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Exploiting the Pascal Distribution Series and Gegenbauer Polynomials to Construct and Study a New Subclass of Analytic Bi-Univalent Functions

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Abstract: In the present analysis, we aim to construct a new subclass of analytic bi-univalent functions defined on symmetric domain by means of the Pascal distribution series and Gegenbauer polynomials. Thereafter, we provide estimates of Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the aforementioned class, and next, we solve the Fekete–Szegö functional problem. Moreover, some interesting findings for new subclasses of analytic bi-univalent functions will emerge by reducing the parameters in our main results.

Keywords: Pascal distribution series; Gegenbauer polynomials; analytic bi-univalent functions; Fekete–Szegö functional problem

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1. Introduction and Preliminaries

In statistics and probability, distributions of random variables play a basic role and are used extensively to describe and model a lot of real life phenomena; they describe the distribution of the probabilities over the random variable values [1]. Some distributions are used in practice and have been given special names to clarify the importance of these distributions and the random experiments behind them. If we have two possible outcomes (success) or (fail) in our random experiment and we are interested in how many independent times we need to repeat this random experiment until we achieve the first success, then the random variable X which represents this number of trials has a geometric distribution. This distribution gets its name from its relationship with the geometric series. The generalization of the geometric distribution is called the negative binomial distribution or Pascal distribution. The name 'negative binomial distribution' results from its relationship to the binomial series expansion with a negative exponent. In Pascal distribution, the random variable X represents the number of trials required to obtain r successes in repeated independent Bernoulli trials.

The probability density function of a discrete random variable X that follows a Pascal distribution reads as

$$P(X=x)=\binom{x-1}{r-1}p^r\,q^{x-r},\quad x=r,r+1,r+2,\cdots, \tag{1}$$

where p is the probability of success in each trial and $q=1-p$.

The above probability density function gives the probability of obtaining the rth success on the xth trial by obtaining $r-1$ successes in the first $x-1$ trials in any order, and then obtaining a success on the xth trial. If, in the probability density function (1), we replace x by $x+r$, then we get

$$P(X=x)=\binom{x+r-1}{r-1}p^r\,q^{x},\quad x=0,1,2,\cdots. \tag{2}$$

More recently, El-Deeb et al. [2] presented the following convergent power series whose coefficients are probabilities of the Pascal distribution.

$$\Lambda_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{U}, \tag{3}$$

where $m \geq 1, 0 \leq q \leq 1$, and the symmetric domain $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is the *open unit disk*.

Orthogonal polynomials have been widely studied in recent years from various perspectives due to their importance in mathematical physics, mathematical statistics, engineering, and probability theory. Orthogonal polynomials that appear most commonly in applications are the classical orthogonal polynomials (Hermite polynomials, Laguerre polynomials, and Jacobi polynomials). The general subclass of Jacobi polynomials is the set of Gegenbauer polynomials, this class includes Legendre polynomials and Chebyshev polynomials as subclasses. To study the basic definitions and the most important properties of the classical orthogonal polynomials, we refer the reader to [3–7]. For a recent connection between the classical orthogonal polynomials and geometric function theory, we mention [8–12].

Gegenbauer polynomials $C_n^\alpha(x)$ for $n = 2, 3, \cdots$, and $\alpha > \frac{-1}{2}$ are defined by the following three-term recurrence formula

$$\begin{aligned} C_0^\alpha(x) &= 1; \\ C_1^\alpha(x) &= 2\alpha x; \\ C_n^\alpha(x) &= \frac{1}{n}\left[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-1}^\alpha(x)\right]. \end{aligned} \tag{4}$$

It is worth mentioning that the Gegenbauer polynomials vanish when $\alpha = 0$, and by setting $\alpha = \frac{1}{2}$ and $\alpha = 1$, we immediately obtain Legendre polynomials $P_n(x) = C_n^{\frac{1}{2}}(x)$ and Chebyshev polynomials of the second kind $U_n(x) = C_n^1(x)$, respectively.

