

Article

Dynamics of Hyperbolically Symmetric Fluids

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Abstract: We study the general properties of dissipative fluid distributions endowed with hyperbolic symmetry. Their physical properties are analyzed in detail. It is shown that the energy density is necessarily negative, and the central region cannot be attained by any fluid element. We describe this inner region by a vacuum cavity around the center. By assuming a causal transport equation some interesting thermodynamical properties of these fluids are found. Several exact analytical solutions, which evolve in the quasi-homologous regime and satisfy the vanishing complexity factor condition, are exhibited.

Keywords: relativistic fluids; non-spherical sources; interior solutions



Citation: Herrera, L.; Di Prisco, A.; Ospino, J. Dynamics of Hyperbolically Symmetric Fluids. *Symmetry* **2021**, *13*, 1568. <https://doi.org/10.3390/sym13091568>

Academic Editors: Mikhail E. Prokhorov and Ilya Kovalenko

Received: 18 May 2021

Accepted: 2 August 2021

Published: 26 August 2021

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1. Introduction

In a recent paper [1], we have presented an approach to describe static, hyperbolically symmetric fluids. The main motivation (but not the only one) behind such an endeavor was the necessity to provide a rigorous description of fluid distributions sourcing the line element

$$ds^2 = -\left(\frac{2M}{R} - 1\right)dt^2 + \frac{dR^2}{\left(\frac{2M}{R} - 1\right)} + R^2d\Omega^2, \quad (1)$$

$$d\Omega^2 = d\theta^2 + \sinh^2\theta d\phi^2,$$

which in its turn is assumed to be the line element at the interior of the horizon, proposed in [2,3] as an alternative global description of the Schwarzschild black hole.

Such a proposal is motivated by the fact that it is impossible to remove the coordinate singularity in the line element, keeping at the same time the static form of the Schwarzschild metric (in the whole space-time) [4]. Thus, the regular extension of the Schwarzschild metric to the whole space-time may be achieved but at the price to admit a non-static space-time inside the horizon [5,6].

Then, from the belief that any dynamic regime should eventually lead to an equilibrium final state, a static solution has to be expected in the whole space-time.

Accordingly, the model proposed in [2] describes the space time as consisting of two four-dimensional manifolds, the outer one described by the usual Schwarzschild metric on the exterior side of the horizon and the inner one described by (1). A change in signature as well as a change in the symmetry at the horizon are required.

The metric (1) is a static solution admitting the four Killing vectors

$$\mathbf{K}_{(0)} = \partial_t, \quad (2)$$

and

$$\begin{aligned} \mathbf{K}_{(2)} &= -\cos \phi \partial_\theta + \coth \theta \sin \phi \partial_\phi, \\ \mathbf{K}_{(1)} &= \partial_\phi, \quad \mathbf{K}_{(3)} = \sin \phi \partial_\theta + \coth \theta \cos \phi \partial_\phi. \end{aligned} \quad (3)$$

Solutions to the Einstein equations endowed with the hyperbolic symmetry (3) have been the subject of research by several authors (see [7–14] and references therein).

Since the fluid that sources the line element (1) is considered as the final state ensuing from a dynamical regime, the obvious question is: What are the general properties of the fluid distribution during this evolving regime, before reaching the equilibrium?

Our purpose in this work is to answer to the above question by carrying on a comprehensive study on the physical properties of evolving fluid distributions in the region inner to the horizon, endowed with the hyperbolic symmetry (3) and that eventually may converge to the static fluid distributions described in [1].

We shall deploy all required equations for a full description of the fluid distribution, including a transport equation. Some specific analytical solutions to these equations will be exhibited. The solutions will be obtained assuming the quasi-homologous condition for their evolution, and the vanishing of the complexity factor.

As we shall see below within the region $r < 2m$, where $m(t, r)$ is a suitable definition of the mass function, the energy density is negative, and the central region cannot be filled with our fluid distribution. Thus, either the center is surrounded by an empty cavity or by a fluid distribution not endowed with hyperbolic symmetry. A discussion about the physical meaning of the obtained results is presented.

2. The General Setup of the Problem: Notation, Variables and Equations

We consider hyperbolically symmetric distributions of evolving fluids, which may be (or not) bounded from outside by a surface Σ^e and, in the case when a cavity is present, are necessarily bounded from inside by a surface Σ^i . The fluid is assumed to be locally anisotropic (principal stresses unequal) and undergoing dissipation in the form of heat flow (diffusion approximation).

Having chosen co-moving coordinates, the general interior metric can be written as

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 (d\theta^2 + \sinh^2 \theta d\phi^2), \quad (4)$$

where A , B and R are assumed positive, and due to the symmetry (3) are functions of t and r . We number the coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \phi$. A and B are dimensionless, whereas R has the same dimension as r .

The energy momentum tensor $T_{\alpha\beta}$ of the fluid distribution may be written as

$$\begin{aligned} T_{\alpha\beta} &= (\mu + P_\perp) V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp) \chi_\alpha \chi_\beta \\ &\quad + q_\alpha V_\beta + V_\alpha q_\beta, \end{aligned} \quad (5)$$

where μ , P_r , P_\perp , q^α , V^α have the usual meaning, and χ^α is unit four-vector along the radial direction. Besides the four-vectors V^α , q^α and χ^α satisfy

$$V^\alpha V_\alpha = -1, \quad V^\alpha q_\alpha = 0, \quad \chi^\alpha \chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0. \quad (6)$$

Since the Lie derivative and the partial derivative commute, then

$$\mathcal{L}_K(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R}) = 8\pi\mathcal{L}_K T_{\alpha\beta} = 0, \quad (7)$$

implying because of (3) that all physical variables only depend on t and r .

We may also express the energy momentum tensor (5) in the equivalent (canonical) form

$$T_{\alpha\beta} = \mu V_\alpha V_\beta + Ph_{\alpha\beta} + \Pi_{\alpha\beta} + q(V_\alpha \chi_\beta + \chi_\alpha V_\beta) \quad (8)$$

with

$$P = \frac{P_r + 2P_\perp}{3}, \quad h_{\alpha\beta} = g_{\alpha\beta} + V_\alpha V_\beta,$$

$$\Pi_{\alpha\beta} = \Pi \left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right), \quad \Pi = P_r - P_\perp.$$

For our comoving observers, we have

$$V^\alpha = A^{-1} \delta_0^\alpha, \quad q^\alpha = q B^{-1} \delta_1^\alpha, \quad \chi^\alpha = B^{-1} \delta_1^\alpha. \quad (9)$$

It is worth noticing that bulk or shear viscosity could be introduced by redefining the radial and tangential pressures. In addition, dissipation in the free streaming approximation can be absorbed in μ , P_r and q .

