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Asymptotic Approximation of the Apostol-Tangent Polynomials Using Fourier Series

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Abstract: Asymptotic approximations of the Apostol-tangent numbers and polynomials were established for non-zero complex values of the parameter λ . Fourier expansion of the Apostol-tangent polynomials was used to obtain the asymptotic approximations. The asymptotic formulas for the cases $\lambda = 1$ and $\lambda = -1$ were explicitly considered to obtain asymptotic approximations of the corresponding tangent numbers and polynomials.

Keywords: tangent polynomials; Bernoulli polynomials; Euler polynomials; Genocchi polynomials; generating functions; asymptotic approximation

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1. Introduction

The tangent polynomials $T_n(z)$ of a degree of n with a complex argument z are defined by the generating function (see [1,2]).

$$\sum_{n=0}^{\infty} T_n(z) \frac{w^n}{n!} = \left(\frac{2}{e^{2w} + 1} \right) e^{zw}, \quad |w| < \frac{\pi}{2} \quad (1)$$

These polynomials can be expressed in polynomial form as

$$T_n(z) = \sum_{k=0}^n \binom{n}{k} T_k z^{n-k}$$

where T_k denotes the tangent numbers defined by

$$\tan w = \sum_{n=0}^{\infty} (-1)^{n+1} T_{2n+1} \frac{w^{2n+1}}{(2n+1)!} \quad (2)$$

It is worth mentioning that tangent numbers are the odd indices of the numbers A_n of alternating permutations known as the Euler zigzag numbers. The first few values of these numbers are as follows:

$$T_0 = 1, \quad T_1 = -1, \quad T_3 = 2, \quad T_5 = -16, \quad T_7 = 272, \quad T_9 = -7936, \quad T_{11} = 353792.$$

Clearly, $T_n := T_n(0)$ for $n \in \mathbb{N}$.

Several mathematicians were attracted to work on tangent polynomials because of the significant properties that they possessed in the field of mathematics and physics (see [3–6]). Analogues, explicit identities, and symmetric properties for tangent polynomials were derived in [2,7,8]. Some interesting Apostol analogues of the classical Bernoulli, Euler, and Genocchi polynomials were investigated by Apostol [9], Corcino, Lou, Srivastava and

Araci (see [10–17]). These analogues are called the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi polynomials of order m defined by the following relations, respectively, (see [18]): For $\lambda \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{Z}^+$,

$$\sum_{n=0}^{\infty} B_n^m(z; \lambda) \frac{w^n}{n!} = \left(\frac{w}{\lambda e^w - 1} \right)^m e^{wz}, |w| < 2\pi \text{ when } \lambda = 1 \quad (3)$$

and $|w + \log \lambda| < 2\pi$ when $\lambda \neq 1$

$$\sum_{n=0}^{\infty} E_n^m(z; \lambda) \frac{w^n}{n!} = \left(\frac{2}{\lambda e^w + 1} \right)^m e^{wz}, |w| < \pi \text{ when } \lambda = 1 \quad (4)$$

and $|w + \log \lambda| < \pi$ when $\lambda \neq 1$

$$\sum_{n=0}^{\infty} G_n^m(z; \lambda) \frac{w^n}{n!} = \left(\frac{2w}{\lambda e^w + 1} \right)^m e^{wz}, |w| < \pi \text{ when } \lambda = 1 \quad (5)$$

and $|w + \log \lambda| < \pi$ when $\lambda \neq 1$

when $m = 1$, the above Equations (3)–(5) give the generating functions for the Apostol–Bernoulli, Apostol–Euler, and Apostol–Genocchi polynomials, respectively (see [19]). We extend the tangent polynomials as follows.

The Apostol–tangent polynomials $T_n(x; \lambda)$ in z are defined by means of the generating function

$$\sum_{n=0}^{\infty} T_n(x; \lambda) \frac{t^n}{n!} = \left(\frac{2}{\lambda e^{2t} + 1} \right) e^{xt}, |2t + \log \lambda| < \pi \quad (6)$$

when $\lambda = 1$, the Equation reduces to the tangent polynomials $T_n(x) = T_n(x; 1)$.

Lopez and Temme [20] used the Fourier series to establish the asymptotic approximations of higher-order Bernoulli and Euler polynomials. C.B. Corcino and R.B. Corcino [21] derived the asymptotics of higher-order Genocchi polynomials by employing the method in [19,22,23]. In the study of Navas et al. [24], Fourier expansion is used to obtain the asymptotic estimates for Apostol–Bernoulli and Apostol–Euler polynomials. In this paper, the asymptotic expansion of Apostol–tangent polynomials is derived using the method of Navas et al. [24].

2. Asymptotic Approximations

Let $\mathcal{T}_\lambda = \left\{ \frac{1}{2}[(2k-1)\pi i - \log \lambda] : k \in \mathbb{Z} \right\}$ be the set of poles of the generating function Equation (6). The Fourier series expansion of the Apostol–tangent polynomials in terms of poles in $\mathcal{T}_\lambda$ is given in the following theorem:

Theorem 1. Let $\lambda \in \mathbb{C} \setminus \{0\}$. For $n \geq 1$, $0 \leq x \leq 1$,

$$\begin{aligned} \frac{T_n(x; \lambda)}{n!} &= \frac{2^{n+1}}{\lambda^{x/2}} \sum_{k \in \mathbb{Z}} \frac{e^{\frac{1}{2}(2k-1)\pi i x}}{[(2k-1)\pi i - \log \lambda]^{n+1}} \\ &= \frac{1}{\lambda^{x/2}} \sum_{k \in \mathbb{Z}} \frac{e^{\frac{1}{2}(2k-1)\pi i x}}{\left\{ \frac{1}{2}[(2k-1)\pi i - \log \lambda] \right\}^{n+1}} \end{aligned} \quad (7)$$

where the logarithm is taken to be the principal branch.

Proof. Consider the integral $\int_{C_N} f_n(z) dz$ where

$$f_n(z) = \frac{2e^{xz}}{(\lambda e^{2z} + 1)z^{n+1}}$$

and the circle C_N is a circle about the origin of radius $\left(\frac{1}{2}(2N-1+\epsilon)\pi \right)$, $N \in \mathbb{Z}^+$ with ϵ being a fixed real number such that $\epsilon\pi i \pm \log \lambda \neq 0 \pmod{\pi i}$.

