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Group Structure and Geometric Interpretation of the Embedded Scator Space

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Abstract: The set of scators was introduced by Fernández-Guasti and Zaldívar in the context of special relativity and the deformed Lorentz metric. In this paper, the scator space of dimension $1+n$ (for $n = 2$ and $n = 3$) is interpreted as an intersection of some quadrics in the pseudo-Euclidean space of dimension 2^n with zero signature. The scator product, nondistributive and rather counterintuitive in its original formulation, is represented as a natural commutative product in this extended space. What is more, the set of invertible embedded scators is a commutative group. This group is isomorphic to the group of all symmetries of the embedded scator space, i.e., isometries (in the space of dimension 2^n) preserving the scator quadrics.

Keywords: scators; fundamental embedding; orthogonal groups; quadrics; Lorentz velocity addition formula

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1. Introduction

The scator algebra was introduced by Fernández-Guasti and Zaldívar in a series of papers, starting from [1]. The elliptic case can be considered as yet another approach to hypercomplex numbers [2] with the corresponding theory of holomorphic functions [3,4], while the hyperbolic case has potential physical applications, usually related to deformations and generalizations of Lorentz symmetries of the special theory of relativity [5,6]; see also [7,8]. In this paper, we confine ourselves to the hyperbolic case, closely related to a specific deformation of the Lorentz symmetry. To be more precise, we consider a real linear space $\mathbb{R}^{1+n}$ with a fixed basis of unit vectors: $\overset{o}{\boldsymbol{e}}_0, \overset{o}{\boldsymbol{e}}_1, \ldots, \overset{o}{\boldsymbol{e}}_n$ (their squares are assumed to be $+1$). An element $\overset{o}{a} \in \mathbb{R}^{1+n}$ is denoted as:

$$\overset{o}{a} = (a_0;\ a_1,\ a_2,\ \ldots,\ a_n) = a_0\overset{o}{\boldsymbol{e}}_0 + a_1\overset{o}{\boldsymbol{e}}_1 + \ldots + a_n\overset{o}{\boldsymbol{e}}_n = a_0 + a_1\overset{o}{\boldsymbol{e}}_1 + \ldots + a_n\overset{o}{\boldsymbol{e}}_n \tag{1}$$

where $a_0, a_1, \ldots, a_2$ are real numbers (a_0 is a scalar component, and $a_1, \ldots, a_n$ are referred to as director components). The unit scalar $\overset{o}{\boldsymbol{e}}_0$ is usually omitted; compare the last equality of (1). The decomposition into scalar and director components is crucial for many properties of scators, including their characteristic nondistributive multiplication.

Scators form a large subset of $\mathbb{R}^{1+n}$ (denoted by $\mathbb{S}^{1+n}$), which consists of all elements with a nonvanishing scalar component and lines along director components. In other words,

$$\mathbb{S}^{1+n} = \mathbb{S}^{1+n}_* \cup \{\overset{o}{a} \in \mathbb{R}^{1+n} : \exists_k\ (\overset{o}{a} = a_k\overset{o}{e}_k)\}\ , \tag{2}$$

where:

$$\mathbb{S}^{1+n}_* := \{\overset{o}{a} \in \mathbb{R}^{1+n} : a_0 \neq 0\}\ . \tag{3}$$

Note that in our earlier papers, $\mathbb{S}^{1+n}_*$ was usually denoted by S'; see [9,10].

Definition 1. *The scator product of two scators,* $\overset{o}{a} = (a_0; a_1, \ldots, a_n) \in \mathbb{S}^{1+n}$ *and* $\overset{o}{b} = (b_0; b_1, \ldots, b_n) \in \mathbb{S}^{1+n}$*, is denoted by* $\overset{o}{a}\overset{o}{b} \equiv (c_0;\ c_1, \ldots, c_n)$*. In the hyperbolic case, it is defined as follows* [11,12]*:*