Amourah et al. in [13] considered the following generating function of Gegenbauer polynomials

$$H_\alpha(x,z) = \frac{1}{(1-2xz+z^2)^\alpha}, \tag{5}$$

where $x \in [-1, 1]$ and $z \in \mathbb{U}$. For fixed x, the function H_α is analytic in $\mathbb{U}$, so it can be expanded in a Taylor–Maclaurin series, as follows

$$H_\alpha(x,z) = \sum_{n=0}^{\infty} C_n^\alpha(x) z^n. \tag{6}$$

Let $\mathcal{A}$ denote the class of all normalized analytic functions f written as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ z \in \mathbb{U}. \tag{7}$$

Definition 1. *Let f and g be in the class $\mathcal{A}$ and given by (7). The function f is said to be subordinate to g, written as $f \prec g$, if there is an analytic function w in $\mathbb{U}$ with the properties*

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$f(z) = g(w(z)).$$

Definition 2. *A single-valued one-to-one function f defined in a simply connected domain is said to be a univalent function.*

Let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$, given by (7), that are univalent in $\mathbb{U}$. Hence, every function $f(z) \in \mathcal{S}$ has an inverse given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{8}$$

Definition 3. *A univalent function $f(z)$ is said to be bi-univalent in $\mathbb{U}$ if its inverse function $f^{-1}(w)$ is univalent in $\mathbb{U}$.*

Let Σ denote the class of all functions $f \in \mathcal{A}$ that are bi-univalent in $\mathbb{U}$ given by (7). For interesting subclasses of functions in the class Σ, see [14–19].

Now, let us define the linear operator

$$\mathcal{I}_q^m(z) : \mathcal{A} \to \mathcal{A}$$

by

$$\mathcal{I}_q^m f(z) = \Lambda_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \qquad z \in \mathbb{U}, \tag{9}$$

where $*$ denotes the Hadamard product.

Motivated essentially by the work of Amourah et al. [13], we construct, in the next section, a new subclass of bi-univalent functions governed by the Pascal distribution series and Gegenbauer polynomials. Then, we investigate the optimal bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ and solve the Fekete–Szegö functional problem for functions in our new subclass.

2. The Class $\mathfrak{G}_\Sigma(x, \alpha, \beta, \gamma)$

In this section, let the function $f \in \Sigma$ given by (7), the function $g = f^{-1}$ is given by (8), and H_α is the generating function of Gegenbauer polynomials given by (6). Now, we are ready to define our new subclass of bi-univalent functions $\mathfrak{G}_\Sigma(x, \alpha, \beta, \gamma)$ as follows.

Definition 4. *A function f is said to be in the class $\mathfrak{G}_\Sigma(x, \alpha, \beta, \gamma)$, if the following subordinations are fulfilled:*

$$(1-\gamma)\frac{\mathcal{I}_q^m f(z)}{z} + \gamma\left(\mathcal{I}_q^m f(z)\right)' + \beta z\left(\mathcal{I}_q^m f(z)\right)'' \prec H_\alpha(x, z), \tag{10}$$

and

$$(1-\gamma)\frac{\mathcal{I}_q^m g(w)}{w} + \gamma\left(\mathcal{I}_q^m g(w)\right)' + \beta w\left(\mathcal{I}_q^m g(w)\right)'' \prec H_\alpha(x, w), \tag{11}$$

where $\alpha > 0$, $\gamma, \beta \geq 0$, and $x \in (\frac{1}{2}, 1]$.

Upon allocating the parameters γ and β, one can obtain several new subclasses of Σ, as illustrated in the following two examples.

Example 1. *A function f is said to be in the class $\mathfrak{G}_\Sigma(x, \alpha, \gamma) = \mathfrak{G}_\Sigma(x, \alpha, 0, \gamma)$, if the following subordinations are fulfilled:*

$$(1-\gamma)\frac{\mathcal{I}_q^m f(z)}{z} + \gamma\left(\mathcal{I}_q^m f(z)\right)' \prec H_\alpha(x, z), \tag{12}$$

and

$$(1-\gamma)\frac{\mathcal{I}_q^m g(w)}{w} + \gamma\left(\mathcal{I}_q^m g(w)\right)' \prec H_\alpha(x, w), \tag{13}$$

where $\alpha > 0$, $\gamma \geq 0$, and $x \in (\frac{1}{2}, 1]$.

Example 2. *A function f is said to be in the class $\mathfrak{G}_\Sigma(x,\alpha) = \mathfrak{G}_\Sigma(x,\alpha,0,1)$, if the following subordinations are fulfilled:*

$$\left(\mathcal{I}_q^m f(z)\right)' \prec H_\alpha(x,z), \tag{14}$$

and

$$\left(\mathcal{I}_q^m g(w)\right)' \prec H_\alpha(x,w), \tag{15}$$

where $\alpha > 0$ and $x \in (\frac{1}{2}, 1]$.

