Article

Convergence Results for the Double-Diffusion Perturbation Equations

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Abstract: We study the structural stability for the double-diffusion perturbation equations. Using the a priori bounds, the convergence results on the reaction boundary coefficients k_1 , k_2 and the Lewis coefficient L_e could be obtained with the aid of some *Poincaré* inequalities. The results showed that the structural stability is valid for the double-diffusion perturbation equations with reaction boundary conditions. Our results can be seen as a version of symmetry in inequality for studying the structural stability.

Keywords: structural stability; double-diffusion perturbation equations; Lewis coefficient; convergence result

1. Introduction

Many papers in the literature have studied the continuous dependence or convergence of solutions of different equations in porous media on construction coefficients. We give these studies a new name. We call these stabilities structural stability. This kind of stability is different from the traditional stability. We do not care about the stability with the initial data, but about their structural stability with the model itself. For an introduction to the nature of this structural stability, please see book [1]. It is important to establish the result of the structural stability in the problem of the continuum mechanics. In [2], the authors studied a variety of equations and obtained many results on structural stability. We think it is very important to study structural stability. In the process of establishing the model, the error always exists. We want to know whether a small error will cause a sharp change in the solution

Straughan in paper [3] proposed a new type of double diffusion perturbation model in porous media. The Darcy approximation is used in the derivation of this type of equation. We usually call this type of equation Darcy equations. Details about such types of equations were introduced in [4,5].

There are many equations that describe fluids in porous media. In books [4,6,7], the authors studied many different types of equations. In [8–10], the saint-venant principle results were studied for the Brinkman, Darcy, and Forchheimer equations. The spatial decay results were obtained. In the literature, many results on the structural stability of equations in porous media have been obtained. Representative papers can be seen by [11–16]. It should be emphasized that some new results have also emerged recently, see [17–48]. These results all belong to the category of the study of structural stability.

In this article, we continue to consider the structural stability of such types of equations. We consider the following double-diffusion perturbation equations with velocity, pressure, temperature, and concentration perturbations:

Citation: Shi, J.; Luo, S. Convergence Results for the Double-Diffusion Perturbation Equations. *Symmetry* **2022**, *14*, 67. <https://doi.org/10.3390/sym14010067>

Academic Editor: Aviv Gibali

Received: 29 November 2021

Accepted: 17 December 2021

Published: 4 January 2022

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$$\begin{cases} \frac{\partial u_i}{\partial t} = C\phi l_i - R\theta l_i + \pi_{,i}, \\ \frac{\partial u_i}{\partial x_i} = 0, \\ \frac{\partial \theta}{\partial t} + u_i \theta_{,i} = u_3 + \Delta \theta, \\ \varepsilon_1 \frac{\partial \varphi}{\partial t} + L_e u_i \varphi_{,i} = u_3 + \Delta \varphi, \end{cases} \quad (1)$$

where u_i , θ , φ , and π are the velocity, temperature, concentration disturbance, and pressure, respectively. Δ is the Laplace operator. In Equation (1), R is the Rayleigh coefficient and C is the salinity Rayleigh coefficient, ε_1 represents the porosity, and L_e is the Lewis coefficient, $l = (0, 0, 1) = (l_1, l_2, l_3)$. The system of Equation (1) is established in the region $\Omega \times [0, \tau]$, where Ω is bounded in the strictly convex region in R^3 , and τ is a given constant and satisfies $0 \leq \tau < \infty$. The boundary conditions are:

$$u_i n_i = 0, \quad \frac{\partial \theta}{\partial n} = k_1 \theta, \quad \frac{\partial \varphi}{\partial n} = k_2 \varphi, \quad (x, t) \in \partial\Omega \times [0, \tau]. \quad (2)$$

The initial conditions are:

$$u_i(x, 0) = u_{i0}(x), \varphi(x, 0) = \varphi_0(x), \theta(x, 0) = \theta_0(x), x \in \Omega. \quad (3)$$

There are significant differences between the double-diffusion perturbation equations and the Brinkman, Forchheimer, Darcy equations. The main difficulty is that we can not get the maximum value of the disturbance as the previous papers [11–16]. In the references, the maximum value of the disturbance is often used to obtain the required structural stability results. In this paper, we can not get the maximum estimates of disturbance. The structural stability results we need will not be obtained by using the previous methods. We must adopt a new method to overcome the difficulty of not getting the maximum value. We adapt the L^4 norm of the disturbance. In addition, since the velocity equation does not contain the Laplacian term, the estimates of the gradient of the velocity is not easy to obtain. There is no Laplacian term, so we can not get the gradient estimation of the velocity according to the conventional method. The estimation of gradient of the velocity is very important in this paper. How to get the gradient estimation of the velocity is the biggest innovation of this paper because the space and the inequalities used in this paper have the property of symmetry. Our results can be regarded as the application of the symmetry in the study of the structural stability. In this article, we will use other estimations to obtain the gradient estimation. The content of this paper is arranged as follows: First, some a prior estimates of the solutions are given, and then based on these a prior estimates, the differential inequality satisfied by the concentration difference of the solution is established, and the convergence results of the solutions are obtained by integrating the inequality. The following notational conventions are adopted in the text: A comma is used to indicate the partial derivative. For example, $,i$ denotes the partial differentiation with respect to x_i . For example $u_{,i} = \frac{\partial u}{\partial x_i}$. The repeated Latin subscripts denotes summation. For example, $u_{i,i} = \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i}$. The symbol $dx = dx_1 dx_2 dx_3$.

2. A Priori Bounds

In the course of producing the results of convergence on the coefficient of (1), we find it is easy if we can derive some a priori bounds for the solutions. We will give some Lemmas that are useful in proving our main results.