- *For* $\overset{o}{a}, \overset{o}{b} \in \mathbb{S}_*^{1+n}$,

$$
\begin{aligned}
c_0 &= a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0}\right)\left(1 + \frac{a_2 b_2}{a_0 b_0}\right) \ldots \left(1 + \frac{a_n b_n}{a_0 b_0}\right) \equiv a_0 b_0 \prod_{j=1}^{n} \left(1 + \frac{a_j b_j}{a_0 b_0}\right) , \\
c_k &= a_0 b_0 \left(\frac{a_k}{a_0} + \frac{b_k}{b_0}\right) \prod_{\substack{j=1 \\ j \neq k}}^{n} \left(1 + \frac{a_j b_j}{a_0 b_0}\right) \qquad (k = 1, \ldots, n) .
\end{aligned}
\tag{4}
$$

- *Other cases are as follows:*

$$
\begin{aligned}
&\left(a_k \overset{o}{\boldsymbol{e}}_k\right)\left(b_0 + \sum_{j=1}^{n} b_j \overset{o}{\boldsymbol{e}}_j\right) = a_k b_k + b_0 a_k \overset{o}{\boldsymbol{e}}_k + \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{a_k b_k b_j}{b_0}\right) \overset{o}{\boldsymbol{e}}_j , \\
&\left(a_k \overset{o}{\boldsymbol{e}}_k\right)\left(b_j \overset{o}{\boldsymbol{e}}_j\right) = a_k b_j \delta_{kj} ,
\end{aligned}
\tag{5}
$$

where δ_{kj} *is Kronecker's delta.*

The above definition implies the commutativity of the scator product. Another useful property is the compatibility of the scator product with the dilation, i.e.,

$$
(\lambda \overset{o}{a})(\mu \overset{o}{b}) = (\lambda \mu)(\overset{o}{a}\overset{o}{b}) \qquad (\text{for } \lambda, \mu \in \mathbb{R}) \tag{6}
$$

The scator product is nondistributive (i.e., usually, $(\overset{o}{x} + \overset{o}{y})\overset{o}{z} \neq \overset{o}{x}\overset{o}{z} + \overset{o}{y}\overset{o}{z}$) and nonassociative (although the associativity holds if all involved factors and their products have nonvanishing scalar components). We point out that scator multiplication admits zero divisors, for instance: the scator products of $(a_0,\ a_1,\ a_0)$ and $(b_0,\ b_1,\ -b_0)$ are equal to zero.

The hypercomplex conjugation of scators maps $\overset{o}{\boldsymbol{e}}_k$ into $-\overset{o}{\boldsymbol{e}}_k$, namely:

$$
\overset{o}{a}{}^* := (a_0, -a_1, \ldots, -a_n) \tag{7}
$$

Similarly, as in the case of the complex numbers, the scator norm is defined by a scator product of the scator and its hypercomplex conjugate:

$$
\|\overset{o}{a}\|^2 = \overset{o}{a}\overset{o}{a}{}^* \tag{8}
$$

Then, the application of Definition 1 yields:

$$
\begin{aligned}
&\|\overset{o}{a}\|^2 = a_0^2 \prod_{k=1}^{n} \left(1 - \frac{a_k^2}{a_0^2}\right) \qquad (\text{if } a_0 \neq 0) , \\
&\|a_k \overset{o}{\boldsymbol{e}}_k\|^2 = -a_k^2 .
\end{aligned}
\tag{9}
$$

In particular, for large $|a_0|$, the scator norm becomes close to the standard Minkowski metric, which is one of main reasons for considering scators as a peculiar extension of special relativity:

$$
\|\overset{o}{a}\|^2 \approx a_0^2 - \sum_{k=1}^{n} a_k^2 \qquad (\text{if } a_0^2 \gg \sum_{k=1}^{n} a_k^2) . \tag{10}
$$

The definition of the scator product presented above is far from being obvious or natural. The first issue we would like to address in the next section is a clear and intuitive motivation for Definition 1.

In our recent paper, we proposed an extension of the scator product in $\mathbb{S}^{1+2}$ on the whole space $\mathbb{R}^{1+2}$ [13]. Here, we follow an alternative path. We show that not only the definition of the scator product, but also its domain (2) is, in a sense, natural.