The function $f_n(z)$ has poles at $z = 0$ of order $n + 1$ and at $z_k = \frac{1}{2}[(2k - 1)\pi i - \log \lambda]$, $k \in \mathbb{Z}$. The poles z_k are simple poles. Using the Cauchy Residue Theorem,

$$\int_{C_N} f_n(z) dz = 2\pi i \operatorname{Res}(f_n(z), z = 0) + 2\pi i \sum_{k \in \mathbb{Z}, k < N} \operatorname{Res}(f_n(z), z = z_k).$$

We observe that, using the basic property of integration,

$$\left| \int_{C_N} \frac{2e^{xz} dz}{(\lambda e^{2z} + 1)z^{n+1}} \right| \leq \int_{C_N} \frac{|2e^{xz}| |dz|}{|\lambda e^{2z} + 1| |z|^{n+1}}.$$

For $0 \leq x \leq 1$, $|\lambda e^{2z} + 1| > |\lambda e^{2z}|$. Let $z = a + bi$,

$$\frac{|e^{xz}|}{|\lambda e^{2z} + 1|} = \frac{|e^{x(a+bi)}|}{|\lambda e^{2z} + 1|} = \frac{e^{ax}}{|\lambda e^{2z} + 1|} \leq \frac{e^{x\Re(z)}}{|\lambda e^{2z}|} \leq \frac{1}{|\lambda|}.$$

Thus,

$$\left| \int_{C_N} \frac{2e^{xz} dz}{(\lambda e^{2z} + 1)z^{n+1}} \right| \leq \frac{2}{|\lambda|} \int_{C_N} \frac{|dz|}{|z|^{n+1}} = \frac{2^{n+1}}{|\lambda|(2N - 1 + \epsilon)\pi)^n}.$$

As $N \rightarrow \infty$, the last expression goes to 0. Hence, as $N \rightarrow \infty$, $n \geq 1$,

$$\int_{C_N} \frac{2e^{xz}}{(\lambda e^{2z} + 1)z^{n+1}} dz \rightarrow 0.$$

This implies that

$$0 = \operatorname{Res}(f_n(z), z = 0) + \sum_{k \in \mathbb{Z}} \operatorname{Res}(f_n(z), z = z_k).$$

Now, the first residue $\operatorname{Res}(f_n(z), z = 0)$ is given as

$$\begin{aligned} \operatorname{Res}(f_n(z), z = 0) &= \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} (z - 0)^{n+1} \frac{1}{z^{n+1}} \left(\frac{2e^{xz}}{\lambda e^{2z} + 1} \right) \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} \left(\frac{2e^{xz}}{\lambda e^{2z} + 1} \right) \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \frac{d^n}{dz^n} \left(\sum_{l=0}^{\infty} T_l(x; \lambda) \frac{z^l}{l!} \right) \\ &= \lim_{z \rightarrow 0} \frac{1}{n!} \sum_{l=n}^{\infty} T_l(x; \lambda) \frac{z^{l-n}}{(l-n)!}. \end{aligned}$$

Note that the limit of each term of the expansion is 0 as $z \rightarrow 0$ except the term when $l = n$. This gives

$$\operatorname{Res}(f_n(z), z = 0) = \frac{T_n(x; \lambda)}{n!}.$$

On the other hand, the residue $\operatorname{Res}(f_n(z), z = z_k)$ is given by

$$\begin{aligned} \operatorname{Res}(f_n(z), z = z_k) &= \lim_{z \rightarrow z_k} (z - z_k) \frac{1}{z^{n+1}} \left(\frac{2e^{xz}}{\lambda e^{2z} + 1} \right) \\ &= \frac{2e^{xz_k}}{z_k^{n+1}} \lim_{z \rightarrow z_k} \left(\frac{z - z_k}{\lambda e^{2z} + 1} \right) = \frac{e^{(x-2)z_k}}{\lambda z_k^{n+1}}. \end{aligned}$$

Since $z_k = \frac{1}{2}[(2k - 1)\pi i - \log \lambda]$,

$$\begin{aligned} \operatorname{Res}(f_n(z), z = z_k) &= \frac{e^{(x-2)\{\frac{1}{2}[(2k-1)\pi i - \log \lambda]\}}}{\lambda \left\{ \frac{1}{2}[(2k-1)\pi i - \log \lambda] \right\}^{n+1}} \\ &= \frac{-2^{n+1} e^{\frac{1}{2}(2k-1)x\pi i}}{\lambda^{\frac{x}{2}} [(2k-1)\pi i - \log \lambda]^{n+1}}. \end{aligned}$$

Combining these residues gives,

$$0 = \frac{T_n(x; \lambda)}{n!} + \sum_{k \in \mathbb{Z}} \frac{-2^{n+1} e^{\frac{1}{2}(2k-1)x\pi i}}{\lambda^{\frac{x}{2}} [(2k-1)\pi i - \log \lambda]^{n+1}}$$

Hence,

$$T_n(x; \lambda) = \frac{2^{n+1} n!}{\lambda^{\frac{x}{2}}} \sum_{k \in \mathbb{Z}} \frac{e^{\frac{1}{2}(2k-1)x\pi i}}{[(2k-1)\pi i - \log \lambda]^{n+1}}.$$

□

Corollary 1. Let $\lambda \in \mathbb{C} \setminus \{0\}$. For $n \geq 1$, the Fourier series of the Apostol-tangent numbers is given by

$$\frac{T_n(0; \lambda)}{n!} = \sum_{k \in \mathbb{Z}} \frac{1}{\left[\frac{1}{2}(2k-1)\pi i - \log \lambda\right]^{n+1}}, \quad (8)$$

where the logarithm is taken to be the principal branch.

Proof. This follows from Theorem 1 by taking $x = 0$. □

Proceeding as in [20], ordering of the poles of the generating function Equation (6) is carried out in the following lemma.

Lemma 1. Let $u_k = \frac{1}{2}[(2k-1)\pi i - \log \lambda]$ with $k \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $\gamma = (\log \lambda)/2\pi i$, where the logarithm is taken to be the principal branch.