3. Main Results

Theorem 1. *If the function f belongs to the class $\mathfrak{G}_\Sigma(x,\alpha,\beta,\gamma)$, then*

$$|a_2| \le \frac{2x\sqrt{2|\alpha|x}}{\sqrt{\left|2\left[(m+1)(1+2\gamma+6\beta)\alpha - m(1+\gamma+2\beta)^2 p^m(1+\alpha)\right]x^2 + m(1+\gamma+2\beta)^2 p^m\right|}},$$

and

$$|a_3| \le \frac{4\alpha^2 x^2}{m^2 q^2 (1+\gamma+2\beta)^2 p^{2m}} + \frac{4|\alpha|x}{mq^2(1+2\gamma+6\beta)(m+1)p^m},$$

where $p = 1-q$.

Proof. Let $f \in \mathfrak{G}_\Sigma(x,\alpha,\beta,\gamma)$. From Definition 4, one can write

$$(1-\gamma)\frac{\mathcal{I}_q^m f(z)}{z} + \gamma\left(\mathcal{I}_q^m f(z)\right)' + \beta z\left(\mathcal{I}_q^m f(z)\right)'' = H_\alpha(x, w(z)), \tag{16}$$

and

$$(1-\gamma)\frac{\mathcal{I}_q^m f(w)}{w} + \gamma\left(\mathcal{I}_q^m f(w)\right)' + \beta w\left(\mathcal{I}_q^m f(w)\right)'' = H_\alpha(x, v(w)), \tag{17}$$

for some analytic functions $w(z) = \sum_{n=1}^{\infty} c_n z^n$ and $v(w) = \sum_{n=1}^{\infty} d_n w^n$ in $\mathbb{U}$ that satisfy the properties mentioned in Definition 1.

From the Equations (16) and (17), we obtain

$$(1-\gamma)\frac{\mathcal{I}_q^m f(z)}{z} + \gamma\left(\mathcal{I}_q^m f(z)\right)' + \beta z\left(\mathcal{I}_q^m f(z)\right)'' = 1 + C_1^\alpha(x)c_1 z + \left[C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2\right]z^2 + \cdots \tag{18}$$

and

$$(1-\gamma)\frac{\mathcal{I}_q^m f(w)}{w} + \gamma\left(\mathcal{I}_q^m f(w)\right)' + \beta w\left(\mathcal{I}_q^m f(w)\right)'' = 1 + C_1^\alpha(x)d_1 w + \left[C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2\right])w^2 + \cdots \tag{19}$$

Next, upon equalizing the corresponding coefficients of z, z^2, w, and w^2 in both sides of Equations (18) and (19), we get

$$mq(1+\gamma+2\beta)(1-q)^m a_2 = C_1^\alpha(x)c_1, \tag{20}$$

$$mq^2\left(\frac{1}{2}+\gamma+3\beta\right)(m+1)(1-q)^m a_3 = C_1^\alpha(x)c_2 + C_2^\alpha(x)c_1^2, \tag{21}$$

$$-mq(1+\gamma+2\beta)(1-q)^m a_2 = C_1^\alpha(x)d_1, \tag{22}$$

and

$$mq^2\left(\frac{1}{2}+\gamma+3\beta\right)(m+1)(1-q)^m\left[2a_2^2 - a_3\right] = C_1^\alpha(x)d_2 + C_2^\alpha(x)d_1^2. \tag{23}$$

By adding Equations (20) and (22) and their squares, we have

$$c_1 = -d_1, \tag{24}$$

and

$$2m^2q^2(1+\gamma+2\beta)^2(1-q)^{2m}a_2^2 = [C_1^\alpha(x)]^2\left(c_1^2+d_1^2\right), \tag{25}$$

respectively.

