Symmetries in Generalized Kaprekar's Routine

Fernando Nuez

Institute for Conservation and Improvement of Valencian Agrobiodiversity (COMAV, UPV), Universitat Politècnica de València, 46022 Valencia, Spain; fnuez@btc.upv.es; Tel.: +34-963-879-421

Abstract: In this paper, algebraic relations were established that determined the invariance of a transformed number after several transformations. The restrictions that determine the group structure of these relationships were analyzed, as was the case of the Klein group. Parametric K' functions associated with the existence of cycles were presented, as well as the role of the number of their links in the grouping of numbers in higher-order equivalence classes. For this, we developed a methodology based on binary equivalence relations and the complete parameterization of the Kaprekar routine using K_i functions of parametric transformation.

Keywords: number theory; arithmetic dynamics; groups

1. Introduction

The process known as Kaprekar's routine consists of sorting the digits in a number in descending order. Then, the opposite arrangement takes place, i.e., in ascending order. These two arrangements are subtracted one from another, resulting in an image of the original number. In 1949, Datlatreya Kaprekar [1] showed that when iterating such a process with four-digit numbers, the result was always 6174.

This routine of transformation has always aroused interest among fans of mathematical games [2]. However, additionally, there is important research regarding this routine has taken place among mathematicians specialized in Number Theory. Much of this effort has been directed to determining constants and cycles in various numbering systems and numbers of digits, as well as some aspects of transformation chains such as maximum distances.

Thus, the work of Prichett et al. stands out [3,4]. This group of researchers established the basic functions of numerical transformation. An n -digit is a constant of Kaprekar if and only if that number is a fixed point under Kaprekar's routine and all numbers are transformed to that number after a finite number of applications of Kaprekar's transformations. They determined the all one-cycle at base 10 and showed that the only decadic Kaprekar's constants are 495 and 6174. In 2005, Walden [5] already presented some new results on the Kaprekar's constant, establishing the unique four-adic, seven-digit and five-adic, nine-digit Kaprekar's constant and showing that 15, 21, 27 or 33-digit Kaprekar constants do not exist. The work on the Kaprekar's constant has continued with the interesting contributions of Dolan [6], Prince [7], Hanover [8] and Thakur [9], among others.

In addition to the study of Kaprekar's constants, the number of transformations necessary to reach the constants or cycles has also been studied. Thus, in a recent work, Yamagami [10] collected data on the number of cycles $N(b, n)$ and their number of links $l(b, n)$ for some bases $2 \leq b \leq 15$ and numbers of digits $2 \leq n \leq 15$, many of which were calculated by the author. He noted: "It is very hard to obtain general results on b -adic n -digit Kaprekar loops for all b and n without observing any case-by-case results". Therefore, he thinks that results with specific n or b values as in the known results above are very valuable to further research. In this article, he considers a case where $b = 2$ and $n = 2$. He proves that any two-adic Kaprekar's constant is the two-adic expression of a product of two suitable Mersenne numbers. He also obtains some formulas for the Kaprekar distances.

Citation: Nuez, F. Symmetries in Generalized Kaprekar's Routine. *Symmetry* **2022**, *14*, 37. <https://doi.org/10.3390/sym14010037>

Academic Editor: Alexei Kanel-Belov

Received: 22 November 2021

Accepted: 15 December 2021

Published: 29 December 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

In 2018, together with Matsui [11], he published other paper on three-adic Kaprekar loops. They obtained the formulas for the number for all three-adic n -digit Kaprekar loops and their lengths in terms of n . Wang and Lu [12] continued Yamagami's studies and completed them with a general solution. Meanwhile, Devlan and Zang [13] determined the maximum number of iterations required to reach a fixed point in the four-digit process.

The constant 6174 has always been surrounded by a halo of mystery [14]. The exploration of Kaprekar's constant is a problem that allows for multiple approaches to investigation. This possibility, together with its mysterious nature, explains why it is frequently used as a didactic resource [15]. The discovery of the patterns produced by Kaprekar's process promotes much excitement and insight into the world of mathematical discovery among students [16].

On the other hand, recent studies have investigated variations of the Kaprekar's transformation [17,18]. In particular, we highlight the interesting paper of Bajorska-Harapinska et al. [19], which presented and discussed some variations of Kaprekar's transformation. They were especially interested in the orbits of such transformations. For such, they understood the sets of all iterations of a transformation function on a fixed point. They provided their full description for small numbers and also proved some general properties. In 2021, Stephen et al. [20] already defined generalized α_s -Meir-Keeler contractions in S_b -metric spaces and used them to examine the existence and uniqueness of fixed points.

There is a current of opinion that understands there are important gaps in the understanding of Kaprekar's process. We are interested in investigating the possible connections with Group Theory. We have not found previous studies on this aspect or on the existing symmetries in the transformation chains. In fact, the verification of this gap in the literature has been the main motivation of our work.

In this work, we approach the study of the transformation in base 10 with generality of the number of digits, and the following questions will be addressed:

- What are the algebraic relations that must exist between the numbers for their images to coincide after one or several transformations? For example:
  - Why do 83,246,529 and 17,487,561 return the same transformed number?
  - Why do 8,178,382,562 and 4,774,473,809 return the same number after two transformations?
  - Why do 5,068,069 and 3,071,934 return the same number after four transformations?
- What algebraic structures underpin these symmetries?
- What is the relation between cycles and the algebraic architecture of the transformation schemes?

2. Design of the Investigation and Methods

Every number has an image through Kaprekar's routine, and that image is unique. The image of a number is uniquely determined by some parameters α that characterize it (Section 3). All numbers with the same parameters give the same image. This leads to the study of Kaprekar's routine in parametric terms.

The scheme of parametric transformations can have one or more components. An example can be seen in Scheme 1, corresponding to a 6-digit case with three components. Any component of a scheme is necessarily coalescent and contains a transformation constant (parameters whose image is themselves) or a cycle of 2 or more links. This is due to the uniqueness of the transformation and to the fact that the set of all numerical images is a subset of the multiples of 9 (Section 3). Its cardinal is much smaller than that of the set of reference numbers.

Scheme 1. Parametric transformation schemes in A_6 . Each number is a class $(\alpha \beta \gamma)$.

Consequently, invariances necessarily appear: sets of numbers that give the same image after one or more transformations. Or, in a dual way, these can be parameters that give the same parametric image. **Our fundamental objective is to find the algebraic relations that determine these invariances**, or, in other words, to identify the symmetries algebraically.

For this, we employed binary equivalence relations. These relationships, characterized by their reflexive, symmetric and transitive properties, allow one to establish partitions in a set of numbers (or of the parameters) so that all the elements of a class have the same image. We thus defined a binary equivalence relation R_r of order r as the one that exists between numbers that give the same image after r transformations (Section 7.1). As a result of what has already been said, the relation R_1 is the one that exists between numbers with the same parameters.

However, how can we determine the R_2 and higher relations? For this, we studied and developed the properties of the R_r relations. Among them, we found (33) $\alpha_1 R_r \alpha_2 \leftrightarrow \alpha'_1 R_{r-1} \alpha'_2$, with α' being the parametric image of α . This relationship is methodologically fundamental. It allows one to obtain the higher-order R_r relations from the lower-order ones. For example, R_3 can be obtained from R_2 . For this, it is necessary to develop K_i functions of parametric transformation such that $K_i(\alpha) = \alpha'$ (Section 4). Each of these functions is associated with a permutation that reorders the digits of the transformed number in decreasing order. However, not all permutations are possible; the restriction must be met (3). This leads to a technical difficulty in the development of the functions K_i .

When parametric transformation functions K_i and e_r functions are associated with equivalences, we can algebraically determine higher-order equivalences. If an equivalence R_r is imposed on the images of two parametric classes α_1 and α_2 , $K_i(\alpha_1)$ and $K_j(\alpha_2)$, the result will be an algebraic relation between α_1 and α_2 so that $\alpha_1 R_{r+1} \alpha_2$. In summary, one of the basic methods used is (51).

$$(e_r \times K_i)(\alpha_1) = K_j(\alpha_2)$$

Other analogous methods derive from the properties of the R_r developed in Section 7.1.

Another methodological contribution of this work is to show how K_i functions can be used to study constants and cycles.

3. The General Process

Let any number n of w digits $n = a_1 a_2 \dots a_w$ belong to set $A_w \subset \mathbb{Z}$, excluding numbers with identical digits. Let numbers of fewer digits be included, as long as they are completed by adding zeros to their left. Let us use the decimal numeral system, so that $0 \leq a_i \leq 9$, $i = 1, \dots, w$.

Let O_d be an operator that sorts the digits of a number in descending order:

$x = O_d(n) = x_1 x_2 \dots x_w$, $x_i \geq x_{i+1}$ and let O_u be another operator which does the opposite: $y = O_u(n) = x_w x_{w-1} \dots x_1$.

Generalized Kaprekar's routine is formalized through an operator K such that

$$K(n) = n' = x - y = O_d(n) - O_u(n) \quad (1)$$

For example, if $n = 83,246,529$, $O_d(n) = x = 98,654,322$

$$O_u(n) = y = 22,345,689, n' = K(n) = x - y = 76,308,633$$

The iteration of the process r times is represented by

$$K^r(n) = K^{r-1}[K(n)] \quad r \geq 1$$

agreeing that $K^0(n) = n$ and writing $K^1 = K$ for convenience.

In our example, $K^2(83,246,529) = 84,326,652$.

Arranging the digits through O_d allows us to define the following parameters, which result from subtracting the digits in x that are symmetrical compared to the central value.

$$\alpha^s = x_s - x_{w-s+1} \quad 1 \leq s \leq h, \quad h = w/2 \text{ if } w = \dot{2}, \quad h = (w-1)/2, \text{ if } w = \dot{2} + 1; \quad (2)$$

$$0 < \alpha^1 \leq 9, 0 \leq \alpha^s \leq 9, \quad 2 \leq s \leq h$$

$$\alpha^s \geq \alpha^{s+1}, \quad s = 1, \dots, h-1 \quad (3)$$

must be verified as a consequence of the arrangements (1) forced by the routine.

Thus, for $w = 4$, $\alpha^1 = x_1 - x_4$, $\alpha^2 = x_2 - x_3$. For simplicity, these parameters are referred to as $\alpha = \alpha^1$ and $\beta = \alpha^2$ in further examples. In a similar way, the same parameters exist in A_5 .

As is shown below, parameters α^s uniquely determine the image of a number. Consequently, any number is characterized by their parameters, which are represented as follows:

$$p(n) = (\alpha^1 \alpha^2 \dots \alpha^h) = \alpha, \quad n \in A_w$$

When the situation is unambiguous, such parameters are written together as a number of h digits.

The transformation functions are the following:

(1) If $0 < \alpha^s \leq 9$, $s = 1, 2, \dots, h$

$$(1.1) \quad w = \dot{2}, \quad h = w/2$$

$$f_1(\alpha^1 \alpha^2 \dots \alpha^h) = (x_1 - x_w \quad x_2 - x_{w-1} \dots x_{w-1} - x_2 \quad x_w - x_1) = (\alpha^1 \alpha^2 \dots \alpha^h - \alpha^h \dots - \alpha^2 - \alpha^1) = \quad (4)$$

$$(\alpha^1 \alpha^2 \dots \alpha^{h-1} \alpha^h - 1 \quad 9 - \alpha^h \quad 9 - \alpha^{h-1} \quad 9 - \alpha^2 \quad 10 - \alpha^1)$$

So, for 6-digit numbers ($w = 6$)

$$f_1(\alpha^1 \alpha^2 \alpha^3) = (\alpha^1 \alpha^2 \alpha^3 - 1 \quad 9 - \alpha^3 \quad 9 - \alpha^2 \quad 10 - \alpha^1)$$

which can be written as follows to avoid using superscripts:

$$f_1(\alpha \beta \gamma) = (\alpha \beta \gamma - 1 \ 9 - \gamma \ 9 - \beta \ 10 - \alpha), \alpha = \alpha^1 > 0, \beta = \alpha^2 > 0, \gamma = \alpha^3 > 0$$

For instance, what is the image of 631,764?

$$O_d(631,764) = 766,431, \alpha = 7-1 = 6, \beta = 6-3 = 3, \gamma = 6-4 = 2$$

$$p(631,764) = 632, f_1(632) = 631,764$$

$$K(631,764) = 631,764$$

Apparently, this number transforms into itself. It is a transformation or balance constant in A_6 .

However, note that any number with these same parameters will yield the same image n_E . For example, $m = 632,000$, $p(m) = 632$, $m' = K(m) = n_E$.

Any permutation P of the digits in

$$(6 + a \ 3 + b \ 2 + c \ a \ b \ c), 0 \leq a \leq 3, 0 \leq b \leq 6, 0 \leq c \leq 7, a \geq b \geq c, a > c$$

will become n_E , $K[P(6 + a \ 3 + b \ 2 + c \ a \ b \ c)] = 631,764 = n_E$.

Digits a'_s in those numbers transformed by f_1 must satisfy the following restrictions:

$$\sum_{s=1}^w a'_s = \dot{9} \rightarrow K(n) = \dot{9}; a'_1 + a'_w = 10; a'_s + a'_{w-s+1} = 9, 1 < s < h \quad (5)$$

$$a'_h + a'_{h+1} = 8$$

$$\text{In the case of } A_6: a'_1 + a'_6 = 10, a'_2 + a'_5 = 9, a'_3 + a'_4 = 8, \sum_{s=1}^6 a_s = 27$$

$$(1.2) \ w = \dot{2} + 1, h = (w - 1)/2$$

In this case, function f_1 is similar to the last one, but it includes a 9 as a central digit in the transformed number

$$f_1(\alpha^1 \alpha^2 \dots \alpha^h) = (\alpha^1 \alpha^2 \dots \alpha^{h-1} \alpha^h - 1 \ 9 \ 9 - \alpha^h \ 9 - \alpha^{h-1} \dots 9 - \alpha^2 \ 10 - \alpha^1) \quad (6)$$

For instance, the image of a telephone number

$$n = 34,326,714,825 \text{ would be}$$

$$O_d = 87,654,433,221, \alpha^1 = 7, \alpha^2 = 5, \alpha^3 = 4, \alpha^4 = 2, \alpha^5 = 1$$

$$f_1(75,421) = (7 \ 5 \ 4 \ 2 \ 1 - 1 \ 9 \ 9 - 1 \ 9 - 2 \ 9 - 4 \ 9 - 5 \ 10 - 7) = 75,420,987,543$$

Notably, for seven-digit numbers

$$f_1(\alpha \beta \gamma) = (\alpha \beta \gamma - 1 \ 9 \ 9 - \gamma \ 9 - \beta \ 10 - \alpha)$$

The digits in the transformed number must satisfy

$$\sum_{s=1}^w a'_s = \dot{9} \rightarrow K(n) = \dot{9}; a'_1 + a'_w = 10; a'_s + a'_{w-s+1} = 9, 1 < s < h \quad (7)$$

$$a_h + a_{w-h+1} = 8, a_{h-1} = 9$$

$$(2) \text{ If } 0 < \alpha^s \leq 9, 1 \leq s < r \text{ and } \alpha^s = 0, s \geq r (r > 2)$$

$$(2.1) \ w = \dot{2}, h = w/2$$

Basic function f_2 is similar to (4) but adding $v = 2(h + 1 - r)$ digits 9 to the middle part (two nines per null parameter):

$$f_2(\alpha^1 \dots \alpha^{r-1} 0 0 \dots 0) = (\alpha^1 \dots \alpha^{r-2} \alpha^{r-1} - 1 \overset{v}{9} \dots 9 \overset{v}{9} - \alpha^{r-1} 9 - \alpha^{r-2} \dots 9 - \alpha^2 10 - \alpha^1) \quad (8)$$

thus satisfying the conditions

$$\sum_{s=1}^w a'_s = 9 \rightarrow K(n) = 9; a'_1 + a'_w = 10; a'_s + a'_{w-s+1} = 9, 2 \leq s \leq r-2 \quad (9)$$

$$a'_{r-1} + a'_{w-r+2} = 8, a'_r = a'_{r+1} = \dots = a'_{w-s+1} = 9$$

Specifically, for A_6

$$f_2(\alpha \beta 0) = (\alpha \beta - 1 \ 9 \ 9 \ 9 - \beta \ 10 - \alpha)$$

$$\sum a'_s = 36 \rightarrow K(n) = 9; a'_1 + a'_6 = 10; a'_2 + a'_5 = 8, a'_3 = a'_4 = 9$$

For example, let $n = 549,945$ have the image

$O_d(n) = 995,544$, $p(n) = 550$, $f_2(550) = 549,945$, the original number itself. This is another constant in A_6 .

