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Some Theorems for Inverse Uncertainty Distribution of Uncertain Processes

Xiumei Chen [1,2,*], Yufu Ning [1,2], Lihui Wang [1,2], Shuai Wang [1,2] and Hong Huang [1,2]

[1] School of Information Engineering, Shandong Youth University of Political Science, Jinan 250103, China; nyf@sdyu.edu.cn (Y.N.); wlh@sdyu.edu.cn (L.W.); wangshuai@sdyu.edu.cn (S.W.); huanghong@sdyu.edu.cn (H.H.)

[2] Key Laboratory of Intelligent Information Processing Technology and Security in Universities of Shandong, Jinan 250103, China

* Correspondence: cxm@sdyu.edu.cn

Abstract: In real life, indeterminacy and determinacy are symmetric, while indeterminacy is absolute. We are devoted to studying indeterminacy through uncertainty theory. Within the framework of uncertainty theory, uncertain processes are used to model the evolution of uncertain phenomena. The uncertainty distribution and inverse uncertainty distribution of uncertain processes are important tools to describe uncertain processes. An independent increment process is a special uncertain process with independent increments. An important conjecture about inverse uncertainty distribution of an independent increment process has not been solved yet. In this paper, the conjecture is proven, and therefore, a theorem is obtained. Based on this theorem, some other theorems for inverse uncertainty distribution of the monotone function of independent increment processes are investigated. Meanwhile, some examples are given to illustrate the results.

Keywords: uncertainty theory; inverse uncertainty distribution; uncertain process

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1. Introduction

"Indeterminacy is absolute, while determinacy is relative" (Liu [1]). It seems that real decisions are usually made in the context of indeterminacy. We should choose proper mathematical tools in order to rationally deal with indeterminate quantity. Probability theory is an appropriate tool to model frequency by dealing with the quantity as a random variable. However, in real life, while the cases that no samples are available or some emergency occurs, the estimated distribution function may not be close enough to the real frequency and may even deviate far from the frequency. If this estimated distribution function is used, probability theory may lead to counterintuitive results. For more research-based answers, interested readers can refer to [2]. While encountering these cases, uncertainty theory is a legitimate approach to model the belief degree by treating the indeterminate quantity as an uncertain variable.

In 2007, Liu [3] founded uncertainty theory, which is a branch of axiomatic mathematics. In uncertainty theory, the uncertain measure is defined for modeling the belief degree of an uncertain event and uncertain variable for depicting the quantity with uncertainty. In order to specify the uncertain variable, uncertainty distribution and inverse uncertainty distribution were put forward. In addition, the expected value of an uncertain variable is defined to represent the average of the uncertain variable. Variance is to provide a degree of the spread of the distribution around its expected value. Compared with density function in probability theory, inverse uncertainty distribution in uncertainty theory is a convenient and useful tool to calculate the expected value and variance of uncertain variables.

Now, uncertainty theory is widespread and applied in many branches, and gratifying results are achieved, such as uncertain programming (Liu [4], Ning et al. [5]), uncertain finance (Chen [6], Zhang et al. [7], Gao et al. [8]), uncertain differential equation

(Chen and Liu [9], Yao [10]), and uncertain statistics (Lio and Liu [11], Yang and Liu [12], Liu and Yang [13], Wang et al. [14]).

Sometimes uncertainty varies over time. For describing this kind of uncertain phenomena, the uncertain process was proposed by Liu [15] in 2008. It is indeed a sequence of uncertain variables indexed by time. Similar to the uncertain variable, the uncertainty distribution and inverse uncertainty distribution of an uncertain process were defined by Liu [16] in order to depict an uncertain process. The operational laws to calculate the inverse uncertainty distribution and uncertainty distribution of independent uncertain processes were proposed. In addition, other definitions were presented, such as independent increment process(Liu [15]), time integral (Liu [15]), and stationary increment process (Chen [17]). On the basis of independent uncertain processes, uncertain calculus (Liu [18], Ye [19]) and the uncertain renewal process ([20]) have been further developed and promoted.

A conjecture proposed by Liu ([2]) for inverse uncertainty distribution of an independent increment process, which has not been solved until now. It is necessary to complete this proof. By using this conjecture, Yao([21]) provided a formula for calculating the inverse uncertainty distribution of the time integral.

In this paper, the proof of the conjecture is given, and relevant theorems are obtained. The rest is organized as follows. Some basic concepts and theorems are introduced in Section 2. The proof of the inverse uncertainty distribution of the uncertain process is presented in Section 3. The other two theorems of the inverse uncertainty distribution of a monotone function of uncertain processes are demonstrated in Section 4. At last, a brief summary is given in Section 5.

2. Preliminaries

In this section, some definitions and theorems, which will be used throughout this paper, in uncertainty theory are introduced. For more details, interested readers should read Liu [2].

