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Symmetric Spaces Approach to Various Cyclic Contractions and Application to Probabilistic Spaces

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Abstract: This paper aims to prove fixed point results for cyclic compatible contraction and Hardy–Rogers cyclic contraction in symmetric spaces. Our results generalize the results of Kumari and Panthi (2016) proved for cyclic compatible contraction and modified Hardy–Rogers cyclic contraction in the generating space of a b-quasi metric family and b-dislocated metric family. After that, as an application, we prove a fixed point result in symmetric pre-probabilistic metric spaces (PPM-spaces).

Keywords: symmetric spaces; fixed point theorems; cyclic contractions; probabilistic spaces

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1. Introduction

Metric spaces are characterized by the applicability of several conventions. Wilson [1] has observed and introduced two such applicable conventions i.e., semi-metric spaces and symmetric spaces. For the first time during 1922 in the area of fixed point theory, Banach brought out the concept of contraction mapping and it was later called the Banach Contraction Principle. Further, during the course of time, many authors like Ciric [2], Reich [3], Kannan [4], etc., have extended and made this principle more widespread in mathematics.

Later in 2003, this Banach Contraction Principle was further generalized by Kirk et al. [5] with the use of cyclic contraction and by Karpagam and Agrawal [6] with the use of the concept cyclic orbital contraction and examined for the existence of fixed points for such maps.

In 2016, Kumari and Panthi [7,8] introduced new versions of Hardy–Rogers type cyclic contraction (known as modified Hardy–Rogers cyclic contraction) and the concept of cyclic compatible contraction and proved fixed point theorems for these contractions in b-dislocated metric family and in the generating space of a b-quasi metric family respectively. In 1976, Cicchese affirmed the first fixed point theorem for contraction mapping in semi-metric spaces. Further, for this class of spaces, fixed point results were attained by Jachymski et al. [9], Hicks and Rhoades [10], Aamri and Moutawakil [11], and the references cited therein.

In this paper, we prove coincidence and fixed point theorems for cyclic compatible contraction and Hardy–Rogers cyclic orbital contraction in symmetric spaces. Our results generalize the results of Kumari and Panthi [7,8] proved for cyclic compatible contraction and modified Hardy–Rogers cyclic contraction in the generating space of a b-quasi metric family and b-dislocated metric family. Additionally, we derive a fixed point result in symmetric pre-probabilistic metric spaces (PPM-spaces).

2. Preliminaries and Definitions

Definition 1 ([12]). *A symmetric space is a pair $(\mathcal{U}, d)$ consisting of a non-empty set U and a non-negative real valued function d defined on $\mathcal{U} \times \mathcal{U}$ such that the following conditions hold for all $\mu, \nu \in \mathcal{U}$,*

(i) $d(\mu, \nu) = 0$ *if and only if* $\mu = \nu$,

(ii) $d(\mu, \nu) = d(\nu, \mu)$.

The open ball having centre $\mu \in \mathcal{U}$ and radius $r > 0$ is outlined by

$$B(\mathcal{U}, r) = \{\nu \in U : d(\mu, \nu) < r\}.$$

Several properties in symmetric spaces are analogous to the properties in metric spaces but not all, due to the absence of the triangle inequality.

A sequence $\{\mu_n\} \subseteq \mathcal{U}$ is forenamed as d-Cauchy sequence if for given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $d(\mu_m, \mu_n) < \epsilon$ for all $m, n \geq N$.

In every symmetric space $(\mathcal{U}, d)$, one may bring up the topology $t(d)$ by defining the family of closed sets as follows: a set $A \subseteq \mathcal{U}$ is closed if and only if for each $\mu \in X, d(\mu, A) = 0$ implies $\mu \in \mathcal{A}$ where $d(\mu, \mathcal{A}) = \inf\{d(\mu, a) : a \in \mathcal{A}\}$.

Let d be a symmetric space on a set $\mathcal{U}$ and for $\epsilon > 0$ and any $\mu \in \mathcal{U}$, let $\mathcal{B}(\mu, \epsilon) = \{\nu \in \mathcal{U} : d(\mu, \nu) < \epsilon\}$. A topology $t(d)$ on $\mathcal{U}$ is given by $G \in t(d)$ if for each $\mu \in G, B(\mu, \epsilon) \subseteq \mathcal{U}$ for some $\epsilon > 0$. If for each $\mu \in \mathcal{U}$ and any $\epsilon > 0, B(\mu, \epsilon)$ is a neighbourhood of μ in the topology $t(d)$ then a symmetric space d is a semi-metric. A sequence is d-Cauchy if it entertains the usual metric condition.

Definition 2 ([13]). *Let $(\mathcal{U}, d)$ be a symmetric space.*

(i) *$(\mathcal{U}, d)$ is S-complete if for every d-Cauchy sequence $\{\mu_n\}$, there exists an element μ in $\mathcal{U}$ with* $\lim_{n\to\infty} d(\mu_n, \mu) = 0$;

(ii) *$(\mathcal{U}, d)$ is d-Cauchy complete if for every d-Cauchy sequence $\{\mu_n\}$, there exists an element μ in $\mathcal{U}$ with $\lim_{n\to\infty} \mu_n = x$ with respect to $t(d)$;*

(iii) *$f : \mathcal{U} \to \mathcal{U}$ is d-continuous if $\lim_{n\to\infty} d(\mu_n, \mu) = 0$ implies $\lim_{n\to\infty} d(f\mu_n, f\mu) = 0$;*

(iv) *$f : \mathcal{U} \to \mathcal{U}$ is $t(d)$-continuous if $\lim_{n\to\infty} \mu_n = \mu$ with respect to $t(d)$ implies $\lim_{n\to\infty} f\mu_n = f\mu$ with respect to $t(d)$.*

If d is a semi-metric on $\mathcal{U}$, then $\lim_{n\to\infty} d(\mu_n, \mu) = 0$ is identical to $\lim_{n\to\infty} \mu_n = \mu$ with respect to $t(d)$, d-continuity of f is identical to $t(d)$ continuity of f, S-completeness of $(\mathcal{U}, d)$ is identical to d-cauchy completeness of $(\mathcal{U}, d)$.