Lemma 1. For the temperature θ and the concentration disturbance φ , we have the following estimates:

$$\int_{\partial\Omega} \theta^2 dS \leq \left(\frac{m_1}{m_0} + \frac{\varepsilon_0 m_2^2}{m_0^2} \right) \int_{\Omega} \theta^2 dS + \frac{1}{\varepsilon_0} \int_{\Omega} \theta_{,i} \theta_{,i} dS, \quad (4)$$

and

$$\int_{\partial\Omega} \varphi^2 dS \leq \left(\frac{m_1}{m_0} + \frac{\varepsilon_0 m_2^2}{m_0^2} \right) \int_{\Omega} \varphi^2 dS + \frac{1}{\varepsilon_0} \int_{\Omega} \varphi_{,i} \varphi_{,i} dS, \quad (5)$$

where m_0, m_1, m_2 are positive constants, and ε_0 is an arbitrary positive constant.

Proof. We defined a function ξ_i on Ω . The function ξ_i satisfies the following conditions:

$$\xi_i n_i \geq m_0 > 0, x \in \partial\Omega, \quad (6)$$

$$|\xi_{i,i}| \leq m_1, x \in \Omega, \quad (7)$$

$$|\xi_i| \leq m_2, x \in \Omega, \quad (8)$$

where n_i is the unit outward normal vector.

From the divergence theorem, we have:

$$\begin{aligned} m_0 \oint_{\partial\Omega} \theta^2 ds &\leq \oint_{\partial\Omega} \xi_i n_i \theta^2 ds \\ &= \int_{\Omega} (\xi_i \theta^2)_{,i} dx \\ &= \int_{\Omega} \xi_{i,i} \theta^2 dx + 2 \int_{\Omega} \xi_i \theta \theta_{,i} dx \\ &\leq m_1 \int_{\Omega} \theta^2 dx + 2m_2 \int_{\Omega} \theta \bar{x} \theta dx. \end{aligned} \quad (9)$$

Using Schwarz's inequality, we have:

$$\oint_{\partial\Omega} \theta^2 ds \leq \left(\frac{m_1}{m_0} + \frac{\varepsilon_0 m_2^2}{m_0^2} \right) \int_{\Omega} \theta^2 dx + \frac{1}{\varepsilon_0} \int_{\Omega} \theta_{,i} \theta_{,i} dx. \quad (10)$$

Following the same procedures, we can also get:

$$\int_{\partial\Omega} \varphi^2 dS \leq \left(\frac{m_1}{m_0} + \frac{\varepsilon_0 m_2^2}{m_0^2} \right) \int_{\Omega} \varphi^2 dS + \frac{1}{\varepsilon_0} \int_{\Omega} \varphi_{,i} \varphi_{,i} dS. \quad (11)$$

□

Lemma 2. For the velocity u_i , temperature θ , and the concentration disturbance φ , we have the following estimates:

$$\int_{\Omega} u_i u_i dx + \int_{\Omega} \theta^2 dx + \int_{\Omega} \varphi^2 dx \leq n_1(t), \quad (12)$$

$$\int_0^t \int_{\Omega} \theta_{,i} \theta_{,i} dx d\eta \leq n_2(t), \quad (13)$$

$$\int_0^t \int_{\Omega} \varphi_{,i} \varphi_{,i} dx d\eta \leq n_3(t), \quad (14)$$

where $n_1(t)$, $n_2(t)$, and $n_3(t)$ are non-negative monotonically increasing functions.

Proof. Multiplying both sides of the Equation (1) by $2u_i$, and integrating over Ω , we can get:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_i u_i dx &= 2C \int_{\Omega} \phi l_i u_i dx - 2R \int_{\Omega} \theta l_i u_i dx + 2 \int_{\Omega} \pi_{,i} u_i dx \\ &\leq 2 \int_{\Omega} u_i u_i dx + C^2 \int_{\Omega} \varphi^2 dx + R^2 \int_{\Omega} \theta^2 dx. \end{aligned} \quad (15)$$

Multiplying both sides of the Equation (1) by 2θ , and integrating over Ω , we can get:

$$\frac{d}{dt} \int_{\Omega} \theta^2 dx = 2 \int_{\Omega} u_3 \theta dx + 2 \int_{\Omega} \theta \Delta \theta dx - 2 \int_{\Omega} u_{,i} \theta_{,i} dx. \quad (16)$$

Using (4) and taking $\varepsilon_0 = 2k_1$, we can get:

$$\int_{\partial\Omega} \theta^2 dS \leq \left[\frac{m_1}{m_0} + \frac{2k_1 m_2^2}{m_0^2} \right] \int_{\Omega} \theta^2 dx + \frac{1}{2k_1} \int_{\Omega} \theta_{,i} \theta_{,i} dx. \quad (17)$$

For the second term on the right side of Equation (16), we have:

$$\begin{aligned} 2 \int_{\Omega} \theta \Delta \theta dx &= 2 \int_{\partial\Omega} \theta \frac{\partial \theta}{\partial n} dS - 2 \int_{\Omega} \theta_{,i} \theta_{,i} dx \\ &= 2k_1 \int_{\partial\Omega} \theta^2 dS - 2 \int_{\Omega} \theta_{,i} \theta_{,i} dx \\ &\leq \left(\frac{2k_1 m_1}{m_0} + \frac{4k_1^2 m_2^2}{m_0^2} \right) \int_{\Omega} \theta^2 dx - \int_{\Omega} \theta_{,i} \theta_{,i} dx. \end{aligned} \quad (18)$$

Combining (16) and (18), and using the Hölder's inequality, we can get:

$$\frac{d}{dt} \int_{\Omega} \theta^2 dx + \int_{\Omega} \theta_{,i} \theta_{,i} dx \leq \int_{\Omega} u_i u_i dx + \left(\frac{2k_1 m_1}{m_0} + \frac{4k_1^2 m_2^2}{m_0^2} + 1 \right) \int_{\Omega} \theta^2 dx. \quad (19)$$