(2.2) $w = 2 + 1$, $h = (w - 1)/2$

Function f_2 is similar to the last one but adds an extra 9 as a middle digit in the transformed number:

$$f_2(\alpha^1 \dots \alpha^{r-1} 0 0 \dots 0) = (\alpha^1 \dots \alpha^{r-2} \alpha^{r-1} - 1 \overset{v+1}{9} \dots 9 \overset{v+1}{9} - \alpha^{r-1} 9 - \alpha^{r-2} \dots 9 - \alpha^2 10 - \alpha^1) \quad (10)$$

$$\text{adding to conditions (5), } a'_{h+1} = 9 \quad (11)$$

For A_7 and $\gamma = 0$, the result is

$$f_2(\alpha \beta 0) = (\alpha \beta - 1 \ 9 \ 9 \ 9 \ 9 - \beta \ 10 - \alpha) \text{ with}$$

$$a'_1 + a'_7 = 10, a'_2 + a'_6 = 8, a'_3 = a'_4 = a'_5 = 9$$

(3) If $0 < \alpha^1 \leq 9$ and $\alpha^s = 0, s \geq 2$

$$f_3(\alpha^1 0 \overset{h-1}{\dots} 0) = (\alpha^1 - 1 \overset{w-2}{9} \dots 9 \ 10 - \alpha^1) \quad (12)$$

which implies the following restrictions for the transformed numbers

$$a'_1 + a'_w = 9, a'_2 = \dots = a'_{w-1} = 9 \quad (13)$$

These two expressions are valid both for even and odd numbers of digits.

For A_6 , the result is

$$f_3(\alpha 0 0) = (\alpha - 1 \ 9 \ 9 \ 9 \ 9 \ 10 - \alpha)$$

and for A_7

$$f_3(\alpha 0 0) = (\alpha - 1 \ 9 \ 9 \ 9 \ 9 \ 9 \ 10 - \alpha)$$

Functions f_1 , f_2 and f_3 are similar to the ones developed by (4). These functions are an epimorphism of A_W in $B_W \subset A_W$, which comprises those multiples of 9 that verify some requirement (5a.1), (5b.1) or (6.1) and that have anti images.

4. Parametric Functions

The previous development returns

$$p(m) = p(n) \leftrightarrow K(m) = K(n) \quad (14)$$

which suggests the transformation process should be analyzed in parametric terms. To this end, it is necessary to know the parameters of the transformed numbers.

One of the questions in the introduction is thus answered. Both numbers 83,246,529 and 17,487,561 return the same number after their transformations because both have the same parameters (7631).

Let us specify the notation used so far:

- K , in a broad sense, refers to the transformation. Its argument is a number and its image is the transformed number $K(n) = n'$.
- f_i is used in a specific sense. It acts upon parameters and yields the numerical image of all numbers whose parameters are those specified.

$$p(n) = \alpha, f_i(\alpha) = n', \alpha = \alpha^1 \alpha^2 \dots \alpha^h$$

- K can also be used broadly, by extension, to mean the parameters of some transformed number, thus transforming parameters into parameters:

$$K(\alpha) = \alpha', \alpha' = p(n')$$

This is the biggest problem in transformations. The digits in the transformed number are not necessarily arranged, so it is not possible to determine its parameters right away. Thus, operators O_d and O_u , which establish a permutation P_i that sorts the digits, must operate. $O_d(n') = P_i(n')$. If P_i is identified, then α' is directly discovered. A specific function $K_i(\alpha) = \alpha'$ can therefore be associated with each permutation P_i .

This is a tedious process, since the possible permutations will grow factorially as the number of digits increases:

- With $w=2$ and $w=3$, there are just two functions $K_i(\alpha)$;
- With $w=4$ and $w=5$, there are 11 functions $K_i(\alpha \beta)$ and 2 functions $K_i(\alpha 0)$;
- With $w=6$ and $w=7$, there are two functions $K_i(\alpha 0 0)$, 11 functions $K_i(\alpha \beta 0)$ and 117 functions $K_i(\alpha \beta \gamma)$;
- With w and $w+1$ digits, the number of functions $K_i(\alpha^1 \alpha^2 \dots \alpha^s \dots \alpha^h)$, $\alpha_s > 0$, $s = 1, \dots, h$ approaches $w!/h!$ by default. This is because in $O_d(n')$, a permutation with α^{r-s} to the left of α^r is, under (3), only possible if $\alpha^r = \alpha^{r-s}$.

Let us see some examples of these functions.

(1) Functions K_i based on (4)

$w = 2, h = w/2, f_1(\alpha)$ defined in (4) and $P_1(1\ 2\ 3 \dots w)$

P_1 is defined as the digit arrangement described in (4).

$$K_1(\alpha) = [\alpha^1 - (10 - \alpha^1) \alpha^2 - (9 - \alpha^2) \dots \alpha^{h-1} - (9 - \alpha^{h-1}) \alpha^h - 1 - (9 - \alpha^h)] = [2\alpha^1 - 10\ 2\alpha^2 - 9 \dots 2\alpha^{h-1} - 9\ 2\alpha^h - 10] \quad (15)$$

The domain of existence of this function $K_1(\alpha) = \alpha'$ is determined by the restrictions imposed by $O_d(n')$, with the digits being sorted in descending order (1) in permutation P_1

$$\alpha^1 \geq \alpha^2, \dots, \alpha^h - 1 \geq 9 - \alpha^h, 9 - \alpha^h \geq 9 - \alpha^{h-1}, \dots, 9 - \alpha^3 \geq 9 - \alpha^2, 9 - \alpha^2 \geq 10 - \alpha^1$$

which implies $\alpha^h \geq 5$, and under (3), $\alpha^s \geq 5, s \geq 1, y \alpha^1 \geq \alpha^2 + 1$

Briefly, the existence conditions for $K_1(\alpha)$ are

$$w = 2; 5 < \alpha^1 \geq \alpha^2 + 1; 5 \leq \alpha^s \leq 9, s \geq 2 \quad (16)$$

For instance,

$n = 181,771,978,221$, $w = 12$, $h = 6$, $p(n) = 877,655 = \alpha$, which satisfies (16);

$K_1(877,655) = 655,310$ and indeed, $f_1(n) = 877,654,443,222 = n'$, $p(n') = 655,310 = \alpha'$.

Note that this α' does not satisfy (16), so it cannot be transformed through K_1 . We need a function K_i based on (8), with $\alpha^h = \alpha^6 = 0$ (see (27) in the last case in Section (3).

$w = 2$, $h = w/2$, $f_1(\alpha)$ defined in (4) and $P_2(h + 1 \dots w - 1 \dots h)$

$$O_d(n') = (9 - \alpha^h \ 9 - \alpha^{h-1} \dots 9 - \alpha^2 \ 10 - \alpha^1 \ \alpha^1 \ \alpha^2 \dots \alpha^{h-1} \ \alpha^h - 1)$$

$$K_2(\alpha) = (10 - 2\alpha^h \ 9 - 2\alpha^{h-1} \dots 9 - 2\alpha^2 \ 10 - 2\alpha^1)$$

$$w = 2; 0 < \alpha^1 \leq 5; \alpha^1 \geq \alpha^2 + 1; 0 < \alpha^s \leq 4, s = 2 \dots h, 2\alpha^h \leq 2\alpha^{h-1} + 1 \quad (17)$$

For instance, in A_6

$$K_2(541) = 810$$

$w = 2$, $h = w/2$ and $P_3(h + 1 \ 1 \ h + 2 \dots w - 2 \ 3 \dots h)$ in (4)

$$O_d(n') = (9 - \alpha^h \ \alpha^1 \ 9 - \alpha^{h-1} \ 9 - \alpha^{h-2} \dots 9 - \alpha^3 \ 9 - \alpha^2 \ 10 - \alpha^1 \ \alpha^2 \dots \alpha^{h-1} \ \alpha^h - 1)$$

$$K_3(\alpha) = (10 - 2\alpha^h \ \alpha^1 - \alpha^{h-1} \ 9 - \alpha^{h-1} - \alpha^{h-2} \dots 9 - \alpha^3 - \alpha^2 \ \alpha^1 - \alpha^2 - 1)$$

$$w = 2; h = w/2; 1 \leq \alpha^1 + \alpha^2 \leq 10; \alpha^s + \alpha^{s+1} \leq 9, s = 2, 3, \dots, h - 2; \\ \alpha^1 \geq \alpha^2 + 1; \alpha^1 \geq \alpha^2 + 1; \alpha^1 + \alpha^{h-1} \geq 9; \alpha^1 + \alpha^h \leq 9;$$

$$\alpha^h \leq 10 + \alpha^{h-1} - 2\alpha^h; 1 \leq \alpha^h \leq 4 \quad (18)$$

This is an important function that generates a family of transformation constants.

(2) Functions K_i based on (6)

The permutations refer to the digits to the right of 9 in (6)

$w = 2 + 1$, $h = (w-1)/2$, $f_1(\alpha)$ defined in (6) and P_1

$$O_d(n') = (9 \ \alpha^1 \ \alpha^2 \dots \alpha^{h-1} \ \alpha^{h-1} \ 9 - \alpha^h \ 9 - \alpha^{h-1} \dots 9 - \alpha^2 \ 10 - \alpha^1)$$

The presence of a 9 completely changes function K_1

$$K_1(\alpha) = (\alpha^1 - 1 \ \alpha^1 + \alpha^2 - 9 \ \alpha^2 + \alpha^3 - 9 \dots \alpha^{h-1} - 1 + \alpha^h - 9) \quad (19)$$

with the existence conditions

$$w = 2 + 1, 5 < \alpha^1 \leq \alpha^2 + 1; 5 \leq \alpha^s \leq 8, s > 1; \alpha^s + \alpha^{s+1} \geq 9, s = 1, \dots, h - 1 \quad (20)$$

For example, $n = 8,650,000$, $p(n) = 865 = \alpha$

$$n' = f_1(n) = 8,649,432, O_d(n') = 9,864,432, p(n') = 752 = \alpha'$$

$\alpha = 865$ satisfies the conditions (20) and, in fact,

$$K_1(\alpha) = (8 - 1 \ 8 + 6 - 9 \ 6 + 5 - 9) = 752$$

$w = 2 + 1$, $h = (w-1)/2$, and P_2 in (6)

$$O_d(n') = (9 \ 9 - \alpha^h \ 9 - \alpha^{h-1} \dots 9 - \alpha^2 \ 10 - \alpha^1 \ \alpha^1 \ \alpha^2 \dots \alpha^{h-1} \ \alpha^h - 1)$$

$$K_2(\alpha) = (10 - \alpha^h \ 9 - \alpha^h - \alpha^{h-1} \dots 9 - \alpha^3 - \alpha^2 \ 9 - \alpha^2 - \alpha^1) \quad (21)$$

$$w = 2 + 1, h = (w - 1)/2; 0 < \alpha^1 \leq 5; \alpha^1 \geq \alpha^2 + 1; \\ \alpha^{s+1} + \alpha^s \leq 9, s = 1, 2, \dots, h - 1$$

$$\alpha^{h-1} \geq 1, 1 < \alpha^h < 5$$

For example, $n = 2,515,324$ $p(n) = 432$

$$w = 7, h = 3 \text{ por } (6), n' = f_1(n) = 4,319,766, p(n') = 842$$

and in fact, since 432 satisfies the existence conditions of K_2

$$K_2(432) = (10-2 \ 9-2-3 \ 9-3-4) = 842$$

w = 2 + 1, h = (w-1)/2 ≥ 4, P₄ (1 h + 2 2 h + 3 ... h + s 3 4 ... s 2 h 2 h + 1 h), s = h-1
en (6)

$$\begin{aligned} O_d(n') &= (9 \ \alpha^1 \ 9-\alpha^h \ \alpha^2 \ 9-\alpha^{h-1} \dots 9-\alpha^3 \ \alpha^3 \dots \alpha^{h-1} \ 9-\alpha^2 \ 10-\alpha^1 \ \alpha^{h-1}) \\ K_4(\alpha) &= (10-\alpha^h \ 2\alpha^1-10 \ \alpha^2-\alpha^h \ \alpha^2-\alpha^{h-1} \ 9-\alpha^{h-2}-\alpha^{h-1} \dots 9-\alpha^4-\alpha^3) \\ h &\geq 4; 9 \leq \alpha^1 + \alpha^h \leq 11, \alpha^2 + \alpha^h \leq 9, \alpha^2 + \alpha^{h-1} \geq 9, \alpha^3 \leq 4 \end{aligned} \quad (22)$$

For example, in $w = 15, h = 7$

$$\begin{aligned} f_1(\alpha) &= (\alpha^1 \ \alpha^2 \dots \alpha^6 \ \alpha^7-1 \ 9 \ 9-\alpha^7 \ 9-\alpha^6 \dots 9-\alpha^2 \ 10-\alpha^1) = n' \\ O_d(n') &= (9 \ \alpha^1 \ 9-\alpha^7 \ \alpha^2 \ 9-\alpha^6 \dots 9-\alpha^3 \ \alpha^3 \dots \alpha^6 \ 9-\alpha^2 \ 10-\alpha^1 \ \alpha^7-1) \\ K_4(\alpha) &= (10-\alpha^7 \ 2\alpha^1-10 \ \alpha^2-\alpha^7 \ \alpha^2-\alpha^6 \ 9-\alpha^5-\alpha^6 \ 9-\alpha^4-\alpha^5 \ 9-\alpha^3-\alpha^4) \end{aligned}$$

$K_4(8,643,332) = 8,643,332$; therefore, this number is a parametric constant in $w = 15$, with the numeric constant of 15 digits, 864,333,197,666,532.

Similarly, for nine digits

$K_4(\alpha \ \beta \ \gamma \ \delta) = (10-\delta \ 2\alpha-10 \ \beta-\delta \ \beta-\gamma)$, $K_4(8642) = 8642$, which is a parametric constant in A_8 .

$K_4(\alpha)$, defined in a general way at $w = 2 + 1$, is an important parametric function from which one-cycles derive.

(3) Functions K_i based on (8)

Functions K_i based on (8), $h = w/2$ are complex, as they depend on the number of null parameters compared to the number of non-null parameters. If $r \leq 1 + h/2$, $\alpha^r = 0$, then the functions' structure remains, since (8) provides as many nines as it provides varying digits. If there are more non-null than null parameters, $r > 1 + h/2$, then functions K_i will vary with each r increment.