A triplet (Γ, L, M) is called an *uncertainty space*, where (Γ, L) is a measurable space and M is an uncertain measure satisfying normality axiom, duality axiom, subadditivity axiom, and product axiom.

By using the axioms of uncertain measure, the monotonicity theorem was derived as follows.

Theorem 1 (Liu [2])**.** *The uncertain measure is a monotone-increasing set function. That is, for any events Λ_1 and Λ_2 with $\Lambda_1 \subseteq \Lambda_2$, we have*

$$M\{\Lambda_1\} \leq M\{\Lambda_2\}.$$

Definition 1 (Gao [22])**.** *An uncertainty space (Γ, L, M) is called continuous if and only if for any events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$M\{\lim_{i\to\infty} \Lambda_i\} = \lim_{i\to\infty} M\{\Lambda_i\}$$

provided that $\lim_{i\to\infty} \Lambda_i$ *exists.*

An uncertain variable ξ is a measurable function from an uncertainty space to the real number set, and its uncertainty distribution is defined by $\Phi(x) = M\{\xi \leq x\}$. It follows from the definition of Φ and duality axiom of uncertain measure that the following theorem holds.

Theorem 2 (Liu [23])**.** *Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$. Then for any real number x, we have*

$$M\{\xi \leq x\} = \Phi(x), \quad M\{\xi > x\} = 1 - \Phi(x).$$

It is remarkable that if the uncertainty distribution Φ is a continuous function, we also have

$$M\{\xi < x\} = \Phi(x), \ \ M\{\xi \geq x\} = 1 - \Phi(x).$$

An uncertainty distribution Φ is *regular*, meaning that it is continuous, strictly increasing, and satisfying $\lim_{i\to-\infty}\Phi(x) = 0, \lim_{i\to+\infty}\Phi(x) = 1$.

An operational law for calculating the inverse uncertainty distribution of strictly monotone function of independent uncertain variables is as follows:

Theorem 3 (Liu [23])**.** *Suppose $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is continuous and strictly increasing concerning $x_1, x_2, \cdots x_m$ and strictly decreasing concerning $x_{m+1}, x_{m+2}, \cdots$, x_n, then*

$$\xi = f(\xi_1, \xi_2, \cdots \xi_n)$$

which posseses an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The uncertain process $X_t(\gamma)$ is proposed to describe the evolution of uncertain phenomena. For each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a *sample path* of X_t , and for each $t \in T$, $X_t(\gamma)$ is an uncertain variable. X_t is said to be *sample-continuous* if almost all sample paths are continuous functions with respect to time t.

Definition 2 (Liu [16])**.** *The uncertainty distribution $\Phi_t(x)$ of uncertain process X_t is defined by*

$$\Phi_t(x) = M\{X_t \leq x\}$$

for any time t and any number x.

Definition 3 (Liu [16])**.** *An uncertainty distribution $\Phi_t(x)$ is said to be regular if at each time t, it is a continuous and strictly increasing function with respect to x, at which $0 < \Phi_t(x) < 1$, and*

$$\lim_{i\to-\infty}\Phi_t(x) = 0, \lim_{i\to+\infty}\Phi_t(x) = 1.$$

Definition 4 (Liu [16])**.** *Suppose X_t is an uncertain process with regular uncertainty distribution $\Phi_t(x)$. Then the inverse function $\Phi_t^{-1}(\alpha)$ is called the inverse uncertainty distribution of X_t.*

Definition 5 (Liu [16])**.** *An uncertain process sequence of $X_{1t}, X_{2t}, \cdots, X_{nt}$ is said to be mutually independent if for any positive integer k and any times $t_1, t_2, \cdots, t_k$, the uncertain vectors*

$$\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}), \ i = 1, 2, \cdots, n$$

are independent, i.e., for any Borel sets $B_1, B_2, \cdots, B_n$ of k-dimensional real vectors, we have

$$M\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} M\{\boldsymbol{\xi}_i \in B_i\}.$$

Similar to Theorem 3, the operational law for the inverse uncertainty distribution of the monotone function of uncertain processes was presented in Liu [16].

Theorem 4 (Liu [16])**.** *Suppose $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t} \cdots, \Phi_{nt}$, respectively. If $f(x_1, x_2 \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2 \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt})$$

has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha), \cdots, \Phi_{nt}^{-1}(1-\alpha)).$$

Definition 6 (Liu [15])**.** *An uncertain process X_t is said to have independent increments if*

$$X_{t_1}, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_k} - X_{t_{k-1}}$$

are independent uncertain variables where $t_1, t_2, \cdots, t_k$ are any times with $t_1 < t_2 < \cdots < t_k$.