Multiplying both sides of Equation (1) by 2φ , and integrating over Ω , we can obtain:

$$\begin{aligned} \varepsilon_1 \frac{d}{dt} \int_{\Omega} \varphi^2 dx + \int_{\Omega} \varphi_{,i} \varphi_{,i} dx \\ \leq \int_{\Omega} u_i u_i dx + \left(\frac{2k_2 m_1}{m_0} + \frac{4k_2^2 m_2^2}{m_0^2} + 1 \right) \int_{\Omega} \varphi^2 dx. \end{aligned} \quad (20)$$

We define a new function:

$$F_1(t) = \int_{\Omega} u_i u_i dx + \int_{\Omega} \theta^2 dx + \int_{\Omega} \varphi^2 dx.$$

Combining (15), (19), and (20), we obtain:

$$F_1(t) \leq m_4 + m_3 \int_0^t F_1(\eta) d\eta, \quad (21)$$

where

$$m_3 = \max \left\{ 3 + \frac{1}{\varepsilon_1}, R^2 + \frac{2k_1 m_1}{m_0} + \frac{4k_1^2 m_2^2}{m_0^2} + 1, C^2 + \frac{2k_2 m_1}{m_0 \varepsilon_1} + \frac{4k_2^2 m_2^2}{m_0^2 \varepsilon_1} + \frac{1}{\varepsilon_1} \right\},$$

and

$$m_4 = \int_{\Omega} u_{i0} u_{i0} dx + \int_{\Omega} \theta_0 \theta_0 dx + \int_{\Omega} \varphi_0 \varphi_0 dx.$$

Using Gronwall's inequality, we can get:

$$F_1(t) \leq m_3 m_4 e^{m_3 t} \int_0^t e^{-m_3 \eta} d\eta = n_1(t). \quad (22)$$

Inserting (22) into (19), we can get:

$$\int_0^t \int_{\Omega} \theta_{,i} \theta_{,i} dx d\eta \leq \left(\frac{2k_1 m_1}{m_0} + \frac{4k_1^2 m_2^2}{m_0^2} + 1 \right) \int_0^t n_1(\eta) d\eta = n_2(t).$$

Inserting (22) into (20), we can get:

$$\int_0^t \int_{\Omega} \varphi_{,i} \varphi_{,i} dx d\eta \leq \left(\frac{2k_2 m_1}{m_0} + \frac{4k_2^2 m_2^2}{m_0^2} + 1 \right) \int_0^t n_1(\eta) d\eta = n_3(t).$$

□

Lemma 3. For velocity u_i , we have the following estimates:

$$\left[\int_{\Omega} (u_i u_i)^2 dx \right]^{\frac{1}{2}} \leq m_1(t), \quad (23)$$

where $m_1(t)$ is a positive function to be defined later.

Proof. We have the following identity:

$$\int_{\Omega} u_{i,j} u_{i,j} dx = \int_{\Omega} u_{i,j} (u_{i,j} - u_{j,i}) dx + \int_{\Omega} u_{i,j} u_{j,i} dx. \quad (24)$$

Since $\partial\Omega$ is bounded, we know from the result of [44]:

$$\left| \int_{\Omega} u_{i,j} u_{j,i} dx \right| \leq k_0 \int_{\partial\Omega} u_i u_i ds,$$

where k_0 is the Gaussian curvature depending on $\partial\Omega$.

Taking $\theta = u_i$, $\varepsilon_0 = 2k_0$ in (4), we have:

$$\begin{aligned} \left| \int_{\Omega} u_{i,j} u_{j,i} dx \right| &\leq k_0 \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) \int_{\Omega} u_i u_i dx + \frac{1}{2} \int_{\Omega} u_{i,j} u_{i,j} dx \\ &\leq k_0 \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) n_1(t) + \frac{1}{2} \int_{\Omega} u_{i,j} u_{i,j} dx. \end{aligned} \quad (25)$$

Combining (24) and (25), we get:

$$\int_{\Omega} u_{i,j} u_{j,i} dx \leq 2 \int_{\Omega} u_{i,j} (u_{i,j} - u_{j,i}) dx + 2k_0 \left(\frac{3}{m} + \frac{2k_0 d^2}{2m^2} \right) n_1(t). \quad (26)$$

Using Equation (1), we obtain:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} u_{i,j} (u_{i,j} - u_{j,i}) dx &= 2 \int_{\Omega} (u_{i,j} - u_{j,i}) u_{i,jt} dx \\ &\leq 4 \int_{\Omega} u_{i,j} (u_{i,j} - u_{j,i}) dx + C^2 \int_{\Omega} \varphi_{,j} \varphi_{,j} dx + R^2 \int_{\Omega} \theta_{,j} \theta_{,j} dx. \end{aligned} \quad (27)$$

We define $E(t) = \int_{\Omega} u_{i,j} (u_{i,j} - u_{j,i}) dx$. From (26), we obtain:

$$E(t) \leq 4e^{4t} \int_0^t m(y) e^{-4y} dy + R^2 n_2(t) + C^2 n_3(t) = m_2(t), \quad (28)$$

with $m(t) = \int_{\Omega} u_{i,j}(x, 0) [(u_{i,j}(x, 0) - u_{j,i}(x, 0))] dx$.