Let us see some examples:

– **w = 2, $r \leq 1 + h/2$, $\alpha^3 = 0$, $h \geq 4$ in (8) and P_1**

$$n' = f_2(\alpha^1 \ \alpha^2 \ 0 \ \dots \dots \ 0) = (\alpha^1 \ \alpha^2 - 1 \ 9 \ \dots \dots \ 9 \ 9 - \alpha^2 \ 10 - \alpha^1)$$

If permutation $P_1(1234)$ is used to the right of numbers 9 in $O_d(n')$

$$O_d(n') = (9 \ \dots \dots \ 9 \ \alpha^1 \ \alpha^2 - 1 \ 9 - \alpha^2 \ 10 - \alpha^1)$$

the associated function is

$$\begin{aligned} K_{21}(\alpha^1 \ \alpha^2 \ 0 \ \dots \dots \ 0) &= (\alpha^1 - 1 \ \alpha^2 \ 10 - \alpha^2 \ 9 - \alpha^1 \ 0 \ \dots \dots \ 0), 6 \leq \alpha^1 \leq 9, \\ 5 \leq \alpha^2 \leq 9, \alpha^1 &\geq \alpha^2 + 1 \end{aligned} \quad (23)$$

For example: $K_{21}(85,000) = 75,510$

- $w = \dot{2}, r \geq 1 + h/2, \alpha^3 = 0, h = 3$ in (8) and P_1

$$n' = f_2(\alpha^1 \alpha^2 0) = (\alpha^1 \alpha^2 - 1 \ 9 \ 9 \ 9 - \alpha^2 \ 10 - \alpha^1)$$

$$O_d(n') = (9 \ 9 \ \alpha^1 \ \alpha^2 - 1 \ 9 - \alpha^2 \ 10 - \alpha^1)$$

$$K_{22}(\alpha^1 \alpha^2 0) = (\alpha^1 - 1 \ \alpha^2 \ \alpha^1 - \alpha^2 + 1), 6 \leq \alpha^1 \leq 9, 5 \leq \alpha^2 \leq 9, \alpha^1 \geq \alpha^2 + 1, 2\alpha^2 \geq \alpha^1 + 1 \quad (24)$$

a function which differs from (23)

For example: $K_{22}(850) = 754$

- $w = \dot{2}, r \leq 1 + h/2, \alpha^3 = 0, h \geq 4$ in (8), $P_2(1423)$

$$O_d(n') = (9 \ \dots \dots \ 9 \ \alpha^1 \ 10 - \alpha^1 \alpha^2 - 1 \ 9 - \alpha^2) K_{23}(\alpha^1 \ \alpha^2 \ 0 \ \dots \dots \ 0) =$$

$$(\alpha^2 10 - \alpha^2 \alpha^1 - 1 \ 9 - \alpha^1 \ 0 \ \dots \dots \ 0), 5 \leq \alpha^1 \leq 9, 5 \leq \alpha^2 \leq 9, \alpha^1 + \alpha^2 \leq 11 \quad (25)$$

For example: $K_{23}(5500) = 5544$

- $w = \dot{2}, r > 1 + h/2, \alpha^3 = 0, h = 3$ in (8), P_2

$$O_d(n') = (9 \ 9 \ \alpha^1 \ 10 - \alpha^1 \ \alpha^2 - 1 \ 9 - \alpha^2)$$

$$K_{24}(\alpha^1 \ \alpha^2 \ 0) = (\alpha^2 \ 10 - \alpha^2 \ 2\alpha^1 - 10), 5 \leq \alpha^1 \leq 9, 5 \leq \alpha^2 \leq 9, \alpha^1 + \alpha^2 \leq 11 \quad (26)$$

which only operates on parameters (550) and (650).

With $r \leq 1 + h/2$, functions differ from those with $r > 1 + h/2$

- $w = \dot{2}, \alpha^h = 0, h = w/2 = 6$ in (8), $P_3 = (6 \ 1 \ 7 \ 2 \ 3 \ 8 \ 9 \ 10 \ 4 \ 5)$

As an example of applying (15), we found:

$n' = 877,654,443,222, \alpha' = p(n') = 655,310$. Such α' cannot be transformed with K_1 . It is a case $w = 12, h = 6, \alpha^6 = 0$ which can be transformed with the function

$$K_{25}(\alpha \ \beta \ \gamma \ \delta \ \varepsilon \ 0) = (10 - \varepsilon \ 9 - \delta \ \alpha - \varepsilon - 1 \ \alpha + \beta - 9 \ \gamma - \delta \ \beta - \gamma),$$

$$\alpha > 1, \alpha \geq \beta + 1, \alpha + \beta \geq 9, 9 \leq \alpha + \delta \leq 10, \alpha + \varepsilon \leq 9$$

$$\alpha + \delta - \varepsilon \leq 10, \beta + \delta \leq 9, \beta + \varepsilon \leq 8, 2\gamma \geq \beta + \delta, \gamma \geq 5, \delta \geq \varepsilon + 1, \alpha + \beta + \delta - \gamma \geq 9, \varepsilon \geq 1 \quad (27)$$

This function is associated with the aforementioned permutation P_3 , in a specific case of (8)

$$f_2(\alpha \ \beta \ \gamma \ \delta \ \varepsilon \ 0) = (\alpha \ \beta \ \gamma \ \delta \ \varepsilon - 1 \ 9 \ 9 \ 9 - \varepsilon \ 9 - \delta \ 9 - \gamma \ 9 - \beta \ 10 - \alpha),$$

$$f_2(655,310) = 655,309,986,444, K_{25}(655,310) = 964,220$$

(4) Functions K_i based on (10)

As previously stated, when the number of digits is odd, $w = \dot{2} + 1, h = (w-1)/2$, function f_2 is similar to the one corresponding for numbers with $w = \dot{2}$, but it includes one more 9 as a middle digit in the transformed number.

Such an addition does not alter functions K_i if $r \leq 1 + h/2, \alpha^r = 0$, although it does change the last parameters of α' if $r > 1 + h/2$.

In parallel with the cases considered in (3):

- $\alpha^3 = 0, w = 7, h = 3$ in (10) and $P_1(1234)$

$$K_{26}(\alpha^1 \ \alpha^2 \ 0) = (\alpha^1 - 1 \ \alpha^2 \ 10 - \alpha^2), 5 \leq \alpha^1 \leq 9, 5 \leq \alpha^2 \leq 9, \alpha^1 \geq \alpha^2 + 1 \quad (28)$$

a closer function to K_{21} than to K_{22}

– $\alpha^3 = 0$, $w = 7$, $h = 3$ in (10) and P_2 (1423)

$$K_{27}(\alpha^1 \alpha^2 0) = (\alpha^2 10 - \alpha^2 \alpha^1 - 1), 5 \leq \alpha^1 \leq 9, 5 \leq \alpha^2 \leq 9, \alpha^1 + \alpha^2 \leq 11 \quad (29)$$

which is also a closer expression to K_{23} than to K_{24}

(5) Functions K_i based on (12)

There are only two functions K_i based on (12):

$$K_{31}(\alpha 0 \dots 0) = (\alpha - 1 10 - \alpha 0 \dots 0), 6 \leq \alpha \leq 9 \quad (30)$$

$$K_{32}(\alpha 0 \dots 0) = (10 - \alpha \alpha - 1 0 \dots 0), 0 \leq \alpha \leq 5 \quad (31)$$

valid both for $w = 2$ with $h = w/2$ and for $w = 2 + 1$ with $h = (w-1)/2$

Tables 1 and 2 show some functions K_i of the different types in A_6 ($w = 6$) and A_7 ($w = 7$).

Table 1. Some functions K_i in A_6 , $w = 6$, $h = 3$.

Permutation $O(n')$	Associated $K_i(\alpha)$	Domain of Existence
P_1 (123,456)	$K_1(\alpha \beta \gamma) = (2\alpha - 10 \ 2\beta - 9 \ 2\gamma - 10)$	$6 \leq \alpha \leq 9, 5 \leq \beta \leq 9, 5 \leq \gamma \leq 9, \alpha \geq \beta + 1$
P_2 (456,123)	$K_2(\alpha \beta \gamma) = (10 - 2\gamma \ 9 - 2\beta \ 10 - 2\alpha)$	$0 < \alpha \leq 5, 0 < \beta \leq 4, 0 < \gamma \leq 4, \alpha \geq \beta + 1, 2\gamma \leq 2\beta + 1$
P_3 (415,623)	$K_3(\alpha \beta \gamma) = (10 - 2\gamma \ \alpha - \beta \ \alpha - \beta - 1)$	$9 \leq \alpha + \beta \leq 10, \alpha \geq \beta + 1, 1 \leq \gamma \leq 4$ $\alpha + \gamma \leq 9, \alpha \leq 10 + \beta - 2\gamma$
P_4 (142,563)	$K_4(\alpha \beta \gamma) = (\alpha - \gamma + 1 \ \alpha - \gamma - 1 \ 2\beta - 9)$	$5 \leq \alpha \leq 9, 5 \leq \beta \leq 8, 1 \leq \gamma \leq 6$ $\alpha \geq \beta + 1, 9 \leq \alpha + \gamma \leq 11, \alpha \geq 2\beta + \gamma - 8$
P_5 (145,263)	$K_5(\alpha \beta \gamma) = (\alpha - \gamma + 1 \ \alpha - \gamma - 1 \ 9 - 2\beta)$	$6 \leq \alpha \leq 9, 1 \leq \beta \leq 4, \alpha + \beta \geq 10$ $9 \leq \alpha + \gamma \leq 11, \alpha \geq 10 - 2\beta + \gamma$
P_6 (412,563)	$K_6(\alpha \beta \gamma) = (10 - 2\gamma \ 2\alpha - 10 \ 2\beta - 9)$	$6 \leq \alpha \leq 9, 5 \leq \beta \leq 9, 1 \leq \gamma \leq 3$ $\alpha \geq \beta + 1, \alpha + \gamma \leq 9$
P_7 (124,536)	$K_7(\alpha \beta \gamma) = (2\alpha - 10 \ \beta - \gamma + 1 \ \beta - \gamma)$	$6 \leq \alpha \leq 9, 5 \leq \beta \leq 8, 2 \leq \gamma \leq 5$ $\alpha + \gamma \geq 11, 9 \leq \beta + \gamma \leq 10, 2\alpha \geq 11 + \beta - \gamma$
P_8 (124,563)	$K_8(\alpha \beta \gamma) = (\alpha - \gamma + 1 \ \alpha + \beta - 10 \ \beta - \gamma)$	$6 \leq \alpha \leq 9, 5 \leq \beta \leq 8, 1 \leq \gamma \leq 5$ $\alpha \geq \beta + 1, \alpha + \beta \geq 10, 10 \leq \alpha + \gamma \leq 11, 9 \leq \beta + \gamma \leq 11$
P_9 (612,453)	$K_9(\alpha \beta \gamma) = (11 - \alpha - \gamma \ \alpha + \beta - 9 \ \beta + \gamma - 9)$	$\alpha = 5, \beta = 5, \gamma = 4$
P_1 (1234)	$K_{22}(\alpha \beta 0) = (\alpha - 1 \ \beta \ \alpha - \beta + 1)$	$6 \leq \alpha \leq 9, 5 \leq \beta \leq 9, \alpha \geq \beta + 1, 2\beta \geq \alpha + 1$
P_2 (1423)	$K_{24}(\alpha \beta 0) = (\beta \ 10 - \beta \ 2\alpha - 10)$	$5 \leq \alpha \leq 9, 5 \leq \beta \leq 9, \alpha + \beta \leq 11$
P_1 (12)	$K_{31}(\alpha 0 0) = (\alpha - 1 \ 10 - \alpha \ 0)$	$6 \leq \alpha \leq 9$
P_2 (21)	$K_{32}(\alpha 0 0) = (10 - \alpha \ \alpha - 1 \ 0)$	$0 \leq \alpha \leq 5$

Table 2. Some functions K_i in A_7 , $w = 7$, $h = 3$.

Permutation (*)	Associated $K_i(\alpha)$	Domain of Existence
P_1 (123,456)	$K_1(\alpha \beta \gamma) = (\alpha-1 \alpha + \beta-9 \beta + \gamma-9)$	$\alpha \geq \beta + 1, \gamma \geq 5$
P_4 (142,563)	$K_4(\alpha \beta \gamma) = (10-\gamma \ 2\alpha-10 \ \beta-\gamma)$	$\alpha \geq \beta + 1, 9 \leq \alpha + \gamma \leq 11, \beta \geq 5, \beta + \gamma \leq 9$
P_5 (145,263)	$K_5(\alpha \beta \gamma) = (10-\gamma \ 2\alpha-10 \ 9-\beta-\gamma)$	$\alpha + \beta \geq 10, \beta \leq 4, 9 \leq \alpha + \gamma \leq 11$
P_6 (412,563)	$K_6(\alpha \beta \gamma) = (10-\gamma \ \alpha-\gamma-1 \ \alpha + \beta-9)$	$\alpha \geq \beta + 1, \beta \geq 5, \alpha + \gamma \leq 9$
P_7 (124,536)	$K_7(\alpha \beta \gamma) = (\alpha-1 \ \alpha-\gamma + 1 \ 2\beta-9)$	$\alpha + \gamma \geq 11, 9 \leq \beta + \gamma \leq 10$
P_{10} (123,465)	$K_{10}(\alpha \beta \gamma) = (\beta \ 2\alpha-10 \ \beta+\gamma-9)$	$\alpha \geq \gamma + 1, \alpha \leq \beta + 1, \gamma \geq 5$
P_{11} (451,623)	$K_{11}(\alpha \beta \gamma) = (10-\gamma \ 9-\gamma-\beta \ \alpha-\beta-1)$	$\alpha \geq 5, \alpha + \beta \leq 9$
P_{12} (145,236)	$K_{12}(\alpha \beta \gamma) = (\alpha-1 \ \alpha-\gamma + 1 \ 9-\beta-\gamma)$	$\alpha + \gamma \geq 11, \beta \leq 4$
P_{13} (124,365)	$K_{13}(\alpha \beta \gamma) = (\beta \ 2\alpha-10 \ \beta-\gamma + 1)$	$\alpha \leq \beta + 1, \alpha + \gamma \geq 11, \gamma \leq 5$
P_{22} (1234)	$K_{22}(\alpha \beta 0) = (\alpha-1 \ \beta \ 10-\beta)$	$\alpha \geq \beta + 1, \beta \geq 5$
P_{23} (1243)	$K_{23}(\alpha \beta 0) = (\beta \ \alpha-1 \ 10-\beta)$	$\alpha \leq \beta + 1, \alpha + \beta \leq 11$
P_{25} (1324)	$K_{25}(\alpha \beta 0) = (\alpha-1 \ 10-\beta \ \beta)$	$\alpha + \beta \leq 11, \beta \leq 5$
P_{26} (1342)	$K_{26}(\alpha \beta 0) = (10-\beta \ \alpha-1 \ \beta)$	$\alpha \geq \beta + 1, 9 \leq \alpha + \beta \leq 11$
P_{27} (3142)	$K_{27}(\alpha \beta 0) = (10-\beta \ \alpha-1 \ 9-\alpha)$	$\alpha \geq 5, \alpha + \beta \leq 9$
P_{31} (12)	$K_{31}(\alpha 0 0) = (\alpha-1 \ 10-\alpha \ 0)$	$6 \leq \alpha \leq 9$
P_{32} (21)	$K_{32}(\alpha 0 0) = (10-\alpha \ \alpha-1 \ 0)$	$1 \leq \alpha \leq 5$

* Permutation of $O(n')$, excluding fixed nines. Thus, $P_1(9 \ \alpha \ \beta \ \gamma-1 \ 9-\gamma \ 9-\beta \ 10-\alpha)$, $P_{22} = (9 \ 9 \ 9 \ \alpha \ \beta-1 \ 9-\beta \ 10-\alpha)$, $P_{31} = (9 \ 9 \ 9 \ 9 \ \alpha-1 \ 10-\alpha)$ and $P_6 = (9 \ 9-\gamma \ \alpha \ \beta \ 9-\beta \ 10-\alpha \ \gamma-1)$.

5. Balance: Transformation Constants

Let balance exist when there is an α_E and a $K_i(\alpha)$ such that $K_i(\alpha_E) = \alpha_E$ or, equivalently, $K(n_E) = n_E$, $p(n_E) = \alpha_E$.

As a corollary, $K^r(n_E) = n_E$, $K^r(\alpha_E) = \alpha_E$, $\forall r \geq 1$.

Let n_E be called **transformation constant** and α_E **parametric constant**. If all the numbers $n \in A_w$, by reiterating their transformation they become n_E ; we will say that n_E is a Kaprekar constant. This is equivalent to the existence of a constant and a one-component scheme in A_w .