The conditions mentioned below can be used as partial replacements for the triangle inequality's absence in the symmetric space $(\mathcal{U}, d)$:

(W) $\lim_{n\to\infty} d(\mu_n, \nu_n) = 0$ and $\lim_{n\to\infty} d(\nu_n, \xi_n) = 0 \implies \lim_{n\to\infty} d(\mu_n, \xi_n) = 0$;

(W_3) $\lim_{n\to\infty} d(\mu_n, \mu) = 0$ and $\lim_{n\to\infty} d(\mu_n, \nu) = 0 \implies \mu = \nu$;

(MT) there exists $s \geq 1$ such that for any $\mu, \nu, \xi \in \mathcal{U}$, $d(\mu, \xi) \leq s(d(\mu, \nu) + d(\nu, \xi))$.

The property W_3 was induced by Wilson [1], W by Mihet [14], and MT by Czerwik [15].

Definition 3. *Let f be any self mapping defined on a non-empty set $\mathcal{U}$ then $\mu \in \mathcal{U}$ is said to be a fixed point of f if $f\mu = \mu$.*

Definition 4 ([7]). *Let P and Q be non-empty subsets of a set $\mathcal{U}$. A map $\mathcal{A} : P \cup Q \to P \cup Q$ is said to be a cyclic map if $\mathcal{A}P \subseteq Q$ and $\mathcal{A}Q \subseteq P$.*

In the following, since P and Q will be always considered as closed sets and $(\mathcal{U}, d)$ a S-complete (d-Cauchy complete) symmetric (semi-metric) space, then, without loss of generality, we can suppose $\mathcal{U} = P \cup Q$. Indeed, closed subsets of S-complete (d-Cauchy complete) symmetric (semi-metric) spaces define S-complete (d-Cauchy complete) symmetric (semi-metric) subspaces.

Definition 5 ([8]). *Let $(\mathcal{U}, d)$ be a S-complete symmetric space and T, S be two mappings. Then T and S are said to be weakly compatible if they commute at their coincidence point $\mu \in \mathcal{U}$, that is, $T\mu = S\mu$ implies $TS\mu = ST\mu$.*

3. Main Results

In this section, we prove some fixed point theorems in the relation of a symmetric space.

Definition 6. *Let $\mathcal{U} = P \cup Q$ be a symmetric space. Suppose $\mathcal{A}, \mathcal{B} : \mathcal{U} \to \mathcal{U}$ are cyclic mappings such that $\mathcal{AU} \subseteq \mathcal{BU}$ and there exists $\kappa \in (0, 1)$ such that*

$$d(\mathcal{A}^{2n}\mu, \mathcal{A}\nu) \leq \kappa d(\mathcal{A}^{2n-1}\mu, \mathcal{B}\nu) \tag{1}$$

for any $\mu, \nu \in P$ and $n \in \mathbb{N}$. Then $\mathcal{A}$ and $\mathcal{B}$ are forenamed as cyclic compatible contraction.

Theorem 1. *Let d be a bounded symmetric (semi-metric) having property W and $(\mathcal{U}, d)$ is S-complete (d-Cauchy complete). Presume that P and Q are non-empty closed sets and contained in $\mathcal{U}$. Suppose $\mathcal{A}, \mathcal{B} : P \cup Q \to P \cup Q$ is a cyclic compatible contraction and $\mathcal{BU}$ is closed, $\mathcal{BU} \subseteq U$. Then, $\mathcal{A}$ and $\mathcal{B}$ have a point of coincidence in $P \cap Q$. In addition, weakly compatibility of mappings $\mathcal{A}$ and $\mathcal{B}$ will give exactly one common fixed point in $P \cap Q$.*

Proof. Let $\mu_0 \in P$ be an arbitrary point. Since $\mathcal{AU} \subseteq \mathcal{BU}$, we may define $\gamma_0 \in \mathcal{U}$ such that $A\mu_0 = B\gamma_0$ and $\mu_1 = B\gamma_0$ where $\mu_1 \in \mathcal{U}$. Hence we can outline the sequence $\{\mu_n\}$ and $\{\gamma_n\}$ in $\mathcal{U}$ by

$$\mathcal{A}\mu_n = \mathcal{B}\gamma_n, \mathcal{B}\gamma_n = \mu_{n+1} \tag{2}$$

for $n \in \mathbb{N} \cup \{0\}$. Then $\{\mu_{2n}\} \in P$ and $\{\mu_{2n+1}\} \in Q$ for any $n \in \mathbb{N} \cup \{0\}$. Here,

$$\mathcal{A}^n\mu_0 = \mathcal{A}^{n-1}\mathcal{B}\gamma_0 = A^{n-1}\mu_1 = \cdots = \mathcal{A}\mu_n.$$

$$\begin{aligned}
d(\mu_{2n}, \mu_{2n+1}) &= d(\mathcal{B}\gamma_{2n-1}, \mathcal{B}\gamma_{2n}) \\
&= d(\mathcal{A}\mu_{2n-1}, \mathcal{A}\mu_{2n}) \\
&= d(\mathcal{A}\mu_{2n}, \mathcal{A}\mu_{2n-1}) \\
&= d(\mathcal{A}^{2n}\mu_0, \mathcal{A}\mu_{2n-1}) \\
&\leq \kappa d(\mathcal{A}^{2n-1}\mu_0, \mathcal{B}\mu_{2n-1}) \\
&= \kappa d(\mathcal{B}\gamma_{2n-1}, \mathcal{B}\mu_{2n-1}) \\
&= \kappa d(\mu_{2n}, \mu_{2n-1}).
\end{aligned}$$

Similarly

$$\begin{aligned}
d(\mu_{2n+1}, \mu_{2n+2}) &= d(\mathcal{B}\gamma_{2n}, \mathcal{B}\gamma_{2n+1}) \\
&= d(\mathcal{A}\mu_{2n}, \mathcal{A}\mu_{2n+1}) \\
&= d(\mathcal{A}^{2n}\mu_0, \mathcal{A}\mu_{2n+1}) \\
&\leq \kappa d(\mathcal{A}^{2n-1}\mu_0, \mathcal{B}\mu_{2n+1}) \\
&= \kappa d(\mathcal{B}\gamma_{2n-1}, \mathcal{B}\mu_{2n+1}) \\
&= \kappa d(\mu_{2n}, \mu_{2n+1}).
\end{aligned}$$

Inductively, for every $n \in \mathbb{N}$, we get

$$d(\mu_n, \mu_{n+1}) \leq \kappa^n d(\mu_0, \mu_1). \tag{3}$$

Since $\kappa \in (0,1)$ therefore $\kappa^n \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = 0$.

