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Fuzzy Sawi Decomposition Method for Solving Nonlinear Partial Fuzzy Differential Equations

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Abstract: The main goal of this paper is to propose a new decomposition method for finding solutions to nonlinear partial fuzzy differential equations (NPFDE) through the fuzzy Sawi decomposition method (FSDM). This method is a combination of the fuzzy Sawi transformation and Adomian decomposition method. For this purpose, two new theorems for fuzzy Sawi transformation regarding fuzzy partial gH-derivatives are introduced. The use of convex symmetrical triangular fuzzy numbers creates symmetry between the lower and upper representations of the fuzzy solution. To demonstrate the effectiveness of the method, a numerical example is provided.

Keywords: fuzzy Sawi decomposition method; fuzzy partial gH-derivative; nonlinear partial fuzzy differential equations

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1. Introduction

A fundamental problem in the process of modeling phenomena is the immense quantity and quality of information that has to be included, such that it is as representative as possible of the real system. The process of the derivation of mathematical models is set to limitations, such as correct understanding, ambiguity in the accuracy and uncertainty of the data, and measurement errors that lead to uncertainties in the model. Fuzzy modeling is an effective method that enables researchers to express scientific issues.

The modeling of many physical phenomena, such as dynamical and magnetic systems, engineering, biological and environmental issues, and humanities phenomena, result in the use of differential equations. Partial differential equations are mathematical equations that appear in a number of fields, such as physics, engineering, chemistry and biology. Many authors have developed analytical and numerical methods for solving different kinds of partial differential equations; see [1–4].

In order to apply fuzzy differential equations as a modeling tool for dynamical systems, some authors have extended the concept of derivatives in the fuzzy context. This allows to define differential equation in the fuzzy context, which was studied by some authors, such as [5–8].

In some cases, partial differential equations are not the best option when dealing with real-life phenomena. To model dynamic systems, we need to collect information from a variety of sources. Such data sets are often uncertain. The modeling of these systems with uncertain data has promoted fuzzy partial differential equations to become one of the main topics of modern mathematical analysis, attracting the attention of many authors [9–14]. In [15], the reduced differential transformation method was successfully applied for solving fuzzy nonlinear partial differential equations under gH-differentiability.

Integral transformations are the first choice of researchers when finding solutions to critical problems. In [16], the Laplace transformation was applied on mathematical models of population growth and decay. Many scholars [17–20] applied different integral transforms (Mahgoub; Aboodh; Elzaki transforms) on important problems in mechanics, physical chemistry and life science for finding their exact solutions.

Mahgub [21] proposed the Sawi transformation and determined the primitives of constant coefficient ordinary linear differential equations. Aggarwal and Gupta [22] presented a relationship between Sawi and other fundamental transforms. Singh and Aggarwal [23] applied Sawi transformation in finding solutions to biological problems of growth and decay.

The fuzzy integral transforms are very useful in solving linear partial differential equations because they convert the original function into a function that is simpler to solve [24–26]. They do not function well in real applications and can only be used for solving fuzzy linear problems.

The objective of the present paper is to propose a stylish combination of the Adomian decomposition method [27,28] and fuzzy Sawi transformation that can solve nonlinear partial fuzzy differential equations. By using fuzzy Sawi transform, equations are reduced to an algebraic equation. Then, the method of Adomian is used to handle the nonlinear parts of the equation for obtaining the solution. The new decomposition method is then called the fuzzy Sawi decomposition method.

In this paper, we consider symmetric fuzzy triangular numbers. From their parametric form, we obtain the parametric form of the fuzzy functions and we establish symmetry between their upper and lower representations. We also observe symmetry in the parametric representation of fuzzy Sawi transform. The symmetry is also preserved in the application of the fuzzy Sawi transformation of the partial derivatives of the fuzzy functions. When applying the fuzzy Sawi decomposition method to solve the nonlinear partial equation, we obtain a symmetry between the lower and upper representations of of the fuzzy solution.

This paper is organized as follows: In Section 2, definitions on a fuzzy number, fuzzy-valued function and gH-Hukuhara differentiability are given. In Section 3, the definition of fuzzy Sawi transform is introduced. Fuzzy Sawi transformation for the fuzzy partial gH-derivative is proposed. In Section 4, the fuzzy Sawi decomposition method is applied to solve nonlinear partial fuzzy differential equations. Section 5 provides a numerical example to demonstrate the proposed method. Finally, Section 6 consists of conclusions.

2. Basic Concepts

In this section, we review some notions and results of fuzzy numbers, fuzzy-number-valued functions and strongly generalized Hukuhara differentiability.

Definition 1 ([29])**.** *A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ that satisfies the following properties:*

(i) *u is upper semi-continuous on $\mathbb{R}$;*
(ii) *$u(x) = 0$ outside of some interval $[c,d]$;*
(iii) *there are $a,\ b \in \mathbb{R}$ with $c \le a \le b \le d$ such that u is increasing on $[c,a]$, and decreasing on $[b,d]$ and $u(x) = 1$ for each $x \in [a,b]$;*
(iv) *$u(rx + (1-r)y) \ge \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R},\ r \in [0,1]$.*

Denote E^1 the set of all fuzzy numbers. If $a \in \mathbb{R}$, then it can be interpreted as a fuzzy number; $\tilde{a} = \chi_{\{a\}}$ is characteristic function and therefore $\mathbb{R} \subset E^1$.

Definition 2 ([30])**.** *For $0 < r \le 1$ and $u \in E^1$, the r-level set of u is the crisp set*

$$[u]^r = \{x \in \mathbb{R} : u(x) \ge r\}.$$

Then, any r-level set is a bounded and closed interval and denoted by $[\underline{u}(r), \overline{u}(r)]$ for all $0 \le r \le 1$, where $\underline{u}, \overline{u} : [0,1] \to \mathbb{R}$ are the lower and upper bounds of $[u]^r$, respectively.

Definition 3 ([30])**.** *A parametric form of fuzzy number u is an ordered pair $u = (\underline{u}(r), \overline{u}(r))$ of functions $\underline{u}(r)$ and $\overline{u}(r)$ for any $0 \le r \le 1$, which satisfies the following conditions:*

(i) *The function* $\underline{u}(r)$ *is a bounded left continuous monotonic increasing in* $[0,1]$*;*
(ii) *The function* $\overline{u}(r)$ *is a bounded left continuous monotonic decreasing in* $[0,1]$*;*
(iii) $\underline{u}(r) \leq \overline{u}(r)$.

