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Parallel Locally Strictly Convex Surfaces in Four-Dimensional Affine Space Contained in Hyperquadrics

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Abstract: Locally strictly convex surfaces in four-dimensional affine space are studied from a perspective of the affine structure invented by Nuño-Ballesteros and Sánchez, which is especially suitable in convex geometry. The surfaces that are embedded in locally strictly convex hyperquadrics are classified under assumptions that the second fundamental form is parallel with respect to the induced connection and the normal connection is compatible with a metric on the transversal bundle. Both connections are induced by a canonical transversal plane bundle, which is defined by certain symmetry conditions. The obtained surfaces are always products of an ellipse and a conical planar curve.

Keywords: affine metric; strictly convex surface; symmetric equiaffine transversal bundle; antisymmetric equiaffine transversal bundle; hyperquadric

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1. Introduction

The main purpose of this paper is to characterize a class of surfaces embedded in locally strictly convex hyperquadrics contained in the four-dimensional affine space. Every such surface is locally strictly convex itself because it locally lies on one side of a tangent space of the hyperquadric and has a non-degenerate contact with it.

In Section 2, we recall a notion of a locally strictly convex surface. Every point of such a surface has a neighborhood that lies on one side of a supporting hyperplane, which is tangent to the surface. We also recall basic equations of surfaces of codimension two.

In Section 3, we prove that there is a collection of supporting hyperplanes that vary smoothly on an oriented, locally strictly convex surface (compare [1] in the Euclidean case). Simultaneously, we explain the existence of a smooth transversal vector field, which, together with the tangent plane, spans a supporting hyperplane at every point of the surface. Such a transversal field is called a metric field.

In order to study natural geometric properties, we choose a geometric structure that was invented by Nuño-Ballesteros and Sánchez in [2]. The construction starts with a family of metric fields that define affine metrics. Those metrics are definite on an oriented, locally strongly convex surface (unlike the Bursting-Meyer metric, see [3]), [4]. For every metric of the family, two canonical equiaffine transversal bundles, a symmetric and an antisymmetric one are defined in [2]. A transversal bundle is called equiaffine if the metric volume element is parallel with respect to the affine connection induced on the surface by the bundle. We recall briefly these concepts in Section 4.

If the surface is embedded in a locally strictly convex hyperquadric then there is a metric from the aforementioned family that coincides with the Blaschke metric induced from the hyperquadric. This special choice of a metric implies that both equiaffine transversal bundles coincide. Moreover, the Blaschke normal of the hyperquadric belongs to them, which meets our expectations, but it is not generally true for other structures, for example the well-known Nomizu–Vrancken affine normal bundle defined in [4]. In Section 5, we give a complete proof of the last property because there is a certain omission in its proof in [2]. The property itself is so important that it justifies the whole construction.

In Section 6, we prove the main theorem of the paper (Theorem 4) based on the affine structure invented in [2]. We classify surfaces embedded in locally strictly convex hyperquadrics of $\mathbb{R}^4$ under the assumptions that the cubic form vanishes (such submanifolds are called parallel because the second fundamental form is parallel with respect to the induced connections) and a normal connection is compatible with a metric canonically defined on the transversal bundle. We prove that if a surface satisfies the above properties and has no inflection points, then it is locally affine equivalent to an open part of one of three types of surfaces: a product of two ellipses (affine equivalent to the Clifford torus), the product of an ellipse and a hyperbola or the product of an ellipse and a parabola. Thus, we obtained another affine characterization of the Clifford torus as a parallel locally strictly convex surface contained in the ellipsoid (see Opozda's work [5]). We note that in [6], a topological torus is characterized as a compact surface with the indefinite Burstin–Mayer metric.

2. Preliminaries

In this section, we assume that M is a connected surface immersed in $\mathbb{R}^4$ by a smooth mapping $f : M \rightarrow \mathbb{R}^4$ and that there is a transversal plane bundle σ such that $f_*(T_x M) \oplus \sigma_x = \mathbb{R}^4$ for every $x \in M$. Because of the local character of this work, we can restrict ourselves to embeddings, and we will write:

$$T_x M \oplus \sigma_x = \mathbb{R}^4. \quad (1)$$

Definition 1. Let M be a surface embedded in $\mathbb{R}^4$ and let $p \in M$. M is locally strictly convex at p if there is a hyperplane π_p tangent to M at p , having a non-degenerate contact of order one with M and an open neighborhood U of p such that $U \setminus \{p\}$ lies on one side of π_p . We say that π_p is a non-singular support hyperplane of U . The contact of order one means that the second derivative of any regular curve on M passing through p is transversal to π_p at p .

Remark 1. We note that for a given non-singular support hyperplane and an arbitrary transversal plane, there is a vector in this plane which, together with the tangent space, spans the support hyperplane.

We assume that M is locally strictly convex. We study and develop a theory developed in [2]. Let X, Y be arbitrary tangent vector fields and ξ be a transversal vector field, which is a section of σ . Denoting by D the standard linear connection in $\mathbb{R}^4$, we obtain the following decompositions:

$$\begin{aligned} D_X Y &= \nabla_X Y + h(X, Y), \\ D_X \xi &= -S_\xi X + \nabla_X^\perp \xi, \end{aligned}$$

where $\nabla_X Y$ and $-S_X \xi$ denote the tangent components, and $h(X, Y)$ and $\nabla_X^\perp \xi$ denote the transversal components of the left hand sides of the equations. It is well-known ([3]) that ∇ is a torsion-free affine connection, h is a $C^\infty(M)$ -bilinear, symmetric σ -valued mapping called the second fundamental form, and S_ξ is a $(1,1)$ tensor field. We call $\nabla^\perp$ the normal connection. It is actually a connection in σ . Both connections are said to be induced by σ . For a given local basis $\{\xi_1, \xi_2\}$ of σ , we introduce the following notations for the components of h and the normal connection:

$$h(X, Y) = h^1(X, Y)\xi_1 + h^2(X, Y)\xi_2, \quad (2)$$

$$\nabla_X^\perp \xi_j = \tau_j^1(X)\xi_1 + \tau_j^2(X)\xi_2 \quad (3)$$

for $j = 1, 2$. Thus, h^1 and h^2 are bilinear symmetric forms, and τ_j^i are one-forms. We will also abbreviate S_{ξ_j} by S_j . It is proven in [4] that the following equations of Gauss, Codazzi and Ricci hold (X, Y and Z denote tangent vector fields).

$$R(X, Y)Z = h^1(Y, Z)S_1X + h^2(Y, Z)S_2X - h^1(X, Z)S_1Y - h^2(X, Z)S_2Y, \quad (4)$$

$$\text{mappings } (X, Y, Z) \mapsto (\nabla_X h^i)(Y, Z) + \tau_1^i(X)h^1(Y, Z) + \tau_2^i(X)h^2(Y, Z) \quad (5)$$

are totally symmetric, $i = 1, 2$;

$$(\nabla_X S_i)Y - (\nabla_Y S_i)X = -\tau_1^1(Y)S_1X + \tau_1^1(X)S_1Y - \tau_1^2(Y)S_2X + \tau_1^2(X)S_2Y \quad (6)$$

for $i = 1, 2$;

$$h^1(X, S_1Y) - h^1(Y, S_1X) = d\tau_2^1(X, Y) + \tau_1^2(Y)\tau_2^1(X) - \tau_2^1(Y)\tau_1^2(X), \quad (7)$$

$$h^2(X, S_1Y) - h^2(Y, S_1X) = d\tau_1^1(X, Y) + \tau_1^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_1^1(X) + \tau_1^2(Y)\tau_2^2(X) - \tau_2^2(Y)\tau_1^2(X), \quad (8)$$

$$h^2(X, S_2Y) - h^2(Y, S_2X) = d\tau_2^2(X, Y) + \tau_2^1(Y)\tau_1^2(X) - \tau_1^2(Y)\tau_2^1(X), \quad (9)$$

$$h^1(X, S_2Y) - h^1(Y, S_2X) = d\tau_2^1(X, Y) + \tau_1^1(X)\tau_2^1(Y) - \tau_2^1(X)\tau_1^1(Y) + \tau_1^1(X)\tau_2^2(Y) - \tau_2^2(X)\tau_1^1(Y). \quad (10)$$

We also consider the cubic form K , which involves both connections:

$$K(X, Y, Z) = \nabla_X^\perp h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \quad (11)$$

It is σ -valued and trilinear over $C^\infty(M)$. Its components with respect to the frame $\{\xi_1, \xi_2\}$ are $(3,0)$ tensor fields, and they will be denoted by K^1, K^2 , so the following formulas hold for $i = 1, 2$:

$$K^i(X, Y, Z) = (\nabla_X h^i)(Y, Z) + \tau_1^i(X)h^1(Y, Z) + \tau_2^i(X)h^2(Y, Z). \quad (12)$$

3. Non-Singular Support Hyperplanes

Let M be a locally strictly convex surface in $\mathbb{R}^4$ with a transversal bundle σ . We fix a point $p \in M$. Let π_p be a hyperplane passing through p . Let $(X_1)_p, (X_2)_p$ be a basis of $T_p M$ and $\xi_p \in \sigma_p$ be a transversal vector at p such that π_p is spanned by $\{(X_1)_p, (X_2)_p, \xi_p\}$ (see Remark 1). Define a function $H_p : M \rightarrow \mathbb{R}$:

$$H_p(x) := [(X_1)_p, (X_2)_p, x - p, \xi_p], \quad (13)$$

where $[\dots]$ denotes the usual determinant in $\mathbb{R}^4$. It follows from Definition 1 that the hyperplane π_p locally supports M in a neighborhood of the point p if and only if H_p has a strict local extremum at p .