Definition 7 (Liu [15])**.** *An uncertain process X_t is said to have stationary increments if its increments are identically distributed uncertain variables whenever the time intervals have the same length.*

3. The Proof for Inverse Uncertainty Distribution of Uncertain Process

In this section, we first show a lemma that is needed for the following proof. Subsequently, we give the proof for the inverse uncertainty distribution of an uncertain process. At last, two examples are given.

Lemma 1. *Let (Γ, L, M) be the uncertainty space, and let ξ_1, ξ_2 be independent uncertain variables with regular uncertainty distribution Φ_1, Φ_2, respectively. If uncertain variable $\xi = \xi_1 + \xi_2$ possesses an uncertain distribution Φ, then we have*

$$M\{\xi_2 \leq \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = \alpha,$$

$$M\{\xi_2 > \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = 1 - \alpha.$$

Proof. Since ξ_1, ξ_2 are independent uncertain variables and $f(x_1, x_2) = x_1 + x_2$ is continuous and strictly increasing with respect to x_1 and x_2 , it follows from Theorem 3 that $\xi = \xi_1 + \xi_2$ has the inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha).$$

Thus, $\Phi_2^{-1}(\alpha) = \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)$. Following Theorem 2, we derive

$$M\{\xi_2 \leq \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = M\{\xi_2 \leq \Phi_2^{-1}(\alpha)\} = \alpha,$$

$$M\{\xi_2 > \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = M\{\xi_2 > \Phi_2^{-1}(\alpha)\} = 1 - \alpha.$$

Thus, it is verified. □

Remark 1. *Note that the uncertainty distribution Φ_2 is regular. Thus*

$$M\{\xi_2 < \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = \alpha,$$

$$M\{\xi_2 \geq \Phi^{-1}(\alpha) - \Phi_1^{-1}(\alpha)\} = 1 - \alpha.$$

Theorem 5. *Let (Γ, L, M) be a continuous uncertain space, and let X_t be sample-continuous independent increment process with regular uncertainty distribution $\Phi_t(x)$. Then for any $0 < \alpha < 1$, we have*

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1 - \alpha.$$

Proof. For $0 < \alpha < 1$, we divide four steps to prove.
Step 1 If T is a finite set, we certify

$$M\{\ X_t \leq \Phi_t^{-1}(\alpha), t \in T\} = \alpha,$$

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} = 1-\alpha.$$

Assume $T = \{t_1, t_2, \cdots, t_n\}$, $t_1 < t_2 < \cdots < t_n$ and write $\xi_1 = X_{t_1}$, $\xi_i = X_{t_i} - X_{t_{i-1}} (2 \leq i \leq n)$. Since X_t is an independent increment process, it follows from Definition 6 that ξ_1, $\xi_i (2 \leq i \leq n)$ are independent uncertain variables. Note that in this case

$$\{X_t \leq \Phi_t^{-1}(\alpha), t \in T\} = \{\bigcap_{i=1}^{n} X_{t_i} \leq \Phi_{t_i}^{-1}(\alpha)\}$$

holds. On the one hand, it follows from the monotonicity theorem that

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in T\} \leq M\{X_{t_1} \leq \Phi_{t_1}^{-1}(\alpha)\} = \alpha. \tag{1}$$

On the other hand, due to

$$\{X_t \leq \Phi_t^{-1}(\alpha), t \in T\} \supseteq \left\{ (\xi_1 \leq \Phi_{t_1}^{-1}(\alpha)) \cap (\bigcap_{i=2}^{n} \xi_i \leq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha)) \right\},$$

by using the independence of $\xi_1, \xi_2, \cdots, \xi_n$, monotonicity theorem and Lemma 1, we have

$$\begin{aligned} M\{X_t \leq \Phi_t^{-1}(\alpha), t \in T\} &\geq M\left\{ (\xi_1 \leq \Phi_{t_1}^{-1}(\alpha)) \cap (\bigcap_{i=2}^{n} (\xi_i \leq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha)) \right\} \\ &= M\{\xi_1 \leq \Phi_{t_1}^{-1}(\alpha)\} \wedge \bigwedge_{i=2}^{n} M\{\xi_i \leq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha)\} \\ &= \alpha \wedge \alpha = \alpha. \end{aligned} \tag{2}$$

It follows from the above two inequalities (1) and (2) that

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in T\} = \alpha.$$

Now we prove that

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} = 1-\alpha.$$

Note that

$$\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} = \{\bigcap_{i=1}^{n} (X_{t_i} \geq \Phi_{t_i}^{-1}(\alpha))\}.$$

holds. On the one hand, we get

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} \leq M\{X_{t_1} \geq \Phi_{t_1}^{-1}(\alpha)\} = 1-\alpha. \tag{3}$$