2.1. Einstein Equations and Conservation Laws

The Einstein equations for (4) and (8) are

$$8\pi\mu = -\frac{1}{R^2} - \frac{1}{B^2} \left[-\frac{2B'}{B} \frac{R'}{R} + \left(\frac{R'}{R} \right)^2 + \frac{2R''}{R} \right] + \frac{1}{A^2} \left(\frac{2\dot{B}}{B} \frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} \right), \quad (10)$$

$$4\pi q = -\frac{1}{AB} \left(\frac{R'}{R} \frac{\dot{B}}{B} + \frac{A'}{A} \frac{\dot{R}}{R} - \frac{\dot{R}'}{R} \right), \quad (11)$$

$$8\pi P_r = \frac{1}{R^2} + \frac{1}{B^2} \left[\frac{2A'}{A} \frac{R'}{R} + \left(\frac{R'}{R} \right)^2 \right] + \frac{1}{A^2} \left(\frac{2\dot{A}}{A} \frac{\dot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{2\ddot{R}}{R} \right), \quad (12)$$

$$8\pi P_\perp = \frac{1}{B^2} \left(-\frac{A'}{A} \frac{B'}{B} + \frac{A'}{A} \frac{R'}{R} - \frac{B'}{B} \frac{R'}{R} + \frac{A''}{A} + \frac{R''}{R} \right) + \frac{1}{A^2} \left(\frac{\dot{A}}{A} \frac{\dot{B}}{B} + \frac{\dot{A}}{A} \frac{\dot{R}}{R} - \frac{\dot{B}}{B} \frac{\dot{R}}{R} - \frac{\ddot{B}}{B} - \frac{\ddot{R}}{R} \right), \quad (13)$$

where dots and primes denote the derivative with respect to t and r , respectively. The difference between these equations and the corresponding to the spherically symmetric case are easily identified (see for example Equations (7)–(10) in [15]).

The conservation laws $T_{;\mu}^{\mu\nu} = 0$, as in the spherically symmetric case, have only two independent components, which are displayed in Appendix A.

2.2. Kinematical Variables

The four-acceleration a_α and the expansion Θ of the fluid are given by

$$a_\alpha = V_{\alpha;\beta} V^\beta, \quad \Theta = V^\alpha_{;\alpha}. \quad (14)$$

From which we obtain for the four-acceleration and its scalar a ,

$$a_1 = \frac{A'}{A}, \quad a = \sqrt{a^\alpha a_\alpha} = \frac{A'}{AB} \Rightarrow a^\alpha = a \chi^\alpha, \quad (15)$$

and for the expansion

$$\Theta = \frac{1}{A} \left(\frac{\dot{B}}{B} + 2 \frac{\dot{R}}{R} \right). \quad (16)$$

The shear tensor $\sigma_{\alpha\beta}$ is defined by (the vorticity vanishes identically)

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha} V_{\beta)} - \frac{1}{3} \Theta h_{\alpha\beta}, \quad (17)$$

its non zero components are

$$\sigma_{11} = \frac{2}{3} B^2 \sigma, \quad \sigma_{22} = \frac{\sigma_{33}}{\sinh^2 \theta} = -\frac{1}{3} R^2 \sigma, \quad (18)$$

and its scalar

$$\frac{3}{2} \sigma^{\alpha\beta} \sigma_{\alpha\beta} = \sigma^2, \quad (19)$$

reads

$$\sigma = \frac{1}{A} \left(\frac{\dot{B}}{B} - \frac{\dot{R}}{R} \right). \quad (20)$$

All the expressions above are the same, in terms of the metric functions, as in the spherically symmetric case.

2.3. The Weyl Tensor

Using Maple we may easily obtain the Weyl tensor corresponding to our metric (4). Thus, the magnetic part of the Weyl tensor vanishes, whereas its electric part may be written as

$$E_{\alpha\beta} = \mathcal{E} \left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right), \quad (21)$$

with

$$\begin{aligned} \mathcal{E} = & \frac{1}{2B^2} \left[-\frac{A'}{A} \frac{R'}{R} - \frac{A'}{A} \frac{B'}{B} + \frac{R'}{R} \frac{B'}{B} + \left(\frac{R'}{R} \right)^2 + \frac{A''}{A} - \frac{R''}{R} \right] \\ & + \frac{1}{2A^2} \left[\frac{\dot{A}}{A} \frac{\dot{B}}{B} - \frac{\dot{A}}{A} \frac{\dot{R}}{R} + \frac{\dot{R}}{R} \frac{\dot{B}}{B} - \left(\frac{\dot{R}}{R} \right)^2 - \frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} \right] + \frac{1}{2R^2}. \end{aligned} \quad (22)$$

2.4. The Mass Function

Following [16] we may define the mass function as

$$m(r, t) = -\frac{R}{2} R_{232}^3 = \frac{R}{2} \left[\left(\frac{R'}{B} \right)^2 - \left(\frac{\dot{R}}{A} \right)^2 + 1 \right], \quad (23)$$

where the Riemann tensor component R_{232}^3 is now calculated for (4).

Introducing the proper time derivative D_T , and the proper radial derivative D_R by

$$D_T = \frac{1}{A} \frac{\partial}{\partial t}, \quad (24)$$

$$D_R = \frac{1}{R'} \frac{\partial}{\partial r}, \quad (25)$$

we can define the velocity U as

$$U = D_T R, \quad (26)$$

which must be smaller than 1 (in relativistic units).

Indeed, in Gaussian coordinates, the position of each fluid element may be given as

$$x^\alpha = x^\alpha(y^a, s), \quad (27)$$

where s is the proper time along the world line of the particle, and y^a determines the position of the fluid element on any three-dimensional hypersurface Σ (latin letters running from 1 to 3).

Next, for an infinitesimal variation of the world line we have

$$\delta x^\alpha = \frac{\partial x^\alpha}{\partial y^a} \delta y^a, \quad (28)$$

from which it follows

$$D_T(\delta x^\alpha) = V_{;\beta}^\alpha \delta x^\beta. \quad (29)$$

The position vector of the particle $y^a + \delta y^a$ relative to the particle y^a on Σ is defined by

$$\delta_\perp x^\alpha = h_\beta^\alpha \delta x^\beta, \quad (30)$$

implying that the relative velocity between these two particles is

$$u^\alpha = h_\beta^\alpha D_T(\delta_\perp x^\beta), \quad (31)$$

then (29) and (30) imply

$$u^\alpha = V_{;\beta}^\alpha \delta_\perp x^\beta. \quad (32)$$

Defining the infinitesimal distance between two neighboring points on Σ by

$$\delta l^2 = g_{\alpha\beta} \delta_\perp x^\beta \delta_\perp x^\alpha, \quad (33)$$

then it can be shown (see [17] for details) that

$$\delta l D_T(\delta l) = \delta_\perp x^\beta \delta_\perp x^\alpha \left(\sigma_{\alpha\beta} + \frac{1}{3} h_{\alpha\beta} \Theta \right), \quad (34)$$

or, introducing the spacelike unit vector

$$e^\alpha = \frac{\delta_\perp x^\alpha}{\delta l}, \quad (35)$$

$$\frac{D_T(\delta l)}{\delta l} = e^\alpha e^\beta \sigma_{\alpha\beta} + \frac{\Theta}{3}. \quad (36)$$

The above expressions are completely general. Let us now consider our hyperbolically symmetric line element and apply (36) to two neighboring points on a closed curve (S) along the ϕ direction ($r = \text{constant}; \theta = \text{constant}$). In this case we have $e^\alpha \equiv (0, 0, 0, \frac{1}{R \sinh \theta})$ and using (16), (18) and (20) in (36) we obtain

$$\frac{D_T(\delta l)}{\delta l} = \frac{\dot{R}}{AR} = \frac{U}{R}. \quad (37)$$

Now, the $D_T(\delta l)$ above is the relative velocity between two neighboring points on S . This quantity of course must be smaller than one (in relativistic units). On the other hand the rate of variation of the total length (L) of S per unit of proper time (say V_L) is also a velocity, and thereof must be smaller than 1, and because of the axial symmetry it is just the sum of (37) over all the curve S . Thus, we have

$$\frac{R}{L} V_L = U. \quad (38)$$

For any value of θ , $\frac{R}{L} < 1$ and thereof $U < 1$.