(a) If $\text{Im } \lambda > 0$, then $0 < \Re \gamma < \frac{1}{2}$, and for $k \geq 1$,

$$|u_1| < |u_0| < |u_2| < |u_{-1}| < \cdots < |u_{-k}| < |u_{k+2}| < \quad (9)$$

(b) If $\text{Im } \lambda < 0$, then $-\frac{1}{2} < \Re \gamma < 0$, and for $k \geq 1$,

$$|u_0| < |u_1| < |u_{-1}| < |u_2| < |u_{-2}| < \cdots < |u_{-k}| < |u_{k+1}| < \quad (10)$$

(c) If $\lambda > 0$ (positive real number), then $\Re \gamma = 0$, and for $k \geq 1$,

$$\begin{aligned} |u_0| = |u_1| < |u_{-1}| = |u_2| < |u_{-2}| = |u_3| < |u_{-3}| = |u_4| < \cdots < \\ |u_{-k}| = |u_{k+1}| < |u_{-(k+1)}| = |u_{k+2}| < \end{aligned} \quad (11)$$

(d) If $\lambda < 0$ (negative real number), then $\Re \gamma = \frac{1}{2}$, and for $k \geq 1$,

$$\begin{aligned} |u_1| < |u_0| = |u_2| < |u_{-1}| = |u_3| < |u_{-2}| = |u_4| < \cdots < \\ |u_k| = |u_{-k+2}| = < \end{aligned} \quad (12)$$

Moreover, $|u_k| \geq \pi(|k| - 1)$ if $|k| \geq 1$.

Proof. With the logarithm taken to be the principal branch, γ (as a function of λ) maps $\lambda \in \mathbb{C} \setminus \{0\}$ to the strip $-\frac{1}{2} < \Re \gamma \leq \frac{1}{2}$ (see [20]). To see this, write

$$\gamma = \frac{\theta}{2\pi} - i \frac{\ln|\lambda|}{2\pi}$$

where $\theta = \text{Arg } \lambda$, from which we have

$$\Re \gamma = \frac{\theta}{2\pi} \text{ and } \text{Im } \gamma = -\frac{\ln|\lambda|}{2\pi}.$$

with $-\pi < \theta \leq \pi$, we have

$$\frac{-\pi}{2\pi} \leq \Re \gamma = \frac{\theta}{2\pi} \leq \frac{\pi}{2\pi} \implies -\frac{1}{2} < \Re \gamma \leq \frac{1}{2}$$

where $\Re \gamma = 0$ when $\lambda > 0$ and $\Re \gamma = \frac{1}{2}$ when $\lambda < 0$. If $\text{Im } \lambda > 0$, then $0 < \theta < \pi$. Hence, $0 < \Re \gamma < \frac{1}{2}$. On the other hand, if $\text{Im } \lambda < 0$, then $-\pi < \theta < 0$. Hence, $-\frac{1}{2} < \Re \gamma < 0$.

To verify the chains Equations (9)–(12), let $x = \Re \gamma$ and $y = \text{Im } \gamma$. Then for $k \in \mathbb{Z}$,

$$\begin{aligned} u_k &= \frac{1}{2}[(2k-1)\pi i - \log(\lambda)] \\ &= \left(k - \frac{1}{2}\right)\pi i - \frac{\log(\lambda)}{2} \\ &= \pi i \left[k - \frac{1}{2} - \frac{\log(\lambda)}{2\pi i}\right] \\ &= \pi i \left[k - \frac{1}{2} - (x + iy)\right] \\ &= \pi \left[i\left(k - \frac{1}{2} - x\right) + y\right] \\ &= \pi \sqrt{\left(k - \frac{1}{2} - x\right)^2 + y^2} \end{aligned}$$

Now, we consider two cases:

Case 1. *Im $\lambda > 0$. Then $0 < x < \frac{1}{2}$ and*

$$\begin{aligned} |u_0| &= \pi \sqrt{\left(-\frac{1}{2} - x\right)^2 + y^2} = \pi \sqrt{\left(\frac{1}{2} + x\right)^2 + y^2} \\ |u_1| &= \pi \sqrt{\left(\frac{1}{2} - x\right)^2 + y^2} \\ |u_2| &= \pi \sqrt{\left(\frac{3}{2} - x\right)^2 + y^2} \\ |u_3| &= \pi \sqrt{\left(\frac{5}{2} - x\right)^2 + y^2} \\ |u_{-1}| &= \pi \sqrt{\left(-\frac{3}{2} - x\right)^2 + y^2} = \pi \sqrt{\left(\frac{3}{2} + x\right)^2 + y^2} \\ |u_{-2}| &= \pi \sqrt{\left(-\frac{5}{2} - x\right)^2 + y^2} = \pi \sqrt{\left(\frac{5}{2} + x\right)^2 + y^2} \\ |u_{-3}| &= \pi \sqrt{\left(-\frac{7}{2} - x\right)^2 + y^2} = \pi \sqrt{\left(\frac{7}{2} + x\right)^2 + y^2} \end{aligned}$$

From this, one can see that the order of magnitude of u_k , $k \in \mathbb{Z}$ given in Equation (9) holds.

Case 2. *Im $\lambda < 0$. Thus, $-\frac{1}{2} < x < 0$. The chain of values of u_k can be derived similarly, in which the order of magnitude of u_k , $k \in \mathbb{Z}$ given in Equation (10) holds.*

Case 3. *Im $\lambda = 0$. This means that λ is a real number, which is either positive or negative but not zero. Hence, we have the following subcases:*

Subcase 1. *If $\lambda > 0$, then $\Re \gamma = 0$. For $k \geq 0$,*

$$\begin{aligned} |u_k| &= \pi \sqrt{\left(K - \frac{1}{2}\right)^2 + y^2} \\ |u_0| &= \pi \sqrt{\left(-\frac{1}{2}\right)^2 + y^2} = \pi \sqrt{\left(\frac{1}{2}\right)^2 + y^2} = |u_1| \\ |u_2| &= \pi \sqrt{\left(\frac{3}{2}\right)^2 + y^2} = \pi \sqrt{\left(-\frac{3}{2}\right)^2 + y^2} = |u_{-1}| \\ |u_3| &= \pi \sqrt{\left(\frac{5}{2}\right)^2 + y^2} = \pi \sqrt{\left(-\frac{5}{2}\right)^2 + y^2} = |u_{-2}| \\ |u_4| &= \pi \sqrt{\left(\frac{7}{2}\right)^2 + y^2} = \pi \sqrt{\left(-\frac{7}{2}\right)^2 + y^2} = |u_{-3}| \end{aligned}$$

and so on. Hence,

$$|u_0| = |u_1| < |u_{-1}| = |u_2| < |u_{-2}| = |u_3| < |u_{-3}| = |u_4| < \cdots < \\ |u_{-k}| = |u_{k+1}| < |u_{-(k+1)}| = |u_{k+2}| < \cdots,$$

which is exactly (11).