A transformation constant is a single-link cycle, which is why some authors call the constant a one-cycle [4]. Others call it a fixed point [5,12,13]. In relation with this fixed point and with the cycles, in general, the classic work of Prichett et al. [4] and the more recent, innovative input of Yamagami [10,11] and Bajorska-Harapinska et al. [19] should be highlighted.

In the example of (4), we see how 631,764 transforms into itself, and the same is true for $\alpha = p(n) = 632$, $K(632) = 632$. Therefore, $631,764 = n_E$ and $632 = \alpha_E$.

The constants in base 10 have been exhaustively studied by Prichett et al. [3,4], who showed that in base 10, there are only two Kaprekar constants, 495 in A_3 and 6174 in A_4 . They also determined one-cycles for different w -digits.

Our objective in this section is not to make a detailed study of the constants, but to show how families of constants can be derived directly from the K_i functions. In this regard,

we show two examples: the family $\alpha_E = 6 \ 3 \ \dots \ 3 \ 2$ in $w = 2 \ h$, $h \geq 2$ and the family α_E

$= 8 \ 6 \ 4 \ 3 \ \dots \ 3 \ 2$ in $w = 2 \ h + 1$, $h \geq 7$. We also show two singular constants, $\alpha_E = 550$

in A_6 and $\alpha_E = 5$ in A_3 .

$$\text{Constants } n_E = \underbrace{63 \dots \dots}_{h-2} \quad \underbrace{3176 \dots \dots}_{h-2} \quad \underbrace{64, \alpha_E = 63 \dots \dots}_{h-2} \quad \underbrace{32, w = 2h}_{h-2}$$

$$\text{Indeed, } O_d(n_E) = \underbrace{76 \dots \dots}_{h-2} \quad \underbrace{6643 \dots \dots}_{h-2} \quad 31 \text{ and } O_u(n_E) = \underbrace{13 \dots \dots}_{h-2} \quad 3466$$

$$\underbrace{\dots \dots}_{h-2} \quad 67$$

$$n'_E = O_d(n_E) - O_u(n_E) = \underbrace{63 \dots \dots}_{h-2} \quad \underbrace{3176 \dots \dots}_{h-2} \quad 64 = n_E$$

For this expression to be valid, at least the two extreme digits and the two middle digits must be present. Such is the case with 6174, which is the renowned Kaprekar's constant [1]. The associated parametric constants are $\alpha_E = \underbrace{63 \dots \dots}_{h-2} \quad 32$.

Such constants, which generalize Kaprekar's constant for w -digits, $w = \dot{2}$, derive from function (18) associated with permutation $P_3(h+1 \ 1 \ h+2 \ \dots \ w \ 2 \ 3 \ \dots \ h)$

$$K_3(\alpha) = \alpha \rightarrow \alpha^1 = 10-2\alpha^h, \alpha^2 = \alpha^1 - \alpha^{h-1}, \alpha^3 = 9 - \alpha^{h-1} - \alpha^{h-2} \dots \alpha^h = \alpha^1 - \alpha^2 - 1$$

with the restrictions imposed in K_3 . There is a unique solution which is $\alpha_E = \underbrace{63 \dots \dots}_{h-2} \quad 32$

For instance, for $w = 8, h = 4$

$$K_3(\alpha) = K_3(\alpha \ \beta \ \gamma \ \delta) = (10-2\delta \ \alpha-\gamma \ 9-\beta-\gamma \ \alpha-\beta-1)$$

$$1 \leq 10-2\delta + \alpha-\gamma \leq 10; \beta + \gamma \leq 9; \alpha \geq \beta + 1, \alpha + \gamma \geq 9, \alpha \leq 10 + \gamma-2\delta, 1 \leq \delta \leq 4$$

Imposing the equality condition

$\alpha = 10-2\delta, \beta = \alpha-\gamma, \gamma = 9-\beta-\gamma, \delta = \alpha-\beta-1$, there is only one solution: $\alpha = 6, \beta = 3, \gamma = 3, \delta = 2 \rightarrow \alpha_E = 6332$, which satisfies the existence conditions and $f_1(6332) = 63,317,664 = n_E$.

In the case where $w = 4, h = 2, P_3 = (3142)$, and the associated function is

$$K_3(\alpha \ \beta) = (10-2\beta \ 2\alpha-10) \ \alpha \geq 5, \alpha + \beta \leq 9$$

$$K_3(\alpha \ \beta) = (\alpha \ \beta) \rightarrow \alpha = 10-2\beta, \beta = 2\alpha-10 \rightarrow \alpha = 6, \beta = 2$$

$$f_1(6 \ 2) = 6174, \text{ which is Kaprekar's constant.}$$

This family of constants cannot exist if $w = \dot{2} + 1$, since a 9 being present in (6) completely changes function K_3 . It is not valid for $w = 2$ either, since permutation P_3 does not make sense.

$$\text{Constants } n_E = \underbrace{8643 \dots \dots}_{h-4} \quad \underbrace{31,976 \dots \dots}_{h-4} \quad \underbrace{532, \alpha_E = 8643 \dots \dots}_{h-4} \quad \underbrace{32, w = 2h+1}_{h-4}$$

$$\underline{h \geq 4}$$

They are derived from the function (22) associated with the permutation $P_4(1 \ h+2 \ 2 \ h+3 \ \dots \ h+s \ 3 \ 4 \ \dots \ s \ 2h \ 2h+1)$, $s = h+1, h \geq 4$. De $K_4(\alpha) = \alpha$ and taking into account the domain of existence of P_4 , it results in α_E .

$$P_4(1 \ h+2 \ 2 \ h+3 \ \dots \ h+s \ 3 \ 4 \ \dots \ s \ 2h \ 2h+1)$$

$$\text{Constant } n_E = \underline{549,945 \text{ and } \alpha_E = 550}$$

Derives from K_{24} in A_6 (Table 1)

$$\alpha = \beta, \beta = 10 - \beta, 0 = 2\alpha - 10 \rightarrow \alpha = 5, \beta = 5, \gamma = 0 \leftrightarrow \alpha_E = 550$$

$$f_2(550) = 549,945 = n_E$$

This constant cannot be generalized.

Constant $n_E = 495$ and $\alpha_E = 5$

This is the other Kaprekar's constant. If $w = 3$, $f_1(\alpha) = (\alpha-1 \ 9 \ 10-\alpha)$ and $O(n')$, this allows for two permutations $P_{12} = (9 \ \alpha-1 \ 10-\alpha)$ and $P_{21} = (9 \ 10-\alpha \ \alpha-1)$ as well as two associated functions $K_1(\alpha) = (\alpha-1)$ and $K_2(\alpha) = (10-\alpha)$. $\alpha_E = 5$ results from $K_2(\alpha) = \alpha$ and $f_1(5) = 495$. It is thus another constant that cannot be generalized.

There are other functions of just one parameter. Such is the case of

- $w = 2$ with $f_1(\alpha) = (\alpha-1 \ 10-\alpha)$ and two functions K_i ,
 $K_1(\alpha) = (2\alpha-11)$ and $K_2(\alpha) = (11-2\alpha)$, which obviously cannot satisfy the condition $K_i(\alpha) = \alpha$. In A_2 , there cannot be any transformation constant.
- The same is true for functions

$$\begin{array}{ccc} h-1 & & h-2 & & h-1 \\ K_{31}(\alpha \ 0 \ \dots \dots \ 0) = (\alpha-1 \ 10-\alpha \ 0 \ \dots \dots \ 0) \text{ and } K_{32}(\alpha \ 0 \ \dots \dots \ 0) = \\ & & h-2 \\ (10-\alpha \ \alpha-1 \ 0 \ \dots \dots \ 0) \end{array}$$

where condition $K(\alpha) = \alpha$ leads to impossible results. In these cases, there cannot be any transformation constant either.

6. Cycles: Terminology and Functions $K^r(\alpha)$

There is a cycle of r links if and only if there are r parameters α_{ci} and r functions $K_{ci}(\alpha)$, such that $K_{ci}(\alpha_{ci}) = \alpha_{ci+1}$ being $\alpha_{ci} = \alpha_{cj} \leftrightarrow i \equiv j \pmod{r}$.

This amounts to the existence of r operators K_{ci}^r such that $K_{ci}^r(\alpha_{ci}) = \alpha_{ci}$, $i = 1 \dots r$.

In terms of numbers, it demands the existence of r numbers

$$n_{ci}, K(n_{ci}) = n_{ci+1}, n_{ci} = n_{cj} \leftrightarrow i \equiv j \pmod{r} \text{ or, equivalently, } K^r(n_{ci}) = n_{ci}, i = 1 \dots r.$$

Let us analyze the case A_6 for six-digit numbers. The parametric transformation schemes are shown in Scheme 1.

Main component A, consisting of 201 parametric classes out of the 219 existing classes, is articulated on a seven-class cycle, which can be conventionally called

$$861 = \alpha_{c1}, 863 = \alpha_{c2}, 643 = \alpha_{c3}, 421 = \alpha_{c4}, 852 = \alpha_{c5}, 751 = \alpha_{c6}, 841 = \alpha_{c7}$$

Each class becomes the following one with the functions in Table 1, indicated in the diagram:

Naturally, there are seven operators K^r , $r = 1, 2 \dots 7$, which transform parameters separated by r links. Thus,

$$K_{c1}^7(\alpha \beta \gamma) = K_5 \times K_6 \times K_4 \times K_2 \times K_3 \times K_7 \times K_4(\alpha \beta \gamma) = (20\alpha + 8\beta - 20\gamma - 180 \mid 20\alpha + 8\beta - 20\gamma - 178 \mid 24\alpha - 16\beta - 24\gamma - 71), K_{c1}^7(861) = 861.$$

Any operator $K_{ci}^s(\alpha_{ci})$ built with $s > 7$ will coincide with other $K_{ci}^t(\alpha_{ci})$ $1 \leq t \leq 7$ if and only if $s \equiv t \pmod{7}$. **The existence of the cycle demands the coupling between functions K_i .** The image of one function must belong to the domain of existence of the following one.

In numerical terms, the cycle consists of the following numbers

$$n_{c1} = 840,852, n_{c2} = 860,832, n_{c3} = 862,632, n_{c4} = 642,654, n_{c5} = 420,876, n_{c6} = 851,742,$$

$$n_{c7} = 750,843, \text{ being } p(n_{ci}) = \alpha_{ci}.$$

In A_2 , there is a five-link cycle. In A_3 and A_4 , there is only one articulated on a Kaprekar's constant. In A_5 , there are two four-link cycles and one two-link cycle.

In A_7 , consisting of seven-digit numbers, all 219 existing parametric classes $(\alpha \beta \gamma)$ group with the following cycle formed by eight links:

7. Symmetries

7.1. Equivalence Relations

Let two numbers m and n be equivalent R_r of order r if and only if

$$m R_r n \leftrightarrow K^r(m) = K^r(n) \quad r \geq 0 \quad (32)$$

Since $K^r(m) = K^{r-1}[K(m)]$, the following recurrence equations are found:

$$m R_r n \leftrightarrow K(m) R_{r-1} K(n) \quad r \geq 1 \quad (33)$$

$$m R_r n \leftrightarrow K^s(m) R_{r-s} K^s(n) \quad s \leq r \quad (34)$$

Note that if two numbers are equivalent, they are so for any equivalence of a higher order:

$$m R_r n \rightarrow m R_{r+s} n \quad s \geq 0 \quad (35)$$

It is necessary to distinguish **new equivalences R_r** , those which are not of a lower order:

$$m R_r n \rightarrow m R_{r-1} n \quad r > 0 \quad (36)$$

$$m R_r n \rightarrow m R_{r-s} n \quad s = 1, 2 \dots r \quad (37)$$

from **old** equivalences ${}_aR_r$, those that are also of a lower order:

$$m ({}_aR_r) n \rightarrow m R_{r-1} n \quad r > 0 \quad (38)$$

The notation R_r does not specify whether an equivalence is new or old.

Equivalence of an order are binary equivalence relations that have the following features:

$$\text{reflexivity } m R_r m \quad \forall m \in A_w$$

$$\text{symmetry } m R_r n \leftrightarrow n R_r m \quad \forall m, n$$

$$\text{transitivity } m R_r n \text{ and } n R_r q \rightarrow m R_r q \quad \forall m, n, q$$

They allow for numbers to be grouped in equivalence classes of order r :

$$C_r = \{m, n \in A_w; m R_r n\} \quad (39)$$

These equivalence classes are pairwise disjoint, and their intersection is set A_w , generating a partition in A_w .

Because two numbers that are equivalent are so for any equivalence of a higher order (35), classes C_r include both the new and old equivalences. This hierarchical and inclusive feature of binary relations R_r is essential to better understand the convergence of this process.

$$C_r = \{m, n \in A_w, m R_r n\}, {}_a C_r = \{p, q \in A_w, p R_{r-1} q\} \quad C_r = C_r \cup {}_a C_r \quad (40)$$

The transitive feature and relation (35) yield

$$m R_r n \text{ and } n R_{r+s} q \rightarrow m R_{r+s} q \quad s \geq 0 \quad (41)$$

In our methodological approach to the analysis of the Kaprekar process, it is essential to extend the concept of equivalence R_r defined between numbers to sets of numbers with images in common, to equivalence classes.

Considering the last relationship, if

$$m \in C_r^i, p \in C_s^j \text{ and } m R_t p, t = \max\{r, s\} + u, u \geq 0 \text{ then if}$$

$n \in C_r^i$ and $q \in C_s^j$ results $n R_t q$. This is due to the univocal character of the transformation. All numbers belonging to a class C_v have the same image in the v -th transformation. From here on, any subsequent transformation will have the same image. Consequently, the element that best represents the transformation is not the number, but the equivalence class. Thus, we can agree to define the equivalence relation between classes according to

$$m \in C_r^i, n \in C_s^j \text{ and } m R_t p, t = \max\{r, s\} + u, u \geq 0 \leftrightarrow C_r^i R_t C_s^j \quad (42)$$

The direction of the arrow to the left must be understood in the sense that if two classes C_r^i y C_s^j have a relation R_t , then all numbers m and n belonging to those classes have a relation R_t . In particular, it turns out

$$m \in C_r^i, n \in C_s^j, m R_{r+s} n \leftrightarrow C_r^i R_{r+s} C_s^j \quad s \geq 0 \quad (43)$$

From relation (14), it follows that the C_1 equivalence classes are defined by the parametric classes (we will return to this topic in Section 7.3). α represents the parameters of a number $\alpha = (\alpha^1 \alpha^2 \dots \alpha^h) = p(n)$ and also the set of numbers that has the same parameters. As what characterizes a class C_1 are precisely these parameters, we can agree to specify each C_1 by its corresponding parameters, being able to write $C_1^i = \alpha_i$, if in the context there is no room for confusion. With this convention α has a double meaning: parameters of a number and a set of numbers with the parameters that are specified.

By extension, we say that two parametric classes α_1 and α_2 are equivalent to R_r if and only if the numbers with the respective parameters are equivalent:

$$\alpha_1 R_r \alpha_2 \leftrightarrow m R_r n, p(m) = \alpha_1, p(n) = \alpha_2 \quad (44)$$

Expression (44) allows us to translate some of the previous relationships into a parametric class language. For example, from (43) it results

$$\alpha_1 \subset C_r^i, \alpha_2 \subset C_r^j, \alpha_1 R_{r+s} \alpha_2 \rightarrow C_r^i R_{r+s} C_r^j, s \geq 0 \quad (45)$$

Going back to expression (14) and to definition (32), it results

$$m R_r n \leftrightarrow p \left[K^{r-1}(m) \right] = p \left[K^{r-1}(n) \right] \quad r \geq 1 \quad (46)$$

To any equivalence R_r can be associated a function or operator e_r such that

$$\alpha_1 R_r \alpha_2 \leftrightarrow e_r(\alpha_1) = \alpha_2 \quad (47)$$

with the domain of existence of R_r .

It is a kind of representation that facilitates notation of calculations.