On the other hand, due to

$$\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} \supseteq \left\{ (\xi_1 \geq \Phi_{t_1}^{-1}(\alpha)) \cap (\bigcap_{i=2}^{n} (\xi_i \geq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha)) \right\},$$

by using the independence of $\xi_1, \xi_2, \cdots, \xi_n$, monotonicity theorem and Remark 1, we obtain

$$
\begin{aligned}
M\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} &\geq M\left\{(\xi_1 \geq \Phi_{t_1}^{-1}(\alpha)) \cap (\bigcap_{i=2}^{n}(\xi_i \geq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha))\right\} \\
&= M\{\xi_1 \geq \Phi_{t_1}^{-1}(\alpha)\} \wedge \bigwedge_{i=1}^{n} M\{\xi_i \geq \Phi_{t_i}^{-1}(\alpha) - \Phi_{t_{i-1}}^{-1}(\alpha)\} \qquad (4) \\
&= (1-\alpha) \wedge (1-\alpha) \\
&= 1-\alpha.
\end{aligned}
$$

It follows from the above two inequalities (3) and (4) that

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in T\} = 1 - \alpha.$$

Step 2 If T is a countable set, especially if T is a rational numbers set Q, for simplicity, we denote $T = \{t_1, t_2, \cdots\}$. Since Γ is a continuous uncertain space, by using Step 1, we get

$$
\begin{aligned}
M\{X_t \leq \Phi_t^{-1}(\alpha), t \in Q\} &= M\{\lim_{i\to\infty} \bigcap_{j=1}^{i}((X_{t_j} \leq \Phi_{t_j}^{-1}(\alpha)))\} \\
&= \lim_{i\to\infty} M\{\bigcap_{j=1}^{i}(X_{t_j} \leq \Phi_{t_j}^{-1}(\alpha))\} = \alpha,
\end{aligned}
$$

and then

$$
\begin{aligned}
M\{X_t \geq \Phi_t^{-1}(\alpha), t \in Q\} &= M\{\lim_{i\to\infty} \bigcap_{j=1}^{i}(X_{t_j} \geq \Phi_{t_j}^{-1}(\alpha))\} \\
&= \lim_{i\to\infty} M\{\bigcap_{j=1}^{i}(X_{t_j} \geq \Phi_{t_j}^{-1}(\alpha))\} = 1-\alpha.
\end{aligned}
$$

Step3 For this step, we show that the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$ is continuous with t.

For any $t_0 \in R$, we first prove that $\overline{\lim_{t\to t_0}} \Phi_t^{-1}(\alpha) \leq \Phi_{t_0}^{-1}(\alpha)$. If not, then there exists the real number $\epsilon_0 > 0$, for any positive integer n, there exists t_n, such that $|t_n - t_0| < 1/n$ and $\Phi_{t_n}^{-1}(\alpha) > \Phi_{t_0}^{-1}(\alpha) + \epsilon_0$.

Take $T_0 = \{t_1, t_2, \cdots\}$, $A = \{X_t \geq \Phi_t^{-1}(\alpha), t \in T_0\}$ and $B = \{X_{t_0} \leq \Phi_{t_0}^{-1}(\alpha) + \epsilon_0/2\}$. It follows from Step 2 that $M\{A\} = 1 - \alpha$. Since $\Phi_t(x)$ is regular, we have $M\{B\} > \alpha$ and then $M\{B^c\} = 1 - M\{B\} < 1 - \alpha$.

Due to

$$1-\alpha = M\{A\} = M\{(A \cap B) \cup (A \cap B^c)\} \leq M\{A \cap B\} + M\{B^c\} \leq M\{A \cap B\} + 1 - M\{B\},$$

we have $M\{A \cap B\} \neq 0$. Thus, $A \cap B \neq \phi$.

For any $\gamma \in A \cap B$, we derive that

$$
\begin{gathered}
X_{t_n}(\gamma) - X_{t_0}(\gamma) \\
\geq \Phi_{t_n}^{-1}(\alpha) - (\Phi_{t_0}^{-1}(\alpha) + \epsilon_0/2) > (\Phi_{t_0}^{-1}(\alpha) + \epsilon_0) - (\Phi_{t_0}^{-1}(\alpha) + \epsilon_0/2) = \epsilon_0/2 > 0
\end{gathered}
$$

which implies $|X_{t_n}(\gamma) - X_{t_0}(\gamma)| > \epsilon_0/2 > 0$. This is in contradiction with X_t being sample-continuous. Hence $\overline{\lim_{t\to t_0}} \Phi_t^{-1}(\alpha) \leq \Phi_{t_0}^{-1}(\alpha)$.