2. Fundamental Embedding

Our main tool to understand the scator product and scator geometry is the so-called fundamental embedding, introduced in [9]; see also [10,13]. The fundamental embedding F maps the scator space $\mathbb{S}^{1+n}$ into $\mathbb{A}^{1,n}$, where the space $\mathbb{A}^{1,n}$ is the algebra over $\mathbb{R}$ generated (using addition and multiplication, which is assumed to be commutative, associative, and distributive over addition) by elements $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ satisfying (in the hyperbolic case):

$$\boldsymbol{e}_k\boldsymbol{e}_k = 1 , \qquad \boldsymbol{e}_k\boldsymbol{e}_j = \boldsymbol{e}_j\boldsymbol{e}_k \quad (1 \leqslant j < k \leqslant n) . \tag{11}$$

The basis elements of $\mathbb{A}^{1,n}$ are of the form $\boldsymbol{e}_J$, where J is a multi-index defined by a subset $\{i_1, \ldots, i_m\}$ of $\{1, 2, \ldots, n\}$, i.e., $\boldsymbol{e}_J \equiv \boldsymbol{e}_{i_1 \ldots i_m} \equiv \boldsymbol{e}_{i_1} \ldots \boldsymbol{e}_{i_m}$. In particular, we have the unit element $\boldsymbol{e}_\emptyset \equiv \boldsymbol{e}_0 \equiv 1$, vectors, bivectors, multivectors, and the element $\boldsymbol{e}_{max} = \boldsymbol{e}_1\boldsymbol{e}_2 \ldots \boldsymbol{e}_n$:

$$1, \quad \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n , \quad \boldsymbol{e}_{jk} \equiv \boldsymbol{e}_j\boldsymbol{e}_k \ (j < k) , \ldots , \quad \boldsymbol{e}_{i_1 \ldots i_m} \equiv \boldsymbol{e}_{i_1} \ldots \boldsymbol{e}_{i_m} \ (i_1 < \ldots < i_m) , \ldots , \quad \boldsymbol{e}_{max} . \tag{12}$$

A linear space $\mathbb{A}^{1,n}$ is isomorphic to Clifford and Grassmann algebras (although the multiplicative structures of all these spaces are totally different) [14,15].

In order to the motivate definition of the fundamental embedding, which appears at the end of this section (see Definition 2), we introduce some useful notions. First, we denote by π the natural projection of the vector space $\mathbb{A}^{1,n}$ on the space spanned by $1, \overset{o}{\boldsymbol{e}}_1, \ldots, \overset{o}{\boldsymbol{e}}_n$ (compare (1)):

$$\pi\left(\sum_J a_J \boldsymbol{e}_J\right) := a_0 + \sum_{k=1}^n a_k \overset{o}{\boldsymbol{e}}_k . \tag{13}$$

We consider the following subset of $A^{1,n}$:

$$\tilde{\mathbb{S}}^{1+n} := \{\boldsymbol{a} \in \mathbb{A}^{1,n} : \boldsymbol{a} = \prod_{k=1}^n (a_{0k} + a_{1k}\boldsymbol{e}_k)\} , \tag{14}$$

where $a_{01}, a_{02}, \ldots, a_{0n}$ and $a_{11}, a_{12}, \ldots, a_{1n}$ are real parameters. Then, for $\boldsymbol{a} \in \tilde{\mathbb{S}}^{1+n}$, $\pi(\boldsymbol{a})$ is of the form (13), where:

$$a_0 := \prod_{k=1}^n a_{0k} , \qquad a_k := a_{1k} \prod_{\substack{j=1 \\ j \neq k}}^n a_{0j} . \tag{15}$$

We denote also:

$$\tilde{\mathbb{S}}_*^{1+n} := \{\boldsymbol{a} \in \tilde{\mathbb{S}}^{1+n} : a_0 \neq 0\} , \tag{16}$$

where, obviously, $a_0 = a_{01}a_{02} \ldots a_{0n}$ for $\boldsymbol{a} \in \tilde{\mathbb{S}}^{1+n}$.

Theorem 1. *There exists a one-to-one correspondence between $\pi(\tilde{\mathbb{S}}^{1+n})$ and $\mathbb{S}^{1+n}$ and a one-to-one correspondence between $\tilde{\mathbb{S}}_*^{1+n}$ and $\mathbb{S}_*^{1+n}$.*