By adding Equations (21) and (23), we get

$$mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m a_2^2 = C_1^\alpha(x)(c_2+d_2) + C_2^\alpha(x)\left(c_1^2+d_1^2\right). \tag{26}$$

We deduce, from Equations (25) and (26), that

$$\begin{aligned}[(1+2\gamma+6\beta)(m+1)] - 2m(1+\gamma+2\beta)^2(1-q)^m \frac{C_2^\alpha(x)}{[C_1^\alpha(x)]^2} m(1-q)^m q^2 a_2^2 \\ = C_1^\alpha(x)(c_2+d_2).\end{aligned} \tag{27}$$

Since, $|w(z)| < 1$ and $|v(w)| < 1$, then by calling a known result we have

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{28}$$

By making use of Equations (4), (27) and (28), we obtain the required optimal bound of $|a_2|$, as follows.

$$|a_2| \leq \frac{2x\sqrt{2|\alpha|x}}{\sqrt{\left|2\left[(m+1)(1+2\gamma+6\beta)\alpha - m(1+\gamma+2\beta)^2p^m(1+\alpha)\right]x^2 + m(1+\gamma+2\beta)^2p^m\right|}},$$

where $p = 1-q$.

Next, subtracting (23) from (21) yields

$$mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m\left(a_3-a_2^2\right) = C_1^\alpha(x)(c_2-d_2) + C_2^\alpha(x)\left(c_1^2-d_1^2\right). \tag{29}$$

Then, by making use of Equation (25), then Equation (29) can be written as

$$a_3 = \frac{[C_1^\alpha(x)]^2}{2m^2q^2(1+\gamma+2\beta)^2p^{2m}}\left(c_1^2+d_1^2\right) + \frac{C_1^\alpha(x)}{mq^2(1+2\gamma+6\beta)(m+1)p^m}(c_2-d_2). \tag{30}$$

where $p = 1-q$.

Now, by applying Equations (4) and (28), we conclude that

$$|a_3| \leq \frac{4\alpha^2x^2}{m^2q^2(1+\gamma+2\beta)^2p^{2m}} + \frac{4|\alpha|x}{mq^2(1+2\gamma+6\beta)(m+1)p^m}.$$

This completes the proof Theorem 1. □

The following result addresses the Fekete–Szegö functional problem for functions in the class $\mathfrak{G}_\Sigma(x,\alpha,\beta,\gamma)$.

Theorem 2. *If the function f belongs to the class $\mathfrak{G}_{\Sigma}(x,\alpha,\beta,\gamma)$, then*

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4|\alpha|x}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}, & |\eta-1|\leq\delta, \\ \frac{8\alpha^2x^3(1-\eta)}{\left|2\alpha x^2(1+2\gamma+6\beta)(m+1)-m(1+\gamma+2\beta)^2(1-q)^m(2(1+\alpha)x^2-1)\right|m(1-q)^m q^2}, & |\eta-1|\geq\delta, \end{cases}$$

where

$$\delta=\left|1-\frac{m(1+\gamma+2\beta)^2(1-q)^m\left(2(1+\alpha)x^2-1\right)}{2\alpha x^2(1+2\gamma+6\beta)(m+1)}\right|.$$

Proof. Let $f\in\mathfrak{G}_{\Sigma}(x,\alpha,\beta,\gamma)$. From Equations (27) and (30), we immediately get

$$\begin{aligned} a_3-\eta a_2^2 &= (1-\eta)\frac{\left[C_1^{\alpha}(x)\right]^3(c_2+d_2)}{\left[(1+2\gamma+6\beta)(m+1)\left[C_1^{\alpha}(x)\right]^2-2m(1+\gamma+2\beta)^2(1-q)^m C_2^{\alpha}(x)\right]m(1-q)^m q^2} \\ &+\frac{C_1^{\alpha}(x)}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}(c_2-d_2), \\ &= C_1^{\alpha}(x)\left[h(\eta)+\frac{1}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}\right]c_2 \\ &+\left[h(\eta)-\frac{1}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}\right]d_2, \end{aligned}$$

where

$$h(\eta)=\frac{\left[C_1^{\alpha}(x)\right]^2(1-\eta)}{\left[(1+2\gamma+6\beta)(m+1)\left[C_1^{\alpha}(x)\right]^2-2m(1+\gamma+2\beta)^2(1-q)^m C_2^{\alpha}(x)\right]m(1-q)^m q^2}.$$

Then, in view of (4), we conclude that

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{2\left|C_1^{\alpha}(x)\right|}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m} & 0\leq|h(\eta)|\leq\frac{1}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}, \\ 2\left|C_1^{\alpha}(x)\right||h(\eta)| & |h(\eta)|\geq\frac{1}{mq^2(1+2\gamma+6\beta)(m+1)(1-q)^m}. \end{cases}$$

which completes the proof of Theorem 2. □

4. Corollaries and Consequences

Corresponding essentially to Examples 1 and 2, Theorems 1 and 2 yield the following consequences.