Inserting (28) into (26), we have:

$$\int_{\Omega} u_{i,j} u_{i,j} dx \leq 2m_2(t) + 2k_0 \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) n_1(t). \quad (29)$$

Using the result of (B.17) in [26], we have:

$$\begin{aligned} \left(\int_{\Omega} |u|^4 dx \right)^{\frac{1}{2}} &\leq M \left(\frac{5}{4} \int_{\Omega} |u|^2 dx + \frac{3}{4} \int_{\Omega} |\nabla u|^2 dx \right) \\ &\leq M \left(\frac{5}{4} n_1(t) + \frac{6}{4} m_2(t) \right) + \frac{6k_0}{4} \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) n_1(t) = m_1(t), \end{aligned} \quad (30)$$

where M is a positive constant. $\square$

Lemma 4. For the temperature θ , concentration disturbance φ , we have the following estimates:

$$\left(\int_{\Omega} \theta^4 dx \right)^{\frac{1}{2}} \leq n_4(t), \quad (31)$$

$$\left(\int_{\Omega} \varphi^4 dx \right)^{\frac{1}{2}} \leq n_5(t), \quad (32)$$

with $n_4(t)$ and $n_5(t)$ are all monotonically increasing functions greater than zero.

Proof. Multiplying both sides of the (1)₃ by θ^3 and integrating over Ω , we have:

$$\begin{aligned} \frac{1}{4} \frac{d}{dt} \int_{\Omega} \theta^4 dx + \int_{\Omega} u_i \theta_{,i} \theta^3 dx &= \int_{\Omega} u_3 \theta^3 dx + \int_{\Omega} \theta^3 \theta_{,ii} dx \\ &= \int_{\Omega} u_3 \theta^3 dx - 3 \int_{\Omega} \theta^2 \theta_{,i} \theta_{,i} dx + k_1 \int_{\partial\Omega} \theta^4 dS \\ &\leq \frac{1}{4} \int_{\Omega} u_3^4 dx + \frac{3}{4} \int_{\Omega} \theta^4 dx - 3 \int_{\Omega} \theta^2 \theta_{,i} \theta_{,i} dx + k_1 \int_{\partial\Omega} \theta^4 dx. \end{aligned} \quad (33)$$

Replacing θ by θ^2 and choosing $\varepsilon_0 = 2k_1$ in (4), we get:

$$k_1 \int_{\partial\Omega} \theta^4 dx \leq \frac{k_1 m_1}{m_0} \int_n \theta^4 dx + \frac{2k_1^2 m_2^2}{m_0^2} \int_n \theta^4 dx + \frac{1}{2} \int_{\Omega} (\theta^2)_{,i} (\theta^2)_{,i} dx. \quad (34)$$

Inserting (34) into (33), we obtain:

$$\frac{d}{dt} \int_{\Omega} \theta^4 dx \leq \int_{\Omega} u_3^4 dx + 3 \int_n \theta^4 dx + \frac{4m_1 k_1}{m_0} \int_n \theta^4 dx + \frac{8k_1^2 m_2^2}{m_0^2} \int_n \theta^4 dx. \quad (35)$$

Inserting (23) into (35), we obtain:

$$\frac{d}{dt} \int_{\Omega} \theta^4 dx \leq m_1^2(t) + \left(3 + \frac{4m_1 k_1}{m_0} + \frac{8k_1^2 m_2^2}{m_0^2} \right) \int_n \theta^4 dx. \quad (36)$$

An integration of (36) leads to

$$\int_{\Omega} \theta^4 dx \leq e^{\left(3 + \frac{4m_1 k_1}{m_0} + \frac{8k_1^2 m_2^2}{m_0^2} \right) t} \left[\int_{\Omega} \theta_0^4 dx + \int_0^t (m_1(\eta))^2 d\eta \right]. \quad (37)$$

We obtain:

$$\left(\int_n \theta^4 dx \right)^{\frac{1}{2}} \leq n_4(t). \quad (38)$$

Following the same procedures, we can also get:

$$\left(\int_{\Omega} \psi^4 dx \right)^{\frac{1}{2}} \leq n_5(t), \quad (39)$$

with

$$n_4(t) = e^{\left(3 + \frac{4m_1k_1}{m_0} + \frac{8k_2^2m_2^2}{m_0^2}\right)t} \left[\int_{\Omega} \theta_0^2 dx + \int_0^t (m_1(\eta))^2 d\eta \right],$$

and

$$n_5(t) = e^{\left(3 + \frac{4mk_2}{m_0} + \frac{8k_2^2m_2^2}{m_0^2}\right)t} \left[\int_{\Omega} \psi^4 dx + \int_0^t (m_1(\eta))^2 d\eta \right].$$

□

3. Convergence Result for the Reaction Boundary Coefficients k_1 and k_2

Let $(u_i, \theta, \varphi, \pi)$ be the solution of (1)–(3) with $k_1 = \hat{k}_1$, $k_2 = \hat{k}_2(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the solution of (1)–(3) with $k_1 = 0$, $k_2 = 0$. We define $\omega_i = u_i - u_i^*$, $\hat{\theta} = \theta - \theta^*$, $\hat{\varphi} = \varphi - \varphi^*$, $\hat{\pi} = \pi - \pi^*$, then $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies the following equations:

$$\begin{aligned} \frac{\partial \omega_i}{\partial t} &= C\hat{\varphi}l_i - R\hat{\theta}l_i + \hat{\pi}_{,i}, \\ \frac{\partial \omega_i}{\partial x_i} &= 0, \\ \frac{\partial \hat{\theta}}{\partial t} + \omega_i \theta_{,i} + u_i^* \hat{\theta}_{,i} &= \omega_3 + \Delta \hat{\theta}, \\ \varepsilon_1 \frac{\partial \hat{\varphi}}{\partial t} + L_e(\omega_i \varphi_{,i} + u_i^* \hat{\varphi}_{,i}) &= \omega_3 + \Delta \hat{\varphi}. \end{aligned} \quad (40)$$

The boundary conditions are:

$$\omega_i n_i = 0, \frac{\partial \hat{\theta}}{\partial n} = \hat{k}_1 \theta, \frac{\partial \hat{\varphi}}{\partial n} = \hat{k}_2 \varphi, (x, t) \in \partial\Omega \times [0, \tau]. \quad (41)$$

The initial conditions are:

$$\omega_i(x, 0) = 0, \hat{\varphi}(x, 0) = 0, \hat{\theta}(x, 0) = 0, x \in \Omega. \quad (42)$$

In deducing our main result, we will use the following Lemma.