7.2. Product of Equivalences

Applying an equivalence onto another one normally yields another equivalence. By using the usual representation of the product of functions

$$e^i [e^j(\alpha)] = (e^i \times e^j)(\alpha) = e^k(\alpha)$$

the domains of existence will have to be compatible. Thus, taking equivalences that will be later deduced

$$e_2^2(\alpha \beta) = (10 - \alpha \beta), \alpha + \beta \leq 10; e_2^3(\alpha \beta) = (\alpha 10 - \beta), \alpha + \beta \geq 10$$

$$e_2^4(\alpha \beta) = (e_2^3 \times e_2^2)(\alpha \beta) = (10 - \alpha 10 - \beta), \alpha = \beta$$

However,

$$(e_2^2 \times e_2^4)(\alpha \beta) = (\alpha 10 - \beta) = e_2^3, \text{ where domains } \alpha = \beta \text{ and } \alpha + \beta \leq 10 \text{ lead to } \alpha + \beta \geq 10.$$

Moreover, the product is conditioned by restriction (3). Therefore, if $\alpha = (\alpha \beta \gamma)$, $\alpha \geq \beta \geq \gamma$

$$e_2^2(\alpha) = (\alpha 9 - \beta \gamma), \alpha + \beta \geq 9, \beta + \gamma \leq 9; e_2^3(\alpha) = (\alpha \beta 10 - \gamma), \beta + \gamma \geq 10$$

$e_2^6(\alpha) = (e_2^3 \times e_2^2)(\alpha) = (\alpha 9 - \beta 10 - \gamma)$ not valid because it demands $\gamma \geq \beta + 1$, which contradicts (3).

Additionally, the other way around, the product of non-allowed equivalences can generate an allowed one. Thus,

$$e_2^7(\alpha) = (10 - \alpha 9 - \beta 10 - \gamma), \text{ not valid}$$

$$(e_2^6 \times e_2^7)(\alpha) = (10 - \alpha \beta \gamma) = e_2^1(\alpha) \text{ valid if } \alpha + \beta \leq 10$$

In summary, the product of equivalences demands that the resulting equivalence's domain of existence be determined.

7.3. Symmetries R_0 and R_1

Symmetry R_0 has been conventionally defined as the non-transformation: $m R_0 n \leftrightarrow K^0(m) = K^0(n) \leftrightarrow m = n$. Each equivalence class C_0 consists of a single number. There are as many classes C_0 as there are numbers belonging to A_W .

Due to (14), symmetry R_1 is the one containing those numbers with the same parameters which return the same number after one transformation. Therefore, parametric classes,

understood as the set of numbers with the same parameters, match with equivalence classes C_1 . Scheme 1 shows all these classes and their transformations in the case of A_6 .

As these classes uniquely determine the image of any number belonging to them, they are an essential element when understanding Kaprekar's routine.

7.4. Symmetries R_2

R_2 symmetries are those that exist between numbers whose transformed seconds coincide.

As a result of R_r definition (32)

$$m R_2 n \leftrightarrow K^2(m) = K^2(n) \text{ and because of (46)}$$

$$m R_2 n \leftrightarrow p[K(m)] = p[K(n)] \quad (48)$$

7.4.1. Permutations of the Same Sequence

If two numbers m and n , $p(m) = \alpha_1$, $p(n) = \alpha_2$ return images with transpositions of digits compatible with belonging to B_w , the ordered sequences will be identical $O_d(m') = O_d(n')$ and their parameters will match $\alpha'_1 = \alpha'_2$, as will their numeric images

$$m'' = f(\alpha'_1) = f(\alpha'_2) = n'', \text{ thus verifying the condition of } m R_2 n.$$

Not just any transposition is valid. Only those that respect the conditions (5) or (7) of belonging to B_w are valid.

R_2 equivalences based on (4a) and (4b)

$$m' = f_1(\alpha_1) = (\alpha_1^1 \alpha_1^2 \dots \alpha_1^{h-1} \alpha_1^h - 1 \ 9 - \alpha_1^h \dots 9 - \alpha_1^2 \ 10 - \alpha_1^1), p(m) = \alpha_1 = (\alpha_1^1 \ \alpha_1^2 \dots \alpha_1^h)$$

$$\text{If } p(n) = \alpha_2 = (10 - \alpha_1^1 \ \alpha_1^2 \dots \alpha_1^h) \ n' = f_1(\alpha_2) = (10 - \alpha_1^1 \dots \alpha_1^2 \dots 9 - \alpha_1^2 \ \alpha_1^1)$$

$n' \in B_w$ because it meets the conditions (5) or (7) and the sequence of n' has the same digits as that of m' , with the ends being translocated. Consequently,

$$O_d(m') = O_d(n') \text{ and } p(m') = p(n') \leftrightarrow \alpha_1 R_2 \alpha_2$$

We used f_1 corresponding to $w = 2$, (4). The result for $w = 2 + 1$ does not change, since including a central 9 in the sequences of m' and n' does not modify the reasoning. Likewise, the symmetric digits can be transposed in relation to the middle ones, α^s and $9 - \alpha^s$, $s = 2, 3 \dots h-1$, or the middle digits themselves, α^{h-1} and $9 - \alpha^h$, returning the following binary equivalences by simple transposition, where $\alpha_1 = (\alpha^1 \ \alpha^2 \dots \alpha^h)$.

- **Equivalences $\alpha_1 R_2^{11} \alpha_2$, $\alpha_2 = (10 - \alpha^1 \ \alpha^2 \dots \alpha^h)$, $\alpha^1 + \alpha^2 \leq 10$**
- **Equivalences $\alpha_1 R_2^{12} \alpha_2$, $\alpha_2 = (\alpha^1 \ \alpha^2 \dots \alpha^{s-1} \ 9 - \alpha^s \ \alpha^{s+1} \dots \alpha^h)$, $s = 2, \dots h-1$**

$$\alpha^{s-1} + \alpha^s \geq 9, \alpha^s + \alpha^{s+1} \leq 9$$

For example: $w = 11$, $\alpha_1 = 76,632$, $\alpha_2 = 76,332$

$$f_1(\alpha_1) = 76,631,976,333, f_1(\alpha_2) = 76,331,976,633 \text{ being}$$

$f_1(\alpha_2)$ the permutation $P = (1 \ 2 \ 9 \ 4 \ 5 \ 6 \ 7 \ 8 \ 3 \ 10 \ 11)$ of $f_1(\alpha_1)$ and $K(\alpha_1) = K(\alpha_2) = 84,433$ consistent with (48). Besides, $f_1(8 \ 4 \ 4 \ 3 \ 3) = (8 \ 4 \ 4 \ 3 \ 2 \ 9 \ 6 \ 6 \ 5 \ 2) =$

$$= K(7 \ 6 \ 6 \ 3 \ 1 \ 9 \ 7 \ 6 \ 3 \ 3) = K(7 \ 6 \ 3 \ 3 \ 1 \ 9 \ 7 \ 6 \ 6 \ 3 \ 3).$$

- **Equivalences $\alpha_1 R_2^{13} \alpha_2$, $\alpha_2 = (\alpha^1 \ \alpha^2 \dots \alpha^{h-1} \ 10 - \alpha^h)$, $\alpha^{h-1} + \alpha^h \geq 10$**

However, multiple transpositions are limited by (3):

- **Non-equivalence $\alpha_1 n_2^1 \alpha_2$**

$$\alpha_2 = \left(\alpha^1 \dots \alpha^{s-1} 9 - \alpha^s \alpha^{s+1} \dots \alpha^{s+r-1} 9 - \alpha^{s+r} \alpha^{s+r+1} \dots \alpha^h \right) \text{ impossible, } \alpha_1 R_2 \alpha_2$$

$$\alpha^{s+r-1} \leq 9 - \alpha^s \text{ and } \alpha^{s+r-1} \geq 9 - \alpha^{s+r}$$

$$\alpha^{s+r} \geq 9 - \alpha^{s+r-1} \geq \alpha^s \rightarrow \alpha^{s+r} = \alpha^s = \alpha^{s+i}, i = 0, 1, \dots, r$$

$$9 - \alpha^s \geq \alpha^s \rightarrow \alpha^s \leq 4 \rightarrow \alpha^{s+r-1} = \alpha^{s+r} \leq 4$$

$\alpha^{s+r-1} \geq 9 - 4 \geq 5$, which contradicts the previous expression. Therefore,
 $\alpha_1 R_2 \alpha_2 \leftrightarrow \alpha_1 n_2^1 \alpha_2$

– **Non-equivalence $\alpha_1 n_2^2 \alpha_2$**

$$\alpha_2 = \left(\alpha^1 \dots \alpha^{s-1} 9 - \alpha^s \alpha^{s+1} \dots 10 - \alpha^h \right) \text{ impossible, } \alpha_1 R_2 \alpha_2$$

$$\alpha^{h-1} \leq 9 - \alpha^s \text{ and } \alpha^{h-1} \geq 10 - \alpha^h$$

$$\alpha^h \geq 10 - \alpha^{h-1} \geq 10 - (9 - \alpha^s) = 1 + \alpha^s, \text{ impossible, since } \alpha^h \leq \alpha^s$$

$$\alpha_1 R_2 \alpha_2 \leftrightarrow \alpha_1 n_2^2 \alpha_2$$

– **Equivalence $\alpha_1 R_2^h \alpha_2$, $\alpha_2 = (10 - \alpha^1 \alpha^2 \dots \alpha^{h-1} 10 - \alpha^h) \rightarrow \alpha_2 = (5 \dots 5) = \alpha_1$**

It is a specific identity solution:

$$\left. \begin{aligned} \alpha^{h-1} &\leq 10 - \alpha^1 \text{ y } \alpha^{h-1} \geq 10 - \alpha^h \rightarrow \alpha^h \geq \alpha^1 \rightarrow \alpha^h = \alpha^1 = \alpha^s, s = 1, \dots, h \\ 10 - \alpha^1 &\geq \alpha^1 \rightarrow \alpha^1 \leq 5 \rightarrow \alpha^{h-1} = \alpha^h \leq 5 \\ \alpha^{h-1} &= 5 \rightarrow \alpha^s = 5, s = 1 \dots h \\ \alpha^{h-1} &\geq 10 - \alpha^h \geq 10 - 5 = 5 \end{aligned} \right\}$$

– **Equivalence $\alpha_1 R_2^{15} \alpha_2$, $\alpha_2 = (\alpha^1 \dots \alpha^{s-1} 9 - \alpha^s \dots 9 - \alpha^{s+r} \alpha^{s+r+1} \alpha^h)$**

According to (3), $9 - \alpha^s \geq 9 - \alpha^{s+r} \rightarrow \alpha^{s+r} = \alpha^s$

Domain of existence:

$$\alpha^s \geq 9 - \alpha^{s-1}, \alpha^{s+r+1} \leq 9 - \alpha^{s+r}, \alpha^{s+i} = \alpha^s, i = 0, 1 \dots r+1, 2 \leq s < h$$

For example: $w = 10$:

$\alpha_1 = 75,552$ and $\alpha_2 = 74,442$, $f_1(\alpha_1) = 7,555,174,443$, $f_1(\alpha_2) = 7,444,175,553$, where $f_1(\alpha_2)$ is the permutation $P = (1 \ 9 \ 8 \ 7 \ 5 \ 6 \ 4 \ 3 \ 2 \ 10)$ of $f_1(\alpha_1)$ and $K(\alpha_1) = K(\alpha_2) = 64,111$ according to (48).

– **Equivalence $\alpha_1 R_2^{16} \alpha_2$, $\alpha_2 = (10 - \alpha^1 \alpha^2 9 - \alpha^3 \dots 9 - \alpha^r \alpha^{r+1} \alpha^h)$**

$$9 - \alpha^3 \leq \alpha^2 \leq 10 - \alpha^1 \rightarrow \alpha^2 \leq \alpha^3 \geq \alpha^1 - 1 \rightarrow \alpha^1 = \alpha^2 = \alpha^3 = 5$$

Domain of existence:

$$\alpha^s = 5, s = 1, 2 \dots r, \alpha^{r+1} \leq 9 - \alpha^r$$

For example, $\alpha_1 = 555,532$, $\alpha_2 = 554,432$, $w = 12$:

$$f_1(\alpha_1) = 555,531,764,445, f_1(\alpha_2) = 554,431,765,545$$

where $f_1(\alpha_2)$ is a permutation $P = (1 \ 2 \ 10 \ 9 \ 5 \ 6 \ 7 \ 8 \ 4 \ 3 \ 11 \ 12)$ of $f_1(\alpha_1)$ and $K(\alpha_1) = K(\alpha_2) = 631,110$.

– **Equivalence $\alpha_1 R_2^{17} \alpha_2$, $\alpha_2 = (10 - \alpha^1 9 - \alpha^2 \dots 9 - \alpha^r \alpha^{r+1} \alpha^h)$**

$$\alpha^{1-1} \leq \alpha^2 \leq \alpha^1, \alpha^s = \alpha^2, s = 2, 3, \dots, r$$

For example: $w = 9$, $\alpha_1 = 8772$, $\alpha_2 = 2222$:

$f_1(\alpha_1) = 877,197,222$, $f_1(\alpha_2) = 222,197,778$, where $f_1(\alpha_2)$ is the permutation $P = (9\ 8\ 74\ 5\ 6\ 3\ 2\ 1)$ of $f_1(\alpha_1)$ and $K(\alpha_1) = K(\alpha_2) = 8655$.

R₂ based on (8) or (10)

$$\alpha = (\alpha^1 \dots \alpha^r \overset{h-r}{0} \dots \dots 0) \text{ and functions } f_2(\alpha) \text{ (8) or (10) operate.}$$

If we consider α^r as α^h , the previous equivalences are valid. For example,

$$\alpha_1 R_2^2 (\alpha^1 \dots \alpha^{r-1} \overset{h-r}{10 - \alpha^r \overset{h-r}{0} \dots \dots 0}) \alpha_1 = 97,400 R_2 97,600 = \alpha_2,$$

since $K(\alpha_1) = K(\alpha_2) = 87,642$.

R₂ based on (12)

$$\alpha = (\alpha^1 \overset{h-1}{0} \dots \dots 0) \text{ and the function } f_3(\alpha) = (\alpha^1 - 1 \overset{w-2}{9} \dots \dots 9 \overset{w-2}{10 - \alpha^1}) \text{ operates.}$$

Only the transposition of extreme digits is possible. That is, if

$$\alpha_1 = (\alpha^1 \overset{h-1}{0} \dots \dots 0) \text{ and } \alpha_2 = (11 - \alpha^1 \overset{h-1}{0} \dots \dots 0) \text{ then}$$

$$f_3(\alpha_1) = (\alpha^1 - 1 \overset{w-2}{9} \dots \dots 9 \overset{w-2}{10 - \alpha^1}), f_3(\alpha_2) = (10 - \alpha^1 \overset{w-2}{9} \dots \dots 9 \overset{w-2}{\alpha^1 - 1})$$

which is a permutation of the same digits compatible with (13), so

$$O_d[f_3(\alpha_1)] = O_d[f_3(\alpha_2)] \leftrightarrow \alpha'_1 = K(\alpha_1) = \alpha'_2 = K(\alpha_2) \leftrightarrow \alpha_1 R_2 \alpha_2.$$

$$- \text{Equivalence } (\alpha \overset{h-1}{0} \dots \dots 0) R_2^3 (11 - \alpha \overset{h-1}{0} \dots \dots 0), \alpha \geq 2$$

Example: $\alpha_1 = 300$, $\alpha_2 = 800$, $\alpha_1 R_2 \alpha_2$ since $K(\alpha_1) = K(\alpha_2) = 720$.

7.4.2. Permutation of Different Sequences

There are equivalences where the digits from the sequence do not necessarily remain.

$$- \text{Equivalence } \alpha_1 R_2^4 \alpha_2$$

$$\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h), \alpha_2 = (10 - \alpha^h \ 9 - \alpha^{h-1} \ 9 - \alpha^{h-2} \dots \ 9 - \alpha^3 \ 9 - \alpha^2 \ 10 - \alpha^1),$$

$$\alpha^1 \geq \alpha^2 + 1, w = \dot{2}, h = w/2$$

Under (15) $K_1(\alpha_1) = (2 \ \alpha_1^1 - 10 \ 2 \ \alpha_1^2 - 9 \dots \ 2 \ \alpha_1^{h-1} - 9 \ 2 \ \alpha_1^h - 10)$.

