Article

Lower Porosity on $\mathbb{R}^2$

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Abstract: In this paper, we study the properties of a lower porosity of a set in $\mathbb{R}^2$. It turns out that the properties of the lower and upper porosity are symmetrical, except that the main tools for testing the lower porosity are not balls but cones. New families of topologies on $\mathbb{R}^2$ generated by the lower porosity are defined. Furthermore, by applying the notion of the lower porosity, we introduce the definition of generalized continuity. Using defined topologies, we study properties of this continuity. We show that the properties of topologies generated by the lower and (upper) porosity are symmetrical.

Keywords: porosity; lower porosity; lower porouscontinuity; topology generated by set and lower porosity; maximal additive class; maximal multiplicative class

MSC: 54C30; 26A15; 54C08

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1. Introduction

The porosity of a set, defined in [1], is the notion of smallness more restrictive than nowhere density and meagerness. It can be defined in arbitrary metric space. The main idea is that we modify the “ball” definition of nowhere density by the request that the sizes of holes should be estimated. Usually, the notion of the (upper) porosity of sets is used.

Recently, the concept of porosity is used not only to study subsets of a metric space but also to compare families of functions. By applying the notion of porosity one can describe how “small” one class of function in the other is, (see [2–4]). In the study of these comparison of classes of functions, it was found that it was possible to strengthen results by using the lower porosity instead of the (upper) porosity.

We deal with the lower porosity, which has also been considered in some papers. It is known that there are big differences between the lower and the upper porosities. In [5,6], some properties of the lower porosity in metric spaces are presented, whereas in [7], some properties of some kind of the lower porosity of subsets of $\mathbb{R}$ are presented. The first aim of this paper is to investigate new properties of the lower porosity of subsets of $\mathbb{R}^2$. It turned out that there are many distinctions between porosity on $\mathbb{R}$ and $\mathbb{R}^2$. For example, for every $U \subset \mathbb{R}$ and $x \in \mathbb{R}$, either $\underline{p}(U, x) \leq \frac{1}{2}$ or $\underline{p}(U, x) = 1$, while for every $r \in [0, 1]$ we can find a subset of $\mathbb{R}^2$ whose porosity equals r . We limit our considerations to the case of $\mathbb{R}^2$ because obtained results can be easily extended to the case of arbitrary $\mathbb{R}^n$. Properties of the lower porosity are symmetrical with properties of the (upper) porosity, but we apply cones instead of families of pairwise disjoint balls to estimate porosity.

The second aim of our paper is to describe some properties of lower porouscontinuous functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. Porouscontinuous functions were introduced by J. Borsík and J. Holos in [8]. Their properties and generalizations can be found, for example, in [9–11]. Lower porouscontinuity is much stronger than (upper) porouscontinuity. In [7], some properties of lower porouscontinuity of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ are studied. It turns out that there are big differences between lower porouscontinuity of functions defined on $\mathbb{R}$ and functions defined on $\mathbb{R}^n$. Moreover, we present similarities and differences between lower

porouscontinuity and (upper) porouscontinuity. For example, we show that lower porouscontinuous functions and (upper) porouscontinuous functions have the same maximal additive class and different maximal multiplicative class.

Let $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. For $f: \mathbf{X} \rightarrow \mathbf{Y}$ and $\mathbf{Z} \subset \mathbf{X}$, by $f|_{\mathbf{Z}}$ we mean the restriction of f to $\mathbf{Z}$. The symbols $\text{cl } \mathbf{Z}$, $\text{int } \mathbf{Z}$, and $\text{bd } \mathbf{Z}$ denote the closure, the interior, and the boundary of $\mathbf{Z} \subset \mathbb{R}^2$ with respect to the natural topology $\tau_{\mathbb{N}}$. The open ball with the center $A \in \mathbb{R}^2$ and the radius $\varrho > 0$ is denoted by $\mathcal{B}(A, \varrho)$. Similarly, by $\mathcal{S}(A, \varrho)$ and $\overline{\mathcal{B}}(A, \varrho)$ we denote the sphere and the closed ball with the center A and the radius ϱ , respectively. For points $A, B, C \in \mathbb{R}^2$, $A \neq B$, $B \neq C$, $A \neq C$, we use the following notations:

- AB —the line passing through A and B ;
- $\text{hl}(A, B)$ —the half-line beginning A through B ;
- $|AB|$ —the length of line segment AB ;
- $|\angle(AB, AC)|$ —the measure of the angle between lines AB and AC ;
- $d(A, BC)$ —the distance between A and line BC .

2. Lower Porosity and Generalized Continuity

Now, we recall definitions of the (upper) porosity and the lower porosity. These notions can be defined in an arbitrary metric space but we present them only for subsets of $\mathbb{R}^2$ because only such case is considered in this paper. Let $\mathbf{U} \subset \mathbb{R}^2$, $A \in \mathbb{R}^2$ and $R > 0$. Then, according to [1,6], by $\gamma(A, R, \mathbf{U})$ we denote the supremum of the set of all $\varrho > 0$ for which there exists $B \in \mathbb{R}^2$ such that $\mathcal{B}(B, \varrho) \subset \mathcal{B}(A, R) \setminus \mathbf{U}$. The number $p(\mathbf{U}, A) = 2 \limsup_{R \rightarrow 0^+} \frac{\gamma(A, R, \mathbf{U})}{R}$ is called the (upper) porosity of $\mathbf{U}$ at A . Obviously, $p(\mathbf{U}, A) = p(\text{cl } \mathbf{U}, A)$ for $\mathbf{U} \subset \mathbb{R}^2$ and $A \in \mathbb{R}^2$. The number $\underline{p}(\mathbf{U}, A) = 2 \liminf_{R \rightarrow 0^+} \frac{\gamma(A, R, \mathbf{U})}{R}$ is called the lower porosity of $\mathbf{U}$ at A . Clearly, $\underline{p}(\mathbf{U}, A) = \underline{p}(\text{cl } \mathbf{U}, A)$ and $\underline{p}(\mathbf{U}, A) \leq p(\mathbf{U}, A)$ for $\mathbf{U} \subset \mathbb{R}^2$ and $A \in \mathbb{R}^2$. Moreover, for $\mathbf{U} \subset \mathbf{V} \subset \mathbb{R}^2$ we have $\underline{p}(\mathbf{V}, A) \leq \underline{p}(\mathbf{U}, A)$, $p(\mathbf{U}, A) \in [0, 2]$ and $\underline{p}(\mathbf{U}, A) \in [0, 1]$ if $A \in \text{cl } \mathbf{U}$.

Example 1. Let $(x_n)_{n \geq 2}$ be a decreasing sequence of positive reals convergent to 0 such that $x_{n+1} < \frac{x_n}{n(n+1)}$ for each $n \geq 2$. Denote $\mathbf{X} = \bigcup_{n=2}^{\infty} \mathcal{B}((x_n, 0), \frac{n}{n+1}x_n) \subset \mathbb{R}^2$. Since $x_{n+1} + \frac{n+1}{n+2}x_{n+1} = x_{n+1} \left(1 + \frac{n+1}{n+2}\right) < \frac{x_n}{n(n+1)} \frac{2n+3}{n+2} < x_n \frac{1}{n+1} \frac{2n+3}{n(n+2)} < x_n \frac{1}{n+1} = x_n - \frac{n}{n+1}x_n$ for $n \geq 2$, we conclude that balls $\mathcal{B}((x_n, 0), \frac{n}{n+1}x_n)$ for $n \geq 2$ are pairwise disjoint. Since $\lim_{n \rightarrow \infty} \frac{2 \frac{n}{n+1}x_n}{x_n + \frac{n}{n+1}x_n} = 1$, by Theorem 1.2 in [11] we obtain $p(\mathbb{R}^2 \setminus \mathbf{X}, (0, 0)) = 1$. On the other hand,

$$\begin{aligned} \frac{2\gamma((0, 0), \frac{x_n}{n+1}, \mathbb{R}^2 \setminus \mathbf{X})}{\frac{x_n}{n+1}} &= \frac{2 \frac{n+1}{n+2}x_{n+1}}{\frac{x_n}{n+1}} = \frac{2(n+1)^2}{n+2} \frac{x_{n+1}}{x_n} \\ &< \frac{2(n+1)^2}{n+2} \frac{1}{n(n+1)} = \frac{2(n+1)}{n(n+2)} \end{aligned}$$

for every $n \geq 2$. Thus $\underline{p}(\mathbb{R}^2 \setminus \mathbf{X}, (0, 0)) = 0$.

Example 2. Let $a, b \in \mathbb{R}$, $\mathbf{X} = \{(x, y) \in \mathbb{R}^2: y \leq ax + b\}$ and $\mathbf{Y} = \{(x, y) \in \mathbb{R}^2: y = ax + b\}$. Then $\underline{p}(\mathbb{R}^2 \setminus \mathbf{X}, A) = 1$ for every $A \in \mathbf{Y}$.

Theorem 1. Let $\mathbf{X} \subset \mathbb{R}^2$, $A \in \mathbb{R}^2$ and $\underline{p}(\mathbf{X}, A) > 0$. Then there exists a sequence of closed balls $(\overline{\mathcal{B}}(A_n, \varrho_n))_{n \geq 1}$, not necessary pairwise disjoint, such that $\lim_{n \rightarrow \infty} A_n = A$, $\varrho_n \leq \frac{1}{n}$ for $n \geq 1$, $\bigcup_{n=1}^{\infty} \overline{\mathcal{B}}(A_n, \varrho_n) \cap \mathbf{X} = \emptyset$ and

$$\underline{p}(\mathbf{X}, A) = \underline{p}\left(\mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \overline{\mathcal{B}}(A_n, \varrho_n), A\right) = \liminf_{n \rightarrow \infty} 2n\varrho_n.$$

Proof. For every $n \geq 1$ put $\gamma_n = \sup \left\{ \varrho : \exists B \in \mathbb{R}^2 \left(\overline{B}(B, \varrho) \subset \overline{B}(A, \frac{1}{n}) \setminus \mathbf{X} \right) \right\}$ and choose a closed ball $\overline{B}(A_n, \varrho_n) \subset \overline{B}(A, \frac{1}{n}) \setminus \mathbf{X}$ such that $\varrho_n > \gamma_n \left(1 - \frac{1}{n^2} \right)$. Denote $\mathbf{Y} = \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \overline{B}(A_n, \varrho_n)$. Since $\mathbf{X} \subset \mathbf{Y}$, we obtain $\underline{p}(\mathbf{Y}, A) \leq \underline{p}(\mathbf{X}, A)$. Fix $n > 1$ and choose any $R \in \left(\frac{1}{n+1}, \frac{1}{n} \right]$. Then

$$\frac{2\gamma(A, R, \mathbf{X})}{R} \leq \frac{2\gamma_n}{\frac{1}{n+1}} < \frac{2\varrho_n}{\left(1 - \frac{1}{n^2} \right) \frac{1}{n+1}} = \frac{2\varrho_n n^2}{n-1}$$

and

$$\frac{2\gamma(A, R, \mathbf{Y})}{R} \geq \frac{2\varrho_n}{\frac{1}{n}} = 2n\varrho_n.$$

We have shown that for each $n > 1$ and for each $R \in \left(\frac{1}{n+1}, \frac{1}{n} \right]$ the following inequalities

$$\frac{2\gamma(A, R, \mathbf{X})}{R} < 2\varrho_n n \frac{n}{n-1} \quad \text{and} \quad \frac{2\gamma(A, R, \mathbf{Y})}{R} \geq 2n\varrho_n$$

are true. Hence

$$\underline{p}(\mathbf{X}, A) \leq \liminf_{n \rightarrow \infty} 2n\varrho_n n \frac{n}{n-1} = \liminf_{n \rightarrow \infty} 2n\varrho_n \cdot \lim_{n \rightarrow \infty} \frac{n}{n-1} = \liminf_{n \rightarrow \infty} 2n\varrho_n$$

and

$$\underline{p}(\mathbf{Y}, A) = \liminf_{R \rightarrow 0} \frac{2\gamma(A, R, \mathbf{Y})}{R} \geq \liminf_{n \rightarrow \infty} 2n\varrho_n.$$

Finally, $\underline{p}(\mathbf{X}, A) = \underline{p}(\mathbf{Y}, A) = \liminf_{n \rightarrow \infty} 2n\varrho_n$. $\square$

Definition 1. Let $A \neq B$, $A, B \in \mathbb{R}^2$ and $\varphi \in (0, \frac{\pi}{2})$. The set

$$c(A, B, \varphi) = \left\{ C \in \mathbb{R}^2 : |\angle(AB, AC)| \leq \varphi \right\}$$

is called a cone appointed by two distinct points A , B , and the angle φ (see Figure 1).