Corollary 1. *If the function f belongs to the class $\mathfrak{G}_{\Sigma}(x,\alpha,\gamma)$, then*

$$|a_2|\leq\frac{2x\sqrt{2|\alpha|x}}{\sqrt{\left|2\left[(m+1)(1+2\gamma)\alpha-m(1+\gamma)^2(1-q)^m(1+\alpha)\right]x^2+m(1+\gamma)^2(1-q)^m\right|}},$$

$$|a_3|\leq\frac{4\alpha^2x^2}{m^2q^2(1+\gamma)^2(1-q)^{2m}}+\frac{4|\alpha|x}{mq^2(1+2\gamma)(m+1)(1-q)^m},$$

and

$$\left|a_3-\eta a_2^2\right| \leq \begin{cases} \frac{4|\alpha|x}{mq^2(1+2\gamma)(m+1)(1-q)^m}, & |\eta-1|\leq\tau, \\ \frac{8\alpha^2x^3(1-\eta)}{\left|2\alpha x^2(1+2\gamma)(m+1)-m(1+\gamma)^2(1-q)^m(2(1+\alpha)x^2-1)\right|m(1-q)^m q^2}, & |\eta-1|\geq\tau, \end{cases}$$

where

$$\tau = \left|1 - \frac{m(1+\gamma)^2(1-q)^m\left(2(1+\alpha)x^2-1\right)}{2\alpha x^2(1+2\gamma)(m+1)}\right|.$$

Proof. Set $\beta = 0$ in the proof of Theorems 1 and 2. $\square$

Corollary 2. *If the function f belongs to the class $\mathfrak{G}_\Sigma(x,\alpha)$, then*

$$|a_2| \leq \frac{2x\sqrt{2|\alpha|x}}{\sqrt{\left|2[(m+1)\alpha - m(1-q)^m(1+\alpha)]x^2 + m(1-q)^m\right|}},$$

$$|a_3| \leq \frac{4\alpha^2x^2}{m^2q^2(1-q)^{2m}} + \frac{4|\alpha|x}{mq^2(m+1)(1-q)^m},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{4|\alpha|x}{mq^2(m+1)(1-q)^m}, & |\eta-1| \leq \left|1 - \frac{m(1-q)^m\left(2(1+\alpha)x^2-1\right)}{2\alpha x^2(m+1)}\right|, \\ \frac{8\alpha^2x^3(1-\eta)}{\left|2\alpha x^2(m+1)-m(1-q)^m(2(1+\alpha)x^2-1)\right|m(1-q)^mq^2}, & |\eta-1| \geq \left|1 - \frac{m(1-q)^m\left(2(1+\alpha)x^2-1\right)}{2\alpha x^2(m+1)}\right|. \end{cases}$$

Proof. Set $\beta = 0$ and $\gamma = 1$ in the proof of Theorems 1 and 2. $\square$

5. Concluding Remarks

In the present work, we have constructed a new subclass $\mathfrak{G}_\Sigma(x,\alpha,\beta,\gamma)$ of normalized analytic and bi-univalent functions governed with the Pascal distribution series and Gegenbauer polynomials. For functions belonging to this class, we have made estimates of Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, and solved the Fekete–Szegö functional problem. Furthermore, by suitably specializing the parameters β and γ, one can deduce the results for the subclasses $\mathfrak{G}_\Sigma(x,\alpha,\gamma)$ and $\mathfrak{G}_\Sigma(x,\alpha)$ which are defined, respectively, in Examples 1 and 2.

The results offered in this paper would lead to other different new results for the classes $\mathfrak{G}_\Sigma(x,1/2,\beta,\gamma)$ for Legendre polynomials and $\mathfrak{G}_\Sigma(x,1,\beta,\gamma)$ for Chebyshev polynomials.

It remains an open problem to derive estimates on the bounds of $|a_n|$ for $n \geq 4; n \in \mathbb{N}$ for the subclasses that have been introduced here.

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