By W, we have $\lim_{n\to\infty} d(\mu_n, \mu_{n+p}) = 0$, therefore $\{\mu_n\}$ is a d-Cauchy sequence in the S-complete symmetric space $\mathcal{U}$. Then, there exists subsequences $\{\mathcal{B}^{2n}\gamma_0\} \in P$ and $\{\mathcal{B}^{2n-1}\gamma_0\} \in Q$ such that both converge to some β in $\mathcal{U}$. Since P and Q are closed in $\mathcal{U}$, therefore $\beta \in P \cap Q$.

Due to the closeness of $\mathcal{BU}$, there is $\nu \in \mathcal{U}$ such that

$$\mathcal{B}\nu = \beta. \tag{4}$$

From the above argument, property W and (4), there exists sequences $\{\mathcal{A}^{2n-1}\mu_0\}$ in P and $\{\mathcal{A}^{2n-2}\mu_0\}$ in Q such that both converge to β.

Consider

$$d(\mathcal{A}^{2n-1}\mu_0, \mathcal{A}\nu) \leq \kappa d(\mathcal{A}^{2n-2}\mu_0, \mathcal{B}\nu).$$

By letting $n \to \infty$,

$$d(\beta, \mathcal{A}\nu) \leq \kappa d(\beta, \mathcal{B}\nu).$$

This returns $d(\beta, \mathcal{A}\nu) = 0$. Thus

$$\beta = \mathcal{A}\nu. \tag{5}$$

From (4) and (5), we have $\mathcal{A}\nu = \mathcal{B}\nu = \beta$. Thus ν is a point of coincidence for $\mathcal{A}$ and $\mathcal{B}$.

From the weak compatibility, we get

$$\mathcal{A}\beta = \mathcal{B}\beta. \tag{6}$$

Consider,

$$\begin{aligned} d(\mathcal{B}\beta, \beta) &= \lim_{n\to\infty} d(\mathcal{A}\beta, \mathcal{A}^{2n-1}\mu_0) \\ &\leq \kappa \lim_{n\to\infty} d(\mathcal{A}^{2n-2}\mu_0, \mathcal{B}\beta) \\ &= \kappa d(\beta, \mathcal{B}\beta) \end{aligned}$$

This implies $(1-\kappa)d(\beta, \mathcal{B}\beta) \leq 0$. Therefore, $d(\beta, \mathcal{B}\beta) = 0$.
Thus $\beta = \mathcal{B}\beta$.
From (6), we get $\mathcal{A}\beta = \mathcal{B}\beta = \beta$. Hence β is a common fixed point of $\mathcal{A}$ and $\mathcal{B}$.
Concerning uniqueness, let β_1 and β_2 be two common fixed points of $\mathcal{A}$ and $\mathcal{B}$.
Consider,

$$\begin{aligned} d(\beta_1, \beta_2) &= \lim_{n\to\infty} d(\mathcal{A}^{2n-1}\mu_0, \mathcal{A}\beta_2) \\ &\leq \kappa \lim_{n\to\infty} d(\mathcal{A}^{2n-2}\mu_0, \mathcal{B}\beta_2) \\ &= \kappa d(\beta_1, \mathcal{B}\beta_2) \\ &= \kappa d(\beta_1, \beta_2) \end{aligned}$$

This implies $(1-\kappa)d(\beta_1, \beta_2) \leq 0$.
Since $0 < \kappa < 1$, therefore $\beta_1 = \beta_2$.
This finalizes the proof. □

Theorem 2. *Let d be a bounded symmetric (semi-metric) having property W and $(\mathcal{U}, d)$ is S-complete (d-Cauchy complete). Presume that P and Q are closed sets and contained in $\mathcal{U}$. Suppose $\mathcal{A}, \mathcal{B} : P \cup Q \to P \cup Q$ are cyclic mappings such that range of $\mathcal{A}$ contained in the range of $\mathcal{B}$ and $\mathcal{BU}$ is closed, $\mathcal{BU} \subseteq \mathcal{U}$ where $\mathcal{A}, \mathcal{B} \subseteq \mathcal{U}$ are non-empty and closed. Suppose for any $\mu, \nu \in P$, $n \in \mathbb{N}$ and $\kappa \in (0,1)$ there exists*

$$\omega = \omega(\mu, \nu, n) \in \{d(\mathcal{B}\mu, \mathcal{B}\nu), d(\mathcal{A}^{n-1}\mu, \mathcal{B}\mu), d(\mathcal{A}^{n-1}\nu, \mathcal{B}\nu), \frac{d(\mathcal{A}^{n-1}\mu, \mathcal{B}\nu) + d(\mathcal{A}^{n-1}\nu, \mathcal{B}\mu)}{2}\}$$

such that $d(\mathcal{B}^n\mu, \mathcal{B}\nu) \leq \kappa\omega$ for $n \in \mathbb{N}$ and $\nu \in P$. Then, $\mathcal{A}$ and $\mathcal{B}$ have a point of coincidence in $P \cap Q$. In addition, weakly compatibility of mappings $\mathcal{A}$ and $\mathcal{B}$ will give exactly one common fixed point in $P \cap Q$.

Proof. Let $\mu_0 \in P$ be an arbitrary point. Since $\mathcal{AU} \subseteq \mathcal{BU}$, we may define $\gamma_0 \in \mathcal{U}$ such that $A\mu_0 = B\gamma_0$ and $\mu_1 = B\gamma_0$ where $\mu_1 \in \mathcal{U}$. Hence we can outline the sequence $\{\mu_n\}$ and $\{\gamma_n\}$ in $\mathcal{U}$ by

$$\mathcal{A}\mu_n = \mathcal{B}\gamma_n, \mathcal{B}\gamma_n = \mu_{n+1} \tag{7}$$

for $n \in \mathbb{N} \cup \{0\}$. Then $\{\mu_{2n}\} \in P$ and $\{\mu_{2n+1}\} \in Q$ for any $n \in \mathbb{N} \cup \{0\}$.