Then, since $U < 1$, it follows at once from (23) that m is a positive defined quantity. Additionally, (23) can be rewritten as

$$E \equiv \frac{R'}{B} = \left(\frac{2m}{R} + U^2 - 1 \right)^{1/2}. \quad (39)$$

Using (23) with (24) and (25) we obtain

$$D_T m = 4\pi(P_r U + qE)R^2, \quad (40)$$

and

$$D_R m = -4\pi \left(\mu + q \frac{U}{E} \right) R^2, \quad (41)$$

which implies

$$m = -4\pi \int_0^r \left(\mu + q \frac{U}{E} \right) R^2 R' dr, \quad (42)$$

satisfying the regular condition $m(t, 0) = 0$.

Integrating (42) we find

$$\frac{3m}{R^3} = -4\pi\mu + \frac{4\pi}{R^3} \int_0^r R^3 \left(D_R \mu - 3q \frac{U}{RE} \right) R' dr. \quad (43)$$

Since any causal transport equation is based on the assumption that the fluid is not very far from thermal equilibrium, then $q \ll |\mu|$. This implies from (42) that μ is necessarily negative, if we assume the condition $R' > 0$ to avoid shell crossing, and remind that $m > 0$ and E is a regular function within the fluid distribution.

Furthermore, it follows from (42) that whenever the energy density is regular, then $m \sim r^3$ as r tends to zero. However, in this same limit $U \sim 0$, and $R \sim r$ implying because of (39) that the central region cannot be filled with the fluid distribution under consideration. Among the many possible scenarios we shall assume here that the center is surrounded by a vacuum cavity. However, it should be clear that this is just one of the possible choices that, even if having implications on specific models, does not affect the general properties of the fluids endowed with hyperbolical symmetry.

The two above-mentioned features of the fluid appear also in the static case [1].

Before concluding this section it is worth discussing some detail of Equation (A7) and comparing it with the corresponding equation for the spherically symmetric case (see Equation (C6) in [15]).

First of all let us notice that it has the “Newtonian” form $Force = Mass\ density \times Acceleration$. Let us next analyze the different terms in (A7). The first term on the right represents the gravitational interaction, it is the product of the passive gravitational mass density (p.g.m.d) $(\mu + P_r)$, which due to the fact that the energy density is negative, would be negative, and the active gravitational mass (a.g.m) $(4\pi P_r R^3 - m)$ which would also be negative for most equations of state. Thus the gravitational term has the same sign as in the spherically symmetric case. However, its effect is the inverse of the latter case. Indeed, since the p.g.m.d is negative, so is the inertial mass density as consequence of the equivalence principle; thus, the gravitational term tends to increase $D_T U$, i.e., it acts as a repulsive force instead of an attractive one, as in (C6) of [15]. In the same order of ideas we see that a negative pressure gradient would tend to push any fluid element inwardly, i.e., everything happens as if force terms switch their roles, as compared with the spherically symmetric case.

3. The Transport Equation

The treatment of dissipative processes requires the adoption of a heat transport equation. In order to ensure causality we shall resort to the transport equation obtained from the Müller–Israel–Stewart theory [18–20].

Then, the corresponding transport equation for the heat flux reads

$$\tau h^{\alpha\beta} V^\gamma q_{\beta;\gamma} + q^\alpha = -\kappa h^{\alpha\beta} (T_{,\beta} + Ta_{\beta}) - \frac{1}{2} \kappa T^2 \left(\frac{\tau V^\beta}{\kappa T^2} \right)_{;\beta} q^\alpha, \quad (44)$$

where κ and τ are two parameters denoting the thermal conductivity and relaxation time, respectively, and T denotes temperature.

There is only one non-vanishing independent component of Equation (44), which may be written as

$$\tau D_T q = -q - \frac{\kappa}{AB} (AT)' - \frac{1}{2} \tau \Theta q - \frac{1}{2} \kappa T^2 D_T \left(\frac{\tau}{\kappa T^2} \right) q. \quad (45)$$

In the case $\tau = 0$ we recover the Eckart–Landau equation [21].

Under some circumstances it is possible to adopt the so-called “truncated” version, in which case the last term in (44) is neglected [22],

$$\tau h^{\alpha\beta} V^\gamma q_{\beta;\gamma} + q^\alpha = -\kappa h^{\alpha\beta} (T_{,\beta} + Ta_{\beta}), \quad (46)$$

and whose only non-vanishing independent component becomes

$$\tau \dot{q} + qA = -\frac{\kappa}{B} (TA)'. \quad (47)$$

Let us now analyze in some detail the changes appearing in the condition for thermal equilibrium, as compared with the spherically symmetric case.

As it was pointed out by Tolman many years ago [23], since all forms of energy have inertia, so should be for thermal energy. This implies, because of the equivalence principle, that thermal energy would tend to displace to regions of lower gravitational potential, thereby modifying the condition of thermal equilibrium in the presence of a gravitational field. In other words, now a temperature gradient is necessary to ensure thermal equilibrium.

Thus, the Tolman condition reads as

$$(TA)' = 0 \Rightarrow T' = -\frac{T}{A} A' = -TaB. \quad (48)$$

However, as it follows from (A3), if $m > 4\pi P_r R^3$, in equilibrium $a < 0$ (the four-acceleration is now directed radially inwardly), implying the existence of a repulsive gravitational force, leading to a positive temperature gradient in order to assure thermal equilibrium. This situation is different from the one observed in the spherically symmetric case, where a negative temperature gradient is required to assure thermal equilibrium.

Before concluding this section it is worth discussing the physical implications of (A8). This equation comes out from the combination of the dynamical Equation (A7) and the transport equation. It brings out the thermal effect on the p.g.m.d., and by virtue of the equivalence principle, on the effective inertial mass density as well. A similar effect was pointed out for the first time for the spherically symmetric case in [24] (see also [25] for a discussion on this effect). In our case the term $\frac{\kappa T}{\tau}$ increases the absolute value of the effective p.g.m.d (which is negative), thereby increasing the absolute value of the effective inertial mass density (the term in the bracket on the left of (A8)), as a result of which any hydrodynamic force directed outward tends to push the fluid element inward, weaker than it does in the non-dissipative case, due to the term $\frac{\kappa T}{\tau}$. On the other hand the gravitational term, which is negative, pushes any fluid element as it does in the non-dissipative case. Overall, the thermal effect enhances the tendency to expansion, as in the spherically

symmetric case, but different terms in the equation playing different roles as compared with this latter case.

In order to obtain specific solutions to the Einstein equations we shall need to impose additional restrictions. In this work we shall assume that the fluid evolves in the quasi-homologous regime and satisfies the vanishing complexity factor condition. The next two sections are devoted to explain these conditions in some detail.

4. The Structure Scalars and the Complexity Factor

The complexity factor is a scalar function intended to measure the degree of complexity of a self-gravitating system (in some cases more than one scalar function may be required). For a static, hyperbolically symmetric fluid distribution it was assumed in [1] (following the arguments developed in [26]) that the simplest system corresponds to a homogeneous (in the energy density), locally isotropic fluid distribution (principal stresses equal). Thus, a zero value of the complexity factor was assumed for such a distribution. Furthermore, it was shown that a single scalar function (hereafter referred to as Y_{TF}) describes the modifications introduced by the energy density inhomogeneity and pressure anisotropy, to the Tolman mass, with respect to its value for the zero complexity case.

This scalar belongs to a set of variables named structure scalars and defined in [27], and which appear in the orthogonal splitting of the Riemann tensor [28–31]. For the sake of completeness we shall highlight the main steps leading to their acquisition (for the spherically symmetric case see [27,32] for details). For our purpose here, we shall need only one of the five structure scalars characterizing our fluid distribution.