Lemma 5. For the difference of the velocity ω_i , we can get the following estimates:

$$\int_{\Omega} \omega_{i,j} \omega_{i,j} dx \leq 2 \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + 2k_0 \left[\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right] \int_{\Omega} \omega_i \omega_i dx, \quad (43)$$

with k_0 as a positive constant.

Proof. We know the fact:

$$\int_{\Omega} \omega_{i,j} \omega_{i,j} dx = \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + \int_{\Omega} \omega_{i,j} \omega_{j,i} dx. \quad (44)$$

Since the boundary of Ω is bounded, we have:

$$\left| \int_{\Omega} \omega_{i,j} \omega_{j,i} dx \right| \leq k_0 \int_{\partial\Omega} \omega_i \omega_i dS, \quad (45)$$

with k_0 as a positive constant depending on the Gaussian curvature of $\partial\Omega$ (see [44]).

Using the result (4) with $\varepsilon_0 = 2k_0$, we can obtain:

$$\int_{\partial\Omega} \omega_i \omega_i dS \leq \left[\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right] \int_{\Omega} \omega_i \omega_i dx + \frac{1}{2k_0} \int_{\Omega} \omega_{i,j} \omega_{i,j} dx. \quad (46)$$

Inserting (45) and (46) into (44), we can get:

$$\int_{\Omega} \omega_{i,j} \omega_{i,j} dx \leq 2 \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + 2k_0 \left[\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right] \int_{\Omega} \omega_i \omega_i dx.$$

In this part, we will get the following Theorem. $\square$

Theorem 1. Let $(u_i, \theta, \varphi, \pi)$ be the classical solution of the initial value problem (1)–(3) with $k_1 = \hat{k}_1$, $k_2 = \hat{k}_2$, and $(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the classical solution of the initial boundary value problem (1)–(3) with $k_1 = 0$, $k_2 = 0$. $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ is the difference of these two solutions. When $\hat{k}_1$ and $\hat{k}_2$ tend to zero, the solution $(u_i, \theta, \varphi, \pi)$ converges to the solution $(u_i^*, \theta^*, \varphi^*, \pi^*)$. The difference of the solution $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies:

$$\begin{aligned} & \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\varphi}^2 dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx \\ & \leq \hat{k}_1^2 m_8 e^{m_8 t} n_6(t) + \hat{k}_2^2 m_8 e^{m_8 t} n_7(t), \end{aligned} \quad (47)$$

where m_8 is a positive constant and $n_6(t)$ and $n_7(t)$ are positive functions.

Proof. Multiplying both sides of Equation (40) by $2\omega_i$, and integrating over Ω , we can get:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \omega_i \omega_i dx &= 2C \int_{\Omega} \hat{\varphi} l_i \omega_i dx - 2R \int_{\Omega} \hat{\theta} l_i \omega_i dx + 2 \int_{\Omega} \hat{\pi}_i \omega_i dx \\ &\leq 2 \int_{\Omega} \omega_i \omega_i dx + C^2 \int_{\Omega} \hat{\varphi}^2 dx + R^2 \int_{\Omega} \hat{\theta}^2 dx. \end{aligned} \quad (48)$$

From Equation (40), we know:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx \\ &= 2 \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \omega_{i,jt} dx \\ &= 2C \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\varphi}_{,j} l_i dx + 2 \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\pi}_{,ij} dx - 2R \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\theta}_{,j} l_i dx. \end{aligned} \quad (49)$$

Using the divergence theorem and Hölder's inequality, we can get:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx \\ & \leq (2C^2 + 2R^2) \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) (\omega_{i,j} - \omega_{j,i}) dx + \frac{1}{2} \int_{\Omega} \hat{\varphi}_{,j} \hat{\varphi}_{,j} dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,j} \hat{\theta}_{,j} dx \\ & = (4C^2 + 4R^2) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + \frac{1}{2} \int_{\Omega} \hat{\varphi}_{,j} \hat{\varphi}_{,j} dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,j} \hat{\theta}_{,j} dx. \end{aligned} \quad (50)$$

Multiplying both sides of Equation (40) by $2\hat{\theta}$, and integrating over Ω , we can get:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx &= 2 \int_{\Omega} \omega_3 \hat{\theta} dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx - 2 \int_{\Omega} \omega_i \theta_{,i} \hat{\theta} dx - 2 \int_{\Omega} u_i^* \hat{\theta}_{,i} \hat{\theta} dx \\ &= 2 \int_{\Omega} \omega_3 \hat{\theta} dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx + 2 \int_{\Omega} \omega_i \theta_{,i} \hat{\theta} dx. \end{aligned} \quad (51)$$

The first term on the right side of Equation (51) can be bounded by:

$$2 \int_{\Omega} \omega_3 \hat{\theta} dx \leq \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx. \quad (52)$$

Using the result (4), and taking $\varepsilon_0 = 1$, we can get:

$$\int_{\partial\Omega} \hat{\theta}^2 dS \leq \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2}\right) \int_{\Omega} \hat{\theta}^2 dx + \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx. \quad (53)$$

We now take the second term on the right side of Equation (51):

$$\begin{aligned} 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx &= 2 \int_{\partial\Omega} \hat{\theta} \frac{\partial \hat{\theta}}{\partial n} dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &= 2 \hat{k}_1 \int_{\partial\Omega} \theta \hat{\theta} dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \int_{\partial\Omega} \hat{\theta}^2 dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2}\right) \int_{\Omega} \hat{\theta}^2 dx - \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx. \end{aligned} \quad (54)$$

Combining (51), (52), and (54), we can get:

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\theta}^2 dx + 2 \int_{\Omega} \omega_i \omega_i \hat{\theta}^2 dx \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dx + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\theta}^2 dx + 2 \left(\int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \hat{\theta}^4 dx \right)^{\frac{1}{2}}, \end{aligned} \quad (55)$$

where $m_5 = \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2}\right) + 1$.