Under (17) $K_2(\alpha_2) = (10 - 2 \ \alpha_2^h \ 9 - 2 \ \alpha_2^{h-1} \dots \ 9 - 2 \ \alpha_2^2 \ 10 - 2 \ \alpha_2^1)$.

If $K_1(\alpha_1) = K_2(\alpha_2) \rightarrow \alpha_1^1 + \alpha_2^h = 10, \alpha_1^{s+1} + \alpha_2^{h-s} = 9, s = 1, 2, \dots, h-2$.

Note that $\alpha_1 + O_u(\alpha_2) = (10 \ 9 \ \dots \ 9 \ 10)$.

For example,

$$m = 8,178,382,562, p(m) = \alpha_1 = 76,641, m' = f_1(\alpha_1) = 7,664,085,333$$

$$n = 4,774,473,809, p(n) = \alpha_2 = 95,333, n' = f_1(\alpha_2) = 9,533,266,641$$

α_1 and α_2 verify the equivalence, and $f_1(\alpha)$ corresponds to (4). m' and n' do not keep the same digits, but $p(m') = p(n') = 84,331$ and $K^2(m) = K^2(n) = 8,433,086,652$. This answers the second question in the introduction.

This equivalence, contrary to what happens with the previous ones, is not valid for odd w . In the previous example, $w = 10$. Let $w = 11$ and keep the same parameters $p(m) = \alpha_1 = 76,641, p(n) = \alpha_2 = 95,333$.

Now, the function f_1 that must be used is the one corresponding to (6):

$$m' = f_1(\alpha_1) = 76,640,985,333, n' = f_1(\alpha_2) = 95,332,966,641$$

which are similar to the previous ones but with a 9 in the middle position:

$$p(m') = 95,432, p(n') = 87,332.$$

Adding a 9 in the sequence of digits makes symmetry break.

This equivalence may generate permutations of digits from the same sequence. Such is the case of $\alpha_1 = 87,662$ and $\alpha_2 = 83,322, \alpha_1 R_2^4 \alpha_2$.

$$m' = f_1(\alpha_1) = 8,766,173,322, n' = f_1(\alpha_2) = 8,332,177,662$$

being $O_d(m') = O_d(n')$. It is just a numeric coincidence, even if for these parameters R_2^4 does not match any of the previous equivalences.

R_2^4 is a very common equivalence given the few restrictions of its domain of existence.

The product of equivalences $R_2^1 \times R_2^4$ creates a family of derivative equivalences. Table 3 shows the equivalences R_2 common in A_6 and A_7 , and Table 4 shows derivative equivalences in A_6 . The superscripts in the equivalences are simplified by using correlative natural numbers. n_2^6 corresponds to a double translocation and n_2^7 to a triple one, but none of them are an equivalence.

Table 3. Equivalences R_2 with three parameters, $\alpha = (\alpha \beta \gamma)$.

Equivalences R_2 Common to A_6 and A_7		
Group	$e_2^i(\alpha_1) = \alpha_2$	Domain of Existence
Set I Based on (4a) or (4b) $\alpha = (\alpha \beta \gamma)$ $1 \leq \alpha \leq 9$ $1 \leq \beta \leq 9$ $1 \leq \gamma \leq 9$	$e_2^0(\alpha) = (\alpha \beta \gamma)$	$1 \leq \alpha \leq 9, 1 \leq \beta \leq 9, 1 \leq \gamma \leq 9$
	$e_2^1(\alpha) = (10 - \alpha \beta \gamma)$	$\alpha + \beta \leq 10$
	$e_2^2(\alpha) = (\alpha \ 9 - \beta \ \gamma)$	$\alpha + \beta \geq 9, \beta + \gamma \leq 9$
	$e_2^3(\alpha) = (\alpha \ \beta \ 10 - \gamma)$	$\beta + \gamma \geq 10$
	$e_2^4(\alpha) = (10 - \alpha \ 9 - \beta \ \gamma)$	$\alpha \leq \beta + 1, \beta + \gamma \leq 9$
	$e_2^5(\alpha) = (10 - \alpha \ \beta \ 10 - \gamma)$	$\alpha = \beta = \gamma = 5$
	$n_2^6(\alpha) = (\alpha \ 9 - \beta \ 10 - \gamma)$	$\alpha + \beta \geq 9, \gamma \geq \beta + 1 \rightarrow \text{Impossible}$
	$n_2^7(\alpha) = (10 - \alpha \ 9 - \beta \ 10 - \gamma)$	$\alpha \leq \beta + 1, \gamma \geq \beta + 1 \rightarrow \text{Impossible}$

Table 3. Cont.

Equivalences R_2 Common to A_6 and A_7		
Group II Based on (5a) or (5b) $\alpha = (\alpha \beta 0)$	$e_2^0(\alpha) = (\alpha \beta 0)$	$1 \leq \alpha \leq 9, 1 \leq \beta \leq 9, \gamma = 0$
	$e_2^8(\alpha) = (10 - \alpha \beta 0)$	$\alpha + \beta \leq 10$
	$e_2^9(\alpha) = (\alpha 10 - \beta 0)$	$\alpha + \beta \geq 10$
	$e_2^{10}(\alpha) = (10 - \alpha 10 - \beta 0)$	$\alpha = \beta$
Group III Based on (6) $\alpha = (\alpha 0 0)$	$e_2^0(\alpha) = (\alpha 0 0)$	$1 \leq \alpha \leq 9$
	$e_2^{11}(\alpha) = (11 - \alpha 0 0)$	$2 \leq \alpha \leq 9$
Equivalences R_2 in A_6		
Group IV $\alpha = (\alpha \beta \gamma)$	$e_2^0(\alpha) = (\alpha \beta \gamma)$	$1 \leq \alpha \leq 9, 1 \leq \beta \leq 9, 1 \leq \gamma \leq 9$
	$e_2^{12}(\alpha) = (10 - \gamma 9 - \beta 10 - \alpha)$	$1 \leq \alpha \leq 9, 1 \leq \beta \leq 8, 1 \leq \gamma \leq 8, \alpha \geq \beta + 1$

Table 4. Derivative equivalences in A_6 , $\alpha = (\alpha \beta \gamma)$.

Equivalence	Domain of Existence	Example
$(e_2^1 \times e_2^{12})(\alpha) = (\gamma 9 - \beta 10 - \alpha)$	$\alpha \geq \beta + 1, \beta + \gamma \geq 9$	963 R_2 331
$(e_2^2 \times e_2^{12})(\alpha) = (10 - \gamma \beta 10 - \alpha)$	$\alpha + \beta \geq 10, \beta + \gamma \leq 10$	963 R_2 761
$(e_2^3 \times e_2^{12})(\alpha) = (10 - \gamma 9 - \beta \alpha)$	$\alpha + \beta \leq 9$	221 R_2 972
$(e_2^4 \times e_2^{12})(\alpha) = (\gamma \beta 10 - \alpha)$	$\alpha + \beta \geq 10, \beta = \gamma$	933 R_2 331
$(n_2^6 \times e_2^{12})(\alpha) = (10 - \gamma \beta \alpha)$	$\alpha = \beta, \beta + \gamma \leq 10$	772 R_2 877
$(n_2^7 \times e_2^{12})(\alpha) = (\gamma \beta \alpha)$	$\alpha = \beta = \gamma$	333 R_2 333

Some of these derivative equivalences are not reciprocal, but

$\alpha_1 R_2^i \alpha_2 \rightarrow \alpha_2 R_2 \alpha_1$, because $K(\alpha_1) = K(\alpha_2)$. Thus, there is always another R_2^j such that $\alpha_2 R_2^j \alpha_1$.

- **Equivalence $\alpha_1 R_2^{51} \alpha_2$, $\alpha_1 = (\alpha 0 0)$, $\alpha_2 = [(\alpha + 9)/2 (19 - \alpha)/2 5]$, $\alpha = 5, 7, 9$, $w = 2$**

$$h - 2$$

Indeed, under (30), $K_{31}(\alpha) = (\alpha_1^1 - 1 10 - \alpha_1^1 0 \dots 0)$ under (15) with (16),

$$K_1(\alpha_2^1 \alpha_2^2 \dots \alpha_2^h) = (2 \alpha_2^1 - 10 2 \alpha_2^2 - 9 \dots 2 \alpha_2^{h-1} - 9 2 \alpha_2^h - 10).$$

$$K_{31}(\alpha_1) = K_1(\alpha_2) \rightarrow \alpha_1^1 - 1 = 2 \alpha_2^1 - 10; 10 - \alpha_1^1 = 2 \alpha_2^2 - 9$$

$2 \alpha_2^s - 9 = 0, s \geq 3, \dots h - 1; 2 \alpha_2^h - 10 = 0$. These conditions are only possible if $h = 3$, yielding

$$\alpha_2^1 = (\alpha_1^1 + 9)/2, \alpha_2^2 = (19 - \alpha_1^1)/2, \alpha_2^3 = 5, \alpha_1^1 = 5, 7, 9$$

$$500 R_2^{51} 775, 700 R_2^{51} 865, 900 R_2^{51} 955.$$

- **Equivalence $\alpha_1 R_2^{52} \alpha_2$, $\alpha_1 = (\alpha 0 0)$, $\alpha_2 = [(10 - (\alpha/2) 4 + (\alpha/2) 5)]$, $\alpha = 2, 4, 6$, $w = 2$**

This results from using $K_{32}(\alpha_1)$ instead of $K_{31}(\alpha_1)$.

As a consequence, $200 R_2^{52} 955, 400 R_2^{52} 865$ and $600 R_2^{52} 775$.

Note that classes 100, 300 and 800 are excluded from R_2^{52} . Class 100 does not have an equivalent R_2 different from itself and $300 R_2^3 800$.

Equivalences R_2^{51} and R_2^{52} do not operate on numbers with w -digits, $w = 2 + 1$.

- In A_w , $w = 2 + 1$, there are specific equivalences R_2 , unshared with $w = 2$. However, such R_2 vary with w , and thus general expressions do not exist. Table 5 shows all those

existing in A_7 and A_5 . Some of these equivalences, such as e_2^{61} , also generate equivalences R_2 from the same sequence, e.g., 977 R_2 973. Furthermore, if the domains of existence of K_i and K_j have classes $(\alpha \beta \gamma)$ in common, they generate equivalences R_1 .

Table 5. Equivalences R_2 in A_w , $w = \dot{2} + 1$, not found in $w = \dot{2}$.

All Equivalences R_2 Unique to A_7 (1)		
Equivalences R_2	Valid for	Derives from $K_i(\alpha_1) = K_j(\alpha_2)$
$e_2^{61}(\alpha \beta \gamma) = [\alpha (\beta + \gamma) / 2 \ 10 - \beta]$	986 R_2 972, 975 R_2 963, 875 R_2 863	K_1, K_7
$e_2^{62}(\alpha \beta \gamma) = [(\alpha + \beta + 1) / 2 \ \beta + \gamma - \alpha + 2 \ 11 - \alpha]$	986 R_2 972, 985 R_2 962, 966 R_2 852, 875 R_2 863	K_1, K_4
$e_2^{63}(\alpha \beta \gamma) = (\alpha \ 10 - \gamma \ 11 - \beta)$	981 R_2 993, 971 R_2 994, 972 R_2 984, 961 R_2 995, 962 R_2 985, 872 R_2 884, 862 R_2 885, 863 R_2 875, 763 R_2 775	K_4, K_{13}
$e_2^{64}(\alpha \beta \gamma) = (\alpha \ 10 - \gamma \ \beta + 2)$	931 R_2 995, 921 R_2 994, 911 R_2 993, 932 R_2 985, 922 R_2 984, 832 R_2 885, 822 R_2 884, 833 R_2 875, 733 R_2 775	K_5, K_{13}
$e_2^{65}(\alpha \beta 0) = (11 - \beta \ \alpha + \beta - 3 \ 21 - 2\alpha - \beta)$	720 R_2 965, 630 R_2 866	K_{27}, K_1
$e_2^{66}(\alpha \beta 0) = (11 - \beta \ 9 - \alpha / 2 \ 13 - \alpha - \beta)$	630 R_2 864	K_{27}, K_7
$e_2^{67}(\alpha \beta 0) = (\alpha + \beta \ 18 - 2\alpha - \beta \ \beta)$	610 R_2 751, 520 R_2 762	K_{27}, K_6
$e_2^{68}(\alpha \beta 0) = (20 - 2\alpha - \beta \ 10 - \alpha - \beta \ \beta)$	710 R_2 521, 620 R_2 622	K_{27}, K_{11}
$e_2^{69}(\alpha \beta 0) = (11 - \beta \ \alpha + \beta - 3 \ 12 - \alpha)$	730 R_2 875, 720 R_2 965, 630 R_2 866	K_{26}, K_1
$e_2^{70}(\alpha \beta 0) = (\alpha \ 19 - \alpha - \beta \ \alpha + 2\beta - 10)$	930 R_2 975, 840 R_2 876	K_{25}, K_1
$e_2^{71}(\alpha \beta 0) = (\beta + 1 \ \alpha - 4 \ \beta - \alpha + 3)$	880 R_2 943, 770 R_2 833	K_{23}, K_{12}
$e_2^{72}(\alpha \beta 0) = [(\alpha + 9) / 2 \ 20 - 2\beta \ 10 - \beta]$	770 R_2 863	K_{23}, K_4
$e_2^{73}(\alpha \beta 0) = (5 + \beta / 2 \ 21 - \alpha - \beta \ 10 - \alpha)$	960 R_2 862	K_{22}, K_4
All equivalences R_2 unique to A_5 (1)		
$e_2^5(\alpha \beta) = [1/2 (\alpha + \beta + 1) \ 11 - \alpha]$	85 R_2 73, 96 R_2 82	K_1, K_{17} (2)
$e_2^6(\alpha 0) = [1/2 (11 - \alpha) \ \alpha]$	30 R_2 43	K_{26}, K_{14}
$e_2^7(\alpha 0) = (11 - \alpha \ 2\alpha - 3)$	40 R_2 75	K_{26}, K_1

(2) Notation for functions K_i in A_5 is independent to that one in A_7 . The permutations associated with K_i in A_7 are shown in Table 2. The permutations associated with K_i in A_3 are: P_1 (1234), P_{17} (1342), P_{14} (4312) and P_{26} (2 1).

(1) There are no more different new R_2 , except for those resulting from the transitivity (product) between R_2 .

7.4.3. Equivalence Classes C_2

Equivalence classes C_2 , which include numbers whose second transformations match, are an essential element in understanding the architecture of transformation trees. As an analogy, were parametric classes C_1 the leaves of a tree, classes C_2 would be the little branches that group the leaves together. The existence of equivalences R_2 contributes to new branches in the transformation schemes by means of cycles.

In Table 6, equivalence classes C_2 in A_6 are shown. Some classes C_1 do not have equivalent R_2 , such as $555 \in C_2^{13}$, but others have up to 5 equivalents. Such is the case of C_2^{74} : any class 863, 833, 762, 732 and 332 belonging to it will become 643. Additionally, any number belonging to such classes will return the same image (642,654) after two transformations. In this example, $863 = \alpha_{c2}$, $643 = \alpha_{c3}$ and $642,654 = n_{c4}$.

Table 6. Equivalence classes C_2 in A_6 .