The first step consists in defining the tensor $Y_{\alpha\beta}$ by

$$Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} V^\gamma V^\delta, \quad (49)$$

which may be split in terms of its trace and its trace-free part as

$$Y_{\alpha\beta} = \frac{1}{3} Y_T h_{\alpha\beta} + Y_{TF} \left(\chi_\alpha \chi_\beta - \frac{1}{3} h_{\alpha\beta} \right). \quad (50)$$

Then, using the field equations and (22) the following expressions can be obtained

$$Y_T = 4\pi(\mu + 3P_r - 2\Pi), \quad Y_{TF} = \mathcal{E} - 4\pi\Pi. \quad (51)$$

On the other hand, combining (12), (13), (22) and (23) we obtain

$$\frac{3m}{R^3} = -4\pi\mu + 4\pi\Pi + \mathcal{E}, \quad (52)$$

or using (43) and (51)

$$Y_{TF} = -8\pi\Pi + \frac{4\pi}{R^3} \int_0^r R^3 \left(D_R \mu - 3q \frac{U}{RE} \right) R' dr. \quad (53)$$

Using (12), (13) and (22), we may express Y_{TF} in terms of the metric functions and their derivatives

$$Y_{TF} = \frac{1}{B^2} \left(\frac{A''}{A} - \frac{A'}{A} \frac{R'}{R} - \frac{A'}{A} \frac{B'}{B} \right) + \frac{1}{A^2} \left(\frac{\dot{A} \dot{B}}{A B} - \frac{\dot{A} \ddot{R}}{A R} - \frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} \right). \quad (54)$$

Following the arguments presented in [1] we shall choose Y_{TF} as the complexity factor. In the dynamic case, however, we still need to provide a criterion for the definition of complexity of the pattern of evolution.

We shall assume here that Y_{TF} is identified with the complexity factor, and we shall consider the quasi-homologous evolution defined in [15] as the simplest mode of evolution.

5. The Quasi-Homologous Condition

In order to provide a rigorous definition of quasi-homologous evolution, let us write (11) as

$$\left(\frac{U}{R}\right)' = 4\pi q B + \sigma \frac{R'}{R}, \quad (55)$$

whose general solution is

$$U = \tilde{a}(t)R + R \int_0^r \left(\frac{4\pi q}{E} + \frac{\sigma}{R}\right) R' dr, \quad (56)$$

where $\tilde{a}(t)$ is an integration function and (39) has been used.

If the fluid distribution is bounded by a surface Σ^e defined by the equation $r = r_{\Sigma^e} = \text{constant}$, we may write

$$U = R \frac{U_{\Sigma^e}}{R_{\Sigma^e}} - R \int_r^{r_{\Sigma^e}} \left(\frac{4\pi q}{E} + \frac{\sigma}{R}\right) R' dr. \quad (57)$$

The quasi-homologous condition reads

$$U = R \frac{U_{\Sigma^e}}{R_{\Sigma^e}}, \quad (58)$$

implying

$$\frac{4\pi q}{E} + \frac{\sigma}{R} = 0. \quad (59)$$

The above condition will be used to obtain specific models, and its assumption is supported, on the one hand, by the fact that it is the relativistic version of the well-known homologous condition widely used in classic astrophysics, and on the other hand by the fact that it qualifies as one of the simplest patterns of evolution (see [15,32] for a discussion on this point).

6. The Exterior Spacetime and Junction Conditions

In the case that the fluid is bounded, then junction conditions on the boundary have to be imposed [33] in order to avoid the presence of thin shells on the boundary. If any specific model violates Darmois conditions then we should relax the continuity of the second fundamental form, implying the presence of thin shells [34].

Thus, outside Σ^e (but inside the horizon) we assume that we have the hyperbolic version of the Vaidya spacetime, described by

$$ds^2 = -\left[\frac{2M(v)}{r} - 1\right] dv^2 - 2dr dv + r^2(d\theta^2 + \sinh^2\theta d\phi^2), \quad (60)$$

where $M(v)$ denotes the total mass, and v is the retarded time.

The continuity of the first fundamental form reads

$$(ds^2)_{\Sigma^e}^- = (ds^2)_{\Sigma^e}^+, \quad (61)$$

where $-$, $+$ means from the inner or the outer side of the boundary surface, respectively.

At the outer side of the boundary, the surface equation reads

$$\Psi \equiv r - r_{\Sigma^e}(v) = 0, \quad (62)$$

whose unit normal vector is defined by

$$n_{\mu}^{+} = \frac{\partial_{\mu}\Psi}{\sqrt{|\partial_{\alpha}\Psi\partial_{\beta}\Psi g^{\alpha\beta}|}}, \quad (63)$$

with components

$$n_{\mu}^{+} = \left(-\beta \frac{d\mathbf{r}_{\Sigma^e}}{dv}, \beta, 0, 0\right), \quad (64)$$

where

$$\beta = \frac{1}{\sqrt{|\frac{2M(v)}{r_{\Sigma^e}} - 1 + \frac{2d\mathbf{r}_{\Sigma^e}}{dv}|}}. \quad (65)$$

At the inner side, the surface equation reads

$$\Phi \equiv r - r_{\Sigma^i} = 0, \quad (66)$$

whose normal unit vector is defined by

$$n_{\mu}^{-} = \frac{\partial_{\mu}\Phi}{\sqrt{|\partial_{\alpha}\Phi\partial_{\beta}\Phi g^{\alpha\beta}|}}, \quad (67)$$

with components

$$n_{\mu}^{-} = (0, B_{\Sigma}, 0, 0), \quad (68)$$

observe that $n_{\mu}^{-} = (\chi_{\mu})_{\Sigma}$.

From (61) it follows that

$$R(t, r_{\Sigma^e}) = \mathbf{r}_{\Sigma^e}(v). \quad (69)$$

Next, instead of calculating the second fundamental form at both sides of the boundary surface we shall impose the continuity of the flux of energy–momentum across Σ^e , which of course implies the absence of thin shells on the boundary surface. For doing so we have to calculate

$$\begin{aligned} (T_{\mu\nu}n^{\nu}n^{\mu})_{\Sigma^e}^{+}; & \quad (T_{\mu\nu}n^{\nu}n^{\mu})_{\Sigma^e}^{-}; \\ (T_{\mu\nu}n^{\nu}V^{\mu})_{\Sigma^e}^{+}; & \quad (T_{\mu\nu}n^{\nu}V^{\mu})_{\Sigma^e}^{-}. \end{aligned} \quad (70)$$

where the vectors $(V^{\mu})^{+}$, $(V^{\mu})^{-}$ have components

$$(V^{\mu})^{+} = \left[\beta, \beta \frac{d\mathbf{r}_{\Sigma^e}(v)}{dv}, 0, 0\right], \quad (71)$$

and

$$(V^{\mu})^{-} = \left[\frac{1}{A}, 0, 0, 0\right]. \quad (72)$$

Next, we have to calculate the energy–momentum tensor corresponding to the line element (60), we obtain

$$T_{\mu\nu}^{(+)} = \frac{1}{4\pi r^2} \frac{dM}{dv} \delta_{\mu}^0 \delta_{\nu}^0. \quad (73)$$

From the above expression it follows at once that the energy density of the null fluid sourcing (60) would be negative for an outgoing flux, which is exactly the inverse of what happens for the usual Vaidya metric.