For fuzzy number $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$ and $k \in \mathbb{R}$, the addition and the scalar multiplication are defined by the following:
$[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$ and
$$[k \odot u]^r = k \cdot [u]^r = \begin{cases} [k\underline{u}(r), k\overline{u}(r)], & k \geq 0 \\ [k\overline{u}(r), k\underline{u}(r)], & k < 0. \end{cases}$$
The neutral element, with respect to $\oplus$ in E^1, is denoted by $\tilde{0} = \chi_{\{0\}}$. For basic algebraic properties of fuzzy numbers, please see ([29]).

We use the Hausdorff metric as a distance between fuzzy numbers.

Definition 4 ([29])**.** *For arbitrary fuzzy numbers* $u = (\underline{u}(r), \overline{u}(r))$ *and* $v = (\underline{v}(r), \overline{v}(r))$*, the quantity*
$$d(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$$
is the distance between u and v.

Definition 5 ([31])**.** *A fuzzy number* $u \in E^1$ *is called to be positive if* $\underline{u}(1) \geq 0$*, strict positive if* $\underline{u}(1) > 0$*, negative if* $\overline{u}(1) \leq 0$ *and strict negative if* $\overline{u}(1) < 0$ *.*

The set of positive (negative) fuzzy numbers is denoted by E^1_+ (E^1_-).

Theorem 1 ([32,33])**.** *Let u and v be positive fuzzy numbers, then* $w = u \odot v$ *defined by* $w(r) = [\underline{w}(r), \overline{w}(r)]$*, where the following holds:*
$$\underline{w}(r) = \underline{u}(r)\underline{v}(1) + \underline{u}(1)\underline{v}(r) - \underline{u}(1)\underline{v}(1)$$
and
$$\overline{w}(r) = \overline{u}(r)\overline{v}(1) + \overline{u}(1)\overline{v}(r) - \overline{u}(1)\overline{v}(1)$$
for every $r \in [0,1]$ *is a positive fuzzy number.*

Let the set D be domain of fuzzy-valued function w. Define the functions $\underline{w}(.,.,r)$, $\overline{w}(.,.,r) : D \to \mathbb{R}$ for all $0 \leq r \leq 1$. These functions are said to be the left and right $r-$ level functions of the function w.

Definition 6 ([34])**.** *A fuzzy-valued function* $w : D \to E^1$ *is said to be continuous at* $(s_0, t_0) \in D$ *if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $d(w(s,t), w(s_0,t_0)) < \varepsilon$ *whenever* $|s - s_0| + |t - t_0| < \delta$*. If w is continuous for each* $(s,t) \in D$*, then we say that w is continuous on D.*

Definition 7 ([35])**.** *Let* $x,\ y \in E^1$ *and exists* $z \in E^1$*, such that the following holds:*

(i) $x = y \oplus z$
or
(ii) $z = x \oplus (-1) \odot y$.

Then, z is said to be the generalized Hukuhara difference $(gH$*- difference) of fuzzy numbers x and y and is given by* $x \ominus_{gH} y$*.*

Now consider $x,\ y \in E^1$, then
$x \ominus_{gH} y = z \Leftrightarrow$
(i) $z = (\underline{x}(r) - \underline{y}(r), \overline{x}(r) - \overline{y}(r))$
or
(ii) $z = (\overline{x}(r) - \overline{y}(r), \underline{x}(r) - \underline{y}(r))$.

The following Lemma shows the connection between the gH-difference and the Hausdor distance.

Lemma 1 ([35]). *For all $u, v \in E^1$, we have the following:*

$$d(u,v) = \sup_{r\in[0,1]} \|[u]^r \ominus_{gH} [v]^r\|,$$

where, for an interval $[a,b]$, the norm is $\|[a,b]\| = \max\{|a|,|b|\}$.

Definition 8 ([36]). *Let $w : D \to E^1$ and $(x_0,t) \in D$. We say that w is strongly generalized Hukuhara differentiable on (x_0,t) (gH-differentiable for short) if there exists an element $\frac{\partial w(x_0,t)}{\partial x} \in E^1$ such that the following holds:*

(i) *For all $h > 0$ sufficiently small, the following gH-differences exist:*

$$w(x_0+h,t) \ominus_{gH} w(x_0,t), \quad w(x_0,t) \ominus_{gH} w(x_0-h,t)$$

and the following limits hold (in the metric d):

$$\lim_{h\to 0} \frac{w(x_0+h,t) \ominus_{gH} w(x_0,t)}{h} = \lim_{h\to 0} \frac{w(x_0,t) \ominus_{gH} w(x_0-h,t)}{h} = \frac{\partial w(x_0,t)}{\partial x}$$

(ii) *For all $h > 0$ sufficiently small, the following gH-differences exist:*

$$w(x_0,t) \ominus_{gH} w(x_0+h,t), \quad w(x_0-h,t) \ominus_{gH} w(x_0,t),$$

and the following limits hold (in the metric d):

$$\lim_{h\to 0} \frac{w(x_0,t) \ominus_{gH} w(x_0+h,t)}{-h} = \lim_{h\to 0} \frac{w(x_0-h,t) \ominus_{gH} w(x_0,t)}{-h} = \frac{\partial w(x_0,t)}{\partial x}$$

Lemma 2 ([37]). *Let $w : D \to E^1$ be a continuous fuzzy-valued function and $w(x,t) = (\underline{w}(x,t,r), \overline{w}(x,t,r))$ for all $r \in [0,1]$. Then, the following holds:*

(i) *If $w(x,t)$ is (i)-partial differentiable for x (i.e., w is partial differentiable for x under the meaning of Definition 8 (i)), then we have the following:*

$$\frac{\partial w(x,t)}{\partial x} = \left(\frac{\partial \underline{w}(x,t,r)}{\partial x}, \frac{\partial \overline{w}(x,t,r)}{\partial x} \right), \tag{1}$$

(ii) *If $w(x,t)$ is (ii)-partial differentiable for x (i.e., w is partial differentiable for x under the meaning of Definition 8 (ii)), then we have the following:*

$$\frac{\partial w(x,t)}{\partial x} = \left(\frac{\partial \overline{w}(x,t,r)}{\partial x}, \frac{\partial \underline{w}(x,t,r)}{\partial x} \right). \tag{2}$$

Theorem 2 ([38]). *Let $w : \mathbb{R}_+ \to E^1$ and for all $r \in [0;1]$.*

(i) *The functions $\underline{w}(t,r)$ and $\overline{w}(x,r)$ are Riemann-integrable on $[0,b]$ for every $b \geq 0$.*

(ii) *There are constants $\underline{M}(r) > 0$ and $\overline{M}(r) > 0$ such that the following holds:*

$$\int_0^b |\underline{w}(t,r)| dx \leq \underline{M}(r), \quad \int_0^b |\overline{w}(t,r)| dx \leq \overline{M}(r),$$

for every $b \geq 0$.