We notice that in the considered case, the differential dH_p vanishes at p so the problem of an extremum of H_p at p comes down to the Hessian of composition of H_p with an arbitrary chart. If we choose a chart (V, ϕ) around a point p with local coordinates u_1, u_2 , the Hessian of $H_p \circ \phi^{-1}$ at p acting on the standard basis e_1, e_2 in $\mathbb{R}^2$ is equal to:

$$\text{Hess}_p(H_p \circ \phi^{-1})(e_i, e_j) = [(X_1)_p, (X_2)_p, \frac{\partial^2(\phi^{-1})}{\partial u_i \partial u_j}(\phi(p)), \xi_p] = [(X_1)_p, (X_2)_p, (D_{\partial_i} \partial_j)_p, \xi_p] \quad (14)$$

where $\partial_i(p) = d_{\phi(p)}(\phi^{-1})(e_i)$ and $i, j = 1, 2$. Let us consider a mapping

$$\Psi_p : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \mapsto [(X_1)_p, (X_2)_p, (D_X Y)_p, \xi_p] \in \mathbb{R} \quad (15)$$

which is bilinear over the ring of smooth functions on M because $\Psi(X, fY) = f\Psi(X, Y)$ as the determinant is independent of the tangent component of $D_X(fY)$, where f is a function. It is also symmetric because $D_Y X = D_X Y - [X, Y]$.

Therefore, we conclude that if mapping Ψ_p is definite then, after fixing a local basis of tangent vector fields associated with a chart ϕ , the matrix $[\Psi_p(\partial_i, \partial_j)]_{ij}$ is definite so

the Hessian of $H_p \circ \phi^{-1}$ is definite. Consequently, $H_p \circ \phi^{-1}$ has an extremum and a non-degenerate critical point at p . This property does not depend on ϕ . The converse is also true because of the bilinearity of Ψ_p . Therefore, we will say that H_p has a non-degenerate critical point at p if Ψ_p is definite.

In the following proposition, we compare notions introduced above.

Proposition 1. Let M , $(X_1)_p$, $(X_2)_p$ and ξ_p be as above. The following conditions are equivalent:

- (i) the hyperplane π_p spanned by $\{(X_1)_p, (X_2)_p, \xi_p\}$ has contact of order one with the surface M at $p \in M$;
- (ii) for every curve γ on M such that $\gamma(0) = p$, $\gamma'(0) \neq 0$ and for every tangent vector field X that extends locally vector field $\gamma'(t)$, $\Psi_p(X, X) \neq 0$;
- (iii) the mapping Ψ_p is definite;
- (iv) the function H_p has a local extremum and a non-degenerate critical point at p ;
- (v) the hyperplane π_p spanned by $(X_1)_p$, $(X_2)_p$ and ξ_p is a non-singular hyperplane supporting a neighborhood of p in M .

Proof. (i) $\Leftrightarrow$ (ii) Looking at (ii), we notice that $\Psi_p(X, X)$ depends only on the values of $\gamma'(t)$ for t in a neighborhood of 0. Since

$$\Psi_p(X, X) = [(X_1)_p, (X_2)_p, D_{\gamma'(0)}\gamma'(t), \xi_p] = [(X_1)_p, (X_2)_p, \gamma''(0), \xi_p], \quad (16)$$

$\Psi_p(X, X) \neq 0$ if and only if $\gamma''(0)$ does not belong to π_p .

(ii) $\Rightarrow$ (iii) It is sufficient to prove that for every tangent vector field X that is non-zero at p , $\Psi_p(X, X) \neq 0$. Let $\psi_t(x)$ be a one-parameter group of diffeomorphisms generated by X in a neighborhood of p . Define a curve $\gamma(t) := \psi_t(p)$. Then, $X_{\gamma(t)} = \gamma'(t)$ and $(D_X X)_p = (D_X X)_{\gamma(0)} = D_{\gamma'(0)}\gamma'(t) = \gamma''(0)$, and (ii) implies that $\Psi_p(X, X) \neq 0$.

(iii) $\Rightarrow$ (ii) It follows from the well-known possibility of extending a vector field from a curve to an open set of a surface.

(iii) $\Leftrightarrow$ (iv) It is a consequence of the fact that $d_p H_p = 0$, as well as a definition of non-degeneracy of the critical point of H_p .

(iv) $\Rightarrow$ (v) Since $H_p(p) = 0$, (iv) implies that H_p has a constant sign in a neighborhood of p in M , and, consequently, this neighborhood lies on one side of π_p . Applying (i), we deduce that π_p is a non-singular supporting hyperplane for U .

Implication (v) $\Rightarrow$ (i) is obvious. $\square$

In the next step, we discuss a question of the local existence of a smooth family of non-singular support hyperplanes for a locally strictly convex surface M . First, we construct a local extension of a transversal non-zero vector, which satisfies property (v) of Proposition 1.

Lemma 1. Let $\eta \in \sigma_p \setminus \{0\}$ be a vector transversal to M . Then, there exists a local non-vanishing vector field ξ in a certain neighborhood U of p such that for every $x \in U$, $\xi_x \in \sigma_x$ and $\xi_p = \eta$.

Proof. Let V be such an open neighborhood of p that there exists a tangent frame $\{X_1, X_2\}$ and a transversal frame $\{\xi_1, \xi_2\}$ in V . Hence, there are smooth functions a, b, c and d in V such that $\eta = aX_1 + bX_2 + c\xi_1 + d\xi_2$ and $a(p) = b(p) = 0$. Then, $\xi := c\xi_1 + d\xi_2$ is a transversal vector field and $\xi_p = \eta \neq 0$, so $\xi \neq 0$ in a neighborhood U of p . Thus, ξ satisfies the assertions of the lemma in U . $\square$

Next, we show that a non-singular support hyperplane can be locally extended in a smooth way.

Proposition 2. Let X_1, X_2 be a tangent frame in a neighborhood of $p \in M$ and $\eta \in \sigma_p$ be such a vector that a hyperplane π_p spanned by $\{(X_1)_p, (X_2)_p, \eta\}$ is a non-singular local support hyperplane for M at p . Then, there exists an open neighborhood U of p and a transversal vector

field ξ defined in U such that $\xi_p = \eta$ and for every $q \in U$, hyperplane π_q spanned by $(X_1)_q, (X_2)_q$ and ξ_q is a non-singular local support hyperplane for M at q .

Proof. Let ξ be a transversal vector field extending η , constructed as in Lemma 1 and defined on an open set $V \subset M$. Then for $q \in V$ we consider a function H_q defined according to Equation (13). A matrix function $V \ni q \mapsto [\Psi_q((X_i)_q, (X_j)_q)]_{i,j=1,2}$ is continuous at p , and a matrix $[\Psi_p((X_i)_p, (X_j)_p)]_{i,j=1,2}$ is definite, so there is an open set $U \subset V$ containing p such that a matrix $[\Psi_q((X_i)_q, (X_j)_q)]_{i,j=1,2}$ is definite for every $q \in U$. Thus, Ψ_q is definite in U and, by Proposition 1, each of the hyperplanes π_q spanned by $(X_1)_q, (X_2)_q$ and ξ_q is a non-singular hyperplane supporting a neighborhood of q in M . Those hyperplanes obviously vary smoothly with q . $\square$

Let M be a locally strictly convex surface in $\mathbb{R}^4$, $p \in M$ and $u = \{X_1, X_2\}$ be a local tangent frame around p . It follows that there is a non-singular hyperplane π_p containing p , supporting locally M , spanned by $(X_1)_p, (X_2)_p$ and a transversal vector $\eta \in \sigma_p$. Applying Propositions 1 and 2, we can extend η to a local transversal vector field ξ such that a bilinear form

$$G_u(Y, Z) = [X_1, X_2, D_Y Z, \xi]$$

is positive definite (compare Equation (15)). In [2], the form G_u serves to define another bilinear symmetric form g_ξ , which turns out to be independent of u up to sign:

$$g_\xi(Y, Z) = \frac{G_u(Y, Z)}{(\det_u G_u)^{1/4}}. \quad (17)$$

A detailed proof is given in [2]. We only notice that $g_u = g_v$ for another tangent frame $v = \{Y_1, Y_2\}$ if the transition matrix between u and v has a positive determinant. We will show that if a surface M is oriented, we can extend ξ to a global transversal vector field such that the definition of g_ξ makes sense.