Figure 1. The cone $c(A, B, \varphi)$.

Theorem 2. Let $c(A, B, \varphi)$ be a cone, $A, B \in \mathbb{R}^2$, $A \neq B$ and $\varphi \in (0, \frac{\pi}{2})$. Then $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = \frac{2 \sin \varphi}{1 + \sin \varphi}$.

Proof. Take any $R > 0$ and let C, D be points on the half-line $hl(A, B)$ such that $|AC| = R$ and $|AD| = \frac{R}{1 + \sin \varphi}$ (see Figure 2). Then $|CD| = R - \frac{R}{1 + \sin \varphi} = \frac{R \sin \varphi}{1 + \sin \varphi}$ and the dis-

tance between D and the boundary of $c(A, B, \varphi)$ equals $|AD| \sin \varphi = \frac{R \sin \varphi}{1 + \sin \varphi}$. Thus $\mathcal{B}\left(D, \frac{R \sin \varphi}{1 + \sin \varphi}\right) \subset c(A, B, \varphi)$. Hence $\gamma(A, R, \mathbb{R}^2 \setminus c(A, B, \varphi)) \geq \frac{R \sin \varphi}{1 + \sin \varphi}$.

Figure 2. The cone $c(A, B, \varphi)$.

Let $\mathcal{B}(E, \varrho) \subset c(A, B, \varphi) \cap \mathcal{B}(A, R)$. Take a point $F \in \text{bd } c(A, B, \varphi)$ such that the distance between E and $\text{bd } c(A, B, \varphi)$ is equal to $|EF|$. Put $\psi = |\angle(AE, AF)|$. Then $\psi \leq \varphi$ and $\varrho \leq |EF| = |AE| \sin \psi$. Since $f: (0, |AE| \sin \psi] \rightarrow \mathbb{R}$ defined by $f(t) = \frac{2t}{c+t}$, where c is an arbitrary positive number, is strictly increasing, we obtain

$$\frac{2\varrho}{R} \leq \frac{2\varrho}{|AE| + \varrho} \leq \frac{2|AE| \sin \psi}{|AE| + |AE| \sin \psi} = \frac{2 \sin \psi}{1 + \sin \psi} \leq \frac{2 \sin \varphi}{1 + \sin \varphi}.$$

Therefore $\frac{\gamma(A, R, \mathbb{R}^2 \setminus c(A, B, \varphi))}{R} = \frac{\sin \varphi}{1 + \sin \varphi}$ for every $R > 0$. Finally,

$$\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = \liminf_{R \rightarrow 0} \frac{2\gamma(A, R, \mathbb{R}^2 \setminus c(A, B, \varphi))}{R} = \frac{2 \sin \varphi}{1 + \sin \varphi}.$$

□

Corollary 1. Let $A \in \mathbb{R}^2$, $\eta > 0$ and $B \in \mathcal{S}(A, \eta)$. Since for every $\varphi \in (0, \frac{\pi}{2})$ there exists $R > 0$ such that $c(B, A, \varphi) \cap \mathcal{B}(B, R) \subset \mathcal{B}(A, \eta)$, we have $\underline{p}(\mathbb{R}^2 \setminus \mathcal{B}(A, \eta), B) = 1$.

Lemma 1. Let $A, B \in \mathbb{R}^2$, $A \neq B$, $\varphi \in (0, \frac{\pi}{2})$. For each $C \in \text{hl}(A, B)$ we can find $R > |AC|$ such that

$$\frac{2\gamma(A, R, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{C\})}{R} = c_\varphi,$$

$$\text{where } c_\varphi = \frac{2 \sin \varphi}{\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} + \sin \varphi} < \frac{2 \sin \varphi}{1 + \sin \varphi}.$$

Proof. Let $F \neq A$ be any point on $\text{bd } c(A, B, \varphi)$. Define $\alpha: (0, \infty) \rightarrow \mathbb{R}$ by

$$\alpha(t) = \sup\{\varrho: \exists E \in \mathbb{R}^2 (\mathcal{B}(E, \varrho) \subset c(A, B, \varphi) \setminus \{C\} \text{ and } |AE| = t)\}.$$

For $t \in (0, \infty)$ let E_t be a point on the half-line $\text{hl}(A, B)$ such that $|AE_t| = t$. It is easy to see that if $d(E_t, AF) \leq |E_t C|$ then

$$\alpha(t) = d(E_t, AF) = t \sin \varphi. \quad (1)$$

Let D be a point on the half-line $\text{hl}(A, C)$ such that $|AD| < |AC|$ and $d(D, AF) = |DC|$ (see Figure 3).

Figure 3. The cone $c(A, B, \varphi)$.

Denote $q_0 = |DC| = d(D, AF)$. Fix a point $G, G \neq D$, lying on a straight line parallel to the straight line AF and passing through the point D such that $|CG| = q_0$. Let t_0 be such that $E_{t_0} = D$. Obviously $\alpha(t_0) = q_0$ and

$$\alpha(t) \leq q_0 \text{ for } t \in [t_0, |AG|]. \quad (2)$$

Let us compute $|AG|$ and q_0 . Since $q_0 = |AD| \sin \varphi$ and $|AD| + q_0 = |AC|$, we obtain $\frac{q_0}{\sin \varphi} + q_0 = |AC|$ and $q_0 = |AC| \frac{\sin \varphi}{1 + \sin \varphi}$. Let H be a point on the half-line $hl(A, F)$ such that $|GH| = q_0$. It is easy to see that $|AH| = |AC| \cos \varphi + q_0 \cos \varphi = q_0 \cos \varphi \left(\frac{1 + \sin \varphi}{\sin \varphi} + 1 \right)$. Moreover, $|AG|^2 = |AH|^2 + q_0^2$. Hence

$$\begin{aligned} |AG| &= \sqrt{|AH|^2 + q_0^2} = \sqrt{q_0^2 \cos^2 \varphi \left(\frac{1 + 2 \sin \varphi}{\sin \varphi} \right)^2 + q_0^2} \\ &= q_0 \sqrt{\frac{\cos^2 \varphi + 4 \cos^2 \varphi \sin \varphi + 4 \cos^2 \varphi \sin^2 \varphi}{\sin^2 \varphi} + 1} \\ &= q_0 \frac{\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)}}{\sin \varphi}. \end{aligned}$$

Put

$$R = |AG| + q_0. \quad (3)$$

By (1)–(3), we obtain

$$\gamma(A, R, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{C\}) = q_0.$$

Moreover,

$$R = |AG| + q_0 = q_0 \frac{\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} + \sin \varphi}{\sin \varphi}.$$

Therefore

$$\begin{aligned} \frac{2\gamma(A, R, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{C\})}{R} &= \frac{2q_0 \sin \varphi}{q_0 \left(\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} + \sin \varphi \right)} \\ &= \frac{2 \sin \varphi}{\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} + \sin \varphi} = c_\varphi. \end{aligned}$$

For fixed $\varphi \in (0, \frac{\pi}{2})$ we have $\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} > 1$. Thus $0 < c_\varphi < \frac{2 \sin \varphi}{1 + \sin \varphi}$. $\square$

Theorem 3. Let $\varphi \in (0, \frac{\pi}{2})$. There exists $\varphi_0 \in (0, \varphi)$ such that for each cone $c(A, B, \varphi)$, $A, B \in \mathbb{R}^2$, $A \neq B$, and for each sequence $(A_n)_{n \geq 1}$ satisfying conditions:

- $\lim_{n \rightarrow \infty} A_n = A$;
- $|\angle(AA_n, AB)| < \varphi_0$ for each $n \geq 1$,

the equality

$$\underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) \leq \frac{c_\varphi + \frac{2 \sin \varphi}{1 + \sin \varphi}}{2} < \frac{2 \sin \varphi}{1 + \sin \varphi},$$

where $c_\varphi = \frac{2 \sin \varphi}{\sqrt{1 + 4 \cos^2 \varphi \sin \varphi (1 + \sin \varphi)} + \sin \varphi}$, is fulfilled.

Proof. Put $\varphi_0 = \frac{\frac{2 \sin \varphi}{1 + \sin \varphi} - c_\varphi}{4} > 0$. Let $(A_n)_{n \geq 1}$ be any sequence of points from $\mathbb{R}^2$ satisfying conditions $\lim_{n \rightarrow \infty} A_n = A$ and $|\angle(AA_n, AB)| < \varphi_0$ for each $n \geq 1$. Fix $n \geq 1$. Take a point D_n on the half-line $hl(A, B)$ such that lines $D_n A_n$ and AA_n are perpendicular. Let R_n be a number from Lemma 1 chosen for the point D_n . Then

$$\frac{2\gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{D_n\})}{R_n} = c_\varphi.$$

Obviously

$$\gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{A_n\}) \leq \gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{D_n\}) + |A_n D_n|.$$

Hence

$$\begin{aligned} \frac{2\gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{A_n\})}{R_n} &\leq c_\varphi + \frac{2|A_n D_n|}{R_n} \\ &\leq c_\varphi + \frac{2|A_n D_n| \sin \varphi_0}{|A_n D_n|} = c_\varphi + 2 \sin \varphi_0 \leq c_\varphi + 2\varphi_0 = \frac{c_\varphi + \frac{2 \sin \varphi}{1 + \sin \varphi}}{2}. \end{aligned}$$

Since $\frac{2\gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{A_n\})}{R_n} \leq \frac{c_\varphi + \frac{2 \sin \varphi}{1 + \sin \varphi}}{2}$ for each $n \geq 1$ and $\lim_{n \rightarrow \infty} A_n = A$, we obtain

$$\begin{aligned} &\underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) \\ &= \liminf_{R \rightarrow 0} \frac{2\gamma(A, R, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \bigcup_{n=1}^{\infty} \{A_n\})}{R} \\ &\leq \liminf_{n \rightarrow \infty} \frac{2\gamma(A, R_n, (\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \{A_n\})}{R_n} \leq \frac{c_\varphi + \frac{2 \sin \varphi}{1 + \sin \varphi}}{2}. \end{aligned}$$

$\square$

In [8], J. Borsík and J. Holos defined families of porouscontinuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Applying their ideas and replacing standard porosity in $\mathbb{R}$ by the lower porosity in $\mathbb{R}^2$ we transfer this concept for real functions defined on $\mathbb{R}^2$.