Here,

$$\mathcal{A}^n\mu_0 = \mathcal{A}^{n-1}\mathcal{B}\gamma_0 = A^{n-1}\mu_1 = \cdots = \mathcal{A}\mu_n.$$

Consider

$$\begin{aligned} d(\mu_2, \mu_1) &= d(\mathcal{B}\gamma_1, \mathcal{B}\gamma_0) \\ &= d(\mathcal{A}\mu_1, \mathcal{A}\mu_0) \\ &\leq \kappa\omega \end{aligned}$$

where

$$\begin{aligned} \omega &\in \{d(\mathcal{B}\mu_1, \mathcal{B}\mu_0), d(\mathcal{A}^0\mu_1, \mathcal{B}\mu_1), d(\mathcal{A}^0\mu_0, \mathcal{B}\mu_0), \frac{d(\mathcal{A}^0\mu_1, \mathcal{B}\mu_0) + d(\mathcal{A}^0\mu_0, \mathcal{B}\mu_1)}{2}\} \\ &= \{d(\mu_1, \mu_0), d(\mu_1, \mu_1), d(\mu_0, \mu_0), \frac{d(\mu_1, \mu_0) + d(\mu_0, \mu_1)}{2}\} \\ &= \{d(\mu_1, \mu_0), 0\} \end{aligned}$$

Therefore,

$$d(\mu_2, \mu_1) \leq \kappa d(\mu_1, \mu_0).$$

Similarly,

$$\begin{aligned} d(\mu_3, \mu_2) &= d(\mathcal{B}\gamma_2, \mathcal{B}\gamma_1) \\ &= d(\mathcal{A}\mu_2, \mathcal{A}\mu_1) \\ &\leq \kappa\omega, \end{aligned}$$

where

$$\begin{aligned} \omega &\in \{d(\mathcal{B}\mu_2, \mathcal{B}\mu_1), d(\mathcal{A}^0\mu_2, \mathcal{B}\mu_2), d(\mathcal{A}^0\mu_1, \mathcal{B}\mu_1), \frac{d(\mathcal{A}^0\mu_2, \mathcal{B}\mu_1) + d(\mathcal{A}^0\mu_1, \mathcal{B}\mu_2)}{2}\} \\ &= \{d(\mu_2, \mu_1), d(\mu_2, \mu_2), d(\mu_1, \mu_1), d(\mu_2, \mu_1)\} \\ &= \{d(\mu_2, \mu_1), 0\} \end{aligned}$$

thus

$$\begin{aligned} d(\mu_3, \mu_2) &\leq \kappa d(\mu_2, \mu_1) \\ &\leq \kappa^2 d(\mu_1, \mu_0) \end{aligned}$$

hence $\forall\ n \in \mathbb{N}$, by using induction, we get

$$d(\mu_{n+1}, \mu_n) \leq \kappa^n d(\mu_1, \mu_0).$$

Since $\kappa \in (0,1)$ therefore $\kappa^n \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} d(\mu_{n+1}, \mu_n) = 0$.

By W, we have $\lim_{n\to\infty} d(\mu_n, \mu_{n+p}) = 0$, therefore $\{\mu_n\}$ is a d-Cauchy sequence in the S-complete symmetric space $\mathcal{U}$. Then, there exists subsequences $\{\mathcal{B}^{2n}\gamma_0\} \in P$ and $\{\mathcal{B}^{2n-1}\gamma_0\} \in Q$ such that both converge to some β in $\mathcal{U}$. Since P and Q are closed in $\mathcal{U}$, therefore $\beta \in P \cap Q$.

Since $\mathcal{BU}$ is closed, there is $\nu \in \mathcal{U}$ such that

$$\mathcal{B}\nu = \beta. \tag{8}$$

From the above argument, property W and Equation (8), there exists sequences $\{\mathcal{A}^{2n-1}\mu_0\}$ in P and $\{\mathcal{A}^{2n-2}\mu_0\}$ in Q such that both converge to β.

Consider

$$\begin{aligned} d(\mathcal{A}\nu, \beta) &= d(\mathcal{B}^2\nu, \mathcal{B}\nu) \\ &\leq \kappa d(\mathcal{B}\nu, \mathcal{B}^0\nu) \\ &= \kappa d(\nu, \nu) = 0, \end{aligned}$$

i.e, $d(\mathcal{A}\nu, \mathcal{B}\nu) = 0$ implies that

$$\mathcal{A}\nu = \mathcal{B}\nu. \tag{9}$$

Thus from (8) and (9), we get $\mathcal{A}\nu = \mathcal{B}\nu = \beta$. Thus ν is a point of coincidence for $\mathcal{A}$ and $\mathcal{B}$.

From the weak compatibility, we get

$$\mathcal{A}\beta = \mathcal{B}\beta. \tag{10}$$

Consider

$$\begin{aligned} d(\mathcal{A}\beta, \beta) &= d(\mathcal{B}^2\beta, \mathcal{B}\beta) \\ &\leq \kappa d(\mathcal{B}\beta, \beta) \\ &= \kappa d(\mathcal{A}\beta, \beta). \end{aligned}$$

This implies $(1-\kappa)d(\mathcal{A}\beta, \beta) \leq 0$.

Therefore, $d(\mathcal{A}\beta, \beta) = 0$ since $(1-\kappa) \geq 0$.

Thus

$$\beta = \mathcal{A}\beta. \tag{11}$$

From (10) and (11), $\mathcal{A}\beta = \mathcal{B}\beta = \beta$.

Concerning uniqueness, let β_1 and β_2 be two common fixed points of $\mathcal{A}$ and $\mathcal{B}$.

Consider,

$$\begin{aligned} d(\beta_1, \beta_2) &= d(\mathcal{A}\beta_1, \mathcal{A}\beta_2) \\ &= d(\mathcal{B}^2\beta_1, \mathcal{B}\beta_2) \\ &\leq \kappa d(\mathcal{B}\beta_1, \beta_2) \\ &= \kappa d(\beta_1, \beta_2). \end{aligned}$$

This implies $(1-\kappa)d(\beta_1, \beta_2) \leq 0$.