Proposition 3. *Let M be an orientable, locally strictly convex surface in $\mathbb{R}^4$. There is an oriented, locally finite atlas $\{(U_j, \phi_j)\}_{j \in J}$ and a family of transversal vector fields $\{\xi_j\}_{j \in J}$, every ξ_j defined on U_j , such that at every point $q \in U_j$, the hyperplane spanned by the tangent plane $T_q M$ and $\xi_j(q)$ supports M locally in a non-singular way. There is also a transversal field ξ on M such that a form g_ξ is positive definite.*

Proof. We choose an oriented atlas $\{(W_i, \psi_i)\}_{i \in I}$ on M . Let $\tilde{\partial}_{i,k} = d\psi_i^{-1}(e_k)$, $k = 1, 2$. According to Propositions 1 and 2, for every point $p \in M$, there exists $i \in I$, an open neighborhood of p , $V_p \subset W_i$ and a transversal vector field ξ^p on V_p such that a quadratic form $X \mapsto [\tilde{\partial}_{i,1}, \tilde{\partial}_{i,2}, D_X X, \xi^p]$ is positive definite. The condition of positivity does not depend on a map ψ_i because for another map ψ_k such that $W_i \cap W_k \neq \emptyset$, the Jacobian of $\psi_i \circ \psi_k^{-1}$ is positive so $[\tilde{\partial}_{k,1}, \tilde{\partial}_{k,2}, D_X X, \xi^p] > 0$ for a non-zero tangent vector field X . Since M is paracompact, there is a locally finite cover $\{U_j\}_{j \in J}$ of M inscribed in $\{V_p\}_{p \in M}$. Thus, for every $j \in J$, there exists $p_j \in M$ and $i_j \in I$ such that $U_j \subset V_{p_j} \subset W_{i_j}$. Let $\phi_j := \psi_{i_j}$ and $\xi_j := \xi^{p_j}$ for every $j \in J$. Now we have an oriented and locally finite atlas $\{(U_j, \psi_j)\}_{j \in J}$ on M . We also know that, for every $j \in J$, a plane spanned by transversal field ξ_j , together with the tangent plane, supports M locally in every point of U_j .

In the next step, we apply a partition of unity $\{\lambda_j\}_{j \in J}$ associated with the cover $\{U_j\}_{j \in J}$ and prove that a transversal vector field $\xi := \sum_{j \in J} \lambda_j \xi_j$ satisfies the last assertion of the proposition. Fix $k \in J$. For every $j \in J$ such that $U_k \cap U_j \neq \emptyset$, we have $[\partial_{k,1}, \partial_{k,2}, D_X X, \xi_j] > 0$ in $U_k \cap U_j$ by a remark in the first part of the proof, where $\partial_{k,l} = d\phi_k^{-1}(e_l)$ ($l = 1, 2$) and X is a non-zero tangent vector field. Therefore, $[\partial_{k,1}, \partial_{k,2}, D_X X, \sum_{j \in J} \lambda_j \xi_j] > 0$ in U_k . It follows that g_ξ is positive definite on M . $\square$

Proposition 3 leads to the following definition, which is a slight refinement of a notion of a metric field from [2]. It depends on an orientation of M up to sign.

Definition 2. A transversal vector field ξ such that the bilinear symmetric form g_ξ defined by Equation (17) is positive definite is called a metric field.

We remark that Equation (17) always yields a definite bilinear form. In the rest of the paper, we will implicitly assume that the tangent frame $u = \{X_1, X_2\}$ is chosen so that g_ξ is positive definite.

4. Equiaffine Transversal Bundles

We recall an important technical theorem from [2].

Theorem 1 ([2], Theorem 3.7). Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ a metric field. Let $u = \{X_1, X_2\}$ be a local orthonormal tangent frame of g_ξ and let σ be an arbitrary transversal plane bundle. Then there exists a unique local frame $\{\xi_1, \xi_2\}$ of σ , such that, according to Equation (2), $[X_1, X_2, \xi_1, \xi_2] = 1$, $h^1(X_1, X_1) = 0$, $\xi + \xi_1$ is tangent to M , $h^2(X_1, X_1) = 1$, $h^2(X_1, X_2) = 0$ and $h^2(X_2, X_2) = 1$.

We remark that the condition that $\xi + \xi_1$ is tangent to M implies that $g_\xi = g_{-\xi_1}$. We also note that the proof of the above theorem implies that the tangent frame u does not need to be positively oriented.

For a given transversal bundle σ and a metric field ξ , a metric $g^\perp$ on σ is given locally in the following way. We take a local g_ξ -orthonormal tangent frame $u = \{X_1, X_2\}$ and the corresponding local transversal frame $\{\xi_1, \xi_2\}$ obtained by Theorem 1. We assume that h^1 does not vanish (we will recall below that it is equivalent to the inexistence of inflection points). Then we define a metric $g_u^\perp$ using the local basis $\{\xi_1, \xi_2\}$:

$$\begin{aligned} g_u^\perp(\xi_1, \xi_1) &= 1, \\ g_u^\perp(\xi_1, \xi_2) &= -\frac{1}{2}h_1(X_2, X_2), \\ g_u^\perp(\xi_2, \xi_2) &= 4h_1(X_1, X_2)^2 + \frac{5}{4}h_1(X_2, X_2)^2. \end{aligned}$$

This metric does not depend on u (see [2]), so we will denote it by $g^\perp$.

For a given metric field ξ , a specific transversal bundle σ is proposed in [2]. It is suitable for locally strictly convex surfaces, which will be explained later. It is equiaffine in the sense that $\nabla\omega_g = 0$, where ∇ is the affine connection induced by σ , and ω_g is the volume element associated with the metric $g = g_\xi$. It is shown in [2] that σ is equiaffine if and only if

$$\begin{aligned} (\nabla g)(X_1, X_1, X_1) + (\nabla g)(X_1, X_2, X_2) &= 0, \\ (\nabla g)(X_2, X_1, X_1) + (\nabla g)(X_2, X_2, X_2) &= 0 \end{aligned} \quad (18)$$

where $\{X_1, X_2\}$ is an orthonormal tangent frame with respect to g . An important assumption that allows the construction of σ in a neighborhood of $p \in M$ is that p is not an inflection point. An equivalent condition is given in [2]: p is an inflection if and only if $h^1(X_1, X_2) = h^1(X_2, X_2) = 0$ for an orthonormal tangent frame $u = \{X_1, X_2\}$ and the associated transversal frame $\{\xi_1, \xi_2\}$ with respect to which h^1 and h^2 are determined. We note that if a surface is contained in a three-dimensional affine subspace of $\mathbb{R}^4$, then it consists of only inflection points. The uniqueness of the construction is possible under the following symmetry conditions.

Definition 3. Let $u = \{X_1, X_2\}$ be a g -orthonormal tangent frame. An equiaffine plane bundle σ is symmetric, if

$$\begin{aligned} (\nabla g)(X_2, X_1, X_1) - (\nabla g)(X_1, X_2, X_1) &= 0, \\ (\nabla g)(X_1, X_2, X_2) - (\nabla g)(X_2, X_1, X_2) &= 0 \end{aligned} \quad (19)$$

and antisymmetric, if

$$\begin{aligned}(\nabla g)(X_2, X_1, X_1) + (\nabla g)(X_1, X_2, X_1) &= 0, \\ (\nabla g)(X_1, X_2, X_2) + (\nabla g)(X_2, X_1, X_2) &= 0.\end{aligned}\quad (20)$$

The following statement comes from [2].

Theorem 2 ([2], Theorems 5.4 and 5.5). *Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface and ξ be a metric field. If $p \in M$ is not an inflection, then there exists a unique antisymmetric equiaffine plane bundle and a unique symmetric equiaffine plane bundle defined on a neighborhood of p .*

5. Locally Strictly Convex Surfaces Contained in Hyperquadrics

In this section, we study locally strictly convex surfaces embedded in hyperquadrics in $\mathbb{R}^4$. We use two specific transversal bundles, which are constructed in [2]. In fact, the bundles coincide in the discussed case. The authors of [2] claim that their bundle is especially well designed for surfaces embedded in hyperquadrics because a canonically determined Blaschke normal of hyperquadric is contained in this bundle, which is not true for Nomizu–Vrancken bundle (see [NV], also [2], Remark 7.4). That statement is a part of Theorem 7.3 in [2]. We prove that the assertion of the theorem is true; however, the original proof contains an essential gap. We restate that part of their conclusions, and then we complete a proof of the statement of the theorem. We also assume that there are no inflection points.

Let M be a surface embedded in a locally strictly convex hypersurface $N \subset \mathbb{R}^4$. We recall that such a surface is locally strictly convex itself. Let $\mathcal{Y}$ be a transversal field on N . It induces an affine connection $\tilde{\nabla}$ on N and the second fundamental form h by the following formula of Gauss:

$$D_X Z = \tilde{\nabla}_X Z + h(X, Z)\mathcal{Y} \quad (21)$$

where X, Z are vector fields tangent to N . In the case of locally strictly convex hypersurfaces, h is non-degenerate, and it can be regarded as a metric on N . We recall a notion of a Blaschke normal (see (NS) for details). Blaschke normal is a transversal field $\mathcal{Y}$, which is determined up to sign by two properties:

- (1) the connection $\tilde{\nabla}$ in Equation (21) is equiaffine, which is equivalent to the fact that $D_X \mathcal{Y}$ is tangent to N for every tangent vector field X ;
- (2) the volume element ω_h associated with metric h is equal to a volume element $\theta(X_1, X_2, X_3) := [X_1, X_2, X_3, \mathcal{Y}]$ induced by $\mathcal{Y}$ on N .