Definition 2. Let $r \in (0, 1)$, $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $A \in \mathbb{R}^2$. The function f will be called:

- $\underline{P}_r$ -continuous at A if there exists a set $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$ and $f|_{\mathbf{U}}$ is continuous at A ;

- $\mathcal{S}_r$ -continuous at A if for each $\varepsilon > 0$ there exists a set $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$ and $f(\mathbf{U}) \subset (f(A) - \varepsilon, f(A) + \varepsilon)$;
- $\mathcal{M}_r$ -continuous at A if there exists a set $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) \geq r$ and $f|_{\mathbf{U}}$ is continuous at A ;
- $\mathcal{N}_r$ -continuous at A if for each $\varepsilon > 0$ there exists a set $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) \geq r$ and $f(\mathbf{U}) \subset (f(A) - \varepsilon, f(A) + \varepsilon)$.

By $\mathcal{P}_r(f)$, $\mathcal{S}_r(f)$, $\mathcal{M}_r(f)$, and $\mathcal{N}_r(f)$ we denote the set of points at which f is $\mathcal{P}_r$ -continuous, $\mathcal{S}_r$ -continuous, $\mathcal{M}_r$ -continuous, and $\mathcal{N}_r$ -continuous, respectively.

Proposition 1. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $A \in \mathbb{R}^2$ and $r \in (0, 1)$. Then

1. $A \in \mathcal{S}_r(f)$ if and only if $\underline{p}(\mathbb{R}^2 \setminus \{C: |f(C) - f(A)| < \varepsilon\}, A) > r$ for every $\varepsilon > 0$;
2. $A \in \mathcal{N}_r(f)$ if and only if $\underline{p}(\mathbb{R}^2 \setminus \{C: |f(C) - f(A)| < \varepsilon\}, A) \geq r$ for every $\varepsilon > 0$.

Similarly as in [8], we can easily check that f is $\mathcal{M}_r$ -continuous at A if and only if it is $\mathcal{N}_r$ -continuous at A .

Proposition 2. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $A \in \mathbb{R}^2$ and $r, r_1, r_2 \in (0, 1)$, $r_1 < r_2$. Then

1. if $A \in \mathcal{P}_r(f)$ then $A \in \mathcal{S}_r(f)$;
2. if $A \in \mathcal{S}_r(f)$ then $A \in \mathcal{N}_r(f)$;
3. if $A \in \mathcal{M}_{r_2}(f)$ then $A \in \mathcal{P}_{r_1}(f)$.

If f is $\mathcal{P}_r$ -continuous, $\mathcal{S}_r$ -continuous, $\mathcal{M}_r$ -continuous at every point of $\mathbb{R}^2$ for some $r \in (0, 1)$ then we say that f is $\mathcal{P}_r$ -continuous, $\mathcal{S}_r$ -continuous, $\mathcal{M}_r$ -continuous, respectively. All of these functions are called lower porouscontinuous functions.

Obviously, if f is continuous at some A then f is lower porouscontinuous (in each sense) at A . We introduce for $r \in (0, 1)$ the following notations:

- $\mathcal{M}_r = \mathcal{N}_r = \{f: \mathcal{M}_r(f) = \mathbb{R}^2\}$;
- $\mathcal{P}_r = \{f: \mathcal{P}_r(f) = \mathbb{R}^2\}$;
- $\mathcal{S}_r = \{f: \mathcal{S}_r(f) = \mathbb{R}^2\}$.

In the sequel, we consider $\mathbb{R}^2$ with several different topologies. Let τ be a topology on $\mathbb{R}^2$ (in particular $\tau = \tau_N$ may be the natural topology). Then we say that $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is τ -continuous at $A \in \mathbb{R}^2$ if it is continuous at A as $f: (\mathbb{R}^2, \tau) \rightarrow (\mathbb{R}, \tau_N)$. Thus τ -continuity of f at A means that for each $\varepsilon > 0$ there exists τ -open set $\mathbf{U}$ such that $A \in \mathbf{U}$ and $f(\mathbf{U}) \subset (f(A) - \varepsilon, f(A) + \varepsilon)$. We say that f is τ -continuous if it is τ -continuous at each point. By $\mathcal{C}(f)$ and $\mathcal{C}_\tau(f)$ we denote the set of points at which f is continuous and f is τ -continuous, respectively. Denote

- $\mathcal{C}_\tau = \{f: \mathcal{C}_\tau(f) = \mathbb{R}^2\}$;
- $\mathcal{C} = \{f: \mathcal{C}(f) = \mathbb{R}^2\}$.

Similarly, by $\text{int}_\tau \mathbf{V}$ and $\text{cl}_\tau \mathbf{V}$ we denote the interior and the closure of $\mathbf{V} \subset \mathbb{R}^2$ in the topology τ . Finally, for any $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ let $\mathbf{N}_f = \{C \in \mathbb{R}^2: f(C) = 0\}$.

3. Maximal Additive Families for Lower Porouscontinuous Functions

It is easily seen that the result of addition or multiplication of functions from one of discussed classes of functions, in general, need not belong to this class. Description of functions that can be represented as a final sum or product of lower porouscontinuous functions is still not known; therefore, we study the following similar notion.

Definition 3 ([12]). Let $\mathcal{F}$ be a family of real functions defined on $\mathbb{R}^2$. A set $\mathfrak{M}_a(\mathcal{F}) = \{g: \mathbb{R}^2 \rightarrow \mathbb{R}: \forall f \in \mathcal{F} (f + g \in \mathcal{F})\}$ is called the maximal additive class for $\mathcal{F}$.

Remark 1. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(C) = 0$ for $C \in \mathbb{R}^2$, be a constant function. Clearly, if $f \in \mathcal{F}$ then $\mathfrak{M}_a(\mathcal{F}) \subset \mathcal{F}$.

Lemma 2. Let $A \in \mathbb{R}^2$, $r \in (0, 1)$, $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ and f be continuous at A . Then

- (1) if $A \in \mathcal{P}_r(g)$ then $A \in \mathcal{P}_r(f + g) \cap \mathcal{P}_r(f \cdot g)$;
- (2) if $A \in \mathcal{S}_r(g)$ then $A \in \mathcal{S}_r(f + g) \cap \mathcal{S}_r(f \cdot g)$;
- (3) if $A \in \mathcal{M}_r(g)$ then $A \in \mathcal{M}_r(f + g) \cap \mathcal{M}_r(f \cdot g)$.

Proof.

- (1) Since g is $\mathcal{P}_r$ -continuous at A , there exists $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $g|_{\mathbf{U}}$ is continuous at A and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$. Since f is continuous at A , $(f + g)|_{\mathbf{U}}$ and $(f \cdot g)|_{\mathbf{U}}$ are continuous at A ; therefore, $A \in \mathcal{P}_r(f + g) \cap \mathcal{P}_r(f \cdot g)$.
- (2) Let $\varepsilon > 0$. Since g is $\mathcal{S}_r$ -continuous at A , there exists $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $g(\mathbf{U}) \subset (g(A) - \frac{\varepsilon}{2}, g(A) + \frac{\varepsilon}{2})$ and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$. By continuity of f at A , we can find $\delta > 0$ such that $|f(B) - f(A)| < \frac{\varepsilon}{2}$ for each $B \in \mathcal{B}(A, \delta)$. Let $\mathbf{V} = \mathbf{U} \cap \mathcal{B}(A, \delta)$. Then $\underline{p}(\mathbb{R}^2 \setminus \mathbf{V}, A) = \underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$ and

$$|(f + g)(B) - (f + g)(A)| \leq |f(B) - f(A)| + |g(B) - g(A)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $B \in \mathbf{V}$. This means that $A \in \mathcal{S}_r(f + g)$.

To prove the second condition, we consider the following two cases.

- $f(A) = 0$. Take any $\varepsilon > 0$. Since g is $\mathcal{S}_r$ -continuous at A , there exists $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$, $g(\mathbf{U}) \subset (g(A) - 1, g(A) + 1)$ and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$. By continuity of f at A , we can find $\delta > 0$ such that

$$|f(B) - f(A)| = |f(B)| < \frac{\varepsilon}{|g(A)| + 1}$$

for each $B \in \mathcal{B}(A, \delta)$. Let $\mathbf{V} = \mathbf{U} \cap \mathcal{B}(A, \delta)$. Then $\underline{p}(\mathbb{R}^2 \setminus \mathbf{V}, A) > r$ and

$$|(f \cdot g)(B) - (f \cdot g)(A)| = |f(B)| \cdot |g(B)| < \frac{\varepsilon}{|g(A)| + 1} (|g(A)| + 1) = \varepsilon$$

for each $B \in \mathbf{V}$. Then $A \in \mathcal{S}_r(f \cdot g)$.

- $f(A) \neq 0$. Let $\varepsilon \in (0, 1)$. Since g is $\mathcal{S}_r$ -continuous at A , there exists $\mathbf{U} \subset \mathbb{R}^2$ such that $A \in \mathbf{U}$,

$$g(\mathbf{U}) \subset \left(g(A) - \min \left\{ \frac{\varepsilon}{2|f(A)|}, \frac{1}{4} \right\}, g(A) + \min \left\{ \frac{\varepsilon}{2|f(A)|}, \frac{1}{4} \right\} \right)$$

and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) > r$. By continuity of f at A , we can find $\delta > 0$ such that $|f(B) - f(A)| < \frac{2\varepsilon}{4|g(A)| + 1}$ for each $B \in \mathcal{B}(A, \delta)$. Let $\mathbf{V} = \mathbf{U} \cap \mathcal{B}(A, \delta)$. Then $\underline{p}(\mathbb{R}^2 \setminus \mathbf{V}, A) > r$ and

$$\begin{aligned} |(f \cdot g)(B) - (f \cdot g)(A)| &\leq |g(B)| \cdot |f(B) - f(A)| + |f(A)| \cdot |g(B) - g(A)| \\ &< \left(|g(A)| + \frac{1}{4} \right) \frac{2\varepsilon}{4|g(A)| + 1} + |f(A)| \frac{\varepsilon}{2|f(A)|} = \varepsilon \end{aligned}$$

for each $B \in \mathbf{V}$. Hence $A \in \mathcal{S}_r(f \cdot g)$.

- (3) The proof is similar to the proof of (1) and we omit it.

□

Lemma 3. Let $\mathbf{X} \subset \mathbb{R}^2$ and $A \in \mathbb{R}^2$. Then there exists a set $\mathbf{Z} \subset \mathbb{R}^2 \setminus \mathbf{X}$ such that

- $\text{cl } \mathbf{Z} \subset \mathbf{Z} \cup \{A\}$;
- $\mathbf{Z}$ is discrete;
- For each $\mathbf{Y} \subset \mathbb{R}^2$, if $\mathbf{Z} \subset \mathbf{Y}$ then $\underline{p}(\mathbf{Y}, A) = \underline{p}(\mathbf{Y} \cup (\mathbb{R}^2 \setminus \mathbf{X}), A)$.