Then, the function $w(t)$ is improper fuzzy Riemann-integrable on $[0,\infty)$ and the following holds:

$$(FR)\int_0^\infty w(t)dt = \left(\int_0^\infty \underline{w}(t,r)dt, \int_0^\infty \overline{w}(t,r)dt\right). \tag{3}$$

3. Fuzzy Sawi Transform

In this part, we give the fuzzy Sawi transform (FST) definition and its inverse. We introduce new results of FST for the fuzzy partial derivative.

Definition 9 ([21,39]). *Let $w : \mathbb{R}_+ \to E^1$ be a continuous fuzzy-valued function and for $\sigma > 0$, the function $\frac{1}{\sigma^2}e^{-\frac{t}{\sigma}} \odot w(t)$ is improper fuzzy Riemann-integrable on $[0,\infty)$. Then, we have the following:*

$$(FR)\int_0^\infty \frac{1}{\sigma^2}e^{-\frac{t}{\sigma}} \odot w(t)dt,$$

which is called FST and is denoted by the following:

$$W(\sigma) = S[w(t)] = (FR)\int_0^\infty \frac{1}{\sigma^2}e^{-\frac{t}{\sigma}} \odot w(t)dt, \tag{4}$$

where the variables σ are used to factor the variable t in the argument of the fuzzy-valued function.

The parametric form of FST is as follows:

$$S[w(t)] = (s[\underline{w}(t,r)], s[\overline{w}(t,r)]), \tag{5}$$

where

$$s[\underline{w}(t,r)] = \frac{1}{\sigma^2}\int_0^\infty e^{-\frac{t}{\sigma}}\underline{w}(t,r)dt, \tag{6}$$

$$s[\overline{w}(t,r)] = \frac{1}{\sigma^2}\int_0^\infty e^{-\frac{t}{\sigma}}\overline{w}(t,r)dt. \tag{7}$$

We can rewrite Equation (4) in the following form:

$$W(\sigma) = S[w(t)] = \frac{1}{\sigma}(FR)\int_0^\infty e^{-t} \odot w(\sigma t)dt. \tag{8}$$

Definition 10 ([21,39]). *The fuzzy inverse Sawi transform can be written as the following formula:*

$$S^{-1}[W(\sigma)] = w(t) = \left(s^{-1}[\underline{W}(\sigma,r)], s^{-1}[\overline{W}(\sigma,r)]\right), \tag{9}$$

where the following holds:

$$s^{-1}[\underline{W}(\sigma,r)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{t}{\sigma}}\underline{W}(\sigma,r)d\sigma,$$

$$s^{-1}[\overline{W}(\sigma,r)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\frac{t}{\sigma}}\overline{W}(\sigma,r)d\sigma.$$

For all $r \in [0,1]$ the functions $\underline{W}(\sigma, r)$ and $\overline{W}(\sigma, r)$ must be analytic functions for all σ in the region defined by the inequalities $Re\sigma \geq \gamma$, where γ is the real constant to be chosen suitably.

In [39], classical Sawi transform is applied on some special functions. Some properties generated by Sawi transform are given.

(i) Let $g(t) = 1$ for $t > 0$, then $s[g(t)] = \frac{1}{\sigma}$.
(ii) Let $g(t) = t^n$, where n are positive integers; then, $s[g(t)] = (n!)\sigma^{n-1}$.

We introduce the results of FST for fuzzy partial gH-derivatives.

Theorem 3. *Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a continuous fuzzy-valued function. Suppose the functions $\frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \odot w(x,t)$, $\frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \odot \frac{\partial^n w(x,t)}{\partial x^n}$ are improper fuzzy Riemann-integrable with respect to t on $[0, \infty)$. Then, we have the following:*

$$S\left[\frac{\partial^n w(x,t)}{\partial x^n}\right] = \frac{\partial^n}{\partial x^n} S[w(x,t)], \tag{10}$$

where $S[w(x,t)]$ denotes the FST of the function w and $n \in \mathbb{N}$.

Proof. Let the function $w(x,t)$ be (i)-differentiable. From (3) and the parametric form of FST (5), we have the following:

$$S\left[\frac{\partial^n w(x,t)}{\partial x^n}\right] = (FR)\int_0^\infty \frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \odot \frac{\partial^n w(x,t)}{\partial x^n} dt$$

$$= \left(\int_0^\infty \frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \frac{\partial^n \underline{w}(x,t,r)}{\partial x^n} dt,\ \int_0^\infty \frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \frac{\partial^n \overline{w}(x,t,r)}{\partial x^n} dt\right)$$

$$= \frac{\partial^n}{\partial x^n}\left(\int_0^\infty \frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \underline{w}(x,t,r) dt,\ \int_0^\infty \frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \overline{w}(x,t,r) dt\right) = \frac{\partial^n}{\partial x^n} S[w(x,t)].$$

□

Theorem 4. *Let $w : \mathbb{R}_+ \times \mathbb{R}_+ \to E^1$ be a fuzzy-valued function. The functions $\frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \odot w(x,t)$, $\frac{1}{\sigma^2} e^{-\frac{t}{\sigma}} \odot \frac{\partial^n w(x,t)}{\partial t^n}$ are improper fuzzy Riemann-integrable with respect to t on $[0, \infty)$. For all $t > 0$ and $n \in \mathbb{N}$, there exist continuous partial gH-derivatives to the $(n-1)-$th order with respect to t, and there exists $\frac{\partial^n w(x,t)}{\partial t^n}$.*