We also recall a fact (see [3]) that N is a hyperquadric if and only if the cubic form $C = \tilde{\nabla}h$ vanishes identically.

In the following proposition, we correct a part of the proof of Theorem 7.3 from [2]. It will be useful to give a complete proof of that result. We recall that a hypersurface is non-degenerate if its second fundamental form is non-degenerate.

Proposition 4. *Let ξ be a metric field on a locally strictly convex connected surface M embedded in a non-degenerate hyperquadric $N \subset \mathbb{R}^4$. We assume that ξ is tangent to N , and there are no inflection points on M . Let X_1, X_2 be a g_ξ -orthonormal frame on M and $f > 0$ a function such that $[X_1, X_2, -\xi, f\mathcal{Y}] = 1$ on M , where $\mathcal{Y}$ is a Blaschke normal on N . Then $\mathcal{Y}$ restricted to M belongs to the equiaffine antisymmetric bundle if and only if f is constant.*

Proof. Since ξ is tangent to N and $\mathcal{Y}$ is transversal to it, then a bundle σ on M is generated by ξ , and $\mathcal{Y}$ is a plane bundle. It is obviously transversal to M . By Theorem 1, there exists a transversal frame $\{\xi_1, \xi_2\}$ in σ such that $\xi_1 + \xi$ is tangent to M , $[X_1, X_2, \xi_1, \xi_2] = 1$, $h^1(X_1, X_1) = 0$ and $h^2(X_i, X_j) = \delta_{ij}$. Our assumption and the proof of the theorem imply that the transversal frame is defined on M . Since $[X_1, X_2, \xi_1, f\mathcal{Y}] = [X_1, X_2, \xi_1, \xi_2] = 1$,

there exists a function λ_1 such that $\xi_2 = \lambda_1 \xi_1 + f\mathcal{Y}$. Denoting by ∇ , as usual, the connection induced by σ on M , we define functions a_1 to a_8 such that

$$\begin{aligned}\nabla_{X_1} X_1 &= a_1 X_1 + a_2 X_2, & \nabla_{X_1} X_2 &= a_3 X_1 + a_4 X_2, \\ \nabla_{X_2} X_1 &= a_5 X_1 + a_6 X_2, & \nabla_{X_2} X_2 &= a_7 X_1 + a_8 X_2.\end{aligned}\quad (22)$$

Next, we make some preparatory computations, using the above facts and notations.

$$\begin{aligned}D_{X_1} X_1 &= a_1 X_1 + a_2 X_2 + \xi_2 \\ &= a_1 X_1 + a_2 X_2 + \lambda_1 \xi_1 + f\mathcal{Y}, \\ D_{X_1} X_2 &= a_3 X_1 + a_4 X_2 + h^1(X_1, X_2) \xi_1, \\ D_{X_2} X_1 &= a_5 X_1 + a_6 X_2 + h^1(X_1, X_2) \xi_1, \\ D_{X_2} X_2 &= a_7 X_1 + a_8 X_2 + h^1(X_2, X_2) \xi_1 + \lambda_1 \xi_1 + f\mathcal{Y}, \\ D_{X_1} \xi_1 &= -S_1 X_1 + \tau_1^1(X_1) \xi_1 + \tau_1^2(X_1) \underbrace{(\lambda_1 \xi_1 + f\mathcal{Y})}_{\xi_2}, \\ D_{X_2} \xi_1 &= -S_1 X_2 + \tau_1^1(X_2) \xi_1 + \tau_1^2(X_2) (\lambda_1 \xi_1 + f\mathcal{Y}).\end{aligned}$$

Taking Equation (21) into consideration and remembering that ξ_1 is tangent to N , we deduce:

$$\begin{aligned}h(X_1, X_1) &= f, & h(X_2, X_2) &= f, & h(X_1, X_2) &= 0, \\ h(X_1, \xi_1) &= \tau_1^2(X_1) f, & h(X_2, \xi_1) &= \tau_1^2(X_2) f.\end{aligned}\quad (23)$$

In the next step, we apply the assumption that the cubic form on N vanishes. We also use Equations (23).

$$\begin{aligned}0 &= C(X_1, X_1, X_1) = X_1(f) - 2h(a_1 X_1 + a_2 X_2 + \lambda_1 \xi_1) \\ &= X_1(f) - 2a_1 f - 2\lambda_1 \tau_1^2(X_1) f,\end{aligned}\quad (24)$$

$$\begin{aligned}0 &= C(X_1, X_1, X_2) = -h(\tilde{\nabla}_{X_1} X_1, X_2) - h(X_1, \tilde{\nabla}_{X_1} X_2) \\ &= -h(a_1 X_1 + a_2 X_2 + \lambda_1 \xi_1, X_2) - h(a_3 X_1 + a_4 X_2 + h^1(X_1, X_2) \xi_1) \\ &= -a_2 f - \lambda_1 \tau_1^2(X_2) f - a_3 f - h^1(X_1, X_2) \tau_1^2(X_1) f,\end{aligned}\quad (25)$$

$$\begin{aligned}0 &= C(X_1, X_2, X_2) = X_1(f) - 2h(\tilde{\nabla}_{X_1} X_2, X_2) \\ &= X_1(f) - 2a_4 f - 2h^1(X_1, X_2) \tau_1^2(X_2) f,\end{aligned}\quad (26)$$

$$\begin{aligned}0 &= C(X_2, X_1, X_1) = X_2(f) - 2h(a_5 X_1 + a_6 X_2 + h^1(X_1, X_2) \xi_1, X_1) \\ &= X_2(f) - 2a_5 f - h^1(X_1, X_2) \tau_1^2(X_1) f,\end{aligned}\quad (27)$$

$$\begin{aligned}0 &= C(X_2, X_1, X_2) = -h(\tilde{\nabla}_{X_2} X_1, X_2) - h(X_1, \tilde{\nabla}_{X_2} X_2) \\ &= -h(a_5 X_1 + a_6 X_2 + h^1(X_1, X_2) \xi_1, X_2) - h(a_7 X_1 + a_8 X_2 + h^1(X_2, X_2) \xi_1 + \lambda_1 \xi_1) \\ &= -a_6 f - h^1(X_1, X_2) \tau_1^2(X_2) f - a_7 f - (h^1(X_2, X_2) + \lambda_1) \tau_1^2(X_1) f,\end{aligned}\quad (28)$$

$$\begin{aligned}0 &= C(X_2, X_2, X_2) = X_2(f) - 2h(a_7 X_1 + a_8 X_2 + h^1(X_2, X_2) \xi_1 + \lambda_1 \xi_1, X_2) \\ &= X_2(f) - 2a_8 f - 2(h^1(X_2, X_2) + \lambda_1) \tau_1^2(X_2) f.\end{aligned}\quad (29)$$

Applying Remark 5.6 from [2], we observe that the antisymmetric transversal bundle is generated by vector fields $\tilde{\xi}_1, \tilde{\xi}_2$ of the form

$$\tilde{\xi}_1 = \xi_1 - aX_1 - bX_2, \quad (30)$$

$$\tilde{\xi}_2 = \xi_2 - cX_1 - dX_2 \quad (31)$$

where

$$a = \frac{-2(a_2 + a_3 + a_5 - a_8)h^1(X_1, X_2) - (a_4 + a_6 + a_7 - a_1)h^1(X_2, X_2)}{4(h^1(X_1, X_2))^2 + (h^1(X_2, X_2))^2} \quad (32)$$

$$b = \frac{-2(a_4 + a_6 + a_7 - a_1)h^1(X_1, X_2) - (a_2 + a_3 + a_5 - a_8)h^1(X_2, X_2)}{4(h^1(X_1, X_2))^2 + (h^1(X_2, X_2))^2}, \quad (33)$$

$$c = -(a_1 + a_4 + bh^1(X_1, X_2)), \quad (34)$$

$$d = -(a_5 + a_8 + ah^1(X_1, X_2) + bh^1(X_2, X_2)). \quad (35)$$

By Equations (24), (26) and (28), we have

$$a_1 - a_4 - a_6 - a_7 = 2h^1(X_1, X_2)\tau_1^2(X_2) + h^1(X_2, X_2)\tau_1^2(X_1)$$

and from Equations (25), (27) and (29) we get

$$a_8 - a_5 - a_2 - a_3 = 2h^1(X_1, X_2)\tau_1^2(X_1) - h^1(X_2, X_2)\tau_1^2(X_2).$$

