generalized fractional integral operators $\mathcal{L}^{\gamma/2}$ are bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$.

7. The Boundedness of $[b, I_\gamma]$

In this section, we show the proof of Theorem 3.

Proof of Theorem 4. Let $f \in (L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$, $B = B(y, r)$, and

$$f = f_1 + f_2 = f\chi_{2B} + f\chi_{(2B)^c}.$$

By the linearity of the commutator operators $[b, I_\gamma]$, we write

$$\begin{aligned} & |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|[b, I_\gamma](f) \chi_{B(y, r)}\|_{L^{\vec{q}}} \\ & \leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|[b, I_\gamma](f_1) \chi_{B(y, r)}\|_{L^{\vec{q}}} \\ & + |B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|[b, I_\gamma](f_2) \chi_{B(y, r)}\|_{L^{\vec{q}}} \\ & := \text{I}(y, r) + \text{II}(y, r). \end{aligned}$$

By Lemma 2 and observing that $\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i} = \frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$,

$$\begin{aligned} \text{I}(y, r) &= |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|[b, I_\gamma](f_1) \chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &\lesssim |2B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|f \chi_{2B(y, r)}\|_{L^{\vec{p}}} \\ &= |2B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f \chi_{2B(y, r)}\|_{L^{\vec{p}}}. \end{aligned}$$

Thus,

$$\sup_{r>0} \|\text{I}(y, r)\|_{L^{\vec{s}}} \lesssim \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \quad (18)$$

Now, let us turn to the estimate of $\text{II}(y, r)$. By the definition of $[b, I_\gamma]$, we have

$$|[b, I_\gamma](f_2)(x)| \leq |b(x) - b_{B(y, r)}| \cdot |I_\gamma(f_2)(x)| + |I_\gamma[(b_{B(y, r)} - b)f_2](x)|.$$

Therefore,

$$\begin{aligned} \text{II}(y, r) &= |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|[b, I_\gamma](f_2) \chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &\leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \| |b - b_{B(y, r)}| |I_\gamma(f_2)| \chi_{B(y, r)} \|_{L^{\vec{q}}} \\ &+ \| |I_\gamma[(b_{B(y, r)} - b)f_2]| \chi_{B(y, r)} \|_{L^{\vec{q}}} \\ &\leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} [\| |b - b_{B(y, r)}| |I_\gamma(f_2)| \chi_{B(y, r)} \|_{L^{\vec{q}}} \\ &+ \| |I_\gamma[(b_{B(y, r)} - b)f_2]| \chi_{B(y, r)} \|_{L^{\vec{q}}}] \\ &:= \text{II}_1(y, r) + \text{II}_2(y, r). \end{aligned}$$

By (14), Lemma 5(ii), and Hölder's inequality,

$$\begin{aligned} \text{II}_1(y, r) &\lesssim |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \sum_{j=1}^{\infty} \frac{\|(b - b_{B(y, r)}) \chi_{B(y, r)}\|_{L^{\vec{q}}}}{|2^{j+1} B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1} B(y, r)} |f(z)| dz \\ &\sim \|b\|_{BMO} \sum_{j=1}^{\infty} |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \frac{1}{|2^{j+1} B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1} B(y, r)} |f(z)| dz \\ &= \|b\|_{BMO} \sum_{j=1}^{\infty} 2^{-j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})} |2^{j+1} B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f \chi_{2^{j+1} B(y, r)}\|_{L^{\vec{p}}}. \end{aligned}$$

Due to the assumption $\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} > 0$,

$$\sum_{j=1}^{\infty} 2^{-j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})} \sim 1 \quad (19)$$

Thus,

$$\sup_{r>0} \|\text{II}(y, r)\|_{L^{\vec{s}}(\mathbb{R}^n)} \lesssim \|b\|_{BMO} \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}(\mathbb{R}^n)}. \quad (20)$$