Proof. Let us take $\boldsymbol{a} \in \tilde{\mathbb{S}}^{1+n}$; see (14). We have two cases. First, $a_0 \neq 0$, which implies $a_{0k} \neq 0$ for $k = 1, \ldots, n$. In this case:

$$a_k = \frac{a_{1k}}{a_{0k}} a_0 , \tag{17}$$

and we can rewrite $\boldsymbol{a} \in \tilde{\mathbb{S}}_*^{1+n}$ as:

$$\boldsymbol{a} = a_0 \prod_{k=1}^{n} \left(1 + \frac{a_{1k}}{a_{0k}} \boldsymbol{e}_k\right) = a_0 \prod_{k=1}^{n} \left(1 + \frac{a_k}{a_0} \boldsymbol{e}_k\right) . \tag{18}$$

Therefore, any element $\boldsymbol{a} \in \tilde{\mathbb{S}}_*^{1+n}$ is uniquely defined by its projection $\pi(\boldsymbol{a})$; see (15) and (18). Hence, there is a bijection between $\tilde{\mathbb{S}}_*^{1+n}$ and $\pi(\tilde{\mathbb{S}}_*^{1+n})$. Moreover, obviously, we can identify $\pi(\tilde{\mathbb{S}}_*^{1+n})$ with $\mathbb{S}_*^{1+n}$. In other words, to any scator $\overset{o}{a} = (a_0, a_1, \ldots, a_n)$ with a nonvanishing scalar component, there corresponds exactly one element $\boldsymbol{a} \in \tilde{\mathbb{S}}_*^{1+n}$ such that $\pi(\boldsymbol{a}) = \overset{o}{a}$. Thus, a one-to-one correspondence between $\tilde{\mathbb{S}}_*^{1+n}$ and $\mathbb{S}_*^{1+n}$ is shown.

In the second case ($a_0 = 0$), the situation is more complicated. Note that $a_0 = 0$ if and only if there exists m such that $a_{0m} = 0$. Then, as a consequence,

$$\prod_{\substack{j=1 \\ j \neq k}}^{n} a_{0j} = 0 \quad \text{for } \; k \neq m \, , \tag{19}$$

and due to (15), $\pi(\boldsymbol{a})$ reduces to:

$$\pi(\boldsymbol{a}) = 0 + a_{1m} \overset{o}{\boldsymbol{e}}_m \prod_{\substack{j=1 \\ j \neq m}}^{n} a_{0j} = a_m \overset{o}{\boldsymbol{e}}_m \, . \tag{20}$$

Thus, for $a_0 \neq 0$, $\pi(\boldsymbol{a})$ has to be proportional to $\overset{o}{\boldsymbol{e}}_m$. We point out that if $a_{0j} = 0$ for any $j \neq m$, then a_m vanishes, and as a consequence, $\pi(\boldsymbol{a}) = 0$. Therefore, the case $a_0 \neq 0$ corresponds to the second part of the scator set (2), which ends the proof. $\square$

A one-to-one correspondence between $\tilde{\mathbb{S}}_*^{1+n}$ and $\mathbb{S}_*^{1+n}$ is realized by the projection π. What is more, the projection π maps the multiplicative structure of $\tilde{\mathbb{S}}_*^{1+n}$ into the scator multiplication. In a sense, this fact can be treated as a derivation of the scator product.

Theorem 2. *The multiplication in the space $\tilde{\mathbb{S}}_*^{1+n}$ (induced from the natural commutative product in $\mathbb{A}^{1,n}$), mapped by the projection π, yields the scator product in the scator space $\mathbb{S}_*^{1+n}$; see (4). In other words,*

$$\pi(\boldsymbol{a})\pi(\boldsymbol{b}) := \pi(\boldsymbol{ab}) \qquad (\textit{for any } \; \boldsymbol{a}, \boldsymbol{b} \in \tilde{\mathbb{S}}_*^{1+n}) \, , \tag{21}$$

can be treated as a definition of the scator product for elements from the space $\mathbb{S}_*^{1+n}$.

Proof. Elements of $\mathbb{S}_*^{1+n}$ are given by Formula (18). Straightforward computation yields:

$$\boldsymbol{ab} = a_0 \prod_{k=1}^{n} \left(1 + \frac{a_k}{a_0} \boldsymbol{e}_k\right) b_0 \prod_{k=1}^{n} \left(1 + \frac{b_k}{b_0} \boldsymbol{e}_k\right) = a_0 b_0 \prod_{k=1}^{n} \left(1 + \frac{a_k b_k}{a_0 b_0} + \left(\frac{a_k}{a_0} + \frac{b_k}{b_0}\right) \boldsymbol{e}_k\right) . \tag{22}$$

Then, the coefficients of $\pi(\boldsymbol{ab})$ by $1, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ yield the Formula (4), which ends the proof. $\square$

The scator product is defined for all scators with a nonvanishing scalar component. However, as shown below, the result of this multiplication can be outside of this set.