Hence, after straightforward computations using Equations (32) and (33), we obtain $a = \tau_1^2(X_1)$ and $b = \tau_1^2(X_2)$. Applying additionally Equations (24), (26) and (34), once again, we get

$$c = \lambda_1\tau_1^2(X_1) - X_1(\log f)$$

and from Equations (27), (29) and (35), we have

$$d = \lambda_1\tau_1^2(X_2) - X_2(\log f).$$

Finally, after substituting the above formulas for a, b, c and d into Equations (30) and (31) we obtain:

$$\begin{aligned} \bar{\xi}_1 &= \xi_1 - \tau_1^2(X_1)X_1 - \tau_1^2(X_2)X_2, \\ \bar{\xi}_2 &= \xi_2 - \lambda_1\tau_1^2(X_1)X_1 + X_1(\log f)X_1 - \lambda_1\tau_1^2(X_2)X_2 + X_2(\log f)X_2 \\ &= \lambda_1\bar{\xi}_1 + f\mathcal{Y} + \lambda_1(\bar{\xi}_1 - \xi_1) + X_1(\log f)X_1 + X_2(\log f)X_2, \end{aligned}$$

hence

$$f\mathcal{Y} = \bar{\xi}_2 - \lambda_1\bar{\xi}_1 - X_1(\log f)X_1 - X_2(\log f)X_2$$

so $\mathcal{Y}$ belongs to the antisymmetric transversal bundle if and only if $X_1(\log f) = 0$ and $X_2(\log f) = 0$, which completes the proof because M is connected. $\square$

Remark 2. Proposition 4 is also true for the symmetric equiaffine bundle. We omit the calculations because they are analogous to those in the antisymmetric case.

Let $\mathfrak{G}$ be the Blaschke metric on a locally strictly convex hyperquadric N containing M (see [4]). It is shown in [2] that for every transversal bundle σ on M , there is a transversal (that means σ -valued) field ξ on M such that $\mathfrak{G}$ restricted to M coincides with g_ξ . We investigate some details of the construction.

Lemma 2. For any point $p \in M$, there is a metric field ξ defined in a connected neighborhood of p such that it is tangent to N and g_ξ coincides with the Blaschke metric $\mathfrak{G}$ on N restricted to M . Moreover, for any g_ξ -orthonormal transversal frame $\{\tilde{X}_1, \tilde{X}_2\}$ and for the Blaschke normal $\mathcal{Y}$, $[\tilde{X}_1, \tilde{X}_2, -\xi, \mathcal{Y}] = 1$.

Proof. Let $p \in M$ and U be a connected, open neighborhood of p . The proof will be carried on in U . Let $u' = \{Y_1, Y_2, Y_3\}$ be a local tangent frame on N and Z, W be arbitrary tangent vector fields on N such that they are also defined in U . Then

$$H_{u'}(Z, W) := [Y_1, Y_2, Y_3, D_Z W]$$

is a bilinear symmetric form, which is non-degenerate because N is locally strictly convex. A bilinear symmetric form

$$\mathfrak{G}(Z, W) := \frac{H_{u'}(Z, W)}{(\det_{u'} H_{u'})^{\frac{1}{5}}},$$

does not depend on u' , and it is called the Blaschke metric.

Let $u = \{X_1, X_2\}$ be a local tangent frame on M . Let X_3 be such a local tangent vector field on N that X_1, X_2 and X_3 form a basis of the tangent space to N at points of M . Denote this frame by u' . It is obvious that X_3 is transversal to M . Take a function

$$\lambda := \frac{(\det_{u'} H_{u'})^{\frac{1}{5}}}{(\det_u G_u)^{\frac{1}{4}}}.$$

It is well-defined on M because we identify $H_{u'}(X_3, X_i)$ with $H_{u'}(X_i, X_3)$ for $i = 1, 2$. We could also extend X_1 and X_2 on N locally with the same effect. Then we obtain a metric field $\xi = -\lambda^{-2} X_3$ such that

$$g_\xi = \mathfrak{G}|_M \quad (36)$$

locally (see [2]). In the next step, we take a g_ξ local orthonormal frame $\tilde{u} = \{\tilde{X}_1, \tilde{X}_2\}$ on M . Then $G_{\tilde{u}}(\tilde{X}_1, \tilde{X}_2) = 0$ and $G_{\tilde{u}}(\tilde{X}_i, \tilde{X}_i) = (\det_{\tilde{u}} G_{\tilde{u}})^{\frac{1}{4}}$ for $i = 1, 2$. Consequently, $\det_{\tilde{u}} G_{\tilde{u}} = G_{\tilde{u}}(\tilde{X}_1, \tilde{X}_1)G_{\tilde{u}}(\tilde{X}_2, \tilde{X}_2)$, so $\det_{\tilde{u}} G_{\tilde{u}} = (\det_{\tilde{u}} G_{\tilde{u}})^{\frac{1}{2}}$, and finally $\det_{\tilde{u}} G_{\tilde{u}} = 1$.

Let $\tilde{X}_3 := -\xi = \lambda^{-2} X_3$. Using the notation $\tilde{u}' = \{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3\}$ and extending λ on an open set in N , we can compute $H_{\tilde{u}'}$ locally on M . The result clearly does not depend on a particular extension of λ because the term containing $X_3(\lambda)X_3$ is canceled in $H_{\tilde{u}'}(\tilde{X}_3, \tilde{X}_3)$. We repeat a construction of a metric field satisfying a condition analogous to Equation (36), starting with $\tilde{u}$ and $\tilde{u}'$. Thus, we define a function

$$\tilde{\lambda} := \frac{(\det_{\tilde{u}'} H_{\tilde{u}'})^{\frac{1}{5}}}{(\det_{\tilde{u}} G_{\tilde{u}})^{\frac{1}{4}}} = (\det_{\tilde{u}'} H_{\tilde{u}'})^{\frac{1}{5}}$$

and obtain a metric field $\frac{\xi}{\tilde{\lambda}^2}$ satisfying

$$g_{\frac{\xi}{\tilde{\lambda}^2}} = \mathfrak{G}|_M = g_\xi.$$

Since $g_{\frac{\xi}{\tilde{\lambda}^2}} = \frac{1}{|\tilde{\lambda}|} g_\xi$, we have $|\tilde{\lambda}| = 1$ and

$$|\det_{\tilde{u}'} H_{\tilde{u}'}| = 1.$$

Let σ be a local transversal bundle on M generated by vector fields ξ and $\mathcal{Y}$, where $\mathcal{Y}$ is a Blaschke normal on N . Since $\{\tilde{X}_1, \tilde{X}_2\}$ is a g_ξ -orthonormal tangent frame, we apply Theorem 1. According to it, there exists a transversal frame $\{\zeta_1, \zeta_2\}$ in σ such that $\xi + \zeta_1$ is tangent to M , $[\tilde{X}_1, \tilde{X}_2, \zeta_1, \zeta_2] = 1$, $h^1(\tilde{X}_1, \tilde{X}_1) = 0$ and $h^2(\tilde{X}_i, \tilde{X}_j) = \delta_{ij}$, $i, j = 1, 2$. It implies that ζ_2 is transversal to N . From the procedure of constructing a Blaschke normal (NS), we know that we can start it with ζ_2 as a transversal field to N and then $\mathcal{Y} = |\det_{\tilde{u}'} H_{\tilde{u}'}| \zeta_2 + Z = \zeta_2 + Z$, where Z is tangent to N . The construction is done at points of M . However, ζ_2 is a linear combination of ζ_1 and $\mathcal{Y}$, so there is a function λ_1 such that $\mathcal{Y} = \zeta_2 + \lambda_1 \xi$. Hence, $[\tilde{X}_1, \tilde{X}_2, -\xi, \mathcal{Y}] = [\tilde{X}_1, \tilde{X}_2, -\xi, \zeta_2] = [\tilde{X}_1, \tilde{X}_2, \zeta_1, \zeta_2] = 1$, which completes the proof. $\square$

The above lemma together with Proposition 4 prove the following theorem from [2]:

Theorem 3. Let $M \subset \mathbb{R}^4$ be a locally strictly convex surface immersed in a hyperquadric N . Then the affine normal field to N belongs to both the antisymmetric and symmetric equiaffine plane bundles of M , with respect to the Blaschke metric $\mathfrak{G}$ restricted to M .

Remark 3. In [2], the proof of the above theorem apparently presupposes that the function f appearing in Proposition 4 is equal to 1, but no verification of this fact is presented.

In the remainder of this section, we present three examples of surfaces contained in strictly convex hyperquadrics. We endow each example in a parametrization that is suitable for presenting the geometric properties of our interest. Only the first one will contain detailed computations.