1. If the function $w(x,t)$ is (i)-differentiable, then the following holds:

$$S\left[\frac{\partial^n w(x,t)}{\partial t^n}\right] = \left(s\left[\frac{\partial^n \underline{w}(x,t,r)}{\partial t^n}\right], s\left[\frac{\partial^n \overline{w}(x,t,r)}{\partial t^n}\right]\right),$$

where

$$s\left[\frac{\partial^n \underline{w}(x,t,r)}{\partial t^n}\right] = \frac{1}{\sigma^n} s[\underline{w}(x,t,r)] - \sum_{j=0}^{n-1} \frac{1}{\sigma^{n-j+1}} \frac{\partial^j \underline{w}(x,0,r)}{\partial t^j}, \tag{11}$$

$$s\left[\frac{\partial^n \overline{w}(x,t,r)}{\partial t^n}\right] = \frac{1}{\sigma^n} s[\overline{w}(x,t,r)] - \sum_{j=0}^{n-1} \frac{1}{\sigma^{n-j+1}} \frac{\partial^j \overline{w}(x,0,r)}{\partial t^j}. \tag{12}$$

2. If the function $w(x,t)$ is (ii)-differentiable then we have the following:

2.1 If $n = 2k-1$, $k = 1,2,\ldots$

$$S\left[\frac{\partial^{2k-1}w(x,t)}{\partial t^{2k-1}}\right] = \left(s\left[\frac{\partial^{2k-1}\overline{w}(x,t,r)}{\partial t^{2k-1}}\right], s\left[\frac{\partial^{2k-1}\underline{w}(x,t,r)}{\partial t^{2k-1}}\right]\right),$$

where

$$\begin{aligned} s\left[\frac{\partial^{2k-1}\overline{w}(x,t,r)}{\partial t^{2k-1}}\right] &= \frac{1}{\sigma^{2k-1}}s[\underline{w}(x,t,r)] - \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)}}\frac{\partial^{2j}\underline{w}(x,0,r)}{\partial t^{2j}} \\ &- \sum_{j=0}^{k-2}\frac{1}{\sigma^{2(k-j)-1}}\frac{\partial^{2j+1}\overline{w}(x,0,r)}{\partial t^{2j+1}}, \end{aligned} \tag{13}$$

$$\begin{aligned} s\left[\frac{\partial^{2k-1}\underline{w}(x,t,r)}{\partial t^{2k-1}}\right] &= \frac{1}{\sigma^{2k-1}}s[\overline{w}(x,t,r)] - \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)}}\frac{\partial^{2j}\overline{w}(x,0,r)}{\partial t^{2j}} \\ &- \sum_{j=0}^{k-2}\frac{1}{\sigma^{2(k-j)-1}}\frac{\partial^{2j+1}\underline{w}(x,0,r)}{\partial t^{2j+1}}. \end{aligned} \tag{14}$$

2.2 If $n = 2k$, $k = 1,2,\ldots$

$$S\left[\frac{\partial^{2k}w(x,t)}{\partial t^{2k}}\right] = \left(s\left[\frac{\partial^{2k}\underline{w}(x,t,r)}{\partial t^{2k}}\right], s\left[\frac{\partial^{2k}\overline{w}(x,t,r)}{\partial t^{2k}}\right]\right),$$

where

$$\begin{aligned} s\left[\frac{\partial^{2k}\underline{w}(x,t,r)}{\partial t^{2k}}\right] &= \frac{1}{\sigma^{2k}}s[\underline{w}(x,t,r)] - \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)+1}}\frac{\partial^{2j}\underline{w}(x,0,r)}{\partial t^{2j}} \\ &- \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)}}\frac{\partial^{2j+1}\overline{w}(x,0,r)}{\partial t^{2j+1}}, \end{aligned} \tag{15}$$

$$\begin{aligned} s\left[\frac{\partial^{2k}\overline{w}(x,t,r)}{\partial t^{2k}}\right] &= \frac{1}{\sigma^{2k}}s[\overline{w}(x,t,r)] - \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)+1}}\frac{\partial^{2j}\overline{w}(x,0,r)}{\partial t^{2j}} - \\ &- \sum_{j=0}^{k-1}\frac{1}{\sigma^{2(k-j)}}\frac{\partial^{2j+1}\underline{w}(x,0,r)}{\partial t^{2j+1}}. \end{aligned} \tag{16}$$

Proof. Let the function $w(x,t)$ be (i)-differentiable. By induction, we prove Equation (11). For $n = 1$, from condition (5), we have the following:

$$S\left[w_t'(x,t)\right] = \left(s[\underline{w}_t'(x,t,r)],\ s[\overline{w}_t'(x,t,r)]\right).$$

By using integration by parts on t, we obtain the following:

$$s[\underline{w}_t'(x,t,r)] = \int_0^\infty \frac{1}{\sigma^2}e^{-\frac{t}{\sigma}}\underline{w}_t'(x,t,r)dt = \frac{1}{\sigma}s[\underline{w}(x,t,r)] - \frac{1}{\sigma^2}\underline{w}(x,0,r).$$

Let for $n = k$ the Equation (11) holds. Hence, for $n = k+1$ we obtain the following:

$$\begin{aligned} s\left[\frac{\partial^{k+1}\underline{w}(x,t,r)}{\partial t^{k+1}}\right] &= \frac{1}{\sigma}s\left[\frac{\partial^{k}\underline{w}(x,t,r)}{\partial t^{k}}\right] - \frac{1}{\sigma^2}\frac{\partial^{k}\underline{w}(x,0,r)}{\partial t^{k}} \\ &= \frac{1}{\sigma^{k+1}}s[\underline{w}(x,t,r)] - \sum_{j=0}^{k-1}\left(\frac{1}{\sigma}\right)^{k-j+2}\frac{\partial^{j}\underline{w}(x,0,r)}{\partial t^{j}} - \frac{1}{\sigma^2}\frac{\partial^{k}\underline{w}(x,0,r)}{\partial t^{k}} \\ &= \frac{1}{\sigma^{k+1}}s[\underline{w}(x,t,r)] - \sum_{j=0}^{k}\left(\frac{1}{\sigma}\right)^{k-j+2}\frac{\partial^{j}\underline{w}(x,0,r)}{\partial t^{j}}. \end{aligned}$$