Subcase 2. If $\lambda < 0$, $\theta = \pi$, and hence, $x = \frac{1}{2}$. For $k \geq 0$,

$$|u_k| = \pi \sqrt{\left(k - \frac{1}{2} - x\right)^2 + y^2}; \text{ when } x = \frac{1}{2} \\ = \pi \sqrt{(k-1)^2 + y^2} \\ = |u_{-k+2}|$$

from which it can easily be observed that

$$|u_1| < |u_0| = |u_2| < |u_{-1}| = |u_3| < |u_{-2}| = |u_4| < \cdots < \\ |u_k| = |u_{-k+2}| < \cdots$$

which is exactly the chain in (12).

Moreover,

$$|u_k| = \pi \left| k - \frac{1}{2} - \gamma \right| \\ = \pi \sqrt{\left(k - \frac{1}{2} - x\right)^2 + y^2} \\ \geq \pi \sqrt{\left(k - \frac{1}{2} - x\right)^2} \\ = \pi \left| k - \frac{1}{2} - x \right| \text{ with } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ = \pi \left| k - \left(x + \frac{1}{2}\right) \right| \\ \geq \pi \left(|k| - \left|x + \frac{1}{2}\right| \right) \\ \geq \pi(|k| - 1).$$

□

The asymptotic expansion of the Apostol-tangent numbers $T_n(0; \lambda)$ is given in the next theorem.

Theorem 2 Given $\lambda \in \mathbb{C} \setminus \{0\}$, let H be a finite subset of $\mathcal{T}_\lambda$ satisfying

$$\max\{|u| : u \in H\} < \min\{|u| : u \in \mathcal{T}_\lambda \setminus H\} := v$$

for all integers $n \geq 2$,

$$\frac{T_n(0; \lambda)}{n!} = \sum_{u \in H} \frac{1}{u^{n+1}} + O\left(v^{-(n+1)}\right)$$

Proof. Write the series (8) as $\sum_{k \in \mathbb{Z}} \frac{1}{(\mu_k)^{n+1}}$. By Lemma 1, we can relabel the set of poles by increasing order of magnitude as

$$|\mu_0| \leq |\mu_1| \leq |\mu_2| \leq \cdots \leq |\mu_M| \leq \cdots$$

Since $|\mu_k| \geq \pi(|k| - 1)$, for $k \geq 2$, the series is absolutely convergent for $n \geq 2$. For any $M > 2$, the tail of the series is

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^{n+1}} = \frac{1}{|\mu_{M+1}|^{n+1}} \sum_{k=M+1}^{\infty} \frac{|\mu_{M+1}|^{n+1}}{|\mu_k|^{n+1}}$$

Since $k > M + 1$, $\left| \frac{\mu_{M+1}}{\mu_k} \right|^{n+1} \leq 1$, we have $\left| \frac{\mu_{M+1}}{\mu_k} \right|^{n+1} \leq \left| \frac{\mu_{M+1}}{\mu_k} \right|^2$ for $n \geq 2$. Hence,

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^{n+1}} \leq \frac{1}{|\mu_{M+1}|^{n+1}} \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^2$$

Let

$$C_{M,\lambda} = \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^2$$

Then,

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^{n+1}} \leq \frac{1}{|\mu_{M+1}|^{n+1}} [C_{M,\lambda}] = \frac{C_{M,\lambda}}{|\mu_{M+1}|^{n+1}}$$

Now, consider $C_{M,\lambda}$:

$$\begin{aligned} C_{M,\lambda} &= \sum_{k=M+1}^{\infty} \left| \frac{\mu_{M+1}}{\mu_k} \right|^2 \\ &= \sum_{k=M+1}^{\infty} \frac{|\mu_{M+1}|^2}{|\mu_k|^2} \\ &= |\mu_{M+1}|^2 \sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^2} \end{aligned}$$

Since

$$\begin{aligned} |\mu_k| &= \pi \left| k - \frac{1}{2} - \gamma \right| \geq \pi(|k| - 1) \\ |\mu_{M+1}| &= \pi \left| M + 1 - \frac{1}{2} - \gamma \right| \geq \pi(|M + 1| - 1) \\ &= \pi \left| M + \frac{1}{2} - \gamma \right| \end{aligned}$$

Then

$$\begin{aligned} C_{M,\lambda} &= \left| M + \frac{1}{2} - \gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{\left| k - \frac{1}{2} - \gamma \right|^2} \\ &\leq 2 \left| M + \frac{1}{2} - \gamma \right|^2 \sum_{k=M+1}^{\infty} \frac{1}{(|k| - 1)^2} \\ &\leq 2 \left| M + \frac{1}{2} - \gamma \right|^2 \left(\frac{1}{M^2} + \sum_{\downarrow=0}^{\infty} \frac{1}{(M + \downarrow)^2} \right) \end{aligned}$$

With

$$\sum_{\downarrow=0}^{\infty} \frac{1}{(M + \downarrow)^2} \leq \int_1^{\infty} \frac{1}{(M + x)^2} dx = \frac{1}{M + 1}$$

So,

$$\begin{aligned} C_{M,\lambda} &\leq 2 \left| M + \frac{1}{2} - \gamma \right|^2 \left(\frac{1}{M^2} + \frac{1}{M+1} \right) \\ &= \frac{2 \left| M + \frac{1}{2} - \gamma \right|^2}{M^2} + \frac{2 \left| M + \frac{1}{2} - \gamma \right|^2}{M+1} \end{aligned}$$

Let

$$\zeta_1 = \frac{\left| M + \frac{1}{2} - \gamma \right|^2}{M^2} \leq \left| \frac{3}{2} - \gamma \right|^2$$

And

$$\xi_2 = \frac{|M + \frac{1}{2} - \gamma|}{M + 1} \leq \frac{|M + 1| + |-\frac{1}{2} - \gamma|}{|M + 1|} \leq 1 + \left| \frac{1}{2} + \gamma \right|.$$