We can now evaluate (70) to obtain

$$(T_{\mu\nu}n^\mu n^\nu)_{\Sigma^e}^- = [P_r]_{\Sigma^e}, \quad (74)$$

$$(T_{\mu\nu}n^\mu V^\nu)_{\Sigma^e}^- = -[q]_{\Sigma^e}, \quad (75)$$

$$(T_{\mu\nu}n^\mu n^\nu)_{\Sigma^e}^+ = \frac{\beta^2}{4\pi r_{\Sigma^e}^2} \frac{dM}{dv}, \quad (76)$$

$$(T_{\mu\nu}n^\mu V^\nu)_{\Sigma^e}^+ = -\frac{\beta^2}{4\pi r_{\Sigma^e}^2} \frac{dM}{dv}. \quad (77)$$

Then, imposing the continuity of the flux of energy–momentum across Σ^e , it follows that

$$q \stackrel{\Sigma^e}{=} P_r. \quad (78)$$

where $\stackrel{\Sigma^e}{=}$ means that both sides of the equation are evaluated on Σ^e .

Finally, following the usual procedure used in the spherically symmetric case, it is a simple matter to check that the continuity of the second fundamental form implies

$$m(t, r) \stackrel{\Sigma^e}{=} M(v). \quad (79)$$

In the cases where the central region is surrounded by an empty vacuole bounded by a surface Σ^i , junction conditions should be considered also at the inner boundary of the fluid distribution. Then, following the same steps as before we find

$$P_r \stackrel{\Sigma^i}{=} 0. \quad (80)$$

and

$$m(t, r) \stackrel{\Sigma^i}{=} 0. \quad (81)$$

7. Some Models

In the following subsections we shall exhibit several families of solutions to the Einstein equations for the hyperbolically symmetric fluids. These solutions will be obtained by assuming quasi-homologous evolution and the vanishing of the complexity factor. This choice, justified by the comments in previous sections, will allow us to compare the behavior of our fluid distributions with similar solutions found for the spherically symmetric case in [15]. Besides, different types of additional restrictions will be imposed in order to obtain specific models. It should be clear that the purpose of the presentation of these models, besides the possible potential of some of them in the study of specific astrophysical scenario, is to illustrate the richness of fluid distributions endowed with hyperbolic symmetry.

7.1. Non-Dissipative Case

Excluding dissipative processes, and assuming the quasi-homologous condition (59), we may write

$$q = 0 \Rightarrow \sigma = 0 \Rightarrow \frac{\dot{B}}{B} = \frac{\dot{R}}{R} \Rightarrow R = rB, \quad (82)$$

and using (56)

$$U = \frac{\dot{R}}{A} = \frac{r\dot{B}}{A} = \tilde{a}(t)rB. \quad (83)$$

Imposing next the condition $Y_{TF} = 0$ we have

$$\frac{A''}{A} - \frac{A'}{A} \frac{B'}{B} - \frac{A'}{A} \frac{R'}{R} = 0. \quad (84)$$

In order to exhibit specific solutions, we shall further assume some additional restrictions.

7.1.1. $\mathcal{E} = 0, \Pi = 0$

We shall assume here that the fluid is conformally flat ($\mathcal{E} = 0$), and the pressure is isotropic ($\Pi = 0$), which combined with $Y_{TF} = 0$ produces $\mu' = 0$ (i.e., the energy density is homogeneous).

From the conditions $\mathcal{E} = 0$ and $\Pi = 0$ we obtain

$$\frac{1}{R^2} + \frac{1}{B^2} \left[\left(\frac{R'}{R} \right)^2 + \frac{B'}{B} \frac{R'}{R} - \frac{R''}{R} \right] - \frac{1}{A^2} \left(\frac{\dot{R}^2}{R^2} - \frac{\dot{B}}{B} \frac{\dot{R}}{R} \right) = 0. \quad (85)$$

Using (82) in (85) and (84) produces

$$1 + r^2 \left[2 \left(\frac{R'}{R} \right)^2 - \frac{1}{r} \frac{R'}{R} - \frac{R''}{R} \right] = 0, \quad (86)$$

and

$$\frac{A''}{A} - \frac{A'}{A} \left(\frac{2R'}{R} - \frac{1}{r} \right) = 0. \quad (87)$$

The solution to the system (86) and (87) is easily found to be

$$R = \frac{\tilde{R}(t)}{\cos[c_1(t) + \ln r]}, \quad (88)$$

$$B = \frac{\tilde{R}(t)}{r \cos[c_1(t) + \ln r]}, \quad (89)$$

$$A = \gamma(t) \tilde{R}^2(t) \tan[c_1(t) + \ln r] + b(t), \quad (90)$$

where $\tilde{R}(t), c_1(t), \gamma(t), b(t)$ are arbitrary functions of their argument. The reader can easily check, using Maple or Mathematica, that the line element (4) with (88)–(90) produces $\mathcal{E} = 0 = \Pi = 0$.

To specify further the solution we shall choose the above functions as follows

$$\dot{c}_1 = \frac{\dot{\tilde{R}}}{\tilde{R}}, \quad b(t) = \gamma(t) \tilde{R}^2, \quad (91)$$

producing

$$\frac{\dot{R}}{R} = \frac{\dot{\tilde{R}}}{\tilde{R}} (1 + \tan u), \quad (92)$$

$$A = \gamma(t) \tilde{R}^2 (1 + \tan u), \quad \Rightarrow \quad A = \frac{\tilde{a} \dot{\tilde{R}}}{\tilde{R}}, \quad (93)$$

with $\tilde{a} = \frac{\gamma(t) \tilde{R}^3}{\dot{\tilde{R}}}$ and $u = c_1(t) + \ln r$. From the above expressions we find for the physical variables and the mass function

$$8\pi\mu = -\frac{3}{\tilde{R}^2} + \frac{3}{\tilde{a}^2}, \quad (94)$$

$$8\pi P_r = 8\pi P_\perp = -\frac{3}{\tilde{a}^2} + \frac{3 \tan u + 1}{\tilde{R}^2 (\tan u + 1)} + \frac{2\tilde{R}\dot{\tilde{a}}}{\tilde{a}^3 \tilde{R} (\tan u + 1)}, \quad (95)$$

$$m = \frac{\tilde{R}}{2 \cos^3 u} \left(1 - \frac{\tilde{R}^2}{\tilde{a}^2} \right). \quad (96)$$

It is a simple matter to check that this solution does not satisfy the Darmois conditions at either boundary surfaces, and therefore we must assume the presence of thin shells there.

If we choose $\tilde{R}(t), c_1(t), \gamma(t)$ such that they tend to a constant as $t \rightarrow \infty$, then the above solution tends to the incompressible isotropic solution found in [14], which is a particular case of the hyperbolically symmetric Bowers–Liang solution found in [1].

The above solution might be considered as a version of the Friedman–Robertson–Walker space–time (FRW) for the hyperbolically symmetric case since they share some similar properties, e.g., $\mathcal{E} = \Pi = \mu' = \sigma = 0$. However it is not geodesic as in the spherically symmetric case. Therefore, we shall next find another version of the hyperbolically symmetric FRW space–time, but satisfying the geodesic condition $A' = 0$.

7.1.2. Geodesic Solutions

If we further impose the geodesic condition on the fluid, then we may put without loss of generality $A = 1$, and the quasi-homologous condition also implies

$$\frac{R_I}{R_{II}} = \text{constant}, \quad (97)$$

where R_I and R_{II} denote the areal radii of two shells (I, II) described by $r = r_I = \text{constant}$, and $r = r_{II} = \text{constant}$, respectively.

From (97) it follows at once that R is a separable function. In the notation of [32], conditions (83) and (97) define the homologous evolution.