Proof. Let $U_n = \overline{B}(A, \frac{1}{n}) \setminus B(A, \frac{1}{n+1})$ for $n \geq 1$. Applying the Zorn lemma, for every n we can choose a discrete set $Z_n \subset U_n \setminus X$ such that $U_n \setminus X \subset \bigcup_{B \in Z_n} B(B, \frac{1}{(n+1)^2})$ and $|B_1 B_2| \geq \frac{1}{(n+1)^2}$ for $B_1, B_2 \in Z_n, B_1 \neq B_2$. Let

$$Z = \bigcup_{n=1}^{\infty} Z_n.$$

Then Z is discrete, $Z \cap X = \emptyset$ and $\text{cl } Z \subset Z \cup \{A\}$. Take any $Y \subset \mathbb{R}^2$ such that $Z \subset Y$. The inequality $\underline{p}(Y, A) \geq \underline{p}(Y \cup (\mathbb{R}^2 \setminus X), A)$ is obvious. If $\underline{p}(Y, A) = 0$ then certainly $\underline{p}(Y \cup (\mathbb{R}^2 \setminus X), A) = 0$. Let $\underline{p}(Y, A) = \alpha > 0$. Choose any β and β_1 such that $0 < \beta < \beta_1 < \alpha$. We can find $n_0 > 1$ such that $\frac{1}{n_0} < \min\{\frac{\beta_1 - \beta}{4}, \frac{\beta_1}{8}\}$. Since $\underline{p}(Y, A) = \alpha > \beta_1$, there exists $B(B, \eta)$ such that $\frac{2\eta}{\eta + |AB|} > \beta_1, |AB| < \frac{1}{4n_0}$ and $B(B, \eta) \cap Y = \emptyset$. Suppose that $B(B, \eta) \not\subset X$ and take any $C \in B(B, \eta) \setminus X$. There exists n_1 such that $\frac{1}{n_1+1} < |CA| \leq \frac{1}{n_1}$, i.e., $C \in U_{n_1}$. Since

$$|CA| \leq |CB| + |BA| < \eta + |BA| < 2|BA| < \frac{1}{2n_0},$$

we obtain $\frac{1}{n_1+1} < \frac{1}{2n_0}$ and $n_1 > n_0$. By construction of Z , there exists $D \in Z_{n_1}$ such that $|CD| \leq \frac{1}{(n_1+1)^2}$. Observe that $D \in Z \subset Y$ and $Y \cap B(B, \eta) = \emptyset$; therefore $D \notin B(B, \eta)$, i.e., $|BD| \geq \eta$. Thus

$$|CB| \geq |BD| - |CD| \geq \eta - \frac{1}{(n_1+1)^2}.$$

This means that $B(B, \eta - \frac{1}{(n_1+1)^2}) \cap [Y \cup (\mathbb{R}^2 \setminus X)] = \emptyset$. By inequality $\frac{2\eta}{\eta + |AB|} > \beta_1$, we obtain $2\eta > \beta_1|AB|$. Hence

$$\eta > \frac{\beta_1|AB|}{2} > \frac{\frac{1}{2}\beta_1|AC|}{2} > \frac{\beta_1}{4(n_1+1)} > \frac{8}{n_0} \frac{1}{4(n_1+1)} > \frac{1}{n_1^2}$$

and $\eta - \frac{1}{(n_1+1)^2} > 0$. Moreover,

$$\begin{aligned} \frac{2(\eta - \frac{1}{(n_1+1)^2})}{\eta - \frac{1}{(n_1+1)^2} + |AB|} &> \frac{2\eta - \frac{2}{(n_1+1)^2}}{\eta + |AB|} \\ &= \frac{2\eta}{\eta + |AB|} - \frac{2}{(n_1+1)^2(\eta + |AB|)} > \beta_1 - \frac{2}{(n_1+1)^2|AC|} \\ &> \beta_1 - \frac{2}{(n_1+1)^2 \frac{1}{n_1+1}} = \beta_1 - \frac{2}{n_1+1} > \beta_1 - \frac{4}{n_0} > \beta_1 - (\beta_1 - \beta) = \beta. \end{aligned}$$

Since $\beta \in (0, \alpha)$ was chosen arbitrarily, $\underline{p}(Y \cup (\mathbb{R}^2 \setminus X), A) \geq \alpha$, which completed the proof. $\square$

Theorem 4. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $A \in \mathbb{R}^2$. The following conditions are equivalent:

- (1) f is continuous at A ;
- (2) $\forall_{r \in (0,1)} \forall_{g: \mathbb{R}^2 \rightarrow \mathbb{R}} (A \in \underline{\mathcal{M}}_r(g) \Rightarrow A \in \underline{\mathcal{M}}_r(f+g))$;
- (3) $\exists_{r \in (0,1)} \forall_{g \in \underline{\mathcal{M}}_r} (A \in \underline{\mathcal{M}}_r(f+g))$;
- (4) $\forall_{r \in (0,1)} \forall_{g: \mathbb{R}^2 \rightarrow \mathbb{R}} (A \in \underline{\mathcal{S}}_r(g) \Rightarrow A \in \underline{\mathcal{S}}_r(f+g))$;
- (5) $\exists_{r \in (0,1)} \forall_{g \in \underline{\mathcal{S}}_r} (A \in \underline{\mathcal{S}}_r(f+g))$;
- (6) $\forall_{r \in (0,1)} \forall_{g: \mathbb{R}^2 \rightarrow \mathbb{R}} (A \in \underline{\mathcal{P}}_r(g) \Rightarrow A \in \underline{\mathcal{P}}_r(f+g))$;
- (7) $\exists_{r \in (0,1)} \forall_{g \in \underline{\mathcal{P}}_r} (A \in \underline{\mathcal{P}}_r(f+g))$.

Proof. Implications (2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), and (6) $\Rightarrow$ (7) are obvious. Implications (1) $\Rightarrow$ (2), (1) $\Rightarrow$ (4), and (1) $\Rightarrow$ (6) follow from Lemma 2.

(3) $\Rightarrow$ (1). For every $s \in (0, 1)$ by φ_s we denote the unique number from $(0, \frac{\pi}{2})$ such that $\frac{2 \sin \varphi_s}{1 + \sin \varphi_s} = s$. Let there exists $r \in (0, 1)$ such that for each $g \in \underline{\mathcal{M}}_r$ we have $A \in \underline{\mathcal{M}}_r(f + g)$. Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = r$ for every $B \in \mathbb{R}^2 \setminus \{A\}$, i.e., $\frac{2 \sin \varphi}{1 + \sin \varphi} = r$. Let $\varphi_0 \in (0, \varphi)$ and $c_\varphi \in (0, \frac{2 \sin \varphi}{1 + \sin \varphi})$ satisfy assertion of Theorem 3.

Suppose that f is not continuous at A . Then there exist $\varepsilon > 0$ and a sequence $(A_n)_{n \geq 1}$ convergent to A such that $|f(A_n) - f(A)| \geq \varepsilon$ for each $n \geq 1$. Clearly, we can find $B \in \mathbb{R}^2 \setminus \{A\}$ such that $c(A, B, \varphi_0)$ contains infinitely many elements of the sequence $(A_n)_{n \geq 1}$. Without loss of generality, we may assume that $A_n \in c(A, B, \varphi_0)$ for every $n \geq 1$. Let $\mathbf{Z}$ be a set from Lemma 3 for $\mathbf{X} = c(A, B, \varphi)$ and A . Then $\mathbf{Z}$ is discrete, $\mathbf{Z} \cap c(A, B, \varphi) = \emptyset$ and for each $\mathbf{Y} \subset \mathbb{R}^2$, if $\mathbf{Z} \subset \mathbf{Y}$ then $\underline{p}(\mathbf{Y}, A) = \underline{p}(\mathbf{Y} \cup (\mathbb{R}^2 \setminus c(A, B, \varphi)), A)$. Define $\tilde{g}: (c(A, B, \varphi) \setminus \{A\}) \cup \mathbf{Z} \rightarrow \mathbb{R}$ by

$$\tilde{g}(C) = \begin{cases} 0 & \text{for } C \in c(A, B, \varphi) \setminus \{A\}, \\ -f(C) + f(A) + \varepsilon & \text{for } C \in \mathbf{Z}. \end{cases}$$

Clearly, $(f + \tilde{g})(C) = f(A) + \varepsilon$ for $C \in \mathbf{Z}$ and $\tilde{g}$ is continuous, because $\mathbf{Z}$ is discrete and $\text{cl } \mathbf{Z} \cap (c(A, B, \varphi) \setminus \{A\}) = \emptyset$. Since $(c(A, B, \varphi) \setminus \{A\}) \cup \mathbf{Z}$ is a closed subset of $\mathbb{R}^2 \setminus \{A\}$, by the Tietze theorem we can find a continuous extension $\bar{g}: \mathbb{R}^2 \setminus \{A\} \rightarrow \mathbb{R}$ of $\tilde{g}$. Finally define $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(C) = \begin{cases} 0 & \text{for } C = A, \\ \bar{g}(C) & \text{for } C \in \mathbb{R}^2 \setminus \{A\}. \end{cases}$$

Obviously, $\mathbb{R}^2 \setminus \{A\} \subset \underline{\mathcal{M}}_r(g)$. Since $\frac{2 \sin \varphi}{1 + \sin \varphi} = r$, $A \in \underline{\mathcal{M}}_r(g)$. Finally, $g \in \underline{\mathcal{M}}_r$.

On the other hand, $(f + g)(A) = f(A)$, $(f + g)(C) = f(A) + \varepsilon$ for every $C \in \mathbf{Z}$ and $|(f + g)(A_n) - (f + g)(A)| = |f(A_n) - f(A)| \geq \varepsilon$ for $n \geq 1$. Let

$$\mathbf{W} = \{C \in \mathbb{R}^2: |(f + g)(C) - (f + g)(A)| < \varepsilon\}.$$

Then $\mathbf{Z} \subset \mathbb{R}^2 \setminus \mathbf{W}$. By Lemma 3,

$$\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) = \underline{p}\left(\left(\mathbb{R}^2 \setminus \mathbf{W}\right) \cup \left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right), A\right) = \underline{p}\left(\mathbb{R}^2 \setminus (\mathbf{W} \cap c(A, B, \varphi)), A\right).$$

Moreover, $\mathbf{W} \cap c(A, B, \varphi) \subset c(A, B, \varphi) \setminus \bigcup_{n=1}^{\infty} \{A_n\}$. By Theorem 3,

$$\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \leq \underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) = c_\varphi < \frac{2 \sin \varphi}{1 + \sin \varphi} = r.$$

Hence, $A \notin \underline{\mathcal{M}}_r(f + g)$, a contradiction.

(5) $\Rightarrow$ (1) and (7) $\Rightarrow$ (1). Assume that there exists $r \in (0, 1)$ such that for each $g \in \underline{\mathcal{S}}_r$ ($g \in \underline{\mathcal{P}}_r$, respectively) we have $A \in \underline{\mathcal{S}}_r(f + g)$ ($A \in \underline{\mathcal{P}}_r(f + g)$, respectively). For every $s \in (0, 1)$ by φ_s we denote a number from $(0, \frac{\pi}{2})$ such that $\frac{2 \sin \varphi_s}{1 + \sin \varphi_s} = s$. There exists $r_1 \in (r, 1)$ such that $c_{\varphi_{r_1}} < r$. Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = r_1$ for every $B \in \mathbb{R}^2 \setminus \{A\}$, i.e., $\frac{2 \sin \varphi}{1 + \sin \varphi} = r_1$. Let $\varphi_0 \in (0, \varphi)$ and $c_\varphi \in (0, \frac{2 \sin \varphi}{1 + \sin \varphi})$ satisfy assertion of Theorem 3.