Consequently,

$$\begin{aligned} C_{M,\lambda} &= 2\xi_1 + \xi_2 \left| M + \frac{1}{2} - \gamma \right| \\ \frac{C_{M,\lambda}}{|\mu_{M+1}|^{n+1}} &\leq \frac{2\xi_1}{|\mu_{M+1}|^{n+1}} + \frac{\xi_2 |M + \frac{1}{2} - \gamma|}{|\mu_{M+1}|^{n+1}} \end{aligned}$$

where

$$|\mu_{M+1}| = \pi \left| M + \frac{1}{2} - \gamma \right| = \sqrt{\left(M + \frac{1}{2} - \operatorname{Re} \gamma \right)^2 + (\operatorname{Im} \gamma)^2} \geq M$$

So,

$$\begin{aligned} \frac{C_{M,\lambda}}{|\mu_{M+1}|^{n+1}} &\leq \frac{2|\frac{3}{2} - \gamma|^2}{\pi^{n+1} |M + \frac{1}{2} - \gamma|^{n+1}} + \frac{2(1 + |\frac{1}{2} + \gamma|) |M + \frac{1}{2} - \gamma|}{\pi^{n+1} |M + \frac{1}{2} - \gamma|^{n+1}} \\ &\leq \frac{2|\frac{3}{2} - \gamma|^2}{\pi^{n+1} |M + \frac{1}{2} - \gamma|^{n+1}} + \frac{2(1 + |\frac{1}{2} + \gamma|)}{\pi^{n+1} |M + \frac{1}{2} - \gamma|^n} \\ &\leq \frac{2|\frac{3}{2} - \gamma|^2}{\pi^{n+1} |M|^{n+1}} + \frac{2(1 + |\frac{1}{2} + \gamma|)}{\pi^{n+1} |M|^n} \\ &\leq \frac{2|\frac{3}{2} - \gamma|^2}{\pi^{n+1} |M|^{n+1}} + \frac{2(1 + |\frac{1}{2} + \gamma|)}{\pi^{n+1} |M|^{n+1}} \\ &\leq \frac{2|\frac{3}{2} - \gamma|^2}{\pi^{n+1}} + \frac{2(1 + |\frac{1}{2} + \gamma|)}{\pi^{n+1}} \end{aligned}$$

We can see that $C_{M,\lambda} \rightarrow 0$ as $n \rightarrow \infty$ for $|M| > 2$. Thus, the tail of the series is

$$\sum_{k=M+1}^{\infty} \frac{1}{|\mu_k|^{n+1}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Moreover, for fixed $M > 2$ and $n \gg 0$, $C_{M,\lambda}$ is bounded and independent of M . Hence, we can replace $C_{M,\lambda}$ with C_λ . This completes the proof of the theorem. $\square$

When $\lambda = 1$, $\log \lambda = 0$, and $u_k = \frac{1}{2}(2k - 1)\pi i$, $k \in \mathbb{Z}$. Take $H = \left\{ \frac{\pi i}{2}, -\frac{\pi i}{2} \right\}$. Then $v = \frac{3\pi}{2}$, and the ordinary tangent numbers $T_n = T_n(0; 1)$ satisfy

$$\begin{aligned} \frac{T_n}{n!} &= \frac{T_n(0; 1)}{n!} \\ &= \frac{1}{(\frac{\pi i}{2})^{n+1}} + \frac{1}{(-\frac{\pi i}{2})^{n+1}} + O\left(\left(\frac{3\pi}{2}\right)^{-(n+1)}\right) \end{aligned} \quad (13)$$

An approximation of $T_n(0; 1)$ is given by

$$\frac{T_n}{n!} \approx \frac{2^{n+1}}{(\pi i)^{n+1}} + \frac{2^{n+1}}{(-\pi i)^{n+1}} \quad (14)$$

For even n , $n \geq 2$, it is known that $T_n = 0$, which is also true when we use Equation (14). Then, we have

$$\frac{T_{2n}}{(2n)!} \approx \frac{2^{2n+1}}{(\pi i)^{2n+1}} + \frac{2^{2n+1}}{(-\pi i)^{2n+1}} = 0.$$

For odd indices,

$$\begin{aligned} \frac{T_{2n-1}}{(2n-1)!} &\approx \frac{2^{2n}}{(\pi i)^{2n}} + \frac{2^{2n}}{(-\pi i)^{2n}} \\ &\approx \frac{2^{2n+1}}{(\pi i)^{2n}} \\ T_{2n-1} &\approx \frac{(-1)^n 2^{2n+1} (2n-1)!}{\pi^{2n}}, \quad n \geq 1. \end{aligned} \quad (15)$$

$$\begin{aligned} T_{2(3)-1} &\approx \frac{(-1)^3 2^{2(3)+1} (2(3)-1)!}{\pi^{2(3)}} \approx \frac{-2^7 5!}{\pi^6} \\ T_5 &\approx -15.97688023. \end{aligned}$$

This value is very close to the exact value of T_5 which is -16 .

It is proved in the next theorem that an asymptotic approximation of the Apostol-tangent polynomials can be obtained from its Fourier series (Theorem 1) by choosing an appropriate subset of $\mathcal{T}_\lambda$.

Theorem 3. Given $\lambda \in \mathbb{C} \setminus \{0\}$, let H be a finite subset of $\mathcal{T}_\lambda$ satisfying

$$\max\{|u| : u \in H\} < \min\{|u| : u \in \mathcal{T}_\lambda \setminus H\} := v.$$

For all integers $n \geq 2$, we have uniformly for x in a compact subset K of $\mathbb{C}$,

$$\frac{T_n(x; \lambda)}{n!} = \sum_{u \in H} \frac{e^{ux}}{u^{n+1}} + O\left(\frac{e^{v|x|}}{v^{n+1}}\right),$$

where the constant implicit in the order term depends on λ , H and K . Moreover, for $n \gg 0$, this constant can be made independent of K , equal to the constant for the Apostol-tangent numbers, corresponding to the case $x = 0$.