Using the result of (B.17) in [26] and (31), we can get:

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\theta}^2 dx + 2 \left(\int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} n_4(t) \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\theta}^2 dx + 2M \left\{ \frac{5}{4} \int_{\Omega} \omega_i \omega_i dx + \frac{3}{4} \int_{\Omega} \omega_{i,j} \omega_{i,j} dx \right\} n_4(t) \\ &\leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + m_6 \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\theta}^2 dx + 3M n_4(\tau) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx, \end{aligned} \quad (56)$$

where $m_6 = M n_4(\tau) \left[3k_0 \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) + \frac{5}{2} \right] + 1$.

Multiplying both sides of Equation (40) by $2\hat{\phi}$, and integrating over Ω we can get:

$$\begin{aligned} \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx &= 2 \int_{\Omega} \omega_3 \hat{\phi} dx + 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx - 2L_e \int_{\Omega} \omega_i \varphi_{,i} \hat{\phi} dx - 2L_e \int_{\Omega} u_i^* \hat{\psi} \hat{\psi}_{,i} dx \\ &= 2 \int_{\Omega} \omega_3 \hat{\phi} dx + 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx + 2L_e \int_{\Omega} \omega_i \varphi \hat{\phi}_{,i} dx. \end{aligned} \quad (57)$$

The first term on the right side of Equation (57) can be bounded by:

$$2 \int_{\Omega} \omega_3 \hat{\phi} dx \leq \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\phi}^2 dx. \quad (58)$$

Using (4), and taking $\varepsilon_0 = 1$, we can get:

$$\int_{\partial\Omega} \hat{\phi}^2 dS \leq \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2}\right) \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx. \quad (59)$$

We now take the second term on the right side of Equation (57). We have:

$$\begin{aligned}
 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx &= 2 \int_{\partial\Omega} \hat{\phi} \frac{\partial \hat{\phi}}{\partial n} dS - 2 \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\
 &= 2 \hat{k}_2 \int_{\partial\Omega} \varphi \hat{\phi} dS - 2 \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\
 &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + \int_{\partial\Omega} \varphi^2 dS - 2 \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\
 &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \hat{\phi}^2 dx - \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx.
 \end{aligned} \tag{60}$$

Combining (57)–(60), we can obtain:

$$\begin{aligned}
 \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\phi}^2 dx + 2L_e^2 \int_{\Omega} \omega_i \omega_i \varphi^2 dx \\
 &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\phi}^2 dx + 2L_e^2 \left(\int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \varphi^4 dx \right)^{\frac{1}{2}}.
 \end{aligned} \tag{61}$$

We can also get:

$$\begin{aligned}
 \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_5 \int_{\Omega} \hat{\phi}^2 dx + 2L_e^2 M \left\{ \frac{5}{4} \int_{\Omega} \omega_i \omega_i dx + \frac{3}{4} \int_{\Omega} \omega_{i,j} \omega_{i,j} dx \right\} n_5(t) \\
 &\leq \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + m_7 \int_{\Omega} \omega_i \omega_i dx + m_7 \int_{\Omega} \hat{\phi}^2 dx + 3ML_e^2 n_5(\tau) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx,
 \end{aligned} \tag{62}$$

where $m_5 = ML_e^2 n_5(\tau) \left[3k_0 \left(\frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right) + \frac{5}{2} \right] + 1$.

Combining (48), (50), (56), (61), and (62), we can get:

$$\begin{aligned}
 \frac{d}{dt} \left[\int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx \right] &\leq (m_6 + m_7 + 2) \int_{\Omega} \omega_i \omega_i + (R^2 + m_5) \int_{\Omega} \hat{\theta}^2 dx + (C^2 + m_5) \int_{\Omega} \hat{\phi}^2 dx \\
 + \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + (3Mn_4(\tau) + 3ML_e^2 n_5(\tau) + 4C^2 + 4R^2) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx.
 \end{aligned} \tag{63}$$

Let

$$F_2(t) = \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx,$$

$$m_8 = \max \left\{ m_6 + m_7 + 2, R^2 + m_5, \frac{c^2 + m_5}{\varepsilon_1}, 3Mn_4(\tau) + 3ML_e^2 n_5(\tau) + 4C^2 + 4R^2 \right\}.$$

From (63), it can be seen that:

$$\frac{d}{dt} F_2(t) \leq \hat{k}_1^2 \int_{\partial\Omega} \theta^2 dS + \hat{k}_2^2 \int_{\partial\Omega} \varphi^2 dS + m_8 F_2(t). \tag{64}$$

Integrating (64), and using (4), (5), (13), and (14), we can get:

$$F_2(t) \leq \hat{k}_1^2 m_8 e^{m_8 t} n_6(t) + \hat{k}_2^2 m_8 e^{m_8 t} n_7(t), \tag{65}$$

with $n_6(t) = \int_0^t \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2} \right) n_1(\eta) d\eta + n_2(t)$ and $n_7(t) = \int_0^t \left(\frac{m_1}{m_0} + \frac{m_2^2}{m_0^2} \right) n_1(\eta) d\eta + n_3(t)$.