Since $0 < \kappa < 1$, therefore $\beta_1 = \beta_2$.

This finalizes the proof. □

Now, before defining the modified Hardy–Rogers cyclic contraction, we recall the property MT.

(MT) there exists $s \geq 1$ such that for any $\mu, \nu, \xi \in \mathcal{U}$, $d(\mu, \xi) \leq s(d(\mu, \nu) + d(\nu, \xi))$.

Definition 7. *Let $(\mathcal{U}, d)$ be a S-complete symmetric space having property MT and let P and Q be non-empty closed subsets of $\mathcal{U}$. A cyclic map $\mathcal{A} : P \cup Q \to P \cup Q$ is forenamed as modified Hardy–Rogers cyclic contraction if we have* $d(\mathcal{A}\mu, \mathcal{A}\nu) \leq \alpha d(\mu, \nu) + \beta d(\mu, \mathcal{A}\nu) + \kappa d(\nu, \mathcal{A}\mu) + \delta d(\nu, \mathcal{A}\nu) + \eta \frac{d(\nu, \mathcal{A}\nu)[1+d(\mu, \mathcal{A}\mu)]}{1+d(\mu, \nu)} + \lambda \frac{d(\nu, \mathcal{A}\nu)+d(\nu, \mathcal{A}\mu)}{1+d(\nu, \mathcal{A}\nu)d(\nu, \mathcal{A}\mu)} + \rho \frac{d(\mu, \mathcal{A}\mu)[1+d(\nu, \mathcal{A}\mu)]}{1+d(\mu, \nu)+d(\nu, \mathcal{A}\nu)}$, $\forall\ \mu, \nu \in \mathcal{U}$ *where* $\alpha, \beta, \kappa, \delta, \eta, \lambda, \rho \geq 0$ *with* $s\alpha + (s^2+s)\beta + 2s^2\kappa + \delta + \eta + \lambda + s\rho < 1$ *and* $s \geq 1$.

Theorem 3. *Let d be a bounded symmetric (semi-metric) having properties W_3, W, MT and $(\mathcal{U}, d)$ is S-complete (d-Cauchy complete). Presume that P and Q be non-empty closed sets and contained in $\mathcal{U}$. Suppose $\mathcal{A} : P \cup Q \to P \cup Q$ is a d-continuous modified Hardy–Rogers cyclic contraction. Then $\mathcal{A}$ has exactly one fixed point in $P \cap Q$.*

Proof. Let μ be an arbitrary point in $\mathcal{U}$. Outline a sequence $\{\mu_n\}$ as $\mu_{n+1} = \mathcal{A}\mu_n \ \forall \ n \in \mathbb{N}$.

$$\begin{aligned}
d(\mathcal{A}\mu, \mathcal{A}^2\mu) &= d(\mathcal{A}\mu, \mathcal{A}(\mathcal{A}\mu)) \\
&\leq \alpha d(\mu, \mathcal{A}\mu) + \beta d(\mu, \mathcal{A}^2\mu) + \kappa d(\mathcal{A}\mu, \mathcal{A}\mu) + \delta d(\mathcal{A}\mu, \mathcal{A}^2\mu) \\
&\quad + \eta \frac{d(\mathcal{A}\mu, \mathcal{A}^2\mu[1 + d(\mu, \mathcal{A}\mu)])}{1 + d(\mu, \mathcal{A}\mu)} + \lambda \frac{d(\mathcal{A}\mu, \mathcal{A}^2\mu) + d(\mathcal{A}\mu, \mathcal{A}\mu)}{1 + d(\mathcal{A}\mu, \mathcal{A}^2\mu)d(\mathcal{A}\mu, \mathcal{A}\mu)} \\
&\quad + \rho \frac{d(\mu, \mathcal{A}\mu)[1 + d(\mathcal{A}\mu, \mathcal{A}\mu)]}{1 + d(\mu, \mathcal{A}\mu) + d(\mathcal{A}\mu, \mathcal{A}^2\mu)} \\
&= \alpha d(\mu, \mathcal{A}\mu) + s\beta[d(\mu, \mathcal{A}\mu) + d(\mathcal{A}\mu, \mathcal{A}^2\mu)] + 2s\kappa d(\mu, \mathcal{A}\mu) \\
&\quad + \delta d(\mathcal{A}\mu, \mathcal{A}^2\mu) + \eta d(\mathcal{A}\mu, \mathcal{A}^2\mu) + \lambda d(\mathcal{A}\mu, \mathcal{A}^2\mu) + \rho d(\mu, \mathcal{A}\mu),
\end{aligned}$$

which implies

$$(1 - s\beta - \delta - \eta - \lambda)d(\mathcal{A}\mu, \mathcal{A}^2\mu) \leq (\alpha + s\beta + 2s\kappa + \rho)d(\mu, \mathcal{A}\mu).$$

Clearly,

$$d(\mathcal{A}\mu, \mathcal{A}^2\mu) \leq kd(\mu, \mathcal{A}\mu).$$

where $k = \frac{\alpha + s\beta + 2s\kappa + \rho}{1 - s\beta - \delta - \eta - \lambda} < 1$.

Similarly,

$$\begin{aligned}
d(\mathcal{A}^2\mu, \mathcal{A}^3\mu) &\leq kd(\mathcal{A}\mu, \mathcal{A}^2\mu) \\
&\leq k^2 d(\mu, \mathcal{A}\mu).
\end{aligned}$$

In general, we get

$$d(\mathcal{A}^n\mu, \mathcal{A}^{n+1}\mu) \leq k^n d(\mu, \mathcal{A}\mu).$$

Since $k \in (0, 1)$ therefore $k^n \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} d(\mathcal{A}^n\mu, \mathcal{A}^{n+1}\mu) = 0$.