The projection π is not an isomorphism between $\tilde{\mathbb{S}}_*^{1+n}$ and $\mathbb{S}_*^{1+n}$, because neither $\tilde{\mathbb{S}}_*^{1+n}$ nor $\mathbb{S}_*^{1+n}$ are closed with respect to the multiplication. In the one-dimensional case, we have:

$$a_0 \left(1 + \frac{a_1}{a_0} \boldsymbol{e}_1\right) b_0 \left(1 + \frac{b_1}{b_0} \boldsymbol{e}_1\right) = a_0 b_0 \left(1 + \frac{a_1 b_1}{a_0 b_0} + \left(\frac{a_1}{a_0} + \frac{b_1}{b_0}\right) \boldsymbol{e}_1\right) . \tag{23}$$

Therefore, in the case $a_1 b_1 = -a_0 b_0$, the above result is proportional to $\boldsymbol{e}_1$, and as a consequence, it does not belong to $\mathbb{S}_*^{1+n}$. In the general case (22), the situation is analogous. If $a_m b_m = -a_0 b_0$, then $\boldsymbol{ab}$ is proportional to $\boldsymbol{e}_m$:

$$\boldsymbol{ab} = (b_0 a_m - a_0 b_m)\boldsymbol{e}_m \prod_{\substack{k=1\\k\neq m}}^{n} \left(1 + \frac{a_k b_k}{a_0 b_0} + \left(\frac{a_k}{a_0} + \frac{b_k}{b_0}\right)\boldsymbol{e}_k\right) . \tag{24}$$

Finally, we would like to address the problem of extending Theorem 2 on all elements of $\tilde{\mathbb{S}}^{1+n}$, including elements of the form (24). Note that:

$$\pi\left(\boldsymbol{e}_k \prod_{\substack{j=1\\j\neq k}}^{n} (1 + \alpha_j \boldsymbol{e}_j)\right) = \overset{o}{\boldsymbol{e}}_k . \tag{25}$$

for any values of $n-1$ parameters α_j ($j \neq k$), which means that the preimage of $\boldsymbol{e}_k$ under π is very large.

Fortunately enough, choosing the simplest element in this preimage, namely $\boldsymbol{e}_k$, we obtain the required extension of Theorem 2. In other words, we embed the scator space $\mathbb{S}^{n+1}$ into $\mathbb{S}^{n+1} \subset A^{1,n}$ in the way leading uniquely to the scator product of Definition 1.

Definition 2. *The fundamental embedding $F : \mathbb{S}^{1+n} \to \mathbb{A}^{1,n}$ is defined as:*

$$\begin{aligned} F(\overset{o}{a}) &= a_0 \prod_{k=1}^{n} \left(1 + \frac{a_k}{a_0}\boldsymbol{e}_k\right) \qquad (\textit{if } \ a_0 \neq 0) , \\ F(a_k \overset{o}{\boldsymbol{e}}_k) &= a_k \boldsymbol{e}_k \qquad (k = 1, \dots, n) . \end{aligned} \tag{26}$$

Corollary 1. *For any a_0 and a_k, we have:*

$$F(a_0 + a_k \overset{o}{\boldsymbol{e}}_k) = a_0 + a_k \boldsymbol{e}_k . \tag{27}$$

Remark 1. *F is a bijection between $\mathbb{S}^{1+n}_*$ and $\tilde{\mathbb{S}}^{1+n}_*$. What is more, π restricted to $\tilde{\mathbb{S}}^{1+n}_*$ coincides with F^{-1}, which means that in this case, $F \circ \pi = \mathrm{id}$.*

The main advantage of the fundamental embedding is a natural motivation for the definition of the scator product proposed by Fernández-Guasti. (Definition 1). Indeed, we can present the following alternative definition of the scator product; see [9].

Theorem 3. *The formula:*

$$\overset{o}{a}\overset{o}{b} = \pi\left(F(\overset{o}{a})F(\overset{o}{b})\right) , \tag{28}$$

is equivalent to Definition 1 of the scator product.