Now let us prove $\underline{\lim}_{t\to t_0} \Phi_t^{-1}(\alpha) \geq \Phi_{t_0}^{-1}(\alpha)$. If not, then there exists $\epsilon_1 > 0$, for any positive integer n, there exists t_n satisfying $|t_n - t_0| < 1/n$ and $\Phi_{t_n}^{-1}(\alpha) < \Phi_{t_0}^{-1}(\alpha) - \epsilon_1$.

Take $A_1 = \{X_t \leq \Phi_t^{-1}(\alpha), t \in T_0\}$ and $B_1 = \{X_{t_0} \geq \Phi_{t_0}^{-1}(\alpha) - \epsilon_1/2\}$. It follows from Step 2 that $M\{A_1\} = \alpha$. Since $\Phi_t(x)$ is regular, we have $M\{B_1\} > M\{X_{t_0} \geq \Phi_{t_0}^{-1}(\alpha)\} = 1-\alpha$ and then $M\{B_1{}^c\} = 1 - M\{B_1\} < \alpha$.

Due to

$$\alpha = M\{A_1\} = M\{(A_1 \cap B_1) \cup (A_1 \cap B_1{}^c)\}$$

$$\leq M\{A_1 \cap B_1\} + M\{B_1{}^c\} \leq M\{A_1 \cap B_1\} + 1 - M\{B_1\},$$

we have $M\{A_1 \cap B_1\} \neq 0$. Thus, $A_1 \cap B_1 \neq \phi$. For any $\gamma \in A_1 \cap B_1$, we derive that $X_{t_0}(\gamma) - X_{t_n}(\gamma) \geq$

$$\Phi_{t_0}^{-1}(\alpha) - \epsilon_1/2 - \Phi_{t_n}^{-1}(\alpha) > \Phi_{t_0}^{-1}(\alpha) - \epsilon_1/2 - (\Phi_{t_0}^{-1}(\alpha) - \epsilon_1) = \epsilon_1/2 > 0$$

which implies $|X_{t_n}(\gamma) - X_{t_0}(\gamma)| > \epsilon_1/2 > 0$. This result is in contradiction with X_t being sample-continuous. So we have

$$\overline{\lim_{t\to t_0}}\Phi_t^{-1}(\alpha) = \underline{\lim}_{t\to t_0} \Phi_t^{-1}(\alpha) = \Phi_{t_0}^{-1}(\alpha).$$

Thus, the continuity is verified.

Step 4 If T is the real number set R, we certify

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha.$$

Let $\Lambda_1 = \{X_t \leq \Phi_t^{-1}(\alpha), t \in Q\}$ and $\Lambda_2 = \{X_t \geq \Phi_t^{-1}(\alpha), t \in Q\}$. On the one hand, it follows from the monotonicity theorem and Step 2 that

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} \leq M\{\Lambda_1\} = \alpha, \tag{5}$$

$$M\{|X_t \geq \Phi_t^{-1}(\alpha), t \in R\} \leq M\{\Lambda_2\} = 1-\alpha. \tag{6}$$

On the other hand, taking any $t_0 \in R$, for any $\gamma \in \Lambda_1$, following the continuity of $\Phi_t^{-1}(\alpha)$ we have

$$X_{t_0}(\gamma) = \lim_{t\to t_0} X_t(\gamma) = \lim_{t\in Q, t\to t_0} X_t(\gamma) \leq \lim_{t\in Q, t\to t_0} \Phi_t^{-1}(\alpha) = \Phi_{t_0}^{-1}(\alpha).$$

For any $\gamma \in \Lambda_2$, we get

$$X_{t_0}(\gamma) = \lim_{t\to t_0} X_t(\gamma) = \lim_{t\in Q, t\to t_0} X_t(\gamma) \geq \lim_{t\in Q, t\to t_0} \Phi_t^{-1}(\alpha) = \Phi_{t_0}^{-1}(\alpha).$$

By the arbitrariness of t_0 and γ, we have

$$\{\gamma \in \Gamma | X_t(\gamma) \leq \Phi_t^{-1}(\alpha), t \in R\} \supseteq \Lambda_1,$$

and

$$\{\gamma \in \Gamma | X_t(\gamma) \geq \Phi_t^{-1}(\alpha), t \in R\} \supseteq \Lambda_2.$$

By using monotonicity theorem, we get

$$M\{\gamma \in \Gamma | X_t(\gamma) \leq \Phi_t^{-1}(\alpha), t \in R\} \geq M\{\Lambda_1\} = \alpha, \tag{7}$$

$$M\{\gamma \in \Gamma | X_t(\gamma) \geq \Phi_t^{-1}(\alpha), t \in R\} \geq M\{\Lambda_2\} = 1-\alpha. \tag{8}$$