Let the function $w(x,t)$ be (ii)-differentiable and $n = 2k$. Then, for $n = 2$, we obtain the following:

$$S\left[\frac{\partial^2 w(x,t)}{\partial t^2}\right] = \left(s\left[\frac{\partial^2 \underline{w}(x,t,r)}{\partial t^2}\right], s\left[\frac{\partial^2 \overline{w}(x,t,r)}{\partial t^2}\right]\right).$$

By using integration by parts on t we obtain the following:

$$\begin{aligned}
&s\left[\frac{\partial^2 \underline{w}(x,t,r)}{\partial t^2}\right] = \frac{1}{\sigma^2}\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial^2 \underline{w}(x,t,r)}{\partial t^2}dt \\
&= -\frac{1}{\sigma^2}\frac{\partial \overline{w}(x,0,r)}{\partial t} + \frac{1}{\sigma^3}\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial \overline{w}(x,t,r)}{\partial t}dt = -\frac{1}{\sigma^2}\frac{\partial \overline{w}(x,0,r)}{\partial t} + \frac{1}{\sigma^3}\int_0^\infty e^{-\frac{t}{\sigma}}d\underline{w}(x,t,r) \\
&= -\frac{1}{\sigma^2}\frac{\partial \overline{w}(x,0,r)}{\partial t} - \frac{1}{\sigma^3}\underline{w}(x,0,r) + \frac{1}{\sigma^4}\int_0^\infty e^{-\frac{t}{\sigma}}\underline{w}(x,t,r)dt \\
&= \frac{1}{\sigma^2}s[\underline{w}(x,t,r)] - \frac{1}{\sigma^3}\underline{w}(x,0,r) - \frac{1}{\sigma^2}\frac{\partial \overline{w}(x,0,r)}{\partial t}.
\end{aligned}$$

Let, for $n = 2k$, Equation (15) hold. Hence, for $n = 2k+2$ we have the following:

$$S\left[\frac{\partial^{2k+2} w(x,t)}{\partial t^{2k+2}}\right] = \left(s\left[\frac{\partial^{2k+2} \underline{w}(x,t,r)}{\partial t^{2k+2}}\right], s\left[\frac{\partial^{2k+2} \overline{w}(x,t,r)}{\partial t^{2k+2}}\right]\right).$$

By using integration by parts on t, we obtain the following:

$$\begin{aligned}
&s\left[\frac{\partial^{2k+2} \underline{w}(x,t,r)}{\partial t^{2k+2}}\right] = \frac{1}{\sigma^2}\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial^{2k+2} \underline{w}(x,t,r)}{\partial t^{2k+2}}dt \\
&= -\frac{1}{\sigma^2}\frac{\partial^{2k+1} \overline{w}(x,0,r)}{\partial t^{2k+1}} + \frac{1}{\sigma^3}\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial^{2k+1} \overline{w}(x,t,r)}{\partial t^{2k+1}}dt \\
&= -\frac{1}{\sigma^2}\frac{\partial^{2k+1} \overline{w}(x,0,r)}{\partial t^{2k+1}} - \frac{1}{\sigma^3}\frac{\partial^{2k} \underline{w}(x,0,r)}{\partial t^{2k}} + \frac{1}{\sigma^2}\frac{1}{\sigma^2}\int_0^\infty e^{-\frac{t}{\sigma}}\frac{\partial^{2k} \underline{w}(x,t,r)}{\partial t^{2k}}dt \\
&= -\frac{1}{\sigma^2}\frac{\partial^{2k+1} \overline{w}(x,0,r)}{\partial t^{2k+1}} - \frac{1}{\sigma^3}\frac{\partial^{2k} \underline{w}(x,0,r)}{\partial t^{2k}} + \frac{1}{\sigma^2}s\left[\frac{\partial^{2k} \underline{w}(x,t,r)}{\partial t^{2k}}\right] \\
&= \frac{1}{\sigma^{2k+2}}s[\underline{w}(x,t,r)] - \sum_{j=0}^{k}\frac{1}{\sigma^{2(k+1-j)+1}}\frac{\partial^{2j} \underline{w}(x,0,r)}{\partial t^{2j}} - \sum_{j=0}^{k}\frac{1}{\sigma^{2(k+1-j)}}\frac{\partial^{2j+1} \overline{w}(x,0,r)}{\partial t^{2j+1}}.
\end{aligned}$$

□

4. Sawi Decomposition Method for Solving NPFDE

In this section, we apply the combined form of FSM and the Adomian decomposition method for solving NPFDE. This equation is defined as follows:

$$\sum_{i=1}^{m} a_i \odot \frac{\partial^i w(x,t)}{\partial x^i} \oplus \sum_{j=0}^{l} b_j \odot \frac{\partial^j w(x,t)}{\partial t^j} \oplus \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} \odot \frac{\partial^k w(x,t)}{\partial x^k} \odot \frac{\partial^p w(x,t)}{\partial x^p} = g(x,t), \quad (17)$$

with initial conditions

$$\frac{\partial^j w(x,0)}{\partial t^j} = \psi_j(x), \quad j = 0,1,\ldots,l-1, \quad (18)$$

where $g,\ w : [0,b]\times[0,d] \to E^1$, $\psi_j : [0,b] \to E^1$ are continuous fuzzy functions, and a_i, $i = 1,2,\ldots,m$, b_j, $j = 1,2,\ldots,l$, c_{kp}, $k = 0,1,2$, $p = 0,1,2$, are positive constants.