Proof. The first part of Definition 1, given by Equation (4), follows from (28) directly by Theorem 2. The second part, given by Equation (5), can be directly computed, as follows.

$$\pi\left(F\left(a_k \overset{o}{\boldsymbol{e}}_k\right) F\left(b_0 + \sum_{j=1}^{n} b_j \overset{o}{\boldsymbol{e}}_j\right)\right) = \pi\left(a_k b_0 \boldsymbol{e}_k \prod_{j=1}^{n}\left(1 + \frac{b_j}{b_0}\boldsymbol{e}_j\right)\right) = a_k\left(b_k + b_0 \overset{o}{\boldsymbol{e}}_k + \sum_{\substack{j=1\\j\neq k}}^{n} \frac{b_k b_j \overset{o}{\boldsymbol{e}}_j}{b_0}\right), \tag{29}$$

where we took into account that:

$$\boldsymbol{e}_k \prod_{j=1}^{n}\left(1 + \frac{b_j}{b_0}\boldsymbol{e}_j\right) = \boldsymbol{e}_k + \sum_{j=1}^{n} \frac{b_j \boldsymbol{e}_j \boldsymbol{e}_k}{b_0} + \frac{1}{2}\sum_{j=1}^{n}\sum_{\substack{i=1\\i\neq j}}^{n} \frac{b_i b_j \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k}{b_0^2} + \dots = \boldsymbol{e}_k + \frac{b_k}{b_0} + \sum_{\substack{j=1\\j\neq k}}^{n} \frac{b_k b_j \boldsymbol{e}_j}{b_0^2} + \dots \tag{30}$$

In these computations, we used the relations (11). Note that the terms replaced by dots are bivectors or multivectors of higher order. Finally,

$$\pi\left(F\left(a_k\overset{o}{\boldsymbol{e}}_k\right)F\left(b_j\overset{o}{\boldsymbol{e}}_j\right)\right) = \pi(a_k b_j \boldsymbol{e}_k \boldsymbol{e}_j) = a_k b_j \delta_{kj} \,, \tag{31}$$

which ends the proof. □

Remark 2. *The scator product is, in general, nonassociative. Indeed,*

$$\begin{aligned}(\overset{o}{a}\overset{o}{b})\overset{o}{c} &= \pi\left(F(\overset{o}{a})F(\overset{o}{b})\right)\overset{o}{c} = \pi\left(F\left(\pi\left(F(\overset{o}{a})F(\overset{o}{b})\right)\right)F(\overset{o}{c})\right) = \pi\left(F\circ\pi\left(F(\overset{o}{a})F(\overset{o}{b})\right)F(\overset{o}{c})\right), \\ \overset{o}{a}(\overset{o}{b}\overset{o}{c}) &= \overset{o}{a}\,\pi\left(F(\overset{o}{b})F(\overset{o}{c})\right) = \pi\left(F(\overset{o}{a})F\left(\pi\left(F(\overset{o}{b})F(\overset{o}{c})\right)\right)\right) = \pi\left(F(\overset{o}{a})\;F\circ\pi\left(F(\overset{o}{b})F(\overset{o}{c})\right)\right).\end{aligned} \tag{32}$$

Both expressions would be identical, equating $\pi\left(F(\overset{o}{a})F(\overset{o}{b})F(\overset{o}{c})\right)$, provided that $F\circ\pi = \mathrm{id}$. The last equality is true only when restricted to $\tilde{\mathbb{S}}_*^{1+n}$; compare Remark 1. For instance, we have:

$$\begin{aligned}(\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_1)\overset{o}{\boldsymbol{e}}_2 &= \pi(\boldsymbol{e}_1\boldsymbol{e}_1)\,\overset{o}{\boldsymbol{e}}_2 = \overset{o}{\boldsymbol{e}}_2\,, \\ \overset{o}{\boldsymbol{e}}_1(\overset{o}{\boldsymbol{e}}_1\overset{o}{\boldsymbol{e}}_2) &= \overset{o}{\boldsymbol{e}}_1\pi(\boldsymbol{e}_1\boldsymbol{e}_2) = 0\,.\end{aligned} \tag{33}$$

In this case, we see clearly that $\pi(\boldsymbol{e}_1\boldsymbol{e}_1)\pi(\boldsymbol{e}_2) = \pi(\boldsymbol{e}_1\boldsymbol{e}_1\boldsymbol{e}_2) \neq \pi(\boldsymbol{e}_1)\pi(\boldsymbol{e}_1\boldsymbol{e}_2)$.