Component A. Branch A													
Equivalence Classes	C ₂ ¹	C ₂ ²	C ₂ ³	C ₂ ⁴	C ₂ ⁵	C ₂ ⁶	C ₂ ⁷	C ₂ ⁸	C ₂ ⁹	C ₂ ¹⁰	C ₂ ¹¹	C ₂ ¹²	
Classes C ₁ = (α β γ) included	966 964 661 631 431	100	832 862	655 554 544	955 900 551 541 200	753 743	810 210	965 531	985 511	970 930	870 830	877 873 772 722 322	
Image of C ₂	832	900	753	210	810	531	970	830	870	873	772	654	
α _{ci} and D *	861;14	863;14	863;14	643;14	643;14	643;14	421;14	421;14	421;14	852;14	852;14	751;14	
Component A. Branch A (Continuation)													
C ₂ ¹³	C ₂ ¹⁴	C ₂ ¹⁵	C ₂ ¹⁶	C ₂ ¹⁷	C ₂ ¹⁸	C ₂ ¹⁹	C ₂ ²⁰	C ₂ ²¹	C ₂ ²²	C ₂ ²³	C ₂ ²⁴		
555	990 910 110	710 310	654 644 444	996 994	981 911 111	961 931	987 983 711 811 311	988 982 881 721 211	977 973 771 721 321	876 874 622 422	854 844 652 642 442		
111	981	961	311	982	977	973	874	876	854	652	621		
841;14	841;14	841;14	841;14	841;14	861;7	861;7	861;7	861;7	863;7	863;7	643;7		
Component A. Branch B													
C ₂ ²⁵	C ₂ ²⁶	C ₂ ²⁷	C ₂ ²⁸	C ₂ ²⁹	C ₂ ³⁰	C ₂ ³¹	C ₂ ³²	C ₂ ³³	C ₂ ³⁴	C ₂ ³⁵	C ₂ ³⁶	C ₂ ³⁷	C ₂ ³⁸
995	980 920	888 882	770 730 330	666 664	775 600 500	963 933 761 731 331	540	665	800 300	853 843 752 742	660 640 440	720 320	954 944 651 641 441
980	882	775	761	331	540	843	640	320	720	641	651	860	821
751;14	841;14	861;14	863;14	863;14	863;14	643;14	643;14	643;14	643;14	421;14	421;14	421;14	852;14
Component A. Branch B (Continuation)													
C ₂ ³⁹	C ₂ ⁴⁰	C ₂ ⁴¹	C ₂ ⁴²	C ₂ ⁴³	C ₂ ⁴⁴	C ₂ ⁴⁵	C ₂ ⁴⁶	C ₂ ⁴⁷	C ₂ ⁴⁸	C ₂ ⁴⁹	C ₂ ⁵⁰	C ₂ ⁵¹	
880 820 220	860 840	951 941	972 922 871 821 221	763 733 333	971 921	865 700 532 400	765 533	975 521	630 430	850	750	754 744 653 643 443	
871	763	971	865	533	975	630	430	850	750	754	653	421	
852;14	852;14	751;7	751;7	751;7	841;7	841;7	841;7	861;7	861;7	863;7	863;7	643;7	
Component A. Branch C													
Equivalence classes	C ₂ ⁵²	C ₂ ⁵³	C ₂ ⁵⁴	C ₂ ⁵⁵	C ₂ ⁵⁶	C ₂ ⁵⁷	C ₂ ⁵⁸	C ₂ ⁵⁹	C ₂ ⁶⁰	C ₂ ⁶¹	C ₂ ⁶²	C ₂ ⁶³	C ₂ ⁶⁴
Classes C ₁ = (α β γ) included	999 991	998 992	997 993	875 522	986 984 611 411	776 774	650	950	777 773	872 822 222	542 552 855	755 553 543	610 410
Image of C ₂	997	986	984	650	872	542	552	855	553	755	610	410	952
α _{ci} and D	861;7	863;7	863;7	643;7	643;7	421;7	421;7	421;7	421;7	421;7	852;7	852;7	751;7
Component A. Cycle zone													
C ₂ ⁶⁵	C ₂ ⁶⁶	C ₂ ⁶⁷	C ₂ ⁶⁸	C ₂ ⁶⁹	C ₂ ⁷⁰	C ₂ ⁷¹	C ₂ ⁷²	C ₂ ⁷³	C ₂ ⁷⁴	C ₂ ⁷⁵			
520	530	510	620 420	886 884	863 833 762 732 332	976 974 621 421	852 842	953 943 751 741	952 942 851 841	962 932 861 831			
842	741	943	851	762	643	852	751	841	861	863			
421;7	852;7	852;7	751;7	861;7	863;7	421;7	852;7	751;7	841;7	861;7			
Component B						Component C							
C ₂ ¹	C ₂ ²	C ₂ ³	C ₂ ⁴	C ₂ ⁵	C ₂ ⁶	C ₂ ¹							
885	887 883	760 740	766 764 663 633 433	960 940	866 864 662 432	550							
760	764	662	432	864	632	550							
632;3	632;3	632;2	632;2	632;2	632;1	550;0							

* D stands for the number of necessary transformations so that any class C_1 belonging to a given C_2 becomes the stated class.

Scheme 2 shows the transformations of classes C_2 in A_6 . When compared to Scheme 1, the structure of schemes stands out.

Scheme 2. Transformation scheme of equivalence classes C_2 in A_6 .

7.5. Groups of Equivalence R_2

In theory, transpositions in those sequences compatible with the transformation functions should form subgroups of symmetric group S_h of all h -element possible permutations. This should be true for the product of those equivalences in Set I based on (4) or (6), those in Set II based on (8) or (10) and those in Set III based on (12).

However, generally speaking, this is not true for sets I and II. This is due to the fact that as h increases, some multiple transpositions are forbidden by (3). Such is the case of n_2^1 and n_2^2 discussed in Section 7.4.1. Specifically, for A_6 and A_7 , the product of equivalences e_2 and transpositions n_2 of Set I are shown in Table 7. Apart from the general associativity in the products of transpositions, as well as the existence of the neutral element e_2^0 , it is also

worth noting the involutory $e_2^i \times e_2^i = e_2^0 \leftrightarrow (e_2^i) = (e_2^i)^{-1}$ and abelian $e_2^i \times e_2^j = e_2^j \times e_2^i$ features. However, the product is no longer a closed binary relation between equivalences, since the product of some of them returns n_2^6 or n_2^7 , which although being transpositions, are not equivalences.

Table 7. Product of equivalences R_2 in A_6 and A_7 based on the transposition of digits from the same sequence. $(f \times g)(\alpha) = f[g(\alpha)]$. (Equivalences e_2 are defined in Table 4).

Group I								
$f \backslash g$	e_2^0	e_2^1	e_2^2	e_2^3	e_2^4	e_2^5	n_2^6	n_2^7
e_2^0	e_2^0	e_2^1	e_2^2	e_2^3	e_2^4	e_2^5	n_2^6	n_2^7
e_2^1	e_2^1	e_2^0	e_2^4	e_2^5	e_2^2	e_2^3	n_2^7	n_2^6
e_2^2	e_2^2	e_2^4	e_2^0	n_2^6	e_2^1	n_2^7	e_2^3	e_2^5
e_2^3	e_2^3	e_2^5	n_2^6	e_2^0	n_2^7	e_2^1	e_2^2	e_2^4
e_2^4	e_2^4	e_2^2	e_2^1	n_2^7	e_2^0	n_2^6	e_2^5	e_2^3
e_2^5	e_2^5	e_2^3	n_2^7	e_2^1	n_2^6	e_2^0	e_2^4	e_2^2
n_2^6	n_2^6	n_2^7	e_2^3	e_2^2	e_2^5	e_2^4	e_2^0	e_2^1
n_2^7	n_2^7	n_2^6	e_2^5	e_2^4	e_2^3	e_2^2	e_2^1	e_2^0
Group II					Group III			
$f \backslash g$	e_2^0	e_2^8	e_2^9	e_2^{10}	$f \backslash g$	e_2^0	e_2^{11}	
e_2^0	e_2^0	e_2^8	e_2^9	e_2^{10}	e_2^0	e_2^0	e_2^{11}	
e_2^8	e_2^8	e_2^0	e_2^{10}	e_2^9				
e_2^9	e_2^9	e_2^{10}	e_2^0	e_2^8	e_2^{11}	e_2^{11}	e_2^0	
e_2^{10}	e_2^{10}	e_2^9	e_2^8	e_2^0				

As h decreases, the forbidden general multiple transpositions n_2^1 and n_2^2 disappear, and all transpositions generate equivalences. This happens when there are two non-null $h - 2$ parameters—classes $(\alpha_1 \alpha_2 0 \dots 0)$ from Set II and $(\alpha_1 \alpha_2)$ from Set I with $h = 2$. Table 7

(Group II) shows the product of equivalences for A_6 and A_7 . The same is true for the four equivalences of Set I in A_4 and A_5 . The product of the equivalence in these cases has an isomorph group structure of Klein group. This group is characterized because all of its elements coincide with their inverses. In addition, it is a subgroup of the symmetric group S_4 of all four-element possible permutations.

When the number of non-null parameters decreases to 1, a cyclical subgroup isomorphic to $\mathbb{Z}_2$ appears. Such is the case of the general equivalence e_2^3 (Section 7.4.1). Table 7 (Group III) shows the products for A_6 and A_7 . This situation also occurs in A_2 and A_3 .

Naturally, any involutory equivalence, along with the identical, form a group isomorphic to $\mathbb{Z}_2$. Such is the case of e_2^{11} in A_6 , which in this case does not exist in A_7 . However, its derivative functions (Table 4), along with the equivalence itself, do not have a subgroup structure, since the product of the former generates equivalences of Set I.

7.6. Symmetries R_3 and Higher

From (44) and (33) it results

$$\alpha_1 R_r \alpha_2 \leftrightarrow \alpha'_1 R_{r-1} \alpha'_2, \alpha'_1 = K_i(\alpha_1), \alpha'_2 = K_j(\alpha_2) \quad (49)$$

$$\text{Specifically: } \alpha_1 R_3 \alpha_2 \leftrightarrow \alpha'_1 R_2 \alpha'_2 \quad (50)$$

One of the approaches to obtain the algebraic expressions of equivalences R_3 is based on this expression. If an equivalence R_2 is imposed on the images of two classes α_1 and α_2 , $K_i(\alpha_1)$ and $K_j(\alpha_2)$, the result will be an algebraic relation between α_1 and α_2 so that $\alpha_1 R_3 \alpha_2$:

$$(e_2 \times K_i)(\alpha_1) = K_j(\alpha_2) \quad (51)$$

As a few examples, we introduce the following equivalences:

Equivalence $\alpha_1 R_3^1 \alpha_2$, $\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h)$, $\alpha_2 = (15 - \alpha^1 \alpha^2 \dots \alpha^h)$

$$w = 2; 6 \leq \alpha^1 \leq 9; \alpha^1 \geq \alpha^2 + 1, \alpha^1 + \alpha^2 \leq 14; 5 \leq \alpha^s \leq 6, s \geq 2; \\ \text{derives from } e_2^{11} \times K_1(\alpha_1) = K_1(\alpha_2)$$

For example, in A_6 :

$$(\alpha \beta \gamma) R_3^1 (15 - \alpha \beta \gamma)$$

Which provides the following equivalences:

$$\text{Component A : } 955 R_3^1 655, 865 R_3^1 765, 855 R_3^1 755;$$

$$\text{Component B : } 866 R_3^1 766$$

or in terms of classes C_2 , for (45)

$$C_2^5 R_3 C_2^4, C_2^{45} R_3 C_2^{46}, C_2^{65} R_3 C_2^{66}, C_2^6 R_3 C_2^4 \text{ (Component B).}$$

Equivalence $\alpha_1 R_3^2 \alpha_2$, $\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h)$, $\alpha_2 = (\alpha^1 \alpha^2 \dots \alpha^{h-1} 15 - \alpha^h)$

$$w = 2, 6 \leq \alpha^h \leq 9, \alpha^{h-1} + \alpha^h \geq 15, \alpha^1 \geq \alpha^2 + 1$$

derives from $e_2^{13} \times K_1(\alpha_1) = K_1(\alpha_2)$.

For example, in A_6 :

$$(\alpha \beta \gamma) R_3^2 (\alpha \beta 15 - \gamma) \text{ valid for}$$

$$988 R_3^2 987, 987 R_3^2 988 \rightarrow C_2^{21} R_3 C_2^{20}$$

Equivalence $\alpha_1 R_3^3 \alpha_2$, $\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h)$, $\alpha_2 = (\alpha^1 \alpha^2 \dots \alpha^{h-1} 28 - 2\alpha^{h-1} - \alpha^h)$

$$w = 2 + 1, 5 < \alpha^1 \leq \alpha^2 + 1, 5 \leq \alpha^s \leq 8, s > 1; \alpha^s + \alpha^{s+1} \geq 9, \\ s = 1, 2 \dots h-1, 3\alpha^{h-1} + \alpha^h \geq 28 \\ 2\alpha^{h-1} + \alpha^h \leq 23$$

derives from $e_2^{13} \times K_1(\alpha_1) = K_1(\alpha_2)$.

For example, in A_7 :

$$(\alpha \beta \gamma) R_3^3 (\alpha \beta 28 - 2\beta - \gamma) \text{ valid for the new equivalences}$$

$$987 R_3^3 985, 985 R_3^3 987 \text{ and for } R_1 986, 977 \text{ y } 877$$

Equivalence $\alpha_1 R_3^4 \alpha_2$, $\alpha_1 = (\alpha 0 0)$, $\alpha_2 = [1/2 (21 - \alpha) 5 5]$

Equivalence R_3 is often limited to a couple of classes. Such is the case of

$$e_2^8 \times K_{31}(\alpha_1) = K_7(\alpha_2) \text{ in } A_6.$$

$$e_2^8 \times K_{31}(\alpha_1 0 0) = (11 - \alpha_1 10 - \alpha_1 0) = (2\alpha_2 - 10\beta_2 - \gamma_2 + 1\beta_2 - \gamma_2) = K_7(\alpha_2)$$

which entails $\gamma_2 = \beta_2$, $\alpha_1 = 9$, $\alpha_2 = 6$, and since K_7 demands $5 \leq \beta \leq 8$ y $2 \leq \gamma \leq 5 \rightarrow \gamma = \beta = 5$.

The equivalence is reduced to $900 R_3^4 655$.

This situation becomes generalized when increasing the order of the equivalence. Thus, in A_7 :

Equivalence $\alpha_1 R_4^1 \alpha_2$, $\alpha_1 = (\alpha \beta \gamma)$, $\alpha_2 = (\alpha 48-4\alpha-\beta + 2\gamma \gamma)$
 derives from $e_3^3 \times K_4 (\alpha_1) = K_4 (\alpha_2)$, only valid for $981 R_4^1 961$, which answers another of the questions in the introduction:

$$5,068,069 R_4 3,071,934 \text{ because } p(5,068,069) = 981 \text{ and } p(3,071,934) = 961$$

Equivalence $533 R_7^1 621$

Indeed, $K^6 (533) = K^6 (621) = 864$. Or, equivalently, $m = 4,687,437$, $p(m) = 533$, $n = 4,693,554$, $p(n) = (621)$, $K^7(m) = K^7(n) = 8,639,532 \rightarrow m R_7 n$.

However, the algebraic relation between 533 and 621, or between m and n , as in the two previous examples, is not extendible to other classes $\alpha_1 R_7 \alpha_2$.

As order r of equivalence R_r increases, it is less interesting to establish the determining algebraic relation.

7.7. Symmetries and Cycles

The schemes are articulated on the cycles, whether they have several links r or only one (transformation constants).

Equivalence classes C_r group numbers with the same image C_{r-1} . In general, $\alpha_1 R_s \alpha_2 \leftrightarrow \alpha'_1 R_{s-1} \alpha'_2$. For example, classes $C_1 963, 933, 761, 731$ and 331 all belong to C_2^{31} because their image is the same, 843 (Table 6). Analogously, C_4^4 and C_5^5 belong to the same class C_5 because they have the same image C_4^6 (Scheme 3). This relation forces the transformation tree to swallow the leaves and little branches as s increases, and therefore, only the big branches and the trunk remain visible, using an analog language.