Applying the fuzzy Sawi transform to both sides of Equation (17) gives the following:

$$\begin{aligned}
&\sum_{i=1}^{m} a_i \odot S\left[\frac{\partial^i w(x,t)}{\partial x^i}\right] \oplus \sum_{j=0}^{l} b_j \odot S\left[\frac{\partial^j w(x,t)}{\partial t^j}\right] \oplus \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} \odot S\left[\frac{\partial^k w(x,t)}{\partial x^k} \odot \frac{\partial^p w(x,t)}{\partial x^p}\right] \\
&= S[g(x,t)].
\end{aligned} \quad (19)$$

Let $\frac{\partial^k w}{\partial x^k}$, $k = 0, 1, 2$ be positive fuzzy-valued functions. Then, the parametric form of Equation (19) is as follows:

$$\begin{aligned}&\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] + \sum_{j=0}^{l} b_j s\left[\frac{\partial^j \underline{w}(x,t,r)}{\partial t^j}\right] + \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right] \\ &= s\left[\underline{g}(x,t,r)\right],\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \overline{w}(x,t,r)}{\partial x^i}\right] + \sum_{j=0}^{l} b_j s\left[\frac{\partial^j \overline{w}(x,t,r)}{\partial t^j}\right] + \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \overline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \overline{w}(x,t,r)}{\partial x^p}\right] \\ &= s[\overline{g}(x,t,r)].\end{aligned} \tag{21}$$

Case 1. Let the function $w(x,t)$ be (i)-partial differentiable of the m-th order with respect to x and l-th order with respect to t.

We consider Equation (20). Then, from (10) and (11) and initial conditions, we have the following:

$$\begin{aligned}\sum_{j=0}^{l} \frac{b_j}{\sigma^j} s[\underline{w}(x,t,r)] &= s\left[\underline{g}(x,t,r)\right] + \sum_{j=1}^{l}\sum_{v=0}^{j-1} \frac{b_j}{\sigma^{j-v+1}} \underline{\psi}_v(x,r) \\ &- \sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] - \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right],\end{aligned}$$

Then

$$\begin{aligned}s[\underline{w}(x,t,r)] &= \left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(s\left[\underline{g}(x,t,r)\right] + \sum_{j=1}^{l}\sum_{v=0}^{j-1} \frac{b_j}{\sigma^{j-v+1}} \underline{\psi}_v(x,r)\right) \\ &- \left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] + \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right]\right).\end{aligned}$$

Applying the inverse fuzzy Sawi transform to both sides of the equation, we obtain the following:

$$\begin{aligned}\underline{w}(x,t,r) &= s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(s\left[\underline{g}(x,t,r)\right] + \sum_{j=1}^{l}\sum_{v=0}^{j-1} \frac{b_j}{\sigma^{j-v+1}} \underline{\psi}_v(x,r)\right)\right] \\ &- s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] + \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right]\right)\right].\end{aligned} \tag{22}$$

Now, apply the Adomain decomposition method (ADM). This method assume an infinite series solution for the following unknowns function:

$$\underline{w}(x,t,r) = \sum_{n=0}^{\infty} \underline{w}_n(x,t,r). \tag{23}$$

The nonlinear terms is represented by an infinite series of the Adomian polynomials $\underline{A}_n^{kp}$ $n \geq 0, k = 0, 1, 2, p = 0, 1, 2$ in the following form:

$$\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p} = \sum_{n=0}^{\infty} \underline{A}_n^{kp}, \tag{24}$$

where

$$\begin{aligned}\underline{A}_0^{kp} &= \frac{\partial^k \underline{w}_0}{\partial x^k}\frac{\partial^p \underline{w}_0}{\partial x^p}, \\ \underline{A}_1^{kp} &= \frac{\partial^k \underline{w}_0}{\partial x^k}\frac{\partial^p \underline{w}_1}{\partial x^p} + \frac{\partial^k \underline{w}_1}{\partial x^k}\frac{\partial^p \underline{w}_0}{\partial x^p}, \\ \underline{A}_2^{kp} &= \frac{\partial^k \underline{w}_0}{\partial x^k}\frac{\partial^p \underline{w}_2}{\partial x^p} + \frac{\partial^k \underline{w}_1}{\partial x^k}\frac{\partial^p \underline{w}_1}{\partial x^p} + \frac{\partial^k \underline{w}_2}{\partial x^k}\frac{\partial^p \underline{w}_0}{\partial x^p}, \\ &\cdots\cdots\cdots\end{aligned}$$

Substituting (23), (24) into (22) leads to the following:

$$\sum_{n=0}^{\infty} \underline{w}_n(x,t,r) = s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(s\left[\underline{g}(x,t,r)\right] + \sum_{j=1}^{l}\sum_{v=0}^{j-1} \frac{b_j}{\sigma^{j-v+1}} \underline{\psi}_v(x,r)\right)\right]$$
$$-s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(\sum_{i=1}^{m} a_i s\left[\sum_{n=0}^{\infty} \frac{\partial^i \underline{w}_n(x,t,r)}{\partial x^i}\right] - \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\sum_{n=0}^{\infty} \underline{A}_n^{kp}\right]\right)\right].$$

The Adomian decomposition method presents for $n \geq 0$ the recursive relation as follows:

$$\begin{aligned}
\underline{w}_0(x,t,r) &= s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(s\left[\underline{g}(x,t,r)\right] + \sum_{j=1}^{l}\sum_{v=0}^{j-1} \frac{b_j}{\sigma^{j-v+1}} \underline{\psi}_v(x,r)\right)\right], \\
\underline{w}_{n+1}(x,t,r) &= -s^{-1}\left[\left(\sum_{j=0}^{l} \frac{b_j}{\sigma^j}\right)^{-1}\left(\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}_n(x,t,r)}{\partial x^i}\right] - \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\underline{A}_n^{kp}\right]\right)\right].
\end{aligned} \tag{25}$$

Case 2. Let function $w(x,t)$ be (i)-partial differentiable of the m-th order with respect to x and (ii)-partial differentiable of the $l = 2q$-th order with respect to t. Then, the parametric form of Equation (19) is the following:

$$\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] + \sum_{j=0}^{q} b_{2j} s\left[\frac{\partial^{2j} \underline{w}(x,t,r)}{\partial t^{2j}}\right] + \sum_{j=1}^{q} b_{2j-1} s\left[\frac{\partial^{2j-1} \underline{w}(x,t,r)}{\partial t^{2j-1}}\right]$$
$$+ \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right] = s\left[\underline{g}(x,t,r)\right],$$

$$\sum_{i=1}^{m} a_i s\left[\frac{\partial^i \overline{w}(x,t,r)}{\partial x^i}\right] + \sum_{j=0}^{q} b_{2j} s\left[\frac{\partial^{2j} \overline{w}(x,t,r)}{\partial t^{2j}}\right] + \sum_{j=1}^{q} b_{2j-1} s\left[\frac{\partial^{2j-1} \overline{w}(x,t,r)}{\partial t^{2j-1}}\right]t$$
$$+ \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \overline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \overline{w}(x,t,r)}{\partial x^p}\right] = s[\overline{g}(x,t,r)]$$