It follows from (5) and (7) that

$$M\{X_t \le \Phi_t^{-1}(\alpha), t \in R\} = \alpha.$$

It follows from (6) and (8) that

$$M\{X_t \ge \Phi_t^{-1}(\alpha), t \in R\} = 1 - \alpha.$$

Therefore, the proof is finished. □

Remark 2. *It is also shown that for any $\alpha \in (0,1)$, the two following two equations are true,*

$$M\{X_t < \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = 1 - \alpha.$$

Example 1. *Take an uncertainty space (Γ, L, M) to be (0,1) with Borel algebra and Lebesgue measure. We can obtain the inverse uncertainty distribution of the uncertain process $X_t(\gamma) = t - \gamma$, $\forall \gamma \in L$ is*

$$\Phi_t^{-1}(\alpha) = t - 1 + \alpha.$$

Thus, we have

$$M\{\gamma \in \Gamma | X_t(\gamma) \le \Phi_t^{-1}(\alpha)\}$$

$$= M\{\gamma \in \Gamma | t - \gamma \le t - 1 + \alpha\} = M\{\gamma \in \Gamma | \gamma \in [1 - \alpha, 1)\} = \alpha,$$

$$M\{\gamma \in \Gamma | X_t(\gamma) > \Phi_t^{-1}(\alpha)\}$$

$$= M\{\gamma \in \Gamma | t - \gamma > t - 1 + \alpha\} = M\{\gamma \in \Gamma | \gamma \in (0, 1 - \alpha)\} = 1 - \alpha.$$

Example 2. *Take X_t to be a linear uncertain process $X_t \sim L(at, bt)$. Its inverse uncertainty distribution is $\Phi_t^{-1}(\alpha) = (1-\alpha)at + \alpha bt$. It follows from Theorem 5 that*

$$M\{X_t \le \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t \le (1-\alpha)at + \alpha bt\} = \alpha,$$

$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t > (1-\alpha)at + \alpha bt\} = 1 - \alpha.$$

4. Theorems for Inverse Uncertainty Distribution of the Monotone Function of Uncertain Processes

In order to deal with complicated problems, the monotone function of uncertain processes is often applied, which is indeed a new uncertain process. In this section, we further consider the inverse uncertainty distribution of the monotone function of uncertain processes and two theorems are derived.

Theorem 6. *Let (Γ, L, M) be a continuous uncertain space, and let Y_t be a sample-continuous independent increment process with regular uncertainty distribution $\Psi_t(x)$. Suppose $f(x)$ is a continuous, strictly monotone function and $X_t = f(Y_t)$ has an uncertainty distribution $\Phi_t(x)$ whose inverse distribution is $\Phi_t^{-1}(\alpha)$. Then for any $0 < \alpha < 1$, we have*

$$M\{X_t \le \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t < \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t \ge \Phi_t^{-1}(\alpha), t \in R\} = 1 - \alpha,$$
$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = 1 - \alpha.$$

Proof. First we certify $M\{X_t \le \Phi_t^{-1}(\alpha), t \in R\} = \alpha$.

Since f is a continuous, strictly monotone function and Y_t is a sample-continuous independent increment process, according to Definitions 5 and 6, we have that X_t is also a sample-continuous independent increment process. It breaks down two cases:

Case I: If f is a strictly increasing function, according to Theorem 4, we have $\Phi_t^{-1}(\alpha) = f(\Psi_t^{-1}(\alpha))$. It follows from Theorem 5 that

$$\begin{aligned} M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} &= M\{f(Y_t) \leq f(\Psi_t^{-1}(\alpha)), t \in R\} \\ &= M\{Y_t \leq \Psi_t^{-1}(\alpha), t \in R\} \\ &= \alpha. \end{aligned}$$

Case II: If f is a strictly decreasing function, according to Theorem 4, we have $\Phi_t^{-1}(\alpha) = f(\Psi_t^{-1}(1-\alpha))$. It follows from Theorem 5 that

$$\begin{aligned} M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} &= M\{f(Y_t) \leq f(\Psi_t^{-1}(1-\alpha)), t \in R\} \\ &= M\{Y_t \geq \Psi_t^{-1}(1-\alpha), t \in R\} \\ &= 1-(1-\alpha) = \alpha. \end{aligned}$$

Hence, if f is strictly increasing or strictly decreasing, we always have

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha.$$

Subsequently, we prove $M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha$. It also breaks down two cases:

Case I: If f is a strictly increasing function, by using Theorem 4, we have $\Phi_t^{-1}(\alpha) = f(\Psi_t^{-1}(\alpha))$. It follows from Theorem 5 that

$$\begin{aligned} M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} &= M\{f(Y_t) \geq f(\Psi_t^{-1}(\alpha)), t \in R\} \\ &= M\{Y_t \geq \Psi_t^{-1}(\alpha), t \in R\} \\ &= 1-\alpha. \end{aligned}$$