Example 1 (Clifford torus). Let $y(u, v, w) = \sqrt{2} \sqrt[8]{2} (\cos u \sin w, \sin u \sin w, \cos v \cos w, \sin v \cos w)$ be a parametrization of a three-dimensional sphere $\mathbb{S}^3$ of a radius $2\sqrt[4]{2}$. Consider a Clifford torus $M_1 = \mathbb{S}^1 \times \mathbb{S}^1$ contained in $\mathbb{S}^3$ parametrized by

$$x(u, v) = y(u, v, \pi/4) = \sqrt[8]{2} (\cos u, \sin u, \cos v, \sin v)$$

where both circles also have the radius $2\sqrt[4]{2}$. First, we determine a frame $u' = \{X_1, X_2, X_3\}$ tangent to $\mathbb{S}^3$ but defined in M_1 :

$$\begin{aligned} X_1 &= x_u = \sqrt[8]{2} (-\sin u, \cos u, 0, 0), \\ X_2 &= x_v = \sqrt[8]{2} (0, 0, \sin v, \cos v), \\ X_3 &= y_w(u, v, \pi/4) = \sqrt[8]{2} (\cos u, \sin u, -\cos v, -\sin v). \end{aligned}$$

Next, we compute $H_{u'}(X_i, X_j)$ for all i, j , taking into account that $D_{X_1} X_1 = x_{uu}$, $D_{X_2} X_2 = x_{vv}$, $D_{X_1} X_2 = D_{X_2} X_1 = x_{uv} = 0$ and $D_{X_3} X_3 = y_{ww}(u, v, \pi/4) = \sqrt[8]{2} (-\cos u, -\sin u, -\cos v, -\sin v)$. We get

$$\det_{u'} H_{u'} = \det \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 2\sqrt{2} \end{bmatrix} = 4\sqrt{2}.$$

We immediately get $\det_u G_u = 2$, where $u = \{X_1, X_2\}$ and G_u is computed with respect to a metric field $-X_3$. According to the proof of Lemma 2, for $\lambda = \sqrt[4]{2}$, we obtain another metric field $\xi = -\lambda^{-2} X_3$ such that g_ξ coincides with the Blaschke metric on $\mathbb{S}^3$ induced on M_1 . Since $g_\xi = \lambda^{-1} g_{-X_3}$, u is a g_ξ -orthonormal frame that can be easily seen. For $\xi_1 = -\xi$ and $\xi_2 = \sqrt[8]{2} (-\cos u, -\sin u, 0, 0)$, we also get $[X_1, X_2, \xi_1, \xi_2] = 1$. Then

$$D_{X_1} X_1 = \xi_2, \quad D_{X_1} X_2 = 0, \quad D_{X_2} X_2 = -\sqrt{2} \xi_1 + \xi_2 \quad (37)$$

which implies that $\{\xi_1, \xi_2\}$ is the transversal frame associated with u according to Theorem 1 in a transversal bundle σ generated by ξ_1 and ξ_2 . In particular, $h^1(X_1, X_2) = 0$ and $h^1(X_2, X_2) = -\sqrt{2}$. From Equation (37), we also deduce that $\nabla_{X_i} X_j = 0$ for all $i, j = 1, 2$. Hence, $\nabla g_\xi = 0$, and by Equations (18)–(20), we conclude that σ is the equiaffine transversal bundle that is both symmetric and antisymmetric. We can also see that all $D_{X_i} \xi_j$ are tangent to M_1 . As a consequence, $K^1 = K^2 = 0$ identically. After straightforward computations, we also obtain $\nabla^\perp g_\xi^\perp = 0$.

Example 2 (The product of an ellipse and a hyperbola). Let $y(u, v, w) = (\cos u \sinh w, \sin u \sinh w, \sinh(\frac{v}{\sqrt{2}}) \cosh w, \cosh(\frac{v}{\sqrt{2}}) \cosh w)$ be a parametrization of a one sheet N_2 of a hyperboloid $y_1^2 + y_2^2 + y_3^2 - y_4^2 = -1$, which is strictly convex. We consider a surface M_2 parametrized by $x(u, v) = y(u, v, w_0)$, where $w_0 = \log(1 + \sqrt{2})$, so $\sinh w_0 = 1$ and $\cosh w_0 = \sqrt{2}$. Therefore,

$$x(u, v) = (\cos u, \sin u, \sqrt{2} \sinh(\frac{v}{\sqrt{2}}), \sqrt{2} \cosh(\frac{v}{\sqrt{2}})).$$

We take vector fields $X_1 = x_u$, $X_2 = x_v$ and $X_3 = y_w(u, v, w_0)$ on M_2 , following the pattern of Example 1. We also need a covariant derivative $D_{X_3}X_3$ restricted to M_2 , so $D_{X_3}X_3 = y_{ww}(u, v, w_0)$. After straightforward calculations, we deduce that $\xi = -X_3$ is a suitable metric field such that g_ξ coincides with the Blaschke metric restricted to M_2 . We also can check that $\{X_1, X_2\}$ is a g_ξ -orthonormal frame. For $\xi_1 = -\xi$ and $\xi_2 = D_{X_1}X_1$, $\{\xi_1, \xi_2\}$ is the associated transversal frame in a bundle σ , which it generates. A transversal bundle σ is the equiaffine symmetric and antisymmetric one. The cubic forms K^1 and K^2 with respect to σ vanish and $\nabla^\perp g_\xi^\perp = 0$.

Example 3 (The product of an ellipse and a parabola). Let $y(u, v, w) = \frac{1}{\sqrt[4]{2}}(w \cos u, w \sin u, v, v^2 + w^2)$ be a parametrization of a paraboloid N_3 : $y_4 = \sqrt[4]{2}(y_1^2 + y_2^2 + y_3^2)$. We consider a surface $x(u, v) = y(u, v, 1) = \frac{1}{\sqrt[4]{2}}(\cos u, \sin u, v, v^2 + 1)$, which is contained in N_3 . We take $u = \{X_1, X_2\}$ as a tangent frame on the surface for $X_1 = \frac{1}{\sqrt[4]{2}}(-\sin u, \cos u, 0, 0)$ and $X_2 = \frac{1}{\sqrt[4]{2}}(0, 0, 1, 2v)$. As a metric field, we choose $\xi = -y_w(u, v, 1) = -\frac{1}{\sqrt[4]{2}}(\cos u, \sin u, 0, 2)$. It is easy to verify, like in previous examples, that g_ξ coincides with the Blaschke metric on N_3 restricted to the surface. It also turns out that u is a g_ξ -orthonormal frame. Let σ be a transversal bundle generated by vector fields $\xi_1 = -\xi$ and $\xi_2 = x_{uu} = \frac{1}{\sqrt[4]{2}}(-\cos u, -\sin u, 0, 0)$. Then, after straightforward computations, we can see that $\{\xi_1, \xi_2\}$ is the transversal frame associated with u in σ . Moreover, σ is the equiaffine symmetric and antisymmetric transversal bundle such that the induced normal connection has the properties $\nabla^\perp g_\xi^\perp = 0$ and $K^1 = K^2 = 0$.

6. Classification of Parallel Surfaces Contained in Locally Strictly Convex Hyperquadrics

The main aim of this paper is to characterize surfaces without inflection points contained in locally strictly convex hyperquadrics. We recall that a surface contained in a locally strictly convex hypersurface in $\mathbb{R}^4$ is always locally strictly convex itself. We choose a metric field satisfying Equation (36). We assume additionally that the cubic form K of a surface vanishes and $\nabla^\perp g_\xi^\perp = 0$. Both connections, ∇ and $\nabla^\perp$, are induced by the equiaffine transversal bundle that is constructed in [2]. We can use either a symmetric or antisymmetric bundle because they are identical in the case of this study. We note that the assumptions given above are invariant under a change of a given tangent frame and a transversal frame.

Let ξ be a metric field on a surface M embedded in a locally strictly convex hyperquadric N of $\mathbb{R}^4$ such that $g = g_\xi$ coincides with the Blaschke metric $\mathfrak{G}$ on N restricted to M . Let σ be the antisymmetric equiaffine plane bundle. Our considerations in this section will be local, so M will denote an open and connected subset of the surface. Let $\{X_1, X_2\}$ be an orthonormal tangent frame on M and $\{\xi_1, \xi_2\}$ be the associated transversal frame satisfying Theorem 1. We adapt notations from Section 3 and from Equation (22). Let $\mathcal{V}$ be a Blaschke normal on N and λ_1 be a function on M such that $\xi_2 = \lambda_1 \xi_1 + \mathcal{V}$. We will also use the following notations:

$$b := h^1(X_1, X_2), \quad c := h^1(X_2, X_2),$$

$$\tau_{jk}^i := \tau_j^i(X_k),$$

We note that conditions Equations (24) up to (29) are satisfied with $f = 1$. Therefore, we immediately obtain the following equations:

$$2a_1 + 2\lambda_1\tau_{11}^2 = 0, \quad (38)$$

$$a_2 + a_3 + \lambda_1\tau_{12}^2 + b\tau_{11}^2 = 0, \quad (39)$$

$$2a_4 + 2b\tau_{12}^2 = 0, \quad (40)$$

$$2a_5 + 2b\tau_{11}^2 = 0, \quad (41)$$

$$a_6 + a_7 + b\tau_{12}^2 + (c + \lambda_1)\tau_{11}^2 = 0, \quad (42)$$

$$2a_8 + 2(c + \lambda_1)\tau_{12}^2 = 0. \quad (43)$$

We can derive more relations from the assumptions about the transversal bundle.