Applying Theorem 4 and initial conditions, we obtain the following system:

$$\begin{aligned}
&As[\underline{w}(x,t,r)] + Bs[\overline{w}(x,t,r)] = s\left[\underline{g}(x,t,r)\right] + F(x,\sigma,r) \\
&- \sum_{i=1}^{m} a_i s\left[\frac{\partial^i \underline{w}(x,t,r)}{\partial x^i}\right] - \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right]
\end{aligned} \tag{26}$$

$$\begin{aligned}
&As[\overline{w}(x,t,r)] + Bs[\underline{w}(x,t,r)] = s\left[\underline{g}(x,t,r)\right] + G(x,\sigma,r) \\
&- \sum_{i=1}^{m} a_i s\left[\frac{\partial^i \overline{w}(x,t,r)}{\partial x^i}\right] - \sum_{k=0}^{2}\sum_{p=k}^{2} c_{kp} s\left[\frac{\partial^k \overline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \overline{w}(x,t,r)}{\partial x^p}\right],
\end{aligned} \tag{27}$$

where

$$A = \sum_{j=0}^{q} \frac{b_{2j}}{\sigma^{2j}}, \quad B = \sum_{j=1}^{q} \frac{b_{2j-1}}{\sigma^{2j-1}},$$

$$F(x,\sigma,r) = \sum_{j=0}^{q} b_{2j}\left(\sum_{v=0}^{j-1} \frac{1}{\sigma^{2(j-v)+1}} \underline{\psi}_{2v}(x,r) + \sum_{v=0}^{j-1} \frac{1}{\sigma^{2(j-v)}} \overline{\psi}_{2v+1}(x,r)\right)$$
$$+ \sum_{j=1}^{q} b_{2j-1}\left(\sum_{v=0}^{j-1} \frac{1}{\sigma^{2(j-v)}} \overline{\psi}_{2v}(x,r) + \sum_{v=0}^{j-2} \frac{1}{\sigma^{2(j-v)-1}} \underline{\psi}_{2v+1}(x,r)\right),$$

$$\begin{aligned}G(x,\sigma,r) = \sum_{j=0}^{q} b_{2j}\left(\sum_{v=0}^{j-1}\frac{1}{\sigma^{2(j-v)+1}}\overline{\psi}_{2v}(x,r) + \sum_{v=0}^{j-1}\frac{1}{\sigma^{2(j-v)}}\underline{\psi}_{2v+1}(x,r)\right) \\ + \sum_{j=1}^{q} b_{2j-1}\left(\sum_{v=0}^{j-1}\frac{1}{\sigma^{2(j-v)}}\underline{\psi}_{2v}(x,r) + \sum_{v=0}^{j-2}\frac{1}{\sigma^{2(j-v)-1}}\overline{\psi}_{2v+1}(x,r)\right),\end{aligned} .$$

From this system, we find $s[\underline{w}(x,t,r)]$ and $s[\overline{w}(x,t,r)]$. Analogous to Case 1, we obtain $w(x,t) = (\underline{w}(x,t,r), \overline{w}(x,t,r)$.

The Sawi decomposition method is illustrated by discussing the following example.

5. Examples

In this section, we consider the following partial fuzzy differential equation:

$$w''_{tt}(x,t) \oplus w'_x(x,t) \odot w''_{xx}(x,t) = g(x,t), \quad x \geq 0, \quad t \geq 0,$$

with initial conditions

$$w(x,0) = \left(\frac{x^2}{2}r, \frac{x^2}{2}(2-r)\right), \quad w'_t(x,0) = (0,0), \quad x > 0$$

and

$$g(x,t) = \left(r + xr^2, 2 - r + x(2-r)^2\right).$$

In this case, $b_2 = 1$, $c_{12} = 1$, $\psi_0(x) = \left(\frac{x^2}{2}r, \frac{x^2}{2}(2-r)\right)$ and $\psi_1(x) = (0,0)$.

Assume that the function $w(x,t)$ is (i)-differentiable of the 2-th order with respect to x.

Case 1. If $w(x,t)$ is (i)-differentiable of the 2-th order with respect to t, using the recursive relation (25) we obtain the following:

$$\underline{w}_0(x,t,r) = s^{-1}\left[\sigma^2 s\left[\underline{g}(x,t,r)\right]\right] + s^{-1}\left[\frac{1}{\sigma}\underline{\psi}_0(x,r)\right],$$

$$\underline{w}_{n+1}(x,t,r) = -s^{-1}\left[\sigma^2 s\left[\underline{A}_n^{12}\right]\right], \quad n \geq 0,$$

where

$$\begin{gathered}\underline{A}_0^{12} = \underline{w}'_{0x}\underline{w}''_{0xx}, \quad \underline{A}_1^{12} = \underline{w}'_{0x}\underline{w}''_{1xx} + \underline{w}'_{1x}\underline{w}''_{0xx}, \\ \underline{A}_2^{12} = \underline{w}'_{0x}\underline{w}''_{2xx} + \underline{w}'_{1x}\underline{w}''_{1xx} + \underline{w}'_{2x}\underline{w}''_{0xx}, \ldots\end{gathered} \tag{28}$$

Then,

$$\underline{w}_0(x,t,r) = \frac{t^2}{2}r + \frac{xt^2}{2}r^2 + \frac{x^2}{2}r, \quad \underline{w}_1(x,t,r) = -\frac{t^4}{4!}r^3 - \frac{xt^2}{2}r^2,$$

$$\underline{w}_2(x,t,r) = \frac{t^4}{4!}r^3, \quad \underline{w}_3(x,t,r) = 0, \ldots$$

Analogously, we obtain the following:

$$\overline{w}_0(x,t,r) = \frac{t^2}{2}(2-r) + \frac{xt^2}{2}(2-r)^2 + \frac{x^2}{2}(2-r), \quad \overline{w}_1(x,t,r) = -\frac{t^4}{4!}(2-r)^3 - \frac{xt^2}{2}(2-r)^2,$$

$$\overline{w}_2(x,t,r) = \frac{t^4}{4!}(2-r)^3, \quad \underline{w}_3(x,t,r) = 0, \ldots$$