Suppose that f is not continuous at A . Then there exist $\varepsilon > 0$ and a sequence $(A_n)_{n \geq 1}$ convergent to A such that $|f(A_n) - f(A)| \geq \varepsilon$ for each $n \geq 1$. Again, we can find $B \in \mathbb{R}^2 \setminus \{A\}$ such that $c(A, B, \varphi_0)$ contains infinitely many elements of the sequence $(A_n)_{n \geq 1}$. Let $\mathbf{Z}$ be a set from Lemma 3 for $\mathbf{X} = c(A, B, \varphi)$ and A . Then $\mathbf{Z}$ is discrete, $\mathbf{Z} \cap$

$c(A, B, \varphi) = \emptyset$ and for each $\mathbf{Y} \subset \mathbb{R}^2$, if $\mathbf{Z} \subset \mathbf{Y}$ then $\underline{p}(\mathbf{Y}, A) = \underline{p}(\mathbf{Y} \cup (\mathbb{R}^2 \setminus c(A, B, \varphi)), A)$. Define $\tilde{g}: (c(A, B, \varphi) \setminus \{A\}) \cup \mathbf{Z} \rightarrow \mathbb{R}$ by

$$\tilde{g}(C) = \begin{cases} 0 & \text{for } C \in c(A, B, \varphi) \setminus \{A\}, \\ -f(C) + f(A) + \varepsilon & \text{for } C \in \mathbf{Z}. \end{cases}$$

Clearly, $(f + \tilde{g})(C) = f(A) + \varepsilon$ for $C \in \mathbf{Z}$ and $\tilde{g}$ is continuous. Since $(c(A, B, \varphi) \setminus \{A\}) \cup \mathbf{Z}$ is a closed subset of $\mathbb{R}^2 \setminus \{A\}$, by the Tietze theorem we can find a continuous extension $\bar{g}: \mathbb{R}^2 \setminus \{A\} \rightarrow \mathbb{R}$ of $\tilde{g}$. Define $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(C) = \begin{cases} 0 & \text{for } C = A, \\ \bar{g}(C) & \text{for } C \in \mathbb{R}^2 \setminus \{A\}. \end{cases}$$

Obviously, $\mathbb{R}^2 \setminus \{A\} \subset \mathcal{P}_r(g)$. Since $\frac{2 \sin \varphi}{1 + \sin \varphi} = r_1 > r$, $A \in \mathcal{P}_r(g)$. Finally, $g \in \mathcal{P}_r$.

On the other hand, $(f + g)(A) = f(A)$, $(f + g)(C) = f(A) + \varepsilon$ for every $C \in \mathbf{Z}$ and $|(f + g)(A_n) - (f + g)(A)| = |f(A_n) - f(A)| \geq \varepsilon$ for $n \geq 1$. Let

$$\mathbf{W} = \{C \in \mathbb{R}^2: |(f + g)(C) - (f + g)(A)| < \varepsilon\}.$$

Then $\mathbf{Z} \subset \mathbb{R}^2 \setminus \mathbf{W}$. By Lemma 3,

$$\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) = \underline{p}\left(\left(\mathbb{R}^2 \setminus \mathbf{W}\right) \cup \left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right), A\right) = \underline{p}\left(\mathbb{R}^2 \setminus (\mathbf{W} \cap c(A, B, \varphi)), A\right).$$

Moreover, $\mathbf{W} \cap c(A, B, \varphi) \subset c(A, B, \varphi) \setminus \bigcup_{n=1}^{\infty} \{A_n\}$. By Theorem 3,

$$\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \leq \underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) = c_{\varphi} < \frac{2 \sin \varphi}{1 + \sin \varphi} = c_{\varphi_1} < r.$$

Hence, $A \notin \underline{\mathcal{S}}_r(f + g)$, a contradiction. $\square$

Corollary 2. For every $r \in (0, 1)$ we have

$$\begin{aligned} \mathfrak{M}_a(\mathcal{M}_r) &= \mathcal{C}, \\ \mathfrak{M}_a(\underline{\mathcal{S}}_r) &= \mathcal{C}, \\ \mathfrak{M}_a(\mathcal{P}_r) &= \mathcal{C}. \end{aligned}$$

Remark 2. One can defined $\mathcal{P}_0$, $\underline{\mathcal{S}}_0$, and $\mathcal{M}_1$. It is easy to see that $\mathfrak{M}_a(\mathcal{P}_0) \neq \mathcal{C}$, $\mathfrak{M}_a(\underline{\mathcal{S}}_0) \neq \mathcal{C}$, and $\mathfrak{M}_a(\mathcal{M}_1) \neq \mathcal{C}$. In [10], classes $\mathfrak{M}_a(\mathcal{P}_0)$, $\mathfrak{M}_a(\underline{\mathcal{S}}_0)$, and $\mathfrak{M}_a(\mathcal{M}_1)$ are described using the notions of topologies p and s introduced in [5,13].

4. Maximal Multiplicative Families for Lower Porouscontinuous Functions

In this section, we describe maximal multiplicative classes for $\underline{\mathcal{S}}_r$ and $\mathcal{M}_r$. It turns out that aiming for this purpose we must define new topologies on $\mathbb{R}^2$ generated by the lower porosity.

First, recall the definition of maximal multiplicative class for a family of functions.

Definition 4 ([12]). Let $\mathcal{F}$ be a family of real functions defined on $\mathbb{R}^2$. A set $\mathfrak{M}_m(\mathcal{F}) = \{g: \mathbb{R}^2 \rightarrow \mathbb{R}: \forall f \in \mathcal{F} (f \cdot g \in \mathcal{F})\}$ is called the maximal multiplicative class for $\mathcal{F}$.

Remark 3. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(C) = 1$ for $C \in \mathbb{R}^2$, be a constant function. If $f \in \mathcal{F}$ then $\mathfrak{M}_m(\mathcal{F}) \subset \mathcal{F}$.

Example 3. Let $\mathbf{U} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, $\mathbf{V} = \{(x, y) \in \mathbb{R}^2 : y < 0\}$, and $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(C) = 0$ for $C \in \mathbf{U}$ and $f(C) = 1$ for $C \in \mathbf{V}$. Clearly, f is not continuous. We will show that $f \in \mathfrak{M}_m(\underline{\mathcal{M}}_r) \cap \mathfrak{M}_m(\underline{\mathcal{S}}_r) \cap \mathfrak{M}_m(\underline{\mathcal{P}}_r)$ for each $r \in (0, 1)$.

Take any $g: \mathbb{R}^2 \rightarrow \mathbb{R}$. Obviously, $(f \cdot g)(C) = 0$ for $C \in \mathbf{U}$; therefore, $\mathbf{U} \subset \underline{\mathcal{M}}_r(f \cdot g) \cap \underline{\mathcal{P}}_r(f \cdot g) \cap \underline{\mathcal{S}}_r(f \cdot g)$. On the other hand, f is continuous on an open set $\mathbf{V}$. Take $A \in \mathbf{V}$ and $r \in (0, 1)$. If $A \in \underline{\mathcal{M}}_r(g)$ (or $A \in \underline{\mathcal{P}}_r(g)$, or $A \in \underline{\mathcal{S}}_r(g)$, respectively) then $A \in \underline{\mathcal{M}}_r(f \cdot g)$ (or $A \in \underline{\mathcal{P}}_r(f \cdot g)$, or $A \in \underline{\mathcal{S}}_r(f \cdot g)$, respectively); therefore, $f \in \mathfrak{M}_m(\underline{\mathcal{M}}_r) \cap \mathfrak{M}_m(\underline{\mathcal{S}}_r) \cap \mathfrak{M}_m(\underline{\mathcal{P}}_r)$ for each $r \in (0, 1)$.

Lemma 4. Let $r \in (0, 1)$, $A \in \mathbb{R}^2$ and $f: \mathbb{R}^2 \rightarrow \mathbb{R}$. If f is not continuous at A and $f(A) \neq 0$ then there exists $g \in \underline{\mathcal{P}}_r$ such that $A \notin \underline{\mathcal{M}}_r(f \cdot g)$.

Proof. As previously, for every $s \in (0, 1)$ by φ_s we denote a number from $(0, \frac{\pi}{2})$ for which $\frac{2 \sin \varphi_s}{1 + \sin \varphi_s} = s$. There exists $r_1 \in (r, 1)$ such that $c_{\varphi_{r_1}} < r$, where

$$c_{\varphi_{r_1}} = \frac{2 \sin \varphi_{r_1}}{\sqrt{1 + 4 \cos^2 \varphi_{r_1} \sin \varphi_{r_1} (1 + \sin \varphi_{r_1}) + \sin \varphi_{r_1}}}.$$

Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = r_1 = \frac{2 \sin \varphi}{1 + \sin \varphi}$ for every $B \in \mathbb{R}^2 \setminus \{A\}$. Let $\varphi_0 \in (0, \varphi)$ and $c_\varphi \in (0, \frac{2 \sin \varphi}{1 + \sin \varphi})$ satisfy assertion of Theorem 3. Assume that $f(A) \neq 0$ and f is not continuous at A . Then there exist $\varepsilon \in (0, \frac{|f(A)|}{2})$ and a sequence $(A_n)_{n \geq 1}$ convergent to A such that $|f(A_n) - f(A)| \geq \varepsilon$ for each $n \geq 1$. We can find $B \in \mathbb{R}^2 \setminus \{A\}$ such that $c(A, B, \varphi_0)$ contains infinitely many elements of the sequence $(A_n)_{n \geq 1}$. Clearly, we may assume that $A_n \in c(A, B, \varphi_0)$ for every $n \geq 1$. Let us define $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(C) = \begin{cases} 1 & \text{for } C \in c(A, B, \varphi), \\ 0 & \text{for } C \notin c(A, B, \varphi). \end{cases}$$

Obviously, $(\mathbb{R}^2 \setminus c(A, B, \varphi)) \cup \text{int}(c(A, B, \varphi)) \subset \mathcal{C}(g) \subset \underline{\mathcal{P}}_r(g)$. By Corollary 1, we obtain $\text{bd}(c(A, B, \varphi)) \setminus \{A\} \subset \underline{\mathcal{P}}_r(g)$. Since $\frac{2 \sin \varphi}{1 + \sin \varphi} = r_1 > r$, $A \in \underline{\mathcal{P}}_r(g)$. Finally, $g \in \underline{\mathcal{P}}_r$.