Inequality (65) shows that when $\hat{k}_1, \hat{k}_2$ simultaneously tend to zero, the energy $F_2(t)$ also tends to zero as the indicated norm. $\square$

4. Convergence Result for the Lewis Coefficient L_e

Let $(u_i, \theta, \varphi, \pi)$ be the solution of (1)–(3) when $Le = \hat{L}e$, and let $(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the solution of (1)–(3) when $Le = 0$. We assume $\omega_i = u_i - u_i^*, \hat{\theta} = \theta - \theta^*, \hat{\varphi} = \varphi - \varphi^*, \hat{\pi} = \pi - \pi^*$, then $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies the following equations:

$$\begin{aligned}\frac{\partial \omega_i}{\partial t} &= C\hat{\varphi}l_i - R\hat{\theta}l_i + \hat{\pi}_{,i}, \\ \frac{\partial \omega_i}{\partial x_i} &= 0, \\ \frac{\partial \hat{\theta}}{\partial t} + \omega_i\theta_{,i} + u_i^*\hat{\theta}_{,i} &= \omega_3 + \Delta\hat{\theta}, \\ \varepsilon_1\frac{\partial \hat{\varphi}}{\partial t} + \hat{L}_e u_i\varphi_{,i} &= \omega_3 + \Delta\hat{\varphi}.\end{aligned}\quad (66)$$

The boundary conditions are:

$$\omega_i n_i = 0, \quad \frac{\partial \hat{\theta}}{\partial n} = k_1 \hat{\theta}, \quad \frac{\partial \hat{\varphi}}{\partial n} = k_2 \hat{\varphi}, \quad (x, t) \in \partial\Omega \times [0, \tau]. \quad (67)$$

The initial conditions are:

$$\omega_i(x, 0) = 0, \hat{\varphi}(x, 0) = 0, \hat{\theta}(x, 0) = 0, x \in \Omega. \quad (68)$$

Theorem 2. Let $(u_i, \theta, \varphi, \pi)$ be the solution of (1)–(3) when $Le = \hat{L}e$, $(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the solution of (1)–(3) when $Le = 0$. We assume $\omega_i = u_i - u_i^*, \hat{\theta} = \theta - \theta^*, \hat{\varphi} = \varphi - \varphi^*, \hat{\pi} = \pi - \pi^*$, then $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies the following estimates:

$$\begin{aligned}\int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\varphi}^2 dx + \int_{\Omega} \omega_{i,j}(\omega_{i,j} - \omega_{j,i}) dx \\ \leq 2\hat{L}_e^2 m_{11} e^{m_{11}t} \int_0^t m_1(\eta) n_5(\eta) e^{-m_{11}\eta} d\eta,\end{aligned}\quad (69)$$

where m_{11} is a constant greater than zero.

Proof. Multiplying both sides of Equation (66) by $2\omega_i$, and integrating over $\Omega \times [0, t]$, we can obtain:

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \omega_i \omega_i dx &= 2C \int_{\Omega} \hat{\varphi} l_i \omega_i dx - 2R \int_{\Omega} \hat{\theta} l_i \omega_i dx + 2 \int_{\Omega} \hat{\pi}_{,i} \omega_i dx \\ &\leq 2 \int_{\Omega} \omega_i \omega_i dx + C^2 \int_{\Omega} \hat{\varphi}^2 dx + R^2 \int_{\Omega} \hat{\theta}^2 dx.\end{aligned}\quad (70)$$

$\square$

Multiplying both sides of Equation (66) by $2\hat{\theta}$, and integrating over Ω , we can get,

$$\begin{aligned}\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx &= 2 \int_{\Omega} \omega_3 \hat{\theta} dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx - 2 \int_{\Omega} \omega_i \theta_{,i} \hat{\theta} dx - 2 \int_{\Omega} u_i^* \hat{\theta}_{,i} \hat{\theta} dx \\ &= 2 \int_{\Omega} \omega_3 \hat{\theta} dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx + 2 \int_{\Omega} \omega_i \theta_{,i} \hat{\theta} dx.\end{aligned}\quad (71)$$

The first term on the right side of Equation (71) can be obtained from Hölder's inequality:

$$2 \int_{\Omega} \omega_3 \hat{\theta} dx \leq \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx. \quad (72)$$

Using a method similar to (4), and taking $\varepsilon_0 = 2k_1$, we can get:

$$\int_{\partial\Omega} \hat{\theta}^2 dS \leq \left(\frac{m_1}{m_0} + 2k_1 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \hat{\theta}^2 dx + \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx. \quad (73)$$

We now deal with the second term on the right side of Equation (71). We have:

$$\begin{aligned} 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} dx &= 2 \int_{\partial\Omega} \hat{\theta} \frac{\partial \hat{\theta}}{\partial n} dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &= 2k_1 \int_{\partial\Omega} \hat{\theta}^2 dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq k_1^2 \int_{\partial\Omega} \hat{\theta}^2 dS - 2 \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq 2k_1 \left(\frac{m_1}{m_0} + 2k_1 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \hat{\theta}^2 dx - \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx. \end{aligned} \quad (74)$$

Combining (71)–(74), we can get:

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \int_{\Omega} \omega_i \omega_i dx + m_9 \int_{\Omega} \hat{\theta}^2 dx + 2 \int_{\Omega} \omega_i \omega_i \hat{\theta}^2 dx \\ &\leq \int_{\Omega} \omega_i \omega_i dx + m_9 \int_{\Omega} \hat{\theta}^2 dx + 2 \left(\int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \hat{\theta}^4 dx \right)^{\frac{1}{2}}, \end{aligned} \quad (75)$$

where $m_9 = 2k_1 \left(\frac{m_1}{m_0} + 2k_1 \frac{m_2^2}{m_0^2} \right) + 1$.