Case II: If f is a strictly decreasing function, by using Theorem 4, we have $\Phi_t^{-1}(\alpha) = f(\Psi_t^{-1}(1-\alpha))$. It follows from Theorem 5 that

$$\begin{aligned} M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} &= M\{f(Y_t) \geq f(\Psi_t^{-1}(1-\alpha)), t \in R\} \\ &= M\{Y_t \leq \Psi_t^{-1}(1-\alpha), t \in R\} \\ &= 1-\alpha. \end{aligned}$$

Hence, if f is strictly increasing or strictly decreasing, we always have

$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha.$$

Similarly, we may prove the other two equations as follows

$$M\{X_t < \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha.$$

Thus, the theorem is proven. □

Theorem 7. *Let (Γ, L, M) be a continuous uncertainty space and let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be a sample-continuous independent increment processes. Assume $y = f(x_1, x_2, \cdots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$. Suppose $X_t = f(X_{1t}, X_{2t} \cdots, X_{nt})$ is an uncertain process with inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. Then for any $0 < \alpha < 1$, we have*

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t < \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha,$$
$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha.$$

Proof. For any $0 < \alpha < 1$, we first prove $M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha$. For simplicity, we only prove the case $n = 2$. Denote X_{1t}, X_{2t} and Y_t, Z_t with uncertainty distributions $\Psi_t^{-1}(\alpha)$ and $\Omega_t^{-1}(\alpha)$, respectively. Assume f is a continuous, strictly increasing with respect to Y_t, and strictly decreasing with respect to Z_t. It follows from Theorem 4 that $X_t = f(Y_t, Z_t)$

has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha) = f(\Psi_t^{-1}(\alpha), \Omega_t^{-1}(1-\alpha)$. Note that we always have

$$\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} \equiv \{f(Y_t, Z_t) \leq f(\Psi_t^{-1}(\alpha), \Omega_t^{-1}(1-\alpha)), t \in R\}.$$

On the one hand, since f is strictly increasing with respect to Y_t, and strictly decreasing with respect to Z_t, we get

$$\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} \supseteq \{Y_t \leq \Psi_t^{-1}(\alpha), t \in R\} \cap \{Z_t \geq \Omega_t^{-1}(1-\alpha)), t \in R\}.$$

Following the monotonicity theorem, independence of Y_t and Z_t and Theorem 5, we have

$$\begin{aligned} M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} &\geq M\{\{Y_t \leq \Psi_t^{-1}(\alpha), t \in R\} \cap \{Z_t \geq \Omega_t^{-1}(1-\alpha), t \in R\}\} \\ &= M\{Y_t \leq \Psi_t^{-1}(\alpha), t \in R\} \wedge M\{Z_t \geq \Omega_t^{-1}(1-\alpha), t \in R\} \\ &= \alpha \wedge (1-(1-\alpha)) \\ &= \alpha. \end{aligned} \tag{9}$$

On the other hand, for any $t_0 \in R$, since f is strictly increasing with respect to Y_t, and strictly decreasing with respect to Z_t, we obtain

$$\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} \subseteq \{X_{t_0} \leq \Phi_{t_0}^{-1}(\alpha)\} \subseteq \{Y_{t_0} \leq \Psi_{t_0}^{-1}(\alpha)\} \cup \{Z_{t_0} \geq \Omega_{t_0}^{-1}(1-\alpha))\}.$$

By using the monotonicity theorem, independence of Y_t and Z_t and Theorem 2, we have

$$\begin{aligned} M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} &\leq M\{\{Y_{t_0} \leq \Psi_{t_0}^{-1}(\alpha)\} \cup \{Z_{t_0} \geq \Omega_{t_0}^{-1}(1-\alpha)\}\} \\ &= M\{Y_{t_0} \leq \Psi_{t_0}^{-1}(\alpha)\} \vee M\{Z_{t_0} \geq \Omega_{t_0}^{-1}(1-\alpha)\} \\ &= \alpha \vee (1-(1-\alpha)) \\ &= \alpha. \end{aligned} \tag{10}$$

It follows from the two inequalities (9) and (10) that $M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = \alpha$.
Now, we prove that $M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha$. Notice that we have

$$\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} \equiv \{f(Y_t, Z_t) \geq f(\Psi_t^{-1}(\alpha), \Omega_t^{-1}(1-\alpha)), t \in R\}.$$

On the one hand, since f is strictly increasing with respect to Y_t and strictly decreasing with respect to Z_t, we get

$$\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} \supseteq \{Y_t \geq \Psi_t^{-1}(\alpha), t \in R\} \cap \{Z_t \leq \Omega_t^{-1}(1-\alpha)), t \in R\}.$$