For the estimates of $\Pi_2(y, r)$, by (14), we have

$$\begin{aligned}\Pi_2(y, r) &= |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{q_i}} \|I_\gamma[(b_{B(y, r)} - b)f_2]\chi_{B(y, r)}\|_{L^{\vec{q}}} \\ &\leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \cdot \sum_{j=1}^{+\infty} \frac{1}{|2^{j+1}B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1}B(y, r)} |f(z)| |b(z) - b_{B(y, r)}| dz \\ &\leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \cdot \sum_{j=1}^{+\infty} \frac{1}{|2^{j+1}B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1}B(y, r)} |f(z)| |b(z) - b_{2^{j+1}B(y, r)}| dz \\ &\quad + |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \cdot \sum_{j=1}^{+\infty} \frac{1}{|2^{j+1}B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1}B(y, r)} |f(z)| dz \cdot |b_{2^{j+1}B(y, r)} - b_{B(y, r)}| \\ &=: \Pi_{21}(y, r) + \Pi_{22}(y, r).\end{aligned}$$

To estimate $\Pi_{21}(y, r)$, applying Hölder's inequality and Lemma 6(ii), we can deduce that

$$\begin{aligned}\Pi_{21}(y, r) &\leq |B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}} \cdot \sum_{j=1}^{+\infty} \frac{\|(b - b_{2^{j+1}B(y, r)})\chi_{2^{j+1}B(y, r)}\|_{L^{\vec{p}'}}}{|2^{j+1}B(y, r)|^{1 - \frac{\gamma}{n}}} \|f\chi_{2^{j+1}B(y, r)}\|_{L^{\vec{p}}} \\ &\sim \|b\|_{BMO} \sum_{j=1}^{+\infty} 2^{j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})} \cdot |2^{j+1}B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f\chi_{2^{j+1}B(y, r)}\|_{L^{\vec{p}}},\end{aligned}$$

where $\frac{1}{\alpha} = \frac{1}{\beta} + \frac{\gamma}{n}$. By (8), we obtain

$$\sup_{r>0} \|\Pi_{21}(y, r)\|_{L^{\vec{s}}} \lesssim \|b\|_{BMO} \cdot \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} \quad (21)$$

Now, we estimate $\Pi_{22}(y, r)$. An application of Lemma 5(i) and Hölder's inequality gives us that

$$\begin{aligned}\Pi_{22}(y, r) &\lesssim \|b\|_{BMO} \sum_{j=1}^{+\infty} \frac{j|B(y, r)|^{\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i}}}{|2^{j+1}B(y, r)|^{1 - \frac{\gamma}{n}}} \int_{2^{j+1}B(y, r)} |f(z)| dz \\ &\leq \|b\|_{BMO} \sum_{j=1}^{+\infty} \frac{j}{2^{j(\frac{1}{\beta} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i})}} \cdot |2^{j+1}B(y, r)|^{\frac{1}{\alpha} - \frac{1}{n} \sum_{i=1}^n \frac{1}{s_i} - \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}} \|f\chi_{2^{j+1}B(y, r)}\|_{L^{\vec{p}}}.\end{aligned}$$

By (19), we obtain

$$\sup_{r>0} \|\Pi_{22}(y, r)\|_{L^{\vec{s}}} \leq \|b\|_{BMO} \cdot \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}. \quad (22)$$

Combining (18), (20)–(22), we conclude that

$$\begin{aligned}\|I_\gamma(f)\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}} &\leq \sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi(\cdot, r)\|_{L^{\vec{s}}} \\ &\leq \sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi_1(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi_2(\cdot, r)\|_{L^{\vec{s}}} \\ &\leq \sup_{r>0} \|I(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi_1(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi_{21}(\cdot, r)\|_{L^{\vec{s}}} + \sup_{r>0} \|\Pi_{22}(\cdot, r)\|_{L^{\vec{s}}} \\ &\leq \|b\|_{BMO} \cdot \|f\|_{(L^{\vec{p}}, L^{\vec{s}})^{\alpha}}.\end{aligned}$$

The proof is completed. $\square$

8. A Characterization of BMO

In this section, we prove Theorem 5. As the consequence of Theorems 4 and 5, the characterization of $BMO(\mathbb{R}^n)$, Corollary 1, is proved.

Proof of Theorem 5. Assume that $[b, I_\alpha]$ is bounded from $(L^{\vec{p}}, L^{\vec{s}})^\alpha(\mathbb{R}^n)$ to $(L^{\vec{q}}, L^{\vec{s}})^\beta(\mathbb{R}^n)$. We use the same method as Janson [37]. Choose $0 \neq z_0 \in \mathbb{R}^n$ such that $0 \notin B(z_0, 2)$. Then for $x \in B(z_0, 2)$, $|x|^{n-\alpha} \in C^\infty(B(z_0, 2))$. Hence, $|x|^{n-\alpha}$ can be written as the absolutely convergent Fourier series:

$$|x|^{n-\alpha} \chi_{B(z_0, 2)}(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2im \cdot x} \chi_{B(z_0, 2)}(x)$$

with $\sum_{m \in \mathbb{Z}^n} |a_m| < \infty$.