Lemma 3. *The transversal bundle σ is equiaffine if and only if*

$$a_1 + a_4 = 0, \quad (44)$$

$$a_5 + a_8 = 0. \quad (45)$$

The transversal bundle σ is antisymmetric if and only if

$$a_2 + a_3 + 2a_5 = 0, \quad (46)$$

$$a_6 + a_7 + 2a_4 = 0. \quad (47)$$

Proof. From Equation (18), we obtain:

$$-2g(\nabla_{X_1}X_1, X_1) - 2g(\nabla_{X_1}X_2, X_2) = 0,$$

$$-2g(\nabla_{X_2}X_1, X_1) - 2g(\nabla_{X_2}X_2, X_2) = 0,$$

which is equivalent to

$$0 = g(a_1X_1 + a_2X_2, X_1) + g(a_3X_1 + a_4X_2, X_2) = a_1 + a_4,$$

$$0 = g(a_5X_1 + a_6X_2, X_1) + g(a_7X_1 + a_8X_2, X_2) = a_5 + a_8.$$

Conditions from Equation (20), equivalent to antisymmetry of σ , yield

$$-2g(\nabla_{X_2}X_1, X_1) - g(\nabla_{X_1}X_2, X_1) - g(X_2, \nabla_{X_1}X_1) = 0,$$

$$-2g(\nabla_{X_1}X_2, X_2) - g(\nabla_{X_2}X_1, X_2) - g(X_1, \nabla_{X_2}X_2) = 0,$$

which is equivalent to

$$0 = 2g(a_5X_1 + a_6X_2, X_1) + g(a_3X_1 + a_4X_2, X_1) + g(X_2, a_1X_1 + a_2X_2) = 2a_5 + a_3 + a_2,$$

$$0 = 2g(a_3X_1 + a_4X_2, X_2) + g(a_5X_1 + a_6X_2, X_2) + g(X_1, a_7X_1 + a_8X_2) = 2a_4 + a_6 + a_7.$$

□

Lemma 4. *If $x \in M$ is not a point of inflection then*

$$\tau_{11}^2 = 0 \text{ and } \tau_{12}^2 = 0 \quad (48)$$

in a neighborhood of x .

Proof. At the beginning, we do computations at point x . We recall that at an inflection point $b = c = 0$ (see [2]). Combining Equations (38), (40) and (44), we get $\lambda_1\tau_{11}^2 + b\tau_{12}^2 = 0$, and from Equations (40), (42) and (47), we get $3b\tau_{12}^2 + (c + \lambda_1)\tau_{11}^2 = 0$. After subtracting these equations, we obtain

$$2b\tau_{11}^2 + c\tau_{12}^2 = 0. \quad (49)$$

Analogously, from Equations (41), (43) and (45), we get $b\tau_{11}^2 + (c + \lambda_1)\tau_{12}^2 = 0$, and from Equations (39), (41) and (46), we get $\lambda_1\tau_{12}^2 + 3b\tau_{11}^2 = 0$. We subtract last two equations and obtain

$$-c\tau_{11}^2 + 2b\tau_{12}^2 = 0. \quad (50)$$

The determinant of the system of Equations (49) and (50), $4b^2 + c^2$, does not vanish in a neighborhood of x because x is not an inflection point. Thus, $\tau_{11}^2 = 0$ and $\tau_{12}^2 = 0$ in a neighborhood of x . $\square$

We now assume that the cubic form K , defined by Equation (11), vanishes on M so $K^1 = K^2 = 0$ on M . After straightforward computations, applying all the triples (X_i, X_j, X_k) to K^1, K^2 and also using Equations (12), (44), (45) and (48), we obtain the following lemma.

Lemma 5. *If the cubic form K is equal to 0 identically, then the following equations hold:*

$$-2a_2b + \tau_{21}^1 = 0, \quad (51)$$

$$X_1(b) - a_2c - a_3 + \tau_{11}^1b = 0, \quad (52)$$

$$X_1(c) - 2a_3b - 2a_4c + \tau_{11}^1c + \tau_{21}^1 = 0, \quad (53)$$

$$-2a_6b + \tau_{22}^1 = 0, \quad (54)$$

$$X_2(b) - a_6c - a_7 + \tau_{12}^1b = 0, \quad (55)$$

$$X_2(c) - 2a_7b - 2a_8c + \tau_{12}^1c + \tau_{22}^1 = 0, \quad (56)$$

$$-2a_1 + \tau_{21}^2 = 0, \quad (57)$$

$$-a_2 - a_3 = 0, \quad (58)$$

$$-2a_4 + \tau_{21}^2 = 0, \quad (59)$$

$$-2a_5 + \tau_{22}^2 = 0, \quad (60)$$

$$-a_6 - a_7 = 0, \quad (61)$$

$$-2a_8 + \tau_{22}^2 = 0. \quad (62)$$

If we make use of Equation (44) up to Equations (47), (58)–(61), we obtain the following

Corollary 1.

$$a_1 = a_4 = a_5 = a_8 = 0,$$

$$a_3 = -a_2, \quad a_7 = -a_6,$$

$$\tau_{21}^2 = \tau_{22}^2 = 0.$$

We finally apply our last assumption $\nabla^\perp g^\perp = 0$.

Lemma 6. *If $\nabla^\perp g^\perp = 0$ then*

$$\tau_{11}^1 = 0, \quad (63)$$

$$\tau_{12}^1 = 0, \quad (64)$$

$$X_1(c) = -2\tau_{21}^1, \quad (65)$$

$$X_2(c) = -2\tau_{22}^1, \quad (66)$$

$$X_1(4b^2 + \frac{5}{4}c^2) + c\tau_{21}^1 = 0, \quad (67)$$

$$X_2(4b^2 + \frac{5}{4}c^2) + c\tau_{22}^1 = 0, \quad (68)$$

$$X_1(b) = a_2(c - 1), \quad (69)$$

$$X_2(b) = a_6(c - 1). \quad (70)$$

Proof. We have

$$0 = (\nabla^\perp g^\perp)(X_1, \xi_1, \xi_1) = -2g^\perp(\tau_{11}^1 \xi_1 + \tau_{11}^2 \xi_2, \xi_1) = -2\tau_{11}^1$$

which gives Equation (63). Similarly,

$$0 = (\nabla^\perp g^\perp)(X_2, \xi_1, \xi_1) = -2g^\perp(\tau_{12}^1 \xi_1 + \tau_{12}^2 \xi_2, \xi_1) = 2\tau_{12}^1$$

which yields Equation (64). Next, we have

$$\begin{aligned} 0 &= (\nabla^\perp g^\perp)(X_1, \xi_1, \xi_2) = -\frac{1}{2}X_1(c) - g^\perp(\tau_{11}^1 \xi_1 + \tau_{11}^2 \xi_2, \xi_2) - g^\perp(\xi_1, \tau_{21}^1 \xi_1 + \tau_{21}^2 \xi_2) \\ &= -\frac{1}{2}X_1(c) + \frac{1}{2}c\tau_{11}^1 - \tau_{21}^1 \end{aligned}$$

which, together with Equation (63) implies Equation (65). We also have

$$\begin{aligned} 0 &= (\nabla^\perp g^\perp)(X_2, \xi_1, \xi_2) = -\frac{1}{2}X_2(c) - g^\perp(\tau_{12}^1 \xi_1 + \tau_{12}^2 \xi_2, \xi_2) - g^\perp(\xi_1, \tau_{22}^1 \xi_1 + \tau_{22}^2 \xi_2) \\ &= -\frac{1}{2}X_2(c) + \frac{1}{2}c\tau_{12}^1 - \tau_{22}^1 \end{aligned}$$

which, together with Equation (64) implies Equation (66). Additionally,

$$0 = (\nabla^\perp g^\perp)(X_1, \xi_2, \xi_2) = X_1(4b^2 + \frac{5}{4}c^2) - 2g^\perp(\tau_{21}^1 \xi_1 + \tau_{21}^2 \xi_2, \xi_2)$$

yields Equation (67) and

$$0 = (\nabla^\perp g^\perp)(X_2, \xi_2, \xi_2) = X_2(4b^2 + \frac{5}{4}c^2) - 2g^\perp(\tau_{22}^1 \xi_1 + \tau_{22}^2 \xi_2, \xi_2)$$

gives Equation (68). The last two equations are a consequence of Equations (52), (55), (58), (61), (63) and (64). $\square$

Corollary 2. The coefficients τ_{jk}^i of the normal connection vanish for all $i, j, k = 1, 2$. Moreover, the function c is constant.

Proof. We are left with a proof that $\tau_{21}^1 = \tau_{22}^1 = 0$. By Equations (51) and (65), we obtain

$$X_1(c) = -4a_2b. \quad (71)$$

By Equations (54) and (66), similarly, we get

$$X_2(c) = -4a_6b. \quad (72)$$

Substituting Equations (69) and (71) into Equation (67), we get $8b(a_2(c-1)) + \frac{5}{2}(-4a_2b) = -2ca_2b$ whence

$$a_2b = 0 \quad (73)$$

and $\tau_{21}^1 = 0$ by Equation (51). Analogously,

$$a_6b = 0 \quad (74)$$

hence $\tau_{22}^1 = 0$. The last assertion is now obvious. $\square$

Lemma 7. After a suitable change of the orthonormal tangent frame function b is equal to zero on the whole surface and function c is a non-zero constant. Moreover, $a_2 = a_6 = 0$ identically.