On the other hand, $(f \cdot g)(A) = f(A)$, $(f \cdot g)(C) = 0$ for every $C \notin c(A, B, \varphi)$ and $|(f \cdot g)(A_n) - (f \cdot g)(A)| = |f(A_n) - f(A)| \geq \varepsilon$ for $n \geq 1$; therefore,

$$\{C \in \mathbb{R}^2 : |(f \cdot g)(C) - (f \cdot g)(A)| < \varepsilon\} \subset c(A, B, \varphi) \setminus \bigcup_{n=1}^{\infty} \{A_n\}.$$

By Theorem 3,

$$\begin{aligned} & \underline{p}(\mathbb{R}^2 \setminus \{C \in \mathbb{R}^2 : |(f \cdot g)(C) - (f \cdot g)(A)| < \varepsilon\}, A) \\ & \leq \underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) \leq c_\varphi < r. \end{aligned}$$

Hence, $A \notin \underline{\mathcal{M}}_r(f \cdot g)$ and the more $A \notin \underline{\mathcal{S}}_r(f \cdot g) \cup \underline{\mathcal{P}}_r(f \cdot g)$. $\square$

Theorem 5. For every $r \in (0, 1)$ and $\mathbf{M} \subset \mathbb{R}^2$ the family of sets $\mathbf{U} \subset \mathbb{R}^2$ satisfying condition:

$$\forall A \in \mathbf{U} \forall \mathbf{W} \subset \mathbb{R}^2, \underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \geq r \quad \left(\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U}) \cup \mathbf{M}], A) \geq r \right)$$

forms a topology. We denote it by $\underline{\mathcal{T}}_r(\mathbf{M})$. The topology $\underline{\mathcal{T}}_r(\mathbf{M})$ is finer than the natural topology.

Proof. Obviously, $\emptyset \in \mathcal{T}_r(\mathbf{M})$. For each $A \in \mathbb{R}^2$ and for each $\mathbf{W} \subset \mathbb{R}^2$ satisfying $\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \geq r$ we obtain

$$\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbb{R}^2) \cup \mathbf{M}], A) = \underline{p}(\mathbb{R}^2 \setminus (\mathbf{W} \cup \mathbf{M}), A) \geq \underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \geq r.$$

Thus $\mathbb{R}^2 \in \mathcal{T}_r(\mathbf{M})$.

Let $\mathbf{U}, \mathbf{V} \in \mathcal{T}_r(\mathbf{M})$, and $A \in \mathbf{U} \cap \mathbf{V}$. Take $\mathbf{W} \subset \mathbb{R}^2$ such that $\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \geq r$. Then $\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U}) \cup \mathbf{M}], A) \geq r$. Hence,

$$\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U} \cap \mathbf{V}) \cup \mathbf{M}], A) = \underline{p}(\mathbb{R}^2 \setminus [((\mathbf{W} \cap \mathbf{U}) \cup \mathbf{M}) \cap \mathbf{V}] \cup \mathbf{M}], A) \geq r,$$

because $\mathbf{V} \in \mathcal{T}_r(\mathbf{M})$. Thus $\mathbf{U} \cap \mathbf{V} \in \mathcal{T}_r(\mathbf{M})$.

Let $\mathbf{U}_t \in \mathcal{T}_r(\mathbf{M})$ for each $t \in T$. Fix $A \in \bigcup_{t \in T} \mathbf{U}_t$. There exists $t_0 \in T$ such that $A \in \mathbf{U}_{t_0}$. Take $\mathbf{W} \subset \mathbb{R}^2$ such that $\underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) \geq r$. Then $\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U}_{t_0}) \cup \mathbf{M}], A) \geq r$ and $\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \bigcup_{t \in T} \mathbf{U}_t) \cup \mathbf{M}], A) \geq \underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U}_{t_0}) \cup \mathbf{M}], A) \geq r$. Thus $\bigcup_{t \in T} \mathbf{U}_t \in \mathcal{T}_r(\mathbf{M})$.

Hence $\mathcal{T}_r(\mathbf{M})$ is a topology on $\mathbb{R}^2$. The remaining part of the proof is obvious. $\square$

Theorem 6. Let $r \in (0, 1)$ and $\mathbf{M} \subset \mathbb{R}^2$. The family of sets $\mathbf{U} \subset \mathbb{R}^2$ satisfying condition:

$$\forall A \in \mathbf{U} \forall \mathbf{W} \subset \mathbb{R}^2, \underline{p}(\mathbb{R}^2 \setminus \mathbf{W}, A) > r \quad \left(\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{U}) \cup \mathbf{M}], A) > r \right)$$

forms a topology. We denote it by $\tau_r(\mathbf{M})$. The topology $\tau_r(\mathbf{M})$ is finer than the natural topology.

Proof. The proof is very similar to the proof of the previous theorem and we omit it. $\square$

Theorem 7. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $r \in (0, 1)$. The following conditions are equivalent:

- (1) $f \in \mathfrak{M}_m(\mathcal{M}_r)$;
- (2) For every $A \in \mathbb{R}^2$, if f is not continuous at A then $f(A) = 0$ and f is $\mathcal{T}_r(\mathbf{N}_f)$ -continuous at A .

Proof. First assume that condition (2) is fulfilled. Take $g \in \mathcal{M}_r$ and $A \in \mathbb{R}^2$. If f is continuous at A then, obviously, $f \cdot g$ is $\mathcal{M}_r$ -continuous at A . Assume that f is not continuous at A . Then, by assumptions, $f(A) = 0$ and f is $\mathcal{T}_r(\mathbf{N}_f)$ -continuous at A . Fix $\varepsilon > 0$. Since $A \in \mathcal{M}_r(g)$, there exists $\mathbf{U} \subset \{C \in \mathbb{R}^2: |g(C) - g(A)| < 1\}$ such that $A \in \mathbf{U}$ and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) \geq r$. By $\mathcal{T}_r(\mathbf{N}_f)$ -continuity of f at A , there is $\mathbf{V} \in \mathcal{T}_r(\mathbf{N}_f)$ for which $A \in \mathbf{V}$ and $|f(C) - f(A)| < \frac{\varepsilon}{|g(A)|+1}$ for each $C \in \mathbf{V}$; therefore $(f \cdot g)(A) = 0$ and $|(f \cdot g)(C)| < \frac{(|g(A)|+1)\varepsilon}{|g(A)|+1} = \varepsilon$ for each $C \in (\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f$. Moreover, $\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f], A) \geq r$. Thus $f \cdot g$ is $\mathcal{M}_r$ -continuous at A . Since A and g were chosen arbitrary, $f \in \mathfrak{M}_m(\mathcal{M}_r)$.

Now assume that $f \in \mathfrak{M}_m(\mathcal{M}_r)$. Take any $A \in \mathbb{R}^2$ and assume that f is not continuous at A . By Lemma 4, $f(A) = 0$. Aiming at a contradiction, suppose that f is not $\mathcal{T}_r(\mathbf{N}_f)$ -continuous at A . Then there exists $\varepsilon > 0$ such that $\mathbf{V} = \{C \in \mathbb{R}^2: |f(C) - f(A)| < \varepsilon\}$ does not contain any $\mathcal{T}_r(\mathbf{N}_f)$ -neighborhood of A . Hence $A \notin \text{int}_{\mathcal{T}_r(\mathbf{N}_f)} \mathbf{V}$. In particular, $\{A\} \cup \text{int} \mathbf{V} \notin \mathcal{T}_r(\mathbf{N}_f)$; therefore, we can find $\mathbf{U} \subset \mathbb{R}^2$ such that $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) \geq r$ and $\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{U} \cap \text{int} \mathbf{V}) \cup \mathbf{N}_f], A) < r$. Observe that if $\mathcal{B}(B, \varrho) \subset (\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f$ then $\mathcal{B}(B, \varrho) \subset (\mathbf{U} \cap \text{int} \mathbf{V}) \cup \mathbf{N}_f$ and $\mathcal{B}(B, \varrho) \subset \mathbf{V}$, because $\mathbf{N}_f \subset \mathbf{V}$. Thus

$$\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f], A) < r. \quad (4)$$

On the other hand, since $f \in \mathfrak{M}_m(\underline{\mathcal{M}}_r)$, we have $f \in \underline{\mathcal{M}}_r$, $A \in \underline{\mathcal{M}}_r(f)$ and $\underline{p}(\mathbb{R}^2 \setminus \mathbf{V}, A) \geq r$. Suppose that $A \in \text{int } \mathbf{U}$. Then

$$\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f], A) = \underline{p}(\mathbb{R}^2 \setminus (\mathbf{V} \cup \mathbf{N}_f), A) = \underline{p}(\mathbb{R}^2 \setminus \mathbf{V}, A) \geq r > 0,$$

which contradicts (4); therefore $A \notin \text{int } \mathbf{U}$. Since $\underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) \geq r$, we have $A \in \text{cl}(\text{int } \mathbf{U})$.

By Theorem 1, there exists a sequence of closed balls $(\overline{B}(A_n, \varrho_n))_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} A_n = A$, $\varrho_n \leq \frac{1}{n}$, $\bigcup_{n \geq 1} \overline{B}(A_n, \varrho_n) \subset \mathbf{U}$ and

$$r \leq \underline{p}(\mathbb{R}^2 \setminus \mathbf{U}, A) = \underline{p}\left(\mathbb{R}^2 \setminus \bigcup_{n \geq 1} \overline{B}(A_n, \varrho_n), A\right). \quad (5)$$

Put $\mathbf{W} = \bigcup_{n=1}^{\infty} \overline{B}(A_n, \varrho_n)$. Then $\mathbf{W} \subset \mathbf{U}$, $\text{cl } \mathbf{W} = \mathbf{W} \cup \{A\}$ and

$$\underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{V}) \cup \mathbf{N}_f], A) < r,$$

because of (4). Let $\mathbf{Z}$ be a set from Lemma 3 for $\mathbf{X} = \mathbf{N}_f \cup \mathbf{W}$ and A . Then $\mathbf{Z}$ is discrete, $\mathbf{Z} \cap (\mathbf{N}_f \cup \mathbf{W}) = \emptyset$, $\text{cl } \mathbf{Z} \subset \mathbf{Z} \cup \{A\}$ and

$$\text{for each } \mathbf{Y} \subset \mathbb{R}^2, \text{ if } \mathbf{Z} \subset \mathbf{Y} \text{ then } \underline{p}(\mathbf{Y}, A) = \underline{p}(\mathbf{Y} \cup (\mathbb{R}^2 \setminus (\mathbf{N}_f \cup \mathbf{W})), A). \quad (6)$$

Since $\mathbf{Z} \cap \mathbf{N}_f = \emptyset$, $f(C) \neq 0$ for $C \in \mathbf{Z}$. Define $\tilde{g}: \mathbf{Z} \cup \mathbf{W} \rightarrow [0, \infty)$ by $\tilde{g}(C) = \frac{2\varepsilon}{|f(C)|}$ for $C \in \mathbf{Z}$ and $\tilde{g}(C) = 1$ for $C \in \mathbf{W}$. Clearly, $|(f \cdot \tilde{g})(C)| = 2\varepsilon$ for $C \in \mathbf{Z}$ and $\tilde{g}$ is continuous, because $\mathbf{Z}$ is discrete and $\text{cl } \mathbf{Z} \cap \text{cl } \mathbf{W} = \{A\}$. Since $\mathbf{Z} \cup \mathbf{W}$ is a closed subset of $\mathbb{R}^2 \setminus \{A\}$, by the Tietze theorem, we can find a continuous extension $\bar{g}: \mathbb{R}^2 \setminus \{A\} \rightarrow [0, \infty)$ of $\tilde{g}$. Finally, define $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$g(C) = \begin{cases} 1 & \text{for } C = A, \\ \bar{g}(C) & \text{for } C \in \mathbb{R}^2 \setminus \{A\}. \end{cases}$$

By construction, g is continuous on an open set $\mathbb{R}^2 \setminus \{A\}$. By (5), we conclude that g is $\underline{\mathcal{M}}_r$ -continuous at A . Hence $g \in \underline{\mathcal{M}}_r$.