Proof. From the generating function in Equation (6), we have

$$\frac{2e^{(x+y)z}}{\lambda e^{2z} + 1} = \sum_{n=0}^{\infty} T_n(x+y; \lambda) \frac{z^n}{n!}.$$

The left-hand side of the equation can be written as

$$\begin{aligned} \frac{2e^{(x+y)z}}{\lambda e^{2z} + 1} &= \frac{2e^{xz} e^{yz}}{\lambda e^{2z} + 1} = \sum_{n=0}^{\infty} T_n(x; \lambda) \frac{z^n}{n!} e^{yz} \\ &= \left(\sum_{n=0}^{\infty} T_n(x; \lambda) \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(yz)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n T_{n-k}(x; \lambda) \frac{z^{n-k}}{(n-k)!} \cdot \frac{(yz)^k}{k!} \cdot \frac{n!}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} T_{n-k}(x; \lambda) y^k \right) \frac{z^n}{n!}. \end{aligned}$$

Hence,

$$T_n(x+y; \lambda) = \sum_{k=0}^n \binom{n}{k} T_{n-k}(x; \lambda) y^k.$$

For $z \in \mathbb{C}$, writing $z = 0 + z$ (here $y = z$, $x = 0$),

$$\begin{aligned} T_n(z; \lambda) &= \sum_{k=0}^n \binom{n}{k} T_{n-k}(0, \lambda) z^k \\ &= \sum_{k=0}^n \frac{n!}{(n-k)!k!} T_{n-k}(0, \lambda) z^k \\ \frac{T_n(z; \lambda)}{n!} &= \sum_{k=0}^n \frac{T_{n-k}(0, \lambda) z^k}{(n-k)!k!} \\ &= \sum_{k=0}^n \frac{T_{n-k}(0, \lambda)}{(n-k)!} \frac{z^k}{k!} \\ &= \sum_{k=0}^n \left(\sum_{u \in H} \frac{1}{u^{n-k+1}} + O\left(v^{-(n-k+1)}\right) \right) \frac{z^k}{k!} \\ &= \sum_{k=0}^n \left(\sum_{u \in H} \frac{1}{u^{n-k+1}} \right) \frac{z^k}{k!} + \sum_{k=0}^n O\left(v^{-(n-k+1)}\right) \frac{z^k}{k!}, \end{aligned}$$

where the implicit constant c in the order term is that corresponding to $z = 0$ and only depends on H and λ . Note also that

$$\begin{aligned} \left| \sum_{k=0}^n O\left(v^{-(n-k+1)}\right) \frac{z^k}{k!} \right| &\leq \sum_{k=0}^n c v^{-(n-k+1)} \frac{|z^k|}{k!} \\ &= c v^{-(n+1)} \sum_{k=0}^n \frac{v^k |z^k|}{k!} \\ &\leq c v^{-(n+1)} e_n(v|z|), \end{aligned}$$

where

$$e_n = \sum_{k=0}^n \frac{w^k}{k!}$$

To complete the proof of the theorem, it remains to show that

$$\frac{e_n^*(uz)}{u^{n+1}} = \frac{e^{uz} - e_n(uz)}{u^{n+1}}$$

is bounded. Using the Mean Value Theorem (MVT) for Banach spaces (see also [20]), we have

$$\begin{aligned} e_n^*(w) &= \frac{w^{n+1}}{(n+1)!} + \frac{w^{n+2}}{(n+2)!} + \cdots \\ &= \frac{w^{n+1}}{(n+1)!} \left\{ 1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots \right\} \end{aligned}$$

from which

$$\begin{aligned} |e_n^*(w)| &\leq \left| \frac{w^{n+1}}{(n+1)!} \right| \left| 1 + \frac{w}{n+2} + \frac{w^2}{(n+3)(n+2)} + \cdots \right| \\ &\leq \frac{|w|^{n+1}}{(n+1)!} e^{\Re^+(w)} \end{aligned}$$

where $\Re^+(w) = \max\{\Re(w), 0\}$. Since $|u| \leq v$, for all $u \in H$, we have

$$\begin{aligned} \frac{|e^*(uz)|}{|u^{n+1}|} &\leq \frac{e^{|uz|} |uz|^{n+1}}{|u^{n+1}| (n+1)!} \\ &= \frac{e^{|uz|} |z|^{n+1} |u^{n+1}|}{|u^{n+1}| (n+1)!} \\ &= \frac{e^{|uz|} |z|^{n+1}}{(n+1)!} \\ &< \frac{e^{v|z|} |z|^{n+1}}{(n+1)!} \end{aligned}$$

so that

$$\begin{aligned} \left| \sum_{u \in H} \frac{e^*(uz)}{u^{n+1}} \right| &\leq \sum_{u \in H} \frac{|e_n^*(uz)|}{|u^{n+1}|} \\ &\leq |H| e^{v|z|} \frac{|z|^{n+1}}{(n+1)!} \end{aligned}$$

where $|H|$ denotes the number of elements in H . We give the argument that

$$|H| e^{v|z|} \frac{|z|^{n+1}}{(n+1)!} < c e^{v|z|} v^{-(n+1)}$$

If

$$|H| \frac{(v|z|)^{n+1}}{n+1} < c,$$

which certainly holds for $n \gg 0$, uniformly for z in a compact subset $K \subset \mathbb{C}$. $\square$

Corollary 2. Let K be an arbitrary compact subset of $\mathbb{C}$. The tangent polynomials satisfy uniformly on K the estimates

$$\begin{aligned}\frac{T_{2n}(x)}{(2n)!} &= \frac{(-1)^n 2^{2n+2} \sin\left(\frac{\pi x}{2}\right)}{\pi^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ \frac{T_{2n-1}(x)}{(2n-1)!} &= \frac{(-1)^n 2^{n+1} \cos\left(\frac{\pi x}{2}\right)}{\pi^{2n}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right)\end{aligned}$$

where the implicit constant in the order term depends on the set K . Moreover, for $n \gg 0$, this constant can be made independent of K , equal to the constant for the tangent numbers, corresponding to the case $x = 0$.