The absorption process occurs in the whole scheme as well as in the cycles. The latter act as black holes that swallow those classes converging with them. Each class $C_s(c_i)$ from the cycle groups with classes C_s which have $C_s(c_i + 1)$ as an image from the cycle, resulting in a new class $C_{s+1}(c_j)$. The process continues until $C_s = C_{s+1}$ for all classes C_s in the scheme. This way, we reach the top of new equivalences $R_s = R_u$, where there are only as many classes C_u as there are links r in the cycle, C_u^i , $i = 1, 2, \dots, r$.

Each class C_u^i belongs to the cycle and groups all classes C_s , $s = 0, 1, 2, \dots, u-1$, whose distance (number of transformations) to α_{ci} is a multiple of r . Thus, the number of links in the cycle establishes the distance pattern for the subsequent groups.

For instance, Table 8 shows the seven classes C_u^i resulting from $u = 13$ transformations in component A from A_6 . As stated several times, component A has a cycle of $r = 7$ links. Class $661 \in C_2^1 \subset C_4^1 \subset C_{13}^1$ and $\alpha_{C1} = 861 \in C_2^{75} \subset C_4^{28} \subset C_{13}^1$ (Table 6). Class 661 needs 14 ($2 \times r$) transformations to become 861—the same number of transformations that C_2^1 needs to reach C_2^{75} (Scheme 2) or C_4^1 to reach C_4^{28} (Scheme 3). Class $430 \in C_2^{48}$ only requires $r = 7$ transformations.

Table 8. Inclusion relations $C_2 \subset C_4 \subset C_{13}$ in A_6 .

C_{13}^1 *	C_{13}^2	C_{13}^3	C_{13}^4	C_{13}^5	C_{13}^6	C_{13}^7
C_2	C_4	C_2	C_4	C_2	C_4	C_2
C_2^1	C_4^1	C_2^2	C_4^2	C_2^3	C_4^3	C_2^4
$C_2^{18}, C_2^{19}, C_2^{20}, C_2^{21}$	C_4^{11}	C_2^3	C_4^3	C_2^6	C_4^5	C_2^{35}, C_2^{36}
C_2^{27}	C_4^{15}	C_2^{28}, C_2^{29}	C_4^{16}	C_2^{24}, C_2^{25}	C_4^{37}	C_2^{37}
C_2^{52}	C_4^{29}	C_2^{30}	C_4^{17}	C_2^{31}, C_2^{32}	C_4^{18}	$C_2^{57}, C_2^{58}, C_2^{59}, C_2^{60}, C_2^{61}$
$C_2^{47}, C_2^{48}, C_2^{49}, C_2^{50}$	C_4^{28}	C_2^{53}, C_2^{54}	C_4^{20}	C_2^{33}, C_2^{34}	C_4^{19}	C_2^{65}, C_2^{71}
		$C_2^{22}, C_2^{23}, C_2^{49}, C_2^{50}, C_2^{70}$	C_4^{12}	C_2^{55}	C_4^{31}	
			C_2^{56}	C_4^{32}		

* Class C_{13}^1 includes class α_{ci} . $C_{13+r}^1 = C_{13}^1$, $r > 0$. Classes C_2 and C_4 containing α_{ci} are marked in bold.

Scheme 3. Transformations between equivalence classes C_4 in A_6 . (C_3 are included in component B, while C_1 appear in component C).

8. Discussion

We understand symmetry as invariance during transformation. This has been the aim of this paper, trying to understand why sets of different numbers return identical images, not only with a single transformation, but with any number r of them. To this end, here, binary equivalence relations of order r were used. Two numbers m and n present an equivalence relation $m R_r n$ if and only if $K^r(m) = K^r(n)$. This simple relationship allows the numbers to be grouped into C_r equivalence classes such that all their numbers give the same transform after r transformations. The use of these equivalences together with the complete parameterization of the transformation by means of K_i functions enabled us to find algebraic relations which explain the invariance of transformations.

The first step was to parameterize numbers. Any number n can have some parameters α which determine its image. We established the general functions f that conduct such transformation $f(\alpha) = n'$, n' being the image of n under Kaprekar's routine. These functions are similar to those established by (4). From here, equivalence classes of order 1 arise—those consisting of numbers with equal parameters and give the same transformed. The transformed numbers are multiples of 9 that satisfy some strict requirements. We analyzed this aspect in detail, as it conditions the existence of symmetries.

The second step was to determine the parameters α of the transformed number. Here lies one of the main problems with transformation. The transformed number need not have its digits sorted in ascending or descending order, which is a prerequisite to determine its parameters. The permutation that arranges the digits must therefore be established. However, this is no easy task, since the number of permutations grows factorially as the number of digits increases. Once both the permutation and the domain of existence have been established, α' is automatically defined. We established the procedure to construct the functions $K_i(\alpha) = \alpha'$, each one associated with a permutation. It is a novel contribution.

With these tools, we were able to approach equivalences R_r of a higher order. For this, the simple relation $\alpha_1 R_r \alpha_2 \leftrightarrow \alpha'_1 R_{r-1} \alpha'_2$ can be very useful. Starting from equivalences R_1 , we can then move onto R_2 , R_3 , etc.

We showed that there are some R_2 —relations which must exist between numbers for their second transformation to match—that are universal. Those in sets I, II and III are valid for numbers with both an even number of digits ($w = 2$) and an odd number ($w = 2 + 1$). Others are only valid for $w = 2$. Thus, for example, the numbers whose parameters are $\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h)$ give the same numbers after two transformations as those with parameters $\alpha_2 = (10 - \alpha^h \ 9 - \alpha^{h-1} \ 9 - \alpha^{h-2} \dots \ 9 - \alpha^3 \ 9 - \alpha^2 \ 10 - \alpha^1)$, $a^1 \geq a^2 + 1$. It is a $\alpha_1 R_2 \alpha_2$ valid for $w = 2$ h, that is, for numbers with an even number of digits. It is an involutorial equivalence.

The products of equivalences I and II only have group algebraic structures for low values of w . The equivalences of set I, for $w = 4$ or 5 , while those of set II, for $w = 6$ or 7 . Therefore, the group is isomorphic of the Klein group. The equivalences of set III are universal, thus generating a group isomorphic to $\mathbb{Z}_2$. The reason why there are not any more groups lies in relation (3), which forces the parameters to be arranged. Many permutations violate this arrangement, resulting in transpositions without a domain of existence in the product of equivalences.

Recently, Bajorska-Harapinska et al. [19] showed interest in these algebraic structures. In the last section of their work, they made an attempt to introduce Kaprekar-style transformation in some algebraic structures. As an example, they defined two transformations in the spirit of Kaprekar on symmetric groups S_n .

The new higher equivalence classes demand increasingly narrow domains of existence. Their algebraic relation is often only valid for a few classes C_i . With this, they are less and less interesting as r increases.

The development of functions K_i allowed us to approach another feature of transformations—the relation between cycles and symmetries. Transformation schemes are articulated on cycles. They can be leafy or not. Thus, in A_6 , component A contains 201 classes C_1 and it is articulated on a seven-link cycle. Component B contains 17 classes, and it is articulated on the constant 631,764, whose parameters are $\alpha_E = 632$. Component C only contains class $\alpha_E = 550$, whose numeric constant is 549,945. In A_7 , the 219 existing parametric classes are grouped on an eight-link cycle. How many classes make up a component of a scheme will depend on the structure of functions K_1 , or equivalently, on the compatibility between successive permutations. This compatibility varies with A_w .

Additionally, dependent on each A_w is the coexistence of cycles and constants—in A_2 , there is a single cycle, in A_3 and A_4 , a scheme of one component with a constant (a Kaprekar constant), in A_5 , three cycles, in A_6 , one cycle and two constants, in A_7 , a single cycle, etc.

The structuration of the schemes in equivalence classes follows a grouping pattern mediated by the number of links in the cycle. This number establishes the “meter” of distances for the successive groupings of the equivalence classes.

In short, Kaprekar’s routine raised the issue of the uniqueness of its constants. Prichett et al. [4] showed that there are only two Kaprekar constants in base 10. Additionally, in this paper, we see that there are several unique facts. The basic transformation applied to numbers in base 10 forces not only the transformed number to be a multiple of 9, but also its digits to satisfy some strict requirements (5), (7) . . . . In many cases, such numeric and parametric restrictions (3) lead to uniqueness. The algebraization of Kaprekar’s routine has proven the existence of universal relations as well. Such is the case of some R_2 symmetries and the existence of universal groups, such as the Klein group.

We hope that our contribution will help arouse interest in this and other numeric transformations.

9. Conclusions

The combination of binary equivalence relations and parametric transformation functions has made it possible to configure a methodology capable of identifying and characterizing symmetries. Thus, we found algebraic relationships that determine these symmetries. Of particular interest are the symmetries associated with R_2 equivalences, that is, those

that generate numerical invariance after two transformations. These equivalences, together with R_1 , are the most frequent. We highlight:

- (a) R_2 equivalences corresponding to transformations of digits of the same sequence (type I)

$$\alpha_1 R_2^{1i} \alpha_{i+1}, \alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h) \quad i = 1, 2, 3$$

$$\alpha_2 = (10 - \alpha^1 \alpha^2 \dots \alpha^h), \alpha^1 + \alpha^2 \leq 10$$

$$\alpha_3 = (\alpha^1 \alpha^2 \dots \alpha^{s-1} 9 - \alpha^s \alpha^{s+1} \dots \alpha^h), s = 2, \dots, h-1 \quad \alpha^{s-1} + \alpha^s \geq 9, \alpha^s + \alpha^{s+1} \leq 9$$

$$\alpha_4 = (\alpha^1 \alpha^2 \dots \alpha^{h-1} 10 - \alpha^h), \alpha^{h-1} + \alpha^h \geq 10$$

where w is the number of digits, and $w = 2h + z$ with $z = 0$ if w is even and $z = 1$ if w is odd.

All the previous equivalences correspond to simple transpositions. Only the multiple transpositions indicated in the text are possible and they lead to $R_2^{14}, R_2^{15}, R_2^{16}$ y R_2^{17} .

Type II

If $\alpha_1 = (\alpha^1 \dots \alpha^r 0 \dots 0)$ then the previous equivalences are valid if we consider

α^r as α^h . This gives rise to a family of equivalences R_2^{2i} .

Type III

If $\alpha_1 = (\alpha 0 \dots 0)$ then $\alpha_1 R_2^3 (11 - \alpha 0 \dots 0)$, $\alpha \geq 2$.

All equivalences are general, regardless of the number of digits.

- (b) Equivalences R_2 corresponding to permutation of different sequences.

R_2^4 stands out for presenting few restrictions. $\alpha_1 R_2^4 \alpha_2$

$$\alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h), \alpha_2 = (10 - \alpha^h 9 - \alpha^{h-1} 9 - \alpha^{h-2} \dots 9 - \alpha^3 9 - \alpha^2 10 - \alpha^1)$$

$$\alpha^1 \geq \alpha^2 + 1, w \text{ even and } h = w/2$$

The R_3 equivalencies explain the numerical invariance after scheme transformations. We highlight:

$$\alpha_1 R_3^i \alpha_{i+1}, \alpha_1 = (\alpha^1 \alpha^2 \dots \alpha^h), i = 1, 2, 3$$

$$\alpha_2 = (15 - \alpha^1 \alpha^2 \dots \alpha^h)$$

$$\alpha_3 = (\alpha^1 \alpha^2 \dots \alpha^{h-1} 15 - \alpha^h)$$

$$\alpha_4 = (\alpha^1 \alpha^2 \dots \alpha^{h-1} 28 - 2\alpha^{h-1} - \alpha^h)$$

with many more restrictions than R_2 , as specified in the text.

More R_3 and higher equivalences are shown in the article. In particular, all the existing R_2 for $w = 6$ and $w = 7$ are shown.

The products of equivalences I and II only have group algebraic structures for low values of w . The equivalences of set I, for $w = 4$ or 5 , while those of set II, for $w = 6$ or 7 . Therefore, the group is isomorphic of the Klein group. The equivalences of set III are universal, thus generating a group isomorphic to $\mathbb{Z}_2$.

The length of the cycles—its number of links—are the pattern of distances which determines how classes are grouped around the cycle's nodes.

Parametric transformation functions have also been useful for studying constants and cycles. We show how families of constants can be derived directly from the K_i functions.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: To María José Díez, professor at the UPV, for her selfless help in the correction of this manuscript.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Kaprekar, D. Another solitaire game. *Scr. Math.* **1949**, *15*, 244–245.
2. Gardner, M. Mathematical games. *Sci. Am.* **1975**, *232*, 112–117. [CrossRef]
3. Lapenta, J.F.; Ludington, A.L.; Prichett, G.D. An algorithm to determine self-producing r -digit g -adic integers. *J. Mathematik. Band* **1979**, *310*, 14.
4. Prichett, G.D.; Ludington, A.L.; Lapenta, J.F. The determination of all decadic Kaprekar constants. *Fibonacci Q.* **1981**, *19*, 45–52.
5. Walden, B.L. Searching for Kaprekar’s constants: Algorithms and results. *Int. J. Math. Math. Sci.* **2005**, *18*, 2999–3004. [CrossRef]
6. Dolan, S. A classification of Kaprekar constants. *Math. Gaz.* **2011**, *95*, 437–443. [CrossRef]
7. Prince, T. Kaprekar constant revised. *Int. J. Math. Arch.* **2013**, *4*, 52–58.
8. Hanover, D. The base dependent behavior of Kaprekar’s routine: A theoretical and computational study revealing new regularities. *arXiv* **2017**, arXiv:1710.06308v1 [math.GM].
9. Thakur, D.S. Kaprekar phenomena. *Proceeding Ropar Conf. RMS-Lect. Notes Ser.* **2019**, *26*, 1–10.
10. Yamagami, A. On 2-adic Kaprekar constants and 2-digit Kaprekar distances. *J. Number Theory* **2018**, *185*, 257–280. [CrossRef]
11. Yamagami, A.; Matsui, Y. On 3-adic Kaprekar loops. *JP J. Algebra Number Theory Appl.* **2018**, *40*, 957–1028. [CrossRef]
12. Wang, Z.; Lu, W. On 2-digit and 3-digit Kaprekar’s routine. *arXiv* **2021**, arXiv:2101.09708.
13. Devlin, P.; Zeng, T. Maximum distances in the four-digit Kaprekar process. *arXiv* **2020**, arXiv:2010.11756v1.
14. Nishiyama, Y. Mysterious Number 6174. 2006. Available online: <https://plus.maths.org/content/mysterious-number-6174> (accessed on 13 November 2021).
15. Peterson, K.; Pulapaka, H. The Kaprekar routine and other digit games for undergraduate exploration. *J. Math. Sci.* **2008**, *10*, 143–156.
16. Deutsch, D.; Goldman, B. Kaprekar’s Constant. *Math. Teach.* **2014**, *98*, 234–242. [CrossRef]
17. Behrends, E. The mystery of the number 1089-how Fibonacci numbers come into play. *Elem. Math.* **2015**, *70*, 144–152. [CrossRef] [PubMed]
18. Phoopha, N.; Pongsriiam, P. Notes on 1089 and a variation of the Kaprekar operator. *Int. J. Math. Comput. Sci.* **2021**, *16*, 1599–1606.
19. Bajorska-Harapińska, B.; Pleszczyński, M.; Różański, M.; Szweda, M.; Witula, R. Generalizations of Kaprekar’s transformation and Kaprekar-style transformations. In *Selected Problems on Experimental Mathematics*; Witula, R., Bajorska-Harapińska, B., Hetmaniok, E., Słota, D., Trawiński, T., Eds.; Wydawnictwo Politechniki Śląskiej: Gliwice, Poland, 2017; pp. 233–258.
20. Stephen, T.; Rohen, Y.; Saleem, N.; Devi, M.B.; Singh, A. Fixed Points of Generalized α -Meir-Keeler Contraction Mappings in S_b -Metric Spaces. *Hindawi J. Funct. Spaces* **2021**, 4684290. [CrossRef]