By W, we have $\lim_{n\to\infty} d(\mathcal{A}^n\mu, \mathcal{A}^{n+p}\mu) = 0$ for $n \in \mathbb{N}$ and $p \geq 1$, therefore $\{\mathcal{A}^n\mu\}$ is a d-Cauchy sequence in the S-complete symmetric space $\mathcal{U}$. Since $(\mathcal{U}, d)$ is S-complete, therefore there exist $\xi \in \mathcal{U}$ with $\lim_{n\to\infty} d(\mathcal{A}^n\mu, \xi) = 0$. Then, there exists subsequences $\{\mathcal{A}^{2n}\mu\} \in P$ and $\{\mathcal{A}^{2n-1}\mu\} \in Q$ such that both converge to ξ in $\mathcal{U}$. Since P and Q are closed in $\mathcal{U}$, therefore $\xi \in P \cap Q$. This gives $P \cap Q \neq \phi$.

Now, we will show that $\mathcal{A}\xi = \xi$.

Since $\mathcal{A}$ is d-continuous therefore $\lim_{n\to\infty} d(\mathcal{A}^{n+1}\mu, \mathcal{A}\xi) = 0$. Now we get that $d(\mathcal{A}\xi, \xi) = 0$ because $(\mathcal{U}, d)$ satisfies W_3.

Thus $\mathcal{A}\xi = \xi$. Hence ξ is a fixed point of $\mathcal{A}$.

Finally, to attain the uniqueness of the fixed point, let ξ_1 and ξ_2 be two fixed points of A.

Then we have

$$\begin{aligned}
d(\xi_1, \xi_2) &= d(\mathcal{A}\xi_1, \mathcal{A}\xi_2) \\
&\leq \alpha d(\xi_1, \xi_2) + \beta d(\xi_1, \mathcal{A}\xi_2) + \kappa d(\xi_2, \mathcal{A}\xi_1) + \delta d(\xi_2, \mathcal{A}\xi_2) \\
&\quad + \eta \frac{d(\xi_2, \mathcal{A}\xi_2)[1 + d(\xi_1, \mathcal{A}\xi_1)])}{1 + d(\xi_1, \mathcal{A}\xi_2)} + \lambda \frac{d(\xi_2, \mathcal{A}\xi_2) + d(\xi_2, \mathcal{A}\xi_1)}{1 + d(\xi_2, \mathcal{A}\xi_2)d(\xi_2, \mathcal{A}\xi_1)} \\
&\quad + \rho \frac{d(\xi_1, \mathcal{A}\xi_1)[1 + d(\xi_2, \mathcal{A}\xi_1)]}{1 + d(\xi_1, \xi_2) + d(\xi_2, \mathcal{A}\xi_2)} \\
&= (\alpha + \beta + \kappa)d(\xi_1, \xi_2) + \lambda d(\xi_2, \xi_1)
\end{aligned}$$

which implies

$$d(\xi_1, \xi_2) \leq (\alpha + \beta + \kappa + \lambda) d(\xi_1, \xi_2),$$

and this implies

$$d(\xi_1, \xi_2)(1 - (\alpha + \beta + \kappa + \lambda)) \leq 0.$$

Thus $d(\xi_1, \xi_2) = 0$, hence $\xi_1 = \xi_2$.

This finalizes the proof. □

Definition 8. *Let $(\mathcal{U}, d)$ be a S-complete symmetric space having property MT and let P and Q be non-empty closed subsets of $\mathcal{U}$. A cyclic map $\mathcal{A} : P \cup Q \to P \cup Q$ is forenamed as Hardy–Rogers cyclic orbital contraction if there is $\mu \in P$ and $\theta \in (0, \frac{1}{1+3s+3s^2})$ such that for any $\nu \in \mathcal{U}$ and $n \in \mathbb{N}$ it holds*

$$d(\mathcal{A}^{2n}\mu, \mathcal{A}\nu) \leq \theta[d(\mathcal{A}^{2n-1}\mu, \mathcal{A}^{2n}\mu) + d(\nu, \mathcal{A}\nu) + d(A^{2n-1}\mu, A\nu) + d(\nu, \mathcal{A}^{2n}\mu) + d(\mathcal{A}^{2n-1}\mu, \nu)].$$

Theorem 4. *Let $(\mathcal{U}, d)$ be a S-complete symmetric space having property W, W_3, MT, P and Q be closed non-empty sets contained in $\mathcal{U}$. Suppose $\mathcal{A} : P \cup Q \to P \cup Q$ is a d-continuous Hardy–Rogers cyclic orbital contraction. Then, $\mathcal{A}$ has exactly one fixed point ξ in $P \cap Q$.*

Proof. Let μ be an arbitrary point in P. Since $\mathcal{A}$ is a Hardy–Rogers cyclic orbital contraction,

$$\begin{aligned} d(\mathcal{A}^2\mu, \mathcal{A}\mu) &\leq \theta[d(\mathcal{A}\mu, \mathcal{A}^2\mu) + d(\mu, \mathcal{A}\mu) + d(\mathcal{A}\mu, \mathcal{A}\mu) + d(\mu, \mathcal{A}^2\mu) + d(\mathcal{A}\mu, \mu)] \\ &\leq \theta[d(\mathcal{A}\mu, \mathcal{A}^2\mu) + d(\mu, \mathcal{A}\mu) + s(d(\mathcal{A}\mu, \mu) + d(\mu, \mathcal{A}\mu)) \\ &\quad + s(d(\mu, \mathcal{A}\mu) + d(\mathcal{A}\mu, \mathcal{A}^2\mu)) + d(\mathcal{A}\mu, \mu)] \\ &= [\theta + (2s + s)\theta + \theta] d(\mu, \mathcal{A}\mu) + (\theta + s\theta) d(\mathcal{A}\mu, \mathcal{A}^2\mu) \end{aligned}$$

which implies

$$d(\mathcal{A}^2\mu, \mathcal{A}\mu) \leq \frac{\theta(2 + 3s)}{1 - \theta(1 + s)} d(\mu, \mathcal{A}\mu).$$