Following the monotonicity theorem, independence of Y_t and Z_t and Theorem 5, we have

$$\begin{aligned} M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} &\geq M\{\{Y_t \geq \Psi_t^{-1}(\alpha), t \in R\} \cap \{Z_t \leq \Omega_t^{-1}(1-\alpha), t \in R\}\} \\ &= M\{Y_t \geq \Psi_t^{-1}(\alpha), t \in R\} \wedge M\{Z_t \leq \Omega_t^{-1}(1-\alpha), t \in R\} \\ &= (1-\alpha) \wedge (1-\alpha) \\ &= 1-\alpha. \end{aligned} \tag{11}$$

On the other hand, for any $t_s \in R$, since f is strictly increasing with respect to Y_t and strictly decreasing with respect to Z_t, we obtain

$$\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} \subseteq \{X_{t_s} \geq \Phi_{t_s}^{-1}(\alpha)\} \subseteq \{Y_{t_s} \geq \Psi_{t_s}^{-1}(\alpha)\} \cup \{Z_{t_s} \leq \Omega_{t_s}^{-1}(1-\alpha)\}.$$

By using the monotonicity theorem, the independence of Y_t and Z_t and Theorem 2 that

$$\begin{aligned} M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} &\leq M\{\{Y_{t_s} \geq \Psi_{t_s}^{-1}(\alpha)\} \cup \{Z_{t_s} \leq \Omega_{t_s}^{-1}(1-\alpha)\}\} \\ &= M\{Y_{t_s} \geq \Psi_{t_s}^{-1}(\alpha)\} \vee M\{Z_{t_s} \leq \Omega_{t_s}^{-1}(1-\alpha)\} \\ &= (1-\alpha) \vee (1-\alpha) \\ &= 1-\alpha. \end{aligned} \tag{12}$$

It follows from the two inequalities (11) and (12) that $M\{X_t \geq \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha$. Similarly, we may prove the other two equations as follows,

$$M\{X_t < \Phi_t^{-1}(\alpha), t \in R\} = \alpha,$$
$$M\{\gamma \in \Gamma | X_t(\gamma) > \Phi_t^{-1}(\alpha), t \in R\} = 1-\alpha.$$

Thus, the theorem is verified. □

Example 3. *Take Y_t to be a linear uncertain process $Y_t \sim L(at, bt)$ whose inverse uncertainty distribution $\Psi_t^{-1}(\alpha) = (1-\alpha)at + \alpha bt$ and Z_t to be normal uncertain process $Z_t \sim N(et, \sigma t)$, whose inverse uncertainty distribution $\Omega_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}$. Assume $f(x_1, x_2) = x_1 + x_2$, which is increasing with respect to x_1 and x_2. Therefore, $X_t = f(Y_t, Z_t) = Y_t + Z_t$ has an inverse uncertainty distribution*

$$\Phi_t^{-1}(\alpha) = (1-\alpha)at + \alpha bt + et + \frac{\sigma t\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}.$$

It follows from Theorem 7 that

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t \leq (1-\alpha)at + \alpha bt + et + \frac{\sigma t\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}\} = \alpha,$$

$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t > (1-\alpha)at + \alpha bt + et + \frac{\sigma t\sqrt{3}}{\pi} ln \frac{\alpha}{1-\alpha}\} = 1-\alpha.$$

Example 4. *In Example 3, assume $g(x_1, x_2) = x_1 - x_2$, which is increasing with respect to x_1 and decreasing with respect to x_2. Therefore, $X_t = g(Y_t, Z_t) = Y_t - Z_t$ has an inverse uncertainty distribution*

$$\Phi_t^{-1}(\alpha) = (1-\alpha)at + \alpha bt - et - \frac{\sigma t\sqrt{3}}{\pi} ln \frac{1-\alpha}{\alpha}.$$

It follows from Theorem 7 that

$$M\{X_t \leq \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t \leq (1-\alpha)at + \alpha bt - et - \frac{\sigma t\sqrt{3}}{\pi} ln \frac{1-\alpha}{\alpha}\} = \alpha,$$

$$M\{X_t > \Phi_t^{-1}(\alpha), t \in R\} = M\{X_t > (1-\alpha)at + \alpha bt - et - \frac{\sigma t\sqrt{3}}{\pi} ln \frac{1-\alpha}{\alpha}\} = 1-\alpha.$$

5. Conclusions

The uncertain process is a sequence of uncertain variables indexed by time for modeling uncertain phenomena. The inverse uncertainty distribution plays an important role in describing uncertain processes. A conjecture of inverse uncertainty distribution of an independent increment process is proven in this paper, and some theorems based on this conjecture are also obtained.

We will continue to study the application of these theorems. We will calculate the inverse uncertainty distributions of some special uncertain processes via these theorems. Based on this, the expected value and variance of uncertain processes may be investigated. We will also study modeling and solving problems with uncertain processes in the future.

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