For any $x_0 \in \mathbb{R}^n$ and $t > 0$, let $B = B(x_0, t)$ and $B_{z_0} = B(x_0 + z_0 t, t)$. Let $s(x) = \frac{\text{sgn}(\int_{B_{z_0}} (b(x) - b(y)) dy)}{\int_{B_{z_0}} (b(x) - b(y)) dy}$. Then,

$$\frac{1}{|B|} \int_B |b(x) - b_{B_{z_0}}| = \frac{1}{|B|} \frac{1}{|B_{z_0}|} \int_B \int_{B_{z_0}} s(x) (b(x) - b(y)) dy dx.$$

If $x \in B$ and $y \in B_{z_0}$, then $\frac{y-x}{t} \in B(z_0, 2)$. Thereby,

$$\begin{aligned} & \frac{1}{|B|} \int_B |b(x) - b_{B_{z_0}}| \\ &= t^{-n-\gamma} \int_B \int_{B_{z_0}} s(x) (b(x) - b(y)) |x - y|^{\alpha-n} \left(\frac{|x - y|}{t} \right)^{n-\gamma} dy dx \\ &= t^{-n-\gamma} \sum_{m \in \mathbb{Z}^n} a_m \int_B \int_{B_{z_0}} s(x) (b(x) - b(y)) |x - y|^{\gamma-n} e^{-2im \cdot \frac{y}{t}} dy \times e^{2im \cdot \frac{x}{t}} dx \\ &= t^{-n-\gamma} \sum_{m \in \mathbb{Z}^n} a_m \int_B [b, I_\gamma] (e^{-2im \cdot \frac{\cdot}{t}} \chi_{B_{z_0}})(x) \times s(x) e^{2im \cdot \frac{x}{t}} dx. \end{aligned}$$

By (5) and Proposition 5,

$$\frac{1}{|B|} \int_B |b(x) - b_{B_{z_0}}| \lesssim t^{-n-\gamma} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\gamma] (e^{-2im \cdot \frac{\cdot}{t}} \chi_{B_{z_0}}) \right\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta} \left\| s \cdot e^{-2im \cdot \frac{\cdot}{t}} \chi_B \right\|_{\mathcal{H}(\vec{q}', \vec{s}', \beta')}.$$

By calculation,

$$\left\| s \cdot e^{-2im \cdot \frac{\cdot}{t}} \chi_B \right\|_{\mathcal{H}(\vec{q}', \vec{s}', \beta')} \lesssim t^{n/\beta'}.$$

Hence,

$$\frac{1}{|B|} \int_B |b(x) - b_{B_{z_0}}| \lesssim t^{-n-\gamma+n/\beta'} \sum_{m \in \mathbb{Z}^n} a_m \left\| [b, I_\gamma] (e^{-2im \cdot \frac{\cdot}{t}} \chi_{B_{z_0}}) \right\|_{(L^{\vec{q}}, L^{\vec{s}})^\beta}.$$

According to the hypothesis

$$\begin{aligned} \frac{1}{|B|} \int_B |b(x) - b_{B_{z_0}}| &\lesssim t^{-n-\gamma+n/\beta'} \|[b, I_\gamma]\| \sum_{m \in \mathbb{Z}^n} a_m \left\| e^{-2im \cdot \frac{\cdot}{t}} \chi_{B_{z_0}} \right\|_{(L^{\vec{p}}, L^{\vec{s}})^\alpha} \\ &\leq t^{-n-\gamma+n/\beta'+n/\alpha} \|[b, I_\gamma]\| \sum_{m \in \mathbb{Z}^n} a_m \\ &\lesssim \|[b, I_\gamma]\|. \end{aligned}$$

Thus, we have

$$\frac{1}{|B|} \int_B |b(x) - b(y)| dx \leq \frac{2}{|Q|} \int_B |b(x) - b_{B_{z_0}}| dx \lesssim \|[b, I_\gamma]\|.$$

Hence $b \in BMO(\mathbb{R}^n)$. $\square$

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