Similarly,

$$\begin{aligned} d(\mathcal{A}^3\mu, \mathcal{A}^2\mu) &\leq \theta[d(\mathcal{A}^2\mu, \mathcal{A}^3\mu) + d(\mathcal{A}\mu, \mathcal{A}^2\mu) + d(\mathcal{A}^2\mu, \mathcal{A}^2\mu) + d(\mathcal{A}\mu, \mathcal{A}^3\mu) + d(\mathcal{A}^2\mu, \mathcal{A}\mu)] \\ &\leq \theta d(\mathcal{A}^2\mu, \mathcal{A}^3\mu) + \theta d(\mathcal{A}^2\mu, \mathcal{A}\mu) + \theta[s(d(\mathcal{A}^2\mu, \mathcal{A}\mu) + d(\mathcal{A}\mu, \mathcal{A}^2\mu)] \\ &\quad + \theta s[d(\mathcal{A}\mu, \mathcal{A}^2\mu) + d(\mathcal{A}^2\mu, \mathcal{A}^3\mu)] + \theta d(\mathcal{A}^2\mu, \mathcal{A}\mu) \\ &= (\theta + 2s\theta + \theta s + \theta) d(\mathcal{A}^2\mu, \mathcal{A}\mu) + (\theta + s\theta) d(\mathcal{A}^3\mu, \mathcal{A}^2\mu) \end{aligned}$$

which gives

$$d(\mathcal{A}^3\mu, \mathcal{A}^2\mu) \leq \frac{\theta(2 + 3s)}{1 - \theta(1 + s)} d(\mathcal{A}^2\mu, \mathcal{A}\mu)$$

That is,

$$\begin{aligned} d(\mathcal{A}^3\mu, \mathcal{A}^2\mu) &\leq k d(\mathcal{A}^2\mu, \mathcal{A}\mu) \\ &\leq k^2 d(\mathcal{A}\mu, \mu) \end{aligned}$$

where $k = \frac{\theta(2+3s)}{1-\theta(1+s)}$.

By continuing the same process, we get

$$d(\mathcal{A}^{n+1}\mu, \mathcal{A}^n\mu) \leq k^n d(\mathcal{A}\mu, \mu).$$

Since $k \in (0, 1)$ therefore $k^n \to 0$ as $n \to \infty$ and hence $\lim_{n\to\infty} d(\mathcal{A}^{n+1}\mu, \mathcal{A}^n\mu) = 0$.

By W, we have $\lim_{n\to\infty} d(\mathcal{A}^n\mu, \mathcal{A}^{n+p}\mu) = 0$ for $n \in \mathbb{N}$, and $p \geq 1$, therefore $\{\mathcal{A}^n\mu\}$ is a d-Cauchy sequence in the S-complete symmetric space $\mathcal{U}$. Since $(\mathcal{U}, d)$ is S-complete, therefore sequence $\{\mathcal{A}^n\mu\}$ converges to some $\xi \in \mathcal{U}$ i.e., $\lim_{n\to\infty} d(\mathcal{A}^n\mu, \xi) = 0$. Then, there exist subsequences $\{\mathcal{A}^{2n}\mu\} \in P$ and $\{\mathcal{A}^{2n-1}\mu\} \in Q$ such that both converge to ξ in $\mathcal{U}$. Since P and Q are closed in $\mathcal{U}$, therefore $\xi \in P \cap Q$. This gives $P \cap Q \neq \phi$.

Now, we will prove that $\mathcal{A}\xi = \xi$.

Since $\mathcal{A}$ is d-continuous therefore $\lim_{n\to\infty} d(\mathcal{A}^{n+1}\mu, \mathcal{A}\xi) = 0$. Now we get that $d(\mathcal{A}\xi, \xi) = 0$ because $(\mathcal{U}, d)$ satisfies W_3.

Thus $\mathcal{A}\xi = \xi$. Hence ξ is a fixed point of $\mathcal{A}$.

Finally, to attain the uniqueness of the fixed point, let ξ_1 and ξ_2 be two fixed points of A.

Then we have

$$\begin{aligned}
d(\xi_1, \xi_2) &= d(\mathcal{A}\xi_1, \mathcal{A}\xi_2) \\
&\leq \alpha d(\xi_1, \xi_2) + \beta d(\xi_1, \mathcal{A}\xi_2) + \kappa d(\xi_2, \mathcal{A}\xi_1) + \delta d(\xi_2, \mathcal{A}\xi_2) \\
&\quad + \eta \frac{d(\xi_2, \mathcal{A}\xi_2)[1 + d(\xi_1, \mathcal{A}\xi_1)])}{1 + d(\xi_1, \mathcal{A}\xi_2)} + \lambda \frac{d(\xi_2, \mathcal{A}\xi_2) + d(\xi_2, \mathcal{A}\xi_1)}{1 + d(\xi_2, \mathcal{A}\xi_2) d(\xi_2, \mathcal{A}\xi_1)} \\
&\quad + \mu \frac{d(\xi_1, \mathcal{A}\xi_1)[1 + d(\xi_2, \mathcal{A}\xi_1)]}{1 + d(\xi_1, \xi_2) + d(\xi_2, \mathcal{A}\xi_2)} \\
&= (\alpha + \beta + \kappa) d(\xi_1, \xi_2) + \lambda d(\xi_2, \xi_1)
\end{aligned}$$

which implies

$$d(\xi_1, \xi_2) \leq (\alpha + \beta + \kappa + \lambda) d(\xi_1, \xi_2),$$

and this implies

$$d(\xi_1, \xi_2)(1 - (\alpha + \beta + \kappa + \lambda)) \leq 0.$$

Thus $d(\xi_1, \xi_2) = 0$, hence $\xi_1 = \xi_2$.

This finalizes the proof. □

4. Applications

Fixed Point Result in PPM-Spaces

We now take into consideration the applications to probabilistic spaces. The definitions mentioned below appeared in Hicks and Rhoades [10].

A real valued function f defined on the set of real numbers is a distribution function if it is non-decreasing, left continuous with $\inf f = 0$ and $\sup f = 1$. H denotes the distribution function defined by $H(t) = 0$ for $t \leq 0$, and $H(t) = 1$ for $t > 0$.