Moreover, $(f \cdot g)(A) = 0$ and $|(f \cdot g)(C)| \geq \varepsilon$ for each $C \in \mathbf{Z}$; therefore $\mathbf{Z} \subset \mathbb{R}^2 \setminus \{C \in \mathbb{R}^2: |(f \cdot g)(C) - (f \cdot g)(A)| < \varepsilon\}$. By (6),

$$\begin{aligned} & \underline{p}(\mathbb{R}^2 \setminus \{C \in \mathbb{R}^2: |(f \cdot g)(A) - (f \cdot g)(C)| < \varepsilon\}, A) \\ &= \underline{p}\left(\left(\mathbb{R}^2 \setminus \{C \in \mathbb{R}^2: |f(A) - f(C)| < \varepsilon\}\right) \cup (\mathbb{R}^2 \setminus (\mathbf{W} \cup \mathbf{N}_f)), A\right) \\ &= \underline{p}\left(\mathbb{R}^2 \setminus \left(\{C \in \mathbb{R}^2: |f(A) - f(C)| < \varepsilon\} \cap (\mathbf{W} \cup \mathbf{N}_f)\right), A\right) \\ &\leq \underline{p}(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{V}) \cup \mathbf{N}_f], A) < r. \end{aligned}$$

Therefore $f \cdot g$ is not $\underline{\mathcal{M}}_r$ -continuous at A , a contradiction. $\square$

Theorem 8. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $r \in (0, 1)$. The following conditions are equivalent:

- (1) $f \in \mathfrak{M}_m(\underline{\mathcal{S}}_r)$;
- (2) For every $A \in \mathbb{R}^2$, if f is not continuous at A then $f(A) = 0$ and f is $\underline{\tau}_r(\mathbf{N}_f)$ -continuous at A .

Moreover, if $f \in \mathfrak{M}_m(\underline{\mathcal{P}}_r)$ then for every $A \in \mathbb{R}^2$, if f is not continuous at A then $f(A) = 0$ and f is $\underline{\tau}_r(\mathbf{N}_f)$ -continuous at A .

Proof. Proof of the implication (2) $\Rightarrow$ (1) is very similar to the analogous part of the proof of the previous theorem and we omit it.

Now assume that $f \in \mathfrak{M}_m(\underline{\mathcal{S}}_r)$ (or $f \in \mathfrak{M}_m(\underline{\mathcal{P}}_r)$, respectively). Take any A and assume that f is not continuous at A . By Lemma 4, $f(A) = 0$. Suppose that f is not $\tau_r(\mathbf{N}_f)$ -continuous at A . Then repeating arguments from the proof of the previous theorem there exists $\varepsilon > 0$ and a sequence of closed balls $(\overline{B}(A_n, \varrho_n))_{n \geq 1}$ such that $\lim_{n \rightarrow \infty} A_n = A$,

$$\underline{p}\left(\mathbb{R}^2 \setminus \bigcup_{n \geq 1} \overline{B}(A_n, \varrho_n), A\right) > r \quad \text{and} \quad \underline{p}\left(\mathbb{R}^2 \setminus [(\mathbf{W} \cap \mathbf{V}) \cup \mathbf{N}_f], A\right) \leq r,$$

where $\mathbf{W} = \bigcup_{n=1}^{\infty} \overline{B}(A_n, \varrho_n)$ and $\mathbf{V} = \{C \in \mathbb{R}^2 : |f(A) - f(C)| < \varepsilon\}$. Again, repeating previous arguments we can construct $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ such that g is continuous on an open set $\mathbb{R}^2 \setminus \{A\}$, $g|_{\mathbf{W} \cup \{A\}} = 1$ and $g(C) = \frac{2\varepsilon}{|f(C)|}$ for $C \in \mathbf{Z}$, where $\mathbf{Z}$ is the set from Lemma 3 for $\mathbf{N}_f \cup \mathbf{W}$ and the point A . Then $g \in \underline{\mathcal{P}}_r$ and

$$\underline{p}(\mathbb{R}^2 \setminus \{C : |(f \cdot g)(C)| < \varepsilon\}, A) \leq \underline{p}(\mathbb{R}^2 \setminus [(\mathbf{U} \cap \mathbf{V}) \cup \mathbf{N}_f], A) \leq r,$$

which implies $A \notin \underline{\mathcal{S}}_r(f \cdot g)$ (and therefore $A \notin \underline{\mathcal{P}}_r(f \cdot g)$). This proves that if $f \in \mathfrak{M}_m(\underline{\mathcal{S}}_r)$ (or $f \in \mathfrak{M}_m(\underline{\mathcal{P}}_r)$, respectively) then f is $\tau_r(\mathbf{N}_f)$ -continuous at A , which completes the proof. $\square$

Problem 1. Does $\mathfrak{M}_m(\underline{\mathcal{P}}_r) = \mathfrak{M}_m(\underline{\mathcal{S}}_r)$?

Remark 4. Again, let us consider families of functions $\underline{\mathcal{P}}_0$, $\underline{\mathcal{S}}_0$ and $\underline{\mathcal{M}}_1$. Probably, in order to determine maximal multiplicative class for these families, we should use analogous of topologies p and s and modify them as in Theorems 5 and 6.

5. Properties of Topologies

We describe some properties of topologies $\underline{\mathcal{T}}_r(\mathbf{M})$ and $\underline{\tau}_r(\mathbf{M})$ for different $r \in (0, 1)$ and for different sets $\mathbf{M} \subset \mathbb{R}^2$. By $\mathbf{M}^d$ we denote the set of accumulation points of $\mathbf{M}$ in the natural topology.

Proposition 3. Let $r \in (0, 1)$, $A \in \mathbb{R}^2$, $\mathbf{M} \subset \mathbb{R}^2$. If $A \notin \mathbf{M}^d$ then for each $\mathbf{U} \subset \mathbb{R}^2$ we have:

- $A \in \text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U}$ if and only if $A \in \text{int } \mathbf{U}$;
- $A \in \text{int}_{\underline{\tau}_r(\mathbf{M})} \mathbf{U}$ if and only if $A \in \text{int } \mathbf{U}$.

Proof. Since $\tau_N \subset \underline{\mathcal{T}}_r(\mathbf{M})$, we obtain $\text{int } \mathbf{U} \subset \text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U}$ for every $\mathbf{U} \subset \mathbb{R}^2$. Hence, if $A \in \text{int } \mathbf{U}$ then $A \in \text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U}$.

Let $A \in \text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U}$ and assume that $A \notin \text{int } \mathbf{U}$. Then we can find a sequence $(A_n)_{n \geq 1}$ convergent to A such that $A_n \notin \mathbf{U}$ for each $n \geq 1$. Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = r$ for every $B \in \mathbb{R}^2 \setminus \{A\}$, i.e., $\frac{2 \sin \varphi}{1 + \sin \varphi} = r$. Let $\varphi_0 \in (0, \varphi)$ and $c_\varphi \in (0, \frac{2 \sin \varphi}{1 + \sin \varphi})$ satisfy assertion of Theorem 3. Since $A \notin \mathbf{M}^d$, we may assume that $A_n \notin \mathbf{M}$ for $n \geq 1$. Clearly, we can find $B \in \mathbb{R}^2 \setminus \{A\}$ such that $c(A, B, \varphi_0)$ contains infinitely many elements of the sequence $(A_n)_{n \geq 1}$. Without loss of generality, we may assume that $A_n \in c(A, B, \varphi_0)$ for every $n \geq 1$. By assumption,

$$\begin{aligned} r &\leq \underline{p}\left(\mathbb{R}^2 \setminus [(c(A, B, \varphi) \cap \mathbf{U}) \cup \mathbf{M}], A\right) \\ &\leq \underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) = c_\varphi < r, \end{aligned}$$

a contradiction. Hence, if $A \in \text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U}$ then $A \in \text{int } \mathbf{U}$, which completed the proof.

The proof of the second statement is analogous and we omit it. $\square$

The following two propositions follow directly from definitions of $\underline{\mathcal{T}}_r(\mathbf{M})$ and $\underline{\tau}_r(\mathbf{M})$.

Proposition 4. Let $r \in (0, 1)$, $\mathbf{M}_1, \mathbf{M}_2 \subset \mathbb{R}^2$. If $\mathbf{M}_1 \subset \mathbf{M}_2$ then

- $\underline{\mathcal{T}}_r(\mathbf{M}_1) \subset \underline{\mathcal{T}}_r(\mathbf{M}_2)$;
- $\underline{\tau}_r(\mathbf{M}_1) \subset \underline{\tau}_r(\mathbf{M}_2)$.

Proposition 5. Let $\mathbf{M}_1, \mathbf{M}_2 \subset \mathbb{R}^2$ and $r \in (0, 1)$. If $\mathbf{M}_2^d = \emptyset$ then

- $\underline{\mathcal{T}}_r(\mathbf{M}_1) = \underline{\mathcal{T}}_r(\mathbf{M}_1 \cup \mathbf{M}_2) = \underline{\mathcal{T}}_r(\mathbf{M}_1 \setminus \mathbf{M}_2)$;
- $\underline{\tau}_r(\mathbf{M}_1) = \underline{\tau}_r(\mathbf{M}_1 \cup \mathbf{M}_2) = \underline{\tau}_r(\mathbf{M}_1 \setminus \mathbf{M}_2)$;
- Both topologies $\underline{\mathcal{T}}_r(\mathbf{M}_2)$ and $\underline{\tau}_r(\mathbf{M}_2)$ are equal to the natural topology.

Proof. It follows immediately from the fact that for every $A \in \mathbb{R}^2$ there exists $\varrho > 0$ such that $\mathbf{M}_2 \cap \mathcal{B}(A, \varrho) \subset \{A\}$. $\square$

Theorem 9. Let $\mathbf{M}_1, \mathbf{M}_2 \subset \mathbb{R}^2$ and $r \in (0, 1)$. If $\mathbf{M}_1^d \setminus \mathbf{M}_2^d \neq \emptyset$ then $\underline{\mathcal{T}}_r(\mathbf{M}_1) \not\subset \underline{\mathcal{T}}_r(\mathbf{M}_2)$ and $\underline{\tau}_r(\mathbf{M}_1) \not\subset \underline{\tau}_r(\mathbf{M}_2)$.

Proof. Let $A \in \mathbf{M}_1^d \setminus \mathbf{M}_2^d$. Then there exist a sequence $(A_n)_{n \geq 1}$ of elements of $\mathbf{M}_1 \setminus \{A\}$ convergent to A and $\delta > 0$ such that $\mathcal{B}(A, \delta) \cap \mathbf{M}_2 \subset \{A\}$ and $A_n \in \mathcal{B}(A, \delta)$ for each $n \geq 1$. Put $\mathbf{U} = \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \{A_n\}$. Obviously, $\mathbf{U} \in \underline{\mathcal{T}}_r(\mathbf{M}_1)$.