Proof. Assume that $b(x_0) \neq 0$ at a point $x_0 \in M$. Then $b \neq 0$ in a neighborhood of x_0 , so Equations (73) and (74) imply that $a_2 = a_6 = 0$ there. From Equations (69) and (70), we deduce that b is constant in this neighborhood of x_0 . It follows that b is equal to $b(x_0) \neq 0$ in an open and closed set in M so is in the whole of M . Then we apply Lemma 3.8 from [2], and we change the tangent frame in the following way (the tildas indicate entries associated with the new bases):

$$\tilde{X}_1 = \cos \theta X_1 + \sin \theta X_2, \quad (75)$$

$$\tilde{X}_2 = -\sin \theta X_1 + \cos \theta X_2 \quad (76)$$

which, according to [2], implies that

$$\tilde{b} := \tilde{h}^1(\tilde{X}_1, \tilde{X}_2) = b \cos 2\theta + \frac{1}{2}c \sin 2\theta \quad (77)$$

$$\tilde{c} := \tilde{h}^1(\tilde{X}_2, \tilde{X}_2) = c \cos 2\theta - 2b \sin 2\theta. \quad (78)$$

If we take $\theta \in \mathbb{R}$ such that $\cot 2\theta = -\frac{c}{2b}$ then Equations (77) and (78) imply $\tilde{b} = 0$ and $\tilde{c}$ being a non-zero constant because there are no inflection points. Now, if $\tilde{c} = 1$, we could replace θ with $\theta + \frac{\pi}{2}$ obtaining $\tilde{b} = 0$ and $\tilde{c} = -1$. Thus, we can always assume that $\tilde{c}$ is never equal to 1. Finally, by Equations (69) and (70), we have $\tilde{a}_2 = \tilde{a}_6 = 0$ on M . We can also see that the lemma is true in the case when $b = 0$ identically. $\square$

From now on, we will apply the assertions of Lemma 7. We introduce functions k_i for $i = 1, \dots, 8$ such that

$$\begin{aligned} S_1 X_1 &= k_1 X_1 + k_2 X_2, & S_1 X_2 &= k_3 X_1 + k_4 X_2, \\ S_2 X_1 &= k_5 X_1 + k_6 X_2, & S_2 X_2 &= k_7 X_1 + k_8 X_2. \end{aligned}$$

Lemma 8. Functions k_i satisfy the following relations:

$$\begin{aligned} k_2 &= k_3 = k_6 = k_7 = k_8 = 0, & k_5 &= -ck_1, \\ X_2(k_1) &= 0, & X_1(k_4) &= 0. \end{aligned}$$

Proof. We consider first the equations of Ricci. From Equation (7), we get $0 = h^1(X_1, S_1 X_2) - h^1(X_2, S_1 X_1) = -ck_2$, so $k_2 = 0$. From Equation (8), we get $0 = h^2(X_1, S_1 X_2) - h^2(X_2, S_1 X_1) = k_3 - k_2$, so $k_3 = 0$. Analogously, by Equations (9) and (10), we have $k_6 = k_7 = 0$.

Applying the equation of Gauss (4) to triples (X_1, X_2, X_1) and (X_1, X_2, X_2) yields $k_8 = 0$ and $k_5 = -ck_1$. We omit the straightforward computations which are simplified because the left hand sides of both equations vanish. Finally we make use of the second Codazzi Equation (6) for S_1 . Since its right hand side is equal to zero and $a_i = 0$, for all i as well, we are left with $X_1(k_3)X_1 + X_1(k_4)X_2 - X_2(k_1)X_1 - X_2(k_2)X_2 = 0$, which immediately yields $X_1(k_4) = 0$ and $X_2(k_1) = 0$. $\square$

Now, we are in a position to prove the main theorem.

Theorem 4. Let M be a surface contained in a locally strictly convex hyperquadric N in $\mathbb{R}^4$. Let ξ be a metric field on M such that the metric g_ξ coincides with the Blaschke metric on N restricted to M . Let σ be the equiaffine antisymmetric transversal bundle with respect to g_ξ . Assume that the following conditions are satisfied:

1. M has no inflection points;
2. $\nabla^\perp g_\xi^\perp = 0$, where $\nabla^\perp$ is the normal connection induced by σ and $g_\xi^\perp$ is a metric in σ associated with ξ ;
3. the cubic form K vanishes on M .

Then, M is locally affinely equivalent to an open part of a Clifford torus, a product of an ellipse and a hyperbola or a product of an ellipse and a parabola.

Proof. Assumptions of the theorem allow us to choose a tangent frame $\{X_1, X_2\}$ satisfying Lemma 7. Let $\{\xi_1, \xi_2\}$ be the associated transversal frame given by Theorem 1. Fix a point $x_0 \in M$. By Corollary 1 and Lemma 7, the Lie bracket $[X_1, X_2]$ vanishes, so there is a local parametrization $x(u, v)$ of M in a neighborhood V of x_0 such that $x_u = X_1$ and $x_v = X_2$. Applying Lemma 8, we can see that k_1 depends only on u and k_4 depends only on v . We can write:

$$\begin{aligned}x_{uu} &= D_{X_1} X_1 = \xi_2, \\x_{uv} &= D_{X_1} X_2 = 0, \\x_{vv} &= D_{X_2} X_2 = c\xi_1 + \xi_2, \\(\xi_1)_u &= D_{X_1} \xi_1 = -k_1(u)x_u, \\(\xi_1)_v &= D_{X_2} \xi_1 = -k_4(v)x_v, \\(\xi_2)_u &= D_{X_1} \xi_2 = ck_1(u)x_u, \\(\xi_2)_v &= D_{X_2} \xi_2 = 0.\end{aligned}$$

It follows that

$$x_{uuu} = ck_1(u)x_u, \quad (79)$$

$$x_{vvv} = -ck_4(v)x_v. \quad (80)$$

Since $x_{uv} = 0$, x is of the form $x(u, v) = y(u) + z(v)$. Differential equations of third order—Equations (79) and (80) are linear so there are functions $y_1(u), y_2(u), z_1(v), z_2(v)$ and vectors $C_i, D_i \in \mathbb{R}^4$ for $i = 1, 2, 3$ such that $y = y_1 C_1 + y_2 C_2 + C_3$ and $z = z_1 D_1 + z_2 D_2 + D_3$. The vectors C_1, C_2, D_1 and D_2 are linearly independent because x_{uu}, x_{vv}, x_{uuu} and x_{vvv} are linearly independent at every point. Therefore, after possibly applying an affine transformation, the parametrization can be written in the form

$$x(u, v) = (y_1(u), y_2(u), z_1(v), z_2(v))$$

for (u, v) in an open rectangle U such that $x(U) \subset V$. The functions y_1, y_2 are linearly independent, and their derivatives are also linearly independent. Otherwise, the surface would be contained in a three-dimensional affine space, and therefore, all its points would be inflection ones. The same is true for z_i . The surface is contained in a hyperquadric, so its coordinates satisfy a quadratic equation, which we write in a simplified form

$$p(u) + q(v) + \sum_{i,j=1}^2 a_{ij} y_i(u) z_j(v) = 0. \quad (81)$$

After differentiating it with respect to u and then v , we obtain:

$$\sum_j \left(\sum_i a_{ij} y'_i(u) \right) z'_j(v) = 0$$

for every u, v , so for $j = 1, 2$ and every u

$$\sum_i a_{ij} y'_i(u) = 0$$

because z'_1 and z'_2 are linearly independent. Finally, $a_{ij} = 0$ for every i, j . It implies that $p(u)$ and $q(v)$ are constants. After applying an affine transformation to $x(u, v)$, we can conclude that the equation that the surface satisfies contains only a sum of first and second powers of y_i and z_i . It is well-known (see [7]) that the only locally strictly convex hyperquadrics in $\mathbb{R}^4$ can be written in the following forms:

$$(a) \quad (x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = 1 \text{ (an ellipsoid),}$$

- (b) $x_4 = (x_1)^2 + (x_2)^2 + (x_3)^2$ (an elliptic paraboloid),
- (c) $(x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2 = -1$ (hyperboloid of two sheets).

Now, we take into account that the coordinates of our surface in a certain order satisfy one of the above equations. Since every such equation contains a sum of three squares of coordinates, two of them have to be functions of the same variable. Thus, without loss of generality, we can assume that $y_1^2 + y_2^2$ is a constant. Then, there are three cases:

- (i) $z_1^2 + z_2^2$ is constant and $x(U)$ is affine equivalent to an open part of the Clifford torus described in Example 1;
- (ii) $z_2 - z_1^2$ is constant and $x(U)$ is affine equivalent to an open part of a product of an ellipse and a parabola given in Example 3;
- (iii) $z_1^2 - z_2^2$ is a constant and $x(U)$ is affine equivalent to an open part of a product of an ellipse and a hyperbola described in Example 2.

The proof of the theorem is now completed. $\square$

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