The conditions $A = 1$ and $q = 0$ imply

$$\frac{\dot{B}}{B} = \frac{\dot{R}'}{R'} \quad (98)$$

where (11) has been used. Since the fluid is shear-free we have $R = Br$, and since R is separable so is B . However, if B is separable, then by a simple reparametrization of r it becomes a function of t alone $B = B(t)$, i.e.,

$$R = rB(t). \quad (99)$$

Then (98) is automatically satisfied, as well as $Y_{TF} = 0$ as it follows from (84). In this case we may write the physical variables and the mass function as

$$8\pi\mu = -\frac{2}{r^2 B^2} + \frac{3\dot{B}^2}{B^2}, \quad (100)$$

$$8\pi P_r = \frac{2}{r^2 B^2} - \frac{\dot{B}^2}{B^2} - \frac{2\ddot{B}}{B}, \quad (101)$$

$$8\pi P_\perp = -\frac{2\dot{B}^2}{B^2} - \frac{2\ddot{B}}{B}, \quad (102)$$

$$m = \frac{rB}{2}(2 - r^2\dot{B}^2). \quad (103)$$

Thus, the fluid is conformally flat, shear-free, geodesic, evolves homologously and satisfies the vanishing complexity factor condition. In this sense it could be considered also as a version of the hyperbolically symmetric FRW space–time. However, unlike the spherically symmetric case, it is anisotropic and the energy density is inhomogeneous.

As in the previous solution, by simple inspection of (101), (103) it can be checked that Darmois conditions cannot be satisfied at either boundary surface.

It is worth analyzing with some detail the differences between this case and the situation in the spherically symmetric case (the usual one). In the latter case we have seen [32] that for a non-dissipative fluid satisfying the homologous condition, the complexity factor vanishes, and there is a single solution characterized by $\Pi = \mu' = a = \mathcal{E} = 0$ (FRW).

However, in the present case, imposing homologous conditions on a geodesic non-dissipative fluid we get a conformally flat, shear-free geodesic fluid with $\Pi, \mu' \neq 0$. If we want to describe an isotropic, homogeneous, shear-free, non-dissipative fluid, then we have to relax the geodesic condition.

Finally, it is instructive to build up a toy model with the above solution, by choosing a particular form for the function B such that asymptotically it leads to a static regime. Thus, let us assume

$$B = \beta(1 + e^{-\alpha t}), \quad (104)$$

where α, β are two positive constants.

Then, it is a simple matter to check that as $t \rightarrow \infty$ we get

$$8\pi\mu = -\frac{2}{r^2\beta^2}, \quad (105)$$

$$8\pi P_r = \frac{2}{r^2\beta^2}, \quad (106)$$

$$8\pi P_\perp = 0, \quad (107)$$

and for the mass function we get asymptotically $m = r\beta$.

Thus, our toy model converges to the static solution corresponding to the stiff equation of state ($P_r = |\mu|$) found in [1] (Equations (138)–(139) in that reference).

We shall next consider dissipative solutions.

7.2. Dissipative Case with $B = 1$

Let us now consider dissipative solutions satisfying the condition $B = 1$. As shown in [35], such a condition is particularly suitable for describing situations when the center is surrounded by an empty cavity, a scenario we expect for the kind of fluid distributions we are dealing with in this work.

Thus, the metric functions for this case read

$$B = 1, \quad A = \frac{\dot{R}}{\tilde{a}(t)R}, \quad (108)$$

and the corresponding Einstein equations may be written as

$$8\pi\mu = -\frac{1}{R^2} - \frac{2R''}{R} - \left(\frac{R'}{R}\right)^2 + \tilde{a}^2, \quad (109)$$

$$4\pi q = \frac{\tilde{a}(t)R'}{R}, \quad (110)$$

$$8\pi P_r = \frac{1}{R^2} - \left(\frac{R'}{R}\right)^2 + \frac{2\dot{R}'R'}{\dot{R}R} - 2\tilde{a}\dot{\tilde{a}}\frac{R}{\dot{R}} - 3\tilde{a}^2, \quad (111)$$

$$8\pi P_\perp = \frac{\dot{R}''}{\dot{R}} - \frac{\dot{R}'}{\dot{R}}\frac{R'}{R} + \left(\frac{R'}{R}\right) - \tilde{a}\dot{\tilde{a}}\frac{R}{\dot{R}} - \tilde{a}^2. \quad (112)$$

We may formally integrate (47) producing for the temperature

$$T(t, r) = \frac{\tilde{a}R}{\dot{R}} \left(f(t) - \frac{\tau\dot{\tilde{a}}}{4\pi\kappa} \ln R - \frac{1}{4\pi\kappa} \int \frac{\dot{R}R'}{\dot{R}R} dr \right) - \frac{\tau\tilde{a}^2}{4\pi\kappa}, \quad (113)$$

where $f(t)$ is a function of integration.

On the other hand the condition $Y_{TF} = 0$ and (59) now read

$$A'' - A'\frac{R'}{R} + A\sigma^2 = \dot{\sigma}, \quad (114)$$

$$-\frac{\dot{R}}{\sigma R} = A. \quad (115)$$

Introducing the intermediate variables (X, Y) ,

$$A = X + \frac{\dot{\sigma}}{\sigma^2} \quad \text{and} \quad R = X'Y, \quad (116)$$

(114) and (115) become

$$-\frac{X'}{X} \frac{Y'}{Y} + \sigma^2 = 0, \quad (117)$$

$$\frac{\dot{X}'}{X'} + \frac{\dot{Y}}{Y} = -\sigma X - \frac{\dot{\sigma}}{\sigma}. \quad (118)$$

Thus, we have a large family of dissipative solutions, among which we shall select some specific ones, by imposing additional restrictions allowing us to integrate (117) and (118).

7.2.1. X Is a Separable Function

If we assume the function X to be separable, then we can integrate the system (117) and (118), obtaining

$$A = \frac{\dot{\sigma}}{2\beta^2\sigma^2} [2\beta^2 - \sigma^2(\beta r + c_1)^2], \quad (119)$$

$$R = \frac{\tilde{R}_0}{\sigma} (\beta r + c_1) e^{\frac{\sigma^2}{4\beta^2}(\beta r + c_1)^2}, \quad (120)$$

$$\tilde{a} = -\sigma, \quad (121)$$

where β , \tilde{R}_0 and c_1 are constants.

The above expressions allow us to write for the physical variables

$$8\pi\mu = -\frac{\sigma^2 e^{-\frac{\sigma^2}{2\beta^2}(\beta r + c_1)^2}}{\tilde{R}_0^2(\beta r + c_1)^2} - \frac{\beta^2}{(\beta r + c_1)^2} - \frac{3\sigma^4}{4\beta^2}(\beta r + c_1)^2 - 3\sigma^2, \quad (122)$$

$$4\pi q = -\frac{\sigma[2\beta^2 + \sigma^2(\beta r + c_1)^2]}{2\beta(\beta r + c_1)}, \quad (123)$$

$$8\pi P_r = \frac{\sigma^2 e^{-\frac{\sigma^2}{2\beta^2}(\beta r + c_1)^2}}{\tilde{R}_0^2(\beta r + c_1)^2} - \frac{4\sigma^2\beta^2}{2\beta^2 - \sigma^2(\beta r + c_1)^2} + \frac{\beta^2}{(\beta r + c_1)^2} + \frac{\sigma^4}{4\beta^2}(\beta r + c_1)^2, \quad (124)$$

$$8\pi P_\perp = -\frac{\sigma^2[2\beta^2 + \sigma^2(\beta r + c_1)^2]^2}{4\beta^2(2\beta^2 - \sigma^2(\beta r + c_1)^2)}, \quad (125)$$