Definition 9. *Let $\mathcal{U}$ be a non-empty set and F a function on $\mathcal{U} \times \mathcal{U}$ such that $F(\mu, \nu) = F_{\mu,\nu}$ is a distribution function. Consider the following conditions:*

(i) $F_{\mu,\nu}(0) = 0$ *for all* μ, ν *in* $\mathcal{U}$*;*
(ii) $F_{\mu,\nu} = H$ *if and only if* $\mu = \nu$*;*
(iii) $F_{\mu,\nu} = F_{\nu,\mu}$*;*
(iv) *If* $F_{\mu,\nu}(\epsilon) = 1$ *and* $F_{\nu,\xi}(\delta) = 1$*, then* $F_{\mu,\xi}(\epsilon + \delta) = 1$.

An F satisfying (i) and (ii) is called a pre-probabilistic metric structure (PPM-structure) on $\mathcal{U}$ and the pair $(\mathcal{U}, F)$ is called a PPM-space. An F satisfying (iii) is forenamed as symmetric. A symmetric PPM-structure F satisfying (iv) is a probabilistic metric structure (PM-structure) and the pair $(\mathcal{U}, F)$ is a probabilistic metric space(PM-space).

Let $(\mathcal{U}, F)$ be a PPM-space. For ϵ, $\lambda > 0$, and $\mu \in \mathcal{U}$, let

$$N_\mu(\epsilon, \lambda) = \{\nu \in \mathcal{U} : F_{\mu,\nu}(\epsilon) > 1 - \lambda\}.$$

A topology $t(F)$ on $\mathcal{U}$ is defined as follows: $G \in t(F)$ if for every μ in G, there is an $\epsilon > 0$ such that $N_\mu(\epsilon, \epsilon) \subset G$. $N_\mu(\epsilon, \epsilon)$ may not be a $t(F)$ neighbourhood of μ. If it is , then $t(F)$ is said to be topological.

Let $(\mathcal{U}, F)$ be a symmetric PPM-space. A sequence $\{\mu_n\}$ is a fundamental sequence if $\lim_{n,m\to\infty} F_{\mu_n,\mu_m}(t) = 1 \; \forall \; t > 0$*. If for every fundamental sequence $\{\mu_n\}$, there is an element $\mu \in \mathcal{U}$ such that* $\lim_{n,m\to\infty} F_{\mu_n,\mu}(t) = 1 \; \forall \; t > 0$*, then the space is complete.*

Let $(\mathcal{U}, F)$ be a symmetric PPM-space. Set

$$d(\mu,\nu) = \begin{cases} 0, & if \quad y \in N_\mu(\epsilon,\epsilon) \quad for\ all \quad \epsilon > 0 \\ sup\{\epsilon : y \notin N_\mu(\epsilon,\epsilon), \epsilon > 0\}, otherwise. \end{cases} \tag{12}$$

Here d is a bounded symmetric for $\mathcal{U}$.

Now, we state a lemma proven by Hicks and Rhoades [10].

Lemma 1. *Let $(\mathcal{U}, F)$ be a symmetric PPM-space. Define d as in Equation (12). Then*

(1) $d(\mu,\nu) < t$ if and only if $F_{\mu,\nu}(t) > 1 - t$;
(2) d is a compatible symmetric space for $t(F)$;
(3) $F_{\mu_n,\mu}(t) \to 1$ for all $t > 0$ if and only if $d(\mu_n,\mu) \to 0$;
(4) $(\mathcal{U}, F)$ is complete if and only if $(\mathcal{U}, d)$ is S-complete;
(5) $t(F)$ is topological if and only if d is a semi-metric.

In the above lemma, it was shown that $F_{\mu_n,\mu}(t) \to 1\ \forall\ t > 0$ iff $d(\mu_n,\mu) \to 0$. Thus, the conditions W, W_3 outlined earlier are equivalent to the under-mentioned conditions respectively:

(PW) $F_{\mu_n,\nu_n}(t) \to 1$ and $F_{\nu_n,\xi_n}(t) \to 1$ for all $t > 0$ imply $F_{\mu_n,\xi_n}(t) \to 1$

(PW_3) $F_{\mu_n,\mu}(t) \to 1$ and $F_{\mu_n,\nu}(t) \to 1$ for all $t > 0$ imply $\mu = \nu$.

Theorem 5. *Let $(\mathcal{U}, F)$ be a complete symmetric PPM-space that satisfies the properties $(PW), (PW_3)$ and let P and Q be non-empty closed subsets of $\mathcal{U}$. Suppose $A, B : P \cup Q \to P \cup Q$ are cyclic mappings such that $A\mathcal{U} \subseteq B\mathcal{U}$ and for $\kappa \in (0,1)$,*

$$F_{A^{2n-1}\mu,B\nu}(t) > 1 - t \implies F_{A^{2n}\mu,A\nu}(\kappa t) > 1 - \kappa t \tag{13}$$

$\forall\ \mu,\nu \in U, n \in N, t > 0$ and $B\mathcal{U}$ is closed, $B\mathcal{U} \subseteq \mathcal{U}$. Then, A and B have a point of coincidence in $P \cap Q$. Additionally, weakly compatibility of mappings A and B will give exactly one common fixed point in $P \cap Q$.

Proof. Let d be as defined in Lemma 1. Now d is a bounded compatible symmetric for $t(F)$. Lemma 1 also gives $(\mathcal{U}, d)$ is S-complete and $d(\mu,\nu) < t$ if $F_{\mu,\nu}(t) > 1 - t$. Presume that A and B are such that Equation (13) holds. Let $\epsilon > 0$ be given, and set $t = d(A^{2n-1}\mu, B\nu) + \epsilon$. Then $t = d(A^{2n-1}\mu, B\nu) < t$ gives $F_{A^{2n-1}\mu,B\nu}(t) > 1 - t$ and (13) yields $F_{A^{2n}\mu,A\nu}(\kappa t) > 1 - \kappa t$. Hence $d(A^{2n}\mu, A\nu) < \kappa t = \kappa d(A^{2n-1}\mu, B\nu) + \kappa\epsilon$. Due to arbitrary ϵ, $d(A^{2n}\mu, A\nu) \leq \kappa d(A^{2n-1}\mu, B\nu)$. We noted above that the fact that F satisfies (PW) and (PW_3) is equivalent to the fact that d satisfies (W) and (W_3). We now put on Theorem 1. Hence the result holds. □

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