The series solution is, therefore, given by the following:

$$w(x,t) = \left(\left(\frac{x^2}{2} + \frac{t^2}{2}\right)r, \left(\frac{x^2}{2} + \frac{t^2}{2}\right)(2-r)\right).$$

Case 2. If $w(x,t)$ is (ii)-differentiable of the 2-th order with respect to t, using Equations (26) and (27), we obtain the following:

$$\frac{1}{\sigma^2}s(\underline{w}(x,t,r)) = s(g(x,t,r)) + \frac{1}{\sigma^3}\underline{\psi_0}(x,r) - s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right], \tag{29}$$

$$\frac{1}{\sigma^2}s(\overline{w}(x,t,r)) = s(g(x,t,r)) + \frac{1}{\sigma^3}\overline{\psi_0}(x,r) - s\left[\frac{\partial^k \overline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \overline{w}(x,t,r)}{\partial x^p}\right] \tag{30}$$

From (29), we have the following:

$$s(\underline{w}(x,t,r)) = \sigma^2 s(g(x,t,r)) + \frac{1}{\sigma}\underline{\psi_0}(x,r) - \sigma^2 s\left[\frac{\partial^k \underline{w}(x,t,r)}{\partial x^k}\frac{\partial^p \underline{w}(x,t,r)}{\partial x^p}\right].$$

Applying the inverse fuzzy Sawi transform to both sides of the equation and by applying Adomian decomposition method, we obtain the following recursive relation:

$$\underline{w}_0(x,t,r) = s^{-1}\left[\sigma^2 s\left[\underline{g}(x,t,r)\right]\right] + s^{-1}\left[\frac{1}{\sigma}\underline{\psi}_0(x,r)\right],$$

$$\underline{w}_{n+1}(x,t,r) = -s^{-1}\left[\sigma^2 s\left[\underline{A}_n^{12}\right]\right], \ \ n \geq 0,$$

Hence, this case equivalent to Case 1.

Case 3. If $w(x,t)$ is (i)-differentiable and $w'_t(x,t)$ is (ii)-differentiable with respect to t, then $S(w'_t(x,t)) = (s(\underline{w}'_t(x,t,r)), s(\overline{w}'_t(x,t,r))), S(w''_{tt}(x,t)) = (s(\overline{w}''_{tt}(x,t,r)), s(\underline{w}''_{tt}(x,t,r)))$.

Using (15) and (16) of Theorem 4 and the initial condition, we obtain the following recursive relation:

$$\underline{w}_0(x,t,r) = s^{-1}\left[\sigma^2 s[\overline{g}(x,t,r)]\right] + s^{-1}\left[\frac{1}{\sigma}\underline{\psi}_0(x,r)\right],$$

$$\underline{w}_{n+1}(x,t,r) = -s^{-1}\left[\sigma^2 s\left[\overline{A}_n^{12}\right]\right], \ \ n \geq 0,$$

$$\overline{w}_0(x,t,r) = s^{-1}\left[\sigma^2 s\left[\underline{g}(x,t,r)\right]\right] + s^{-1}\left[\frac{1}{\sigma}\overline{\psi}_0(x,r)\right],$$

$$\overline{w}_{n+1}(x,t,r) = -s^{-1}\left[\sigma^2 s\left[\underline{A}_n^{12}\right]\right], \ \ n \geq 0,$$

where

$$\underline{A}_0^{12} = \underline{w}'_{0x}\underline{w}''_{0xx}, \ \ \underline{A}_1^{12} = \underline{w}'_{0x}\underline{w}''_{1xx} + \underline{w}'_{1x}\underline{w}''_{0xx},$$
$$\underline{A}_2^{12} = \underline{w}'_{0x}\underline{w}''_{2xx} + \underline{w}'_{1x}\underline{w}''_{1xx} + \underline{w}'_{2x}\underline{w}''_{0xx}, \ldots$$

$$\overline{A}_0^{12} = \overline{w}'_{0x}\overline{w}''_{0xx}, \overline{A}_1^{12} = \overline{w}'_{0x}\overline{w}''_{1xx} + \overline{w}'_{1x}\overline{w}''_{0xx},$$
$$\overline{A}_2^{12} = \overline{w}'_{0x}\overline{w}''_{2xx} + \overline{w}'_{1x}\overline{w}''_{1xx} + \overline{w}'_{2x}\overline{w}''_{0xx}, \ldots$$

Then,

$$\underline{w}_0(x,t,r) = \frac{t^2}{2}(2-r) + \frac{xt^2}{2}(2-r)^2 + \frac{x^2}{2}r, \ \ \underline{w}_1(x,t,r) = -\frac{t^4}{4!}r^2(2-r) - \frac{xt^2}{2}(2-r)^2,$$

$$\underline{w}_2(x,t,r) = \frac{t^4}{4!}r^2(2-r), \underline{w}_3(x,t,r) = 0, \ldots$$

The series solution is, therefore, given by the following:

$$w(x,t,r) = \left(\frac{x^2}{2}r + \frac{t^2}{2}(2-r), \frac{x^2}{2}(2-r) + \frac{t^2}{2}r\right).$$

6. Conclusions and Future Work

The main idea of this work is to provide a simple method for solving the nonlinear partial fuzzy differential equations under gH-differentiability. A combined form of the fuzzy Sawi transformation method and Adomian decomposition method for these equations is applied. New results on fuzzy Sawi transform for fuzzy partial gH-derivatives are proposed. The main advantage of this method is the fact that it provides an analytical solution. Finally, an example to illustrate the proposed method is solved. The results reveal that the method is a powerful and efficient technique for solving nonlinear partial fuzzy differential equations.

For future research, we will apply the fuzzy Sawi decomposition method to fuzzy nonlinear integro-differential equations under generalized Hukuhara differentiability and the fuzzy nonlinear Fitzhugh–Nagumo–Huxley equation, which is an important model in the work of neuron axons [40]. Additionally, one can discuss the application of this method to more complex problems, such as the eigenproblem [41] and maximum likelihood estimation [42].

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