$$m = \frac{\tilde{R}_0(\beta r + c_1)}{2\sigma} e^{\frac{\sigma^2}{4\beta^2}(\beta r + c_1)^2} \left\{ 1 + \frac{\tilde{R}_0^2}{4\sigma^2\beta^2} [4\beta^4 + \sigma^4(\beta r + c_1)^4] e^{\frac{\sigma^2}{2\beta^2}(\beta r + c_1)^2} \right\}, \quad (126)$$

while the expression for the temperature reads in this case as

$$T(t, r) = \frac{2\beta^2\sigma^2}{\dot{\sigma}[2\beta^2 - \sigma^2(\beta r + c_1)^2]} \left\{ f(t) + \frac{\dot{\sigma}\tau}{4\pi\kappa} \left[\frac{\sigma^2}{4\beta^2}(\beta r + c_1)^2 + \ln \left[\frac{\tilde{R}_0}{\sigma}(\beta r + c_1) \right] \right] \right. \\ \left. + \frac{\dot{\sigma}}{4\pi\sigma\kappa} \ln(\beta r + c_1) - \frac{\dot{\sigma}\sigma^3}{64\pi\beta^4\kappa}(\beta r + c_1)^4 \right\} - \frac{\tau\sigma^2}{4\pi\kappa}. \quad (127)$$

7.2.2. $A = A(r)$

Another sub-family of solutions may be obtained by assuming that A only depends on r , then the solution to the system (117) and (118) produces

$$A = \frac{1}{4}(\sqrt{2\sigma_0}r + c_1)^2, \quad \tilde{a} = \sigma_0 t - \sigma_1, \quad (128)$$

$$R = \tilde{R}(r)e^{-\frac{1}{4}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)}, \quad (129)$$

where $\tilde{R}(r)$ is an arbitrary function of its argument, and σ_0, σ_1, c_1 are constants. To obtain a specific model, we shall further assume $\tilde{R} = \tilde{R}_0 = \text{constant}$, in which case we find for the physical variables

$$8\pi\mu = \sigma_1^2 - \frac{3\sigma_0}{2}(\sqrt{2\sigma_0}r + c_1)^2\left(-\frac{\sigma_0}{2}t^2 + \sigma_1 t\right)^2 - \frac{1}{\tilde{R}_0^2}e^{\frac{1}{2}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)}, \quad (130)$$

$$4\pi q = \frac{\sqrt{2\sigma_0}}{2}(\sqrt{2\sigma_0}r + c_1)\left(-\frac{\sigma_0}{2}t^2 + \sigma_1 t\right)(-\sigma_0 t + \sigma_1), \quad (131)$$

$$\begin{aligned} 8\pi P_r &= \frac{1}{\tilde{R}_0^2}e^{\frac{1}{2}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)} - t^2\sigma_0^2 + 2t\sigma_0\sigma_1 - 3\sigma_1^2 - \frac{8\sigma_0}{(\sqrt{2\sigma_0}r + c_1)^2} \\ &+ \frac{\sigma_0}{2}(\sqrt{2\sigma_0}r + c_1)^2\left(-\frac{\sigma_0}{2}t^2 + \sigma_1 t\right)^2, \end{aligned} \quad (132)$$

$$8\pi P_\perp = \frac{1}{2}\sigma_0^2 t^2 - t\sigma_0\sigma_1 - \sigma_1^2 + \frac{\sigma_0}{2}(\sqrt{2\sigma_0}r + c_1)^2\left(-\frac{\sigma_0}{2}t^2 + \sigma_1 t\right)^2, \quad (133)$$

$$\begin{aligned} m &= \frac{\tilde{R}_0}{2}e^{-\frac{1}{4}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)} \\ &\times \left\{1 + \tilde{R}_0^2\left[\frac{\sigma_0}{2}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)^2 - (\sigma_0 t - \sigma_1)^2\right]e^{-\frac{1}{2}(\sqrt{2\sigma_0}r + c_1)^2(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)}\right\}. \end{aligned} \quad (134)$$

For the temperature the corresponding expression reads

$$\begin{aligned} T(t, r) &= \frac{4}{(\sqrt{2\sigma_0}r + c_1)^2} \left\{ f(t) - \frac{\tau\sigma_0}{4\pi\kappa} \left[\ln \tilde{R}_0 - \frac{1}{4}(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)(\sqrt{2\sigma_0}r + c_1)^2 \right] \right\} \\ &- \frac{(-\frac{\sigma_0}{2}t^2 + \sigma_1 t)(-\sigma_0 t + \sigma_1)(\sqrt{2\sigma_0}r + c_1)^2}{32\pi\kappa} - \frac{\tau(-\sigma_0 t + \sigma_1)^2}{4\pi\kappa}. \end{aligned} \quad (135)$$

7.2.3. $\dot{\sigma} = 0$

Finally, we shall obtain a class of solutions by assuming that the shear scalar is constant, in which case the integration of the system (117) and (118) produces

$$A = \beta r - \frac{\beta^2}{\sigma}t + \beta_0, \quad \tilde{a} = -\sigma = \text{const.} \quad (136)$$

$$R = \tilde{R}_0\beta e^{\left(\frac{\sigma^2}{2}r^2 - \sigma\beta tr + \frac{\sigma^2\beta_0}{\beta}r + \frac{\beta^2}{2}t^2 - \sigma\beta_0 t\right)}, \quad (137)$$

where $\tilde{R}_0, \beta, \beta_0$ are constants. The physical variables for this case read

$$\begin{aligned} 8\pi\mu &= -\sigma^2 - 3\left[\sigma^2\left(r + \frac{\beta_0}{\beta}\right) - \sigma\beta t\right]^2 \\ &- \frac{e^{-2\left(\frac{\sigma^2}{2}r^2 - \sigma\beta tr + \frac{\sigma^2\beta_0}{\beta}r + \frac{\beta^2}{2}t^2 - \sigma\beta_0 t\right)}}{\tilde{R}_0^2\beta^2}, \end{aligned} \quad (138)$$

$$4\pi q = -\sigma^3 \left(r - \frac{\beta}{\sigma} t + \frac{\beta_0}{\beta} \right), \quad (139)$$

$$8\pi P_r = -\sigma^2 + \sigma^4 \left(r - \frac{\beta}{\sigma} t + \frac{\beta_0}{\beta} \right)^2 + \frac{e^{-2\left(\frac{\sigma^2}{2}r^2 - \sigma\beta tr + \frac{\sigma^2\beta_0}{\beta}r + \frac{\beta^2}{2}t^2 - \sigma\beta_0 t\right)}}{\tilde{R}_0^2\beta^2}, \quad (140)$$

$$8\pi P_\perp = \sigma^2 + \left[\sigma^2 \left(r + \frac{\beta_0}{\beta} \right) - \sigma\beta t \right]^2, \quad (141)$$

and

$$m = \frac{\tilde{R}_0\beta}{2} e^{\left(\frac{\sigma^2}{2}r^2 - \sigma\beta tr + \frac{\sigma^2\beta_0}{\beta}r + \frac{\beta^2}{2}t^2 - \sigma\beta_0 t\right)} \times \left\{ 1 + \tilde{R}_0^2\beta^2\sigma^2 \left[\frac{\sigma^2}{\beta^2} \left(\beta r - \frac{\beta^2 t}{\sigma} + \beta_0 \right)^2 - 1 \right] e^{2\left(\frac{\sigma^2}{2}r^2 - \sigma\beta tr + \frac{\sigma^2\beta_0}{\beta}r + \frac{\beta^2}{2}t^2 - \sigma\beta_0 t\right)} \right\}, \quad (142)$$

$$T(t, r) = \frac{f(t)}{\left(\beta r - \frac{\beta^2}{\sigma}t + \beta_0\right)} + \frac{\sigma^3}{12\beta^2\pi\kappa} \left(\beta r - \frac{\beta^2}{\sigma}t + \beta_0\right)^2 - \frac{\tau\sigma^2}{4\pi\kappa}. \quad (143)$$

It can be easily verified that none of the above solutions can be matched smoothly on either of the boundary surfaces.