Proof. The tangent polynomials correspond to the case $\lambda = 1$ so that

$$u_k = \frac{1}{2}(2k-1)\pi i, \quad k \in \mathbb{Z}.$$

Thus

$$\mathcal{T}_1 = \left\{ \frac{1}{2}(2k-1)\pi i : k \in \mathbb{Z} \right\}$$

Taking

$$H = \{(2k-1)\pi i : k = -1, 0\} = \left\{ -\frac{\pi i}{2}, \frac{\pi i}{2} \right\},$$

then $v = \frac{|3\pi i|}{2} = \frac{3\pi}{2}$. From Theorem 3,

$$\begin{aligned}\frac{T_n(x;1)}{n!} &= \sum_{u \in H} \frac{e^{ux}}{u^{n+1}} + O\left(\frac{e^{v|x|}}{v^{n+1}}\right) \\ &= \left(\frac{e^{-\frac{\pi i x}{2}}}{\left(-\frac{\pi i}{2}\right)^{n+1}} + \frac{e^{\frac{\pi i x}{2}}}{\left(\frac{\pi i}{2}\right)^{n+1}} \right) + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{n+1}}\right).\end{aligned}$$

For even indices,

$$\begin{aligned}\frac{T_{2n}(x)}{(2n)!} &= \frac{T_{2n}(x;1)}{(2n)!} = \left(\frac{e^{-\frac{\pi i x}{2}}}{\left(-\frac{\pi i}{2}\right)^{2n+1}} + \frac{e^{\frac{\pi i x}{2}}}{\left(\frac{\pi i}{2}\right)^{2n+1}} \right) + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{2n+1} e^{\frac{\pi i x}{2}}}{(\pi i)^{2n+1}} - \frac{2^{n+1} e^{-\frac{\pi i x}{2}}}{(\pi i)^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{2n+1} \left(\cos \frac{\pi x}{2} + i \sin \frac{\pi x}{2} \right) - 2^{n+1} \left(\cos \frac{\pi x}{2} - i \sin \frac{\pi x}{2} \right)}{(\pi i)^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{n+1} \left(\cos \frac{\pi x}{2} + i \sin \frac{\pi x}{2} - \cos \frac{\pi x}{2} + i \sin \frac{\pi x}{2} \right)}{(\pi i)^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{n+1} 2i \sin \frac{\pi x}{2}}{(\pi i)^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{n+2} i \sin \left(\frac{\pi x}{2}\right)}{\pi^{2n+1} i^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{2^{2n+2} \sin\left(\frac{\pi x}{2}\right)}{\pi^{2n+1} (-1)^n} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right) \\ &= \frac{(-1)^n 2^{2n+2} \sin\left(\frac{\pi x}{2}\right)}{\pi^{2n+1}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n+1}}\right).\end{aligned}$$

For odd indices,

$$\begin{aligned}
 \frac{T_{2n-1}(x)}{(2n-1)!} &= \frac{T_{2n-1}(x;1)}{(2n-1)!} = \left(\frac{e^{-\frac{\pi i x}{2}}}{\left(\frac{-\pi i}{2}\right)^{2n}} + \frac{e^{\frac{\pi i x}{2}}}{\left(\frac{\pi i}{2}\right)^{2n}} \right) + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \left(\frac{e^{-\frac{\pi i x}{2}}}{\left(\frac{\pi i}{2}\right)^{2n}} + \frac{e^{\frac{\pi i x}{2}}}{\left(\frac{\pi i}{2}\right)^{2n}} \right) + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \left(\frac{2^n e^{-\frac{\pi i x}{2}}}{(\pi i)^{2n}} + \frac{2^n e^{\frac{\pi i x}{2}}}{(\pi i)^{2n}} \right) + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \frac{2^n (\cos \frac{\pi x}{2} - i \sin \frac{\pi x}{2}) + 2^n (\cos \frac{\pi x}{2} + i \sin \frac{\pi x}{2})}{(\pi i)^{2n}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \frac{2^n (\cos \frac{\pi x}{2} - i \sin \frac{\pi x}{2} + \cos \frac{\pi x}{2} + i \sin \frac{\pi x}{2})}{(\pi i)^{2n}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \frac{2^n \cdot 2 \cos\left(\frac{\pi x}{2}\right)}{(\pi i)^{2n}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \frac{2^{n+1} \cos\left(\frac{\pi x}{2}\right)}{\pi^{2n} (i^2)^n} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right) \\
 &= \frac{(-1)^n 2^{n+1} \cos\left(\frac{\pi x}{2}\right)}{\pi^{2n}} + O\left(\frac{e^{\frac{3\pi}{2}|x|}}{\left(\frac{3\pi}{2}\right)^{2n}}\right).
 \end{aligned}$$

□

3. The Case When λ Is a Negative Real Number

When λ is a negative real number, writing $\lambda = -|\lambda|$, the generating function in Equation (6) can be written as

$$\frac{2e^{xt}}{-|\lambda|e^{2t} + 1} = \sum_{n=0}^{\infty} T_n(x; -|\lambda|) \frac{t^n}{n!} \quad (16)$$

The poles of the generating function (3.1) is given by

$$\mathcal{T}_{-|\lambda|} = \left\{ \frac{1}{2}[(2k+1)\pi i + \ln|\lambda|] : k \in \mathbb{Z} \right\}.$$

The next theorem immediately follows from Theorem 3.

Theorem 4. Given that λ is a negative real number, let F be a finite subset of $\mathcal{T}_{-|\lambda|}$ satisfying

$$\max\{|a| : a \in F\} < \min\{|a| : a \in \mathcal{T}_{-|\lambda|} \setminus F\} =: \xi.$$

For all integers $n \geq 2$, we have uniformly for x in a compact subset K of $\mathbb{C}$,

$$\frac{T_n(x; \lambda)}{n!} = \sum_{a \in F} \frac{e^{ax}}{a^{n+1}} + O\left(\frac{e^{\xi|x|}}{\xi^{n+1}}\right), \quad (17)$$

where the constant implicit in the order term depends on λ , F , and K .

The Apostol-tangent numbers $T_n(0; -1)$ corresponding to the case $\lambda = -1$ have the generating function

$$\frac{2}{-e^{2t} + 1} = \sum_{n=0}^{\infty} T_n(0; -1) \frac{t^n}{n!}, \quad (18)$$

The set of poles is $\mathcal{T}_{-1} = \{k\pi i : k \in \mathbb{Z} \setminus \{0\}\}$. An asymptotic formula for $T_n(0; -1)$ is given in the following theorem.

Theorem 5. For $n \geq 3$, the Apostol-tangent numbers $T_n(0; -1)$ satisfying

$$\frac{T_n(0; -1)}{n!} = \left(\frac{1}{(-\pi i)^{n+1}} + \frac{1}{(\pi i)^{n+1}} \right) + O\left((2\pi)^{-(n+1)}\right). \quad (19)$$

In particular,

$$\frac{T_{2n-1}(0; -1)}{(2n-1)!} = \frac{(-1)^n 2}{\pi^{2n}} + O\left((2\pi)^{-(2n)}\right). \quad (20)$$

Proof. Taking $x = 0$, $F = \{-\pi i, \pi i\}$ in Theorem 4, then $\xi = 2\pi$.