We will show that $\mathbf{U} \notin \underline{\mathcal{T}}_r(\mathbf{M}_2)$. Clearly, $A \in \mathbf{U}$. Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\underline{p}(\mathbb{R}^2 \setminus c(A, B, \varphi), A) = r = \frac{2 \sin \varphi}{1 + \sin \varphi}$ for every $B \in \mathbb{R}^2 \setminus \{A\}$. Let $\varphi_0 \in (0, \varphi)$ and $c_\varphi \in (0, \frac{2 \sin \varphi}{1 + \sin \varphi})$ satisfy assertion of Theorem 3.

We can find $B \in \mathbb{R}^2 \setminus \{A\}$ such that $c(A, B, \varphi_0)$ contains infinitely many elements of the sequence $(A_n)_{n \geq 1}$. Without loss of generality, we may assume that $A_n \in c(A, B, \varphi_0)$ for every $n \geq 1$. Then

$$\underline{p}\left(\mathbb{R}^2 \setminus [(c(A, B, \varphi) \cap \mathbf{U}) \cup \mathbf{M}_2], A\right) \leq \underline{p}\left(\left(\mathbb{R}^2 \setminus c(A, B, \varphi)\right) \cup \bigcup_{n=1}^{\infty} \{A_n\}, A\right) = c_\varphi < r.$$

Hence, $\mathbf{U} \notin \underline{\mathcal{T}}_r(\mathbf{M}_2)$.

In a similar way we can prove the second statement. $\square$

Corollary 3. Let $r \in (0, 1)$, $\mathbf{M}_1 \subset \mathbf{M}_2 \subset \mathbb{R}^2$ and $\mathbf{M}_2^d \setminus \mathbf{M}_1^d \neq \emptyset$. Then $\underline{\mathcal{T}}_r(\mathbf{M}_1) \subsetneq \underline{\mathcal{T}}_r(\mathbf{M}_2)$ and $\underline{\tau}_r(\mathbf{M}_1) \subsetneq \underline{\tau}_r(\mathbf{M}_2)$.

Theorem 10. Let $r \in (0, 1)$, $\mathbf{M} \subset \mathbb{R}^2$, $A \in \mathbb{R}^2$. The following conditions are equivalent:

- (1) $\{A\} \in \underline{\mathcal{T}}_r(\mathbf{M})$;
- (2) $\{A\} \cup \text{int } \mathbf{M} \in \underline{\mathcal{T}}_r(\mathbf{M})$;
- (3) $\underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) \geq r$.

Proof. Implication (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Now assume that $\{A\} \cup \text{int } \mathbf{M} \in \underline{\mathcal{T}}_r(\mathbf{M})$. Then

$$\begin{aligned} r &\leq \underline{p}\left(\mathbb{R}^2 \setminus \left[(\mathbb{R}^2 \cap (\{A\} \cup \text{int } \mathbf{M})) \cup \mathbf{M}\right], A\right) \\ &= \underline{p}(\mathbb{R}^2 \setminus (\{A\} \cup \mathbf{M}), A) = \underline{p}(\mathbb{R}^2 \setminus (\{A\} \cup \text{int } \mathbf{M}), A). \end{aligned}$$

(3) $\Rightarrow$ (1). Assume that $\underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) \geq r$. Take any $\mathbf{X} \subset \mathbb{R}^2$ such that $\underline{p}(\mathbb{R}^2 \setminus \mathbf{X}, A) \geq r$. Then

$$\underline{p}\left(\mathbb{R}^2 \setminus [(\mathbf{X} \cap \{A\}) \cup \mathbf{M}], A\right) \geq \underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) \geq r.$$

Thus $\{A\} \in \underline{\mathcal{T}}_r(\mathbf{M})$. $\square$

In a similar way we can prove the following theorem.

Theorem 11. Let $r \in (0, 1)$, $\mathbf{M} \subset \mathbb{R}^2$ and $A \in \mathbb{R}^2$. The following conditions are equivalent:

- (1) $\{A\} \in \underline{\tau}_r(\mathbf{M})$;
- (2) $\{A\} \cup \text{int } \mathbf{M} \in \underline{\tau}_r(\mathbf{M})$;
- (3) $\underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) > r$.

Example 4. Let $0 < r_1 < r_2 < 1$, $A, B \in \mathbb{R}^2$, $A \neq B$. Choose $\varphi \in (0, \frac{\pi}{2})$ such that $\frac{2 \sin \varphi}{1 + \sin \varphi} = \frac{r_1 + r_2}{2}$. Then, by Theorems 10 and 11, $\{A\}, \{A\} \cup \text{int } c(A, B, \varphi) \in \left(\underline{\mathcal{T}}_{r_1}(c(A, B, \varphi)) \cap \underline{\tau}_{r_1}(c(A, B, \varphi)) \right) \setminus \left(\underline{\mathcal{T}}_{r_2}(c(A, B, \varphi)) \cup \underline{\tau}_{r_2}(c(A, B, \varphi)) \right)$.

Corollary 4. For every $0 < r_1 < r_2 < 1$ there exists $\mathbf{M} \subset \mathbb{R}^2$ such that $\underline{\mathcal{T}}_{r_1}(\mathbf{M}) \not\subseteq \underline{\mathcal{T}}_{r_2}(\mathbf{M})$ and $\underline{\tau}_{r_1}(\mathbf{M}) \not\subseteq \underline{\tau}_{r_2}(\mathbf{M})$.

Example 5. Let $r \in (0, 1)$. Choose $A, B \in \mathbb{R}^2$, $A \neq B$ and $\varphi \in (0, \frac{\pi}{2})$ such that $\frac{2 \sin \varphi}{1 + \sin \varphi} = r$. Then, by Theorems 10 and 11, $\{A\} \in \underline{\mathcal{T}}_r(c(A, B, \varphi)) \setminus \underline{\tau}_r(c(A, B, \varphi))$.

Remark 5. By Example 5, for every $r \in (0, 1)$ we can find $\mathbf{M} \subset \mathbb{R}^2$ such that $\underline{\mathcal{T}}_r(\mathbf{M}) \not\subseteq \underline{\tau}_r(\mathbf{M})$. The authors do not know whether for every $r \in (0, 1)$ we can find $\mathbf{M} \subset \mathbb{R}^2$ such that $\underline{\tau}_r(\mathbf{M}) \not\subseteq \underline{\mathcal{T}}_r(\mathbf{M})$.

Problem 2. Does for every $0 < r_1 < r_2 < 1$ there exist $\mathbf{M} \subset \mathbb{R}^2$ such that $\underline{\mathcal{T}}_{r_2}(\mathbf{M}) \not\subseteq \underline{\mathcal{T}}_{r_1}(\mathbf{M})$ and $\underline{\tau}_{r_2}(\mathbf{M}) \not\subseteq \underline{\tau}_{r_1}(\mathbf{M})$?

Corollary 5. Let $r \in (0, 1)$, $\mathbf{M}, \mathbf{U} \subset \mathbb{R}^2$. Then

- $\text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U} \supset [\text{int}(\mathbf{U} \cup \mathbf{M}) \cap \mathbf{U}] \cup \{A \in \mathbb{R}^2 : \underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) \geq r\}$;
- $\text{int}_{\underline{\tau}_r(\mathbf{M})} \mathbf{U} \supset [\text{int}(\mathbf{U} \cup \mathbf{M}) \cap \mathbf{U}] \cup \{A \in \mathbb{R}^2 : \underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) > r\}$.

Proof. Inclusion $\text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U} \supset \{A \in \mathbb{R}^2 : \underline{p}(\mathbb{R}^2 \setminus \mathbf{M}, A) \geq r\}$ follows from Theorem 10, whereas inclusion $\text{int}_{\underline{\mathcal{T}}_r(\mathbf{M})} \mathbf{U} \supset \text{int}(\mathbf{U} \cup \mathbf{M}) \cap \mathbf{U}$ follows from $(\mathbf{U} \cap \mathbf{V}) \cup \mathbf{M} \supset (\mathbf{U} \cup \mathbf{M}) \cap \mathbf{V}$ for each $\mathbf{V}$.

The proof of the second assertion is similar and we omit it. $\square$

Problem 3. It seems obvious that all properties of the lower porosity and lower porouscontinuity presented in the paper can be extended for sets in $\mathbb{R}^n$ and functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ for every $n \geq 2$. Does these properties can still be extended for the Hilbert spaces?

6. Summary

In the paper, we proved some properties of the lower porosity of subsets of $\mathbb{R}^2$. We found that the most useful tool for this is the notion of the cone. In Theorem 2, we showed that the lower porosity of the complement of a cone at the vertex can be expressed in the term of the angle of the cone. Moreover, we proved that the lower porosity of the union of the complement of the cone and elements of a sequence of points lying “close” to the axis of the cone at the vertex strictly decreases. It turns out that properties of the lower porosity of the complement of the cone are similar to properties of the (upper) porosity of the complement of a union of a sequence of pairwise disjoint balls, presented in [11]. Next, we introduced families of lower porouscontinuous functions. It generalizes the notion of (upper) porouscontinuity introduced by J. Borsík and J. Holos in [8]. There are three parameterized families of lower porouscontinuous functions $\underline{\mathcal{P}}_r$, $\underline{\mathcal{S}}_r$, and $\underline{\mathcal{M}}_r$ for $r \in (0, 1)$.

In Section 3, we showed that maximal additive class for every family of lower porous-continuous functions consists of continuous functions, similar to the case of (upper) porous-continuity.

It turns out that maximal multiplicative classes for lower porouscontinuous functions have more complicated structure. To describe these classes, we introduced two families of new topologies $\mathcal{T}_r(\mathbf{M})$ and $\underline{\tau}_r(\mathbf{M})$ for $r \in (0, 1)$ and $\mathbf{M} \subset \mathbb{R}^2$, symmetrical with topologies $\mathcal{T}_r(A)$ and $\tau_r(A)$ generated by (upper) porosity introduced in [11]. In Theorems 7 and 8, we described maximal multiplicative classes for $\underline{S}_r$ and $\underline{M}_r$ for $r \in (0, 1)$. Moreover, we proved inclusions $\mathcal{C} \subset \mathfrak{M}_m(\mathcal{P}_r) \subset \mathfrak{M}_m(\underline{S}_r)$.

In Section 5, some properties of topologies $\mathcal{T}_r(\mathbf{M})$ and $\underline{\tau}_r(\mathbf{M})$ are presented. It turns out that properties of topologies generated by the lower and (upper) porosity are symmetrical.

The whole work was devoted to sets and functions of the lower porosity from $(0, 1)$. Several natural questions remained unanswered. Moreover, it seems that properties of sets that lower porosity is just greater than 0 or is equal to 1 are more complicated and cones may not be a sufficient tool to characterize these types of porosity. It would be useful to find sets characterizing these kinds of the lower porosity. Similarly, the classes $\underline{M}_1$ as well as $\underline{S}_0$ and $\underline{P}_0$ seem to have much different and more complicated properties than the corresponding classes for $r \in (0, 1)$. Probably, the maximum additive classes of these families of functions consist of not only continuous functions and one need to define new topologies that are equivalent to the p and s topologies introduced by V. Kelar in [13] and L. Zajíček in [5]. Moreover, to determine the maximum multiplicative classes for $\underline{M}_1$, $\underline{S}_0$, and $\underline{P}_0$ it will be necessary to modify these topologies, analogically as in Theorem 5 and Theorem 6.

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