8. Discussion and Conclusions

We have presented a general approach to describe the dynamics of hyperbolically symmetric fluids, including dissipative processes. Although our main motivation was (and still is) to provide a formalism allowing us to study the dynamic regime leading to a static source of the line element (1), the obtained results are sufficiently general as to be applied to any other scenario where we expect hyperbolic symmetry to play a relevant role.

The four more remarkable features of hyperbolically symmetric fluids are as follows:

1. The energy density is necessarily negative.
2. The fluid cannot fill the central region.
3. The Tolman condition for thermodynamic equilibrium implies in this case the presence of a positive temperature gradient.
4. The thermal modification of the inertial mass density reported for the spherically symmetric case in [24], produces an effect that is similar to the one obtained in this latter case (to enhance the tendency to expansion) but comes about through different terms in the equation.

It should be reminded that the first two properties are common to the static and the dynamic regimes.

With respect to the violation of the weak energy condition ($\mu < 0$) it should be stressed that while it is true that at the classic level we do not expect negative energy density in a realistic fluid, the situation is quite different in the quantum regime, where the appearance of negative energy density is possible (see [36–40] and references therein). This confirms our believe that the type of fluids considered in this manuscript might be useful for studying systems where quantum effects are expected to be relevant.

As mentioned in Section 3, this negative energy density implies the appearance of a repulsive gravitational force, which has two important thermodynamic consequences mentioned in point 3 above.

Next, the impossibility of the fluid distribution to fill the central region leaves several possible scenarios. We lean to assume the existence of an empty vacuole surrounding the center; however, many other scenarios may be regarded as well, such as filling the central region with a fluid endowed with a different type of symmetry. At any rate, this impossibility is consistent with the result obtained in [3], according to which test particles are not allowed to reach the center for the line element (1).

The final description of the central region, as well as the fulfillment or not of the Dar-mois conditions at both interfaces, would depend on the specific system under consideration.

After having deployed the set of equations for describing the dynamics of hyperbolically symmetric fluids, we have presented several exact solutions. These were found under the condition of the vanishing complexity factor defined in [26] ($Y_{TF} = 0$) and the quasi-homologous evolution defined by (59).

We first considered the non-dissipative case. Two exact solutions were found for this case. One of them (88)–(95) describes a fluid distribution satisfying conditions $Y_{TF} = \mathcal{E} = \sigma = 0 = \Pi = \mu' = 0$, which is a reminiscence of the usual FRW space–time. However, unlike the latter it is not geodesic. If we impose the geodesic condition, then the quasi-homologous condition becomes homologous, and the solution is described by (99)–(103). This is a geodesic fluid, satisfying also the conditions $Y_{TF} = \mathcal{E} = \sigma = 0$, and therefore is also a good candidate to be regarded as the hyperbolic version of the FRW space–time; however, unlike the latter, it is anisotropic in the pressure and inhomogeneous in the energy density.

In both cases, if the arbitrary functions appearing in the solutions are chosen such that the system tends to a static situation in the limit $t \rightarrow \infty$, then these solutions tend to the static solutions studied in [1].

Thus, alternative cosmological models emerge from the study of hyperbolically symmetric fluids, which could be of interest when seeking for more sophisticated models of the universe (see for example [41] and references therein).

Finally, we have considered the dissipative case. In order to obtain specific models we have restricted ourselves to the case where the condition $B = 1$ is satisfied. Such a condition is suggested by the fact that it appears to be suitable for the description of fluids whose central region is surrounded by a vacuum cavity [35]. The purpose of these solutions, as well as the non-dissipative ones, is not the modeling of any specific astrophysical scenario, but just to illustrate a possible way of finding solutions, some of which might be used for the modeling of hyperbolically symmetric fluids required for describing specific physical situations. Neither of the exhibited models matches smoothly on the boundary surfaces. In order to obtain models satisfying Dar-mois conditions, one could try to extend the general methods developed for the spherically symmetric case in [42–45] to the hyperbolically symmetric case.

In the temperature profiles exhibited for each solution we may identify two types of contributions: the contribution in the stationary dissipative regime (not containing τ) and the contribution from the transient regime (terms proportional to τ).

Author Contributions: All authors contributed equally to this work. Conceptualization, L.H.; methodology, L.H., A.D.P. and J.O.; software, J.O.; formal analysis, L.H., A.D.P. and J.O.; writing—original draft preparation, L.H. writing—review and editing, L.H., A.D.P. and J.O.; funding acquisition, L.H., J.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Ministerio de Ciencia, Innovación y Universidades. Grant number: PGC2018–096038–B–I00, and Junta de Castilla y León. Grant number: SA096P20.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

Appendix A. Conservation Laws $T_{\nu;\mu}^\mu = 0$

In our case the conservation laws have only two independent components which read

$$\dot{\mu} + (\mu + P_r) \frac{\dot{B}}{B} + 2(\mu + P_\perp) \frac{\dot{R}}{R} + q' \frac{A}{B} + 2q \frac{A}{B} \left(\frac{A'}{A} + \frac{R'}{R} \right) = 0, \quad (\text{A1})$$

and

$$P_r' + (\mu + P_r) \frac{A'}{A} + 2(P_r - P_\perp) \frac{R'}{R} + \dot{q} \frac{B}{A} + 2q \frac{B}{A} \left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R} \right) = 0. \quad (\text{A2})$$

Using (12) and (11) we may write

$$D_T U = \frac{m}{R^2} - 4\pi R P_r + aE, \quad (\text{A3})$$

$$D_R \left(\frac{U}{R} \right) = \frac{4\pi q}{E} + \frac{\sigma}{R'}, \quad (\text{A4})$$

which allows to rewrite (A1) and (A2) as

$$D_T \mu + \frac{1}{3}(3\mu + P_r + 2P_\perp)\Theta + \frac{2}{3}(P_r - P_\perp)\sigma + ED_R q + 2q \left(a + \frac{E}{R} \right) = 0, \quad (\text{A5})$$

and

$$ED_R P_r + (\mu + P_r)a + 2(P_r - P_\perp) \frac{E}{R} + D_T q + \frac{2}{3}q(2\Theta + \sigma) = 0. \quad (\text{A6})$$

Finally, combining (A3) with (A6) we find

$$(\mu + P_r)D_T U = -(\mu + P_r)(4\pi R^3 P_r - m) \frac{1}{R^2} - E^2 \left[D_R P_r + \frac{2}{R}(P_r - P_\perp) \right] - E \left[D_T q + \frac{2}{3}q(2\Theta + \sigma) \right]. \quad (\text{A7})$$

The above equation may be transformed further by replacing (45) in (A7), and using (A3)

$$\left(\mu + P_r - \frac{\kappa T}{\tau} \right) D_T U = - \left(\mu + P_r - \frac{\kappa T}{\tau} \right) (4\pi R^3 P_r - m) \frac{1}{R^2} - E^2 \left[D_R P_r + \frac{2}{R}(P_r - P_\perp) - \frac{\kappa}{\tau} D_R T \right] + E q \left[\frac{1}{\tau} + \frac{1}{2} D_T \ln \left(\frac{\tau}{\kappa T^2} \right) - \frac{5}{6} \Theta - \frac{2}{3} \sigma \right] \quad (\text{A8})$$

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