Hence,

$$\begin{aligned} \frac{T_n(x; \lambda)}{n!} &= \sum_{a \in F} \frac{e^{ax}}{a^{n+1}} + O\left(\frac{e^{\xi|x|}}{\xi^{n+1}}\right) \\ \frac{T_n(0; -1)}{n!} &= \left(\frac{1}{(-\pi i)^{n+1}} + \frac{1}{(\pi i)^{n+1}} \right) + O\left((2\pi)^{-(n+1)}\right) \end{aligned} \quad (21)$$

for which Equation (19) follows. For $n \geq 2$, (21) gives $T_{2n}(0; -1) \approx 0$. Indeed $T_{2n}(0; -1) = 0$

$\forall n \geq 1$. For $n \geq 1$,

$$\begin{aligned} \frac{T_{2n-1}(0; -1)}{(2n-1)!} &= \left(\frac{1}{(-\pi i)^{2n}} + \frac{1}{(\pi i)^{2n}} \right) + O\left((2\pi)^{-(n+1)}\right) \\ &= \left(\frac{1}{(\pi i)^{2n}} + \frac{1}{(\pi i)^{2n}} \right) + O\left((2\pi)^{-(n+1)}\right) \\ &= \frac{2}{(\pi i)^{2n}} + O\left((2\pi)^{-(n+1)}\right) \\ &= \left(\frac{2}{(\pi)^{2n} (i)^{2n}} \right) + O\left((2\pi)^{-(n+1)}\right) \\ &= \frac{(-1)^n 2}{(\pi)^{2n}} + O\left((2\pi)^{-(n+1)}\right). \end{aligned}$$

□

Taking $n = 4$,

$$T_7(0; -1) = \frac{2(7!)}{(\pi)^8} \approx 1.06233.$$

The actual value of $T_7(0; -1) = -\frac{2^8 B_8}{8} = \frac{16}{15} \approx 1.06667$.

The Apostol-tangent polynomials $T_n(x; -1)$ correspond to the case $\lambda = -1$. These polynomials have the generating function

$$\frac{2e^{xt}}{-e^{2t} + 1} = \sum_{n=0}^{\infty} T_n(x; -1) \frac{t^n}{n!} \quad (22)$$

We will prove the following theorem.

Theorem 6. Let K be a compact subset of $\mathbb{C}$. The Apostol-tangent polynomials $T_n(x; -1)$ satisfy uniformly on K the estimates

$$\begin{aligned} \frac{T_{2n}(x; -1)}{(2n)!} &= \frac{(-1)^n 2 \sin(\pi x)}{\pi^{2n+1}} = O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}}\right) \\ \frac{T_{2n-1}(x; -1)}{(2n-1)!} &= \frac{(-1)^n 2 \cos(\pi x)}{\pi^{2n}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}}\right). \end{aligned}$$

Proof. Taking $F = \{-\pi i, \pi i\}$, then $\xi = 2\pi$. Hence, it follows from Theorem 4 that

$$\frac{T_n(x; -1)}{n!} = \left(\frac{e^{\pi i x}}{(\pi i)^{n+1}} + \frac{e^{-\pi i x}}{(-\pi i)^{n+1}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{n+1}} \right).$$

For even indices,

$$\begin{aligned} \frac{T_{2n}(x; -1)}{(2n)!} &= \left(\frac{e^{\pi i x}}{(\pi i)^{2n+1}} + \frac{e^{-\pi i x}}{(-\pi i)^{2n+1}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right) \\ &= \left(\frac{e^{\pi i x}}{(\pi i)^{2n+1}} - \frac{e^{-\pi i x}}{(\pi i)^{2n+1}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right) \\ &= \left(\frac{\cos \pi x + i \sin \pi x - \cos \pi x + i \sin \pi x}{(\pi i)^{2n+1}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right) \\ &= \frac{2i \sin \pi x}{(\pi i)^{2n+1}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right) = \frac{2 \sin \pi x}{\pi^{2n+1} (i)^{2n}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right) \\ &= \frac{(-1)^n 2 \sin(\pi x)}{\pi^{2n+1}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n+1}} \right). \end{aligned}$$

For odd indices,

$$\begin{aligned} \frac{T_{2n-1}(x; -1)}{(2n-1)!} &= \left(\frac{e^{\pi i x}}{(\pi i)^{2n}} + \frac{e^{-\pi i x}}{(-\pi i)^{2n}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right) \\ &= \left(\frac{e^{\pi i x}}{(\pi i)^{2n}} + \frac{e^{-\pi i x}}{(\pi i)^{2n}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right) \\ &= \left(\frac{\cos \pi x + i \sin \pi x + \cos \pi x - i \sin \pi x}{(\pi i)^{2n}} \right) + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right) \\ &= \frac{2 \cos \pi x}{(\pi i)^{2n}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right) \\ &= \frac{2 \cos \pi x}{\pi^{2n} (i)^{2n}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right) \\ &= \frac{(-1)^n 2 \cos(\pi x)}{\pi^{2n}} + O\left(\frac{e^{2\pi|x|}}{(2\pi)^{2n}} \right). \end{aligned}$$

□

4. Conclusions and Recommendation

The method of Navas et al. [24] is a clever way to obtain an asymptotic approximation from the Fourier series. In this paper, the method was applied to obtain asymptotic approximations of the Apostol-tangent numbers and polynomials for nonzero complex values of the parameter λ . The case when λ is negative was explicitly considered because the poles are simply in terms of $\frac{1}{2} \ln|\lambda|$ plus odd multiples of $\frac{\pi}{2}i$. Moreover, the cases $\lambda = 1$ and $\lambda = -1$ give beautiful approximations of the corresponding Tangent polynomials in terms of the sine and cosine functions depending on whether n is even or odd.

The author recommends finding Fourier expansion and asymptotic approximations of higher-order Apostol-Tangent numbers and polynomials using the method employed in this paper (see also [25]). Furthermore, one may also try to consider multiple generalized Tangent polynomials and their p-adic interpolation function [26].

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