Using the result of (B.17) in [26] again, we can also get:

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\theta}_{,i} \hat{\theta}_{,i} dx \\ &\leq \int_{\Omega} \omega_i \omega_i dx + m_9 \int_{\Omega} \hat{\theta}^2 dx + 2 \left(\int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} n_4(t) \\ &\leq \int_{\Omega} \omega_i \omega_i dx + m_9 \int_{\Omega} \hat{\theta}^2 dx + 2M \left\{ \frac{5}{4} \int_{\Omega} \omega_i \omega_i dx + \frac{3}{4} \int_{\Omega} \omega_{i,j} \omega_{i,j} dx \right\} n_4(t) \\ &\leq m_6 \int_{\Omega} \omega_i \omega_i dx + m_9 \int_{\Omega} \hat{\theta}^2 dx + 3M n_4(\tau) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx. \end{aligned} \quad (76)$$

Multiplying both sides of Equation (66)₃ by $2\hat{\phi}$, and integrating over $\Omega \times [0, t]$, we can get,

$$\begin{aligned} \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx &= 2 \int_{\Omega} \omega_3 \hat{\phi} dx + 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx - 2\hat{L}_e \int_{\Omega} u_i \varphi_{,i} \hat{\phi} dx \\ &= 2 \int_{\Omega} \omega_3 \hat{\phi} dx + 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx + 2\hat{L}_e \int_{\Omega} u_i \varphi \hat{\phi}_{,i} dx. \end{aligned} \quad (77)$$

The first term on the right side of Equation (77) can be bounded by:

$$2 \int_{\Omega} \omega_3 \hat{\phi} dx \leq \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\phi}^2 dx. \quad (78)$$

Using the result (4), and taking $\varepsilon_0 = 2k_2$, we can get:

$$\int_{\partial\Omega} \hat{\phi}^2 dS \leq \left(\frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx. \quad (79)$$

The second term on the right side of (77) can be bounded as follows:

$$\begin{aligned} 2 \int_{\Omega} \hat{\phi} \Delta \hat{\phi} dx &= 2 \int_{\partial\Omega} \hat{\phi} \frac{\partial \hat{\phi}}{\partial n} dS - 2 \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\ &= 2k_2 \int_{\partial\Omega} \hat{\phi}^2 dS - 2 \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\ &\leq 2k_2 \left(\frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \hat{\phi}^2 dx - \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx. \end{aligned} \quad (80)$$

Combining (77), (78), and (80), we can obtain:

$$\begin{aligned} \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\ \leq \int_{\Omega} \omega_i \omega_i dx + m_{10} \int_{\Omega} \hat{\phi}^2 dx + 2\hat{L}_e^2 \int_{\Omega} u_i u_i \varphi^2 dx \\ \leq \int_{\Omega} \omega_i \omega_i dx + m_{10} \int_{\Omega} \hat{\phi}^2 dx + 2\hat{L}_e^2 \left(\int_{\Omega} (u_i u_i)^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \varphi^4 dx \right)^{\frac{1}{2}}, \end{aligned} \quad (81)$$

where $m_{10} = 2k_2 \left(\frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) + 1$.

Using the results (23) and (32), we can obtain:

$$\begin{aligned} \varepsilon_1 \frac{d}{dt} \int_{\Omega} \hat{\phi}^2 dx + \frac{1}{2} \int_{\Omega} \hat{\phi}_{,i} \hat{\phi}_{,i} dx \\ \leq \int_{\Omega} \omega_i \omega_i dx + m_{10} \int_{\Omega} \hat{\phi}^2 dx + 2\hat{L}_e^2 m_1(t) n_5(t). \end{aligned} \quad (82)$$

A combination of (70), (76), (82), and (50) gives:

$$\begin{aligned} \frac{d}{dt} \left[\int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx \right] \\ \leq (m_6 + 3) \int_{\Omega} \omega_i \omega_i dx + (R^2 + m_9) \int_{\Omega} \hat{\theta}^2 dx + (C^2 + m_{10}) \int_{\Omega} \hat{\phi}^2 dx \\ + (3Mn_4(\tau) + 4C^2 + 4R^2) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + 2\hat{L}_e^2 m_1(t) n_5(t). \end{aligned} \quad (83)$$

Let,

$$\begin{aligned} F_3(t) &= \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \hat{\theta}^2 dx + \varepsilon_1 \int_{\Omega} \hat{\phi}^2 dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx, \\ m_{11} &= \max \left\{ m_6 + 3, \quad R^2 + m_9, \quad \frac{C^2 + m_{10}}{\varepsilon_1}, \quad 3Mn_4(\tau) + 4C^2 + 4R^2 \right\}. \end{aligned}$$

From (83), it can be seen that:

$$\frac{d}{dt} F_3(t) \leq 2\hat{L}_e^2 m_1(t) n_5(t) + m_{11} F_2(t). \quad (84)$$

by an integration of (84) leads to:

$$F_3(t) \leq 2\hat{L}_e^2 m_{11} e^{m_{11}t} \int_0^t m_1(\eta) n_5(\eta) e^{-m_{11}\eta} d\eta. \quad (85)$$

Inequality (85) shows that when $\hat{L}_e$ tends to zero, the energy $F_3(t)$ also tends to zero.

5. Conclusions

In this paper, we studied the convergence results for the double-diffusion perturbation equations in a bounded domain. The convergence result of solutions were gained for

the reaction boundary coefficients k_1 , k_2 and the Lewis coefficient L_e . Using the method in this paper, similar results for other coefficients could also be gained. Our method is useful for studying the structural stability of bounded regions. However, for unbounded regions, because the regions become more complex, and the inequalities that can be used in bounded regions cannot be used in unbounded regions, essential difficulties will arise. Methods of dealing with stress terms will be the biggest difficulty in unbounded areas. It is an open problem now that we could solve by constructing special functions in relevant future research. In this paper, we only give a theoretical proof and a numerical simulation will be given in another paper.

Author Contributions: Writing—original draft, S.L. and J.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by key projects of universities in the Guangdong Province (Natural Science) (2019KZDXM042), "Thirteenth Five-Year Plan" 2020 research project approval (2020JKDY040), and the Research Foundations of Guangzhou Huashang College (2021HSKT01, 2020HSDS16).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to deeply thank all the reviewers for their insightful and constructive comments.

Conflicts of Interest: The authors declare no conflict of interest.

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