Nonassociativity is usually related to the quantum aspects of physical systems (see [16–18]). An attempt to involve quantum effects has been made also in the case of scators [19].

3. Group Structure of the Embedded Scator Space

The set $\tilde{\mathbb{S}}^{1+n}$ is closed under multiplication. Indeed, for any $\boldsymbol{a},\boldsymbol{b} \in \tilde{\mathbb{S}}^{1+n}$, we have:

$$\boldsymbol{ab} = \prod_{k=1}^{n}((a_{0k} + a_{1k}\boldsymbol{e}_k)(b_{0k} + b_{1k}\boldsymbol{e}_k)) = \prod_{k=1}^{n}(a_{0k}b_{0k} + a_{1k}b_{1k} + (a_{0k}b_{1k} + a_{1k}b_{0k})\boldsymbol{e}_k)\,. \tag{34}$$

In order to determine the group structure, we have to consider the invertibility of the elements of $\tilde{\mathbb{S}}^{1+n}$. As usual, conjugate elements (compare (7)) are useful in this context. Hypercomplex conjugation at the level of the space $A^{1,n}$ is realized by the reflection $\boldsymbol{e}_k \to -\boldsymbol{e}_k$ ($k = 1,\dots,n$). Thus:

$$\boldsymbol{aa}^* = \prod_{j=1}^{n}((a_{0j} + a_{1j}\boldsymbol{e}_j)(a_{0j} - a_{1j}\boldsymbol{e}_j)) = \prod_{j=1}^{n}(a_{0j}^2 - a_{1j}^2) \tag{35}$$

Hence, if $|a_{0j}| \neq |a_{1j}|$ for all $j = 1,\dots,n$, then:

$$\boldsymbol{a}^{-1} = \frac{\boldsymbol{a}^*}{\prod_{j=1}^{n}(a_{0j}^2 - a_{1j}^2)}\,. \tag{36}$$

The following elementary equality is very helpful.

$$\varepsilon^2 = 1 \quad\Longrightarrow\quad a_{0k}b_{0k} + a_{1k}b_{1k} - \varepsilon(a_{0k}b_{1k} + a_{1k}b_{0k}) \equiv (a_{0k} - \varepsilon_k a_{1k})(b_{0k} - \varepsilon b_{1k}) \tag{37}$$

Therefore, the product of an element proportional to $1 \pm \boldsymbol{e}_k$ (i.e., such that $a_{0k} - \varepsilon_k a_{1k} = 0$) by any element of the set $\tilde{\mathbb{S}}^{1+n}$ is proportional to $1 \pm \boldsymbol{e}_k$, as well.

Corollary 2. *An element of $\tilde{\mathbb{S}}^{1+n}$ is noninvertible if and only if it is proportional to $1 \pm \boldsymbol{e}_k$ for at least one value of k ($1 \leqslant k \leqslant n$).*

The next corollary is another direct consequence of (37).

Corollary 3. *The product of two invertible elements is invertible. Invertible elements of* $\tilde{\mathbb{S}}^{1+n}$ *form a multiplicative group, which is denoted by* $\tilde{\mathbb{S}}^{1+n}_{inv}$.

Theorem 4. *The set defined by:*

$$\tilde{\mathbb{S}}^{1+n}_{+} := \{\boldsymbol{a} \in \mathbb{A}^{1,n} : \boldsymbol{a} = a_0 \prod_{k=1}^{n} (1 + \alpha_k \boldsymbol{e}_k) \,, \quad a_0 > 0 \,, \quad |\alpha_k| < 1 \text{ for } k = 1, \ldots, n\} \,, \tag{38}$$

is a commutative, simply connected, multiplicative group.

Proof. We compute the product of two elements of the form of (38) (parameters corresponding to the second element are marked with a prime).

$$\boldsymbol{a}\boldsymbol{a}' = a_0 a_0' \prod_{k=1}^{n} (1 + \alpha_k \boldsymbol{e}_k)(1 + \alpha_k' \boldsymbol{e}_k) = a_0 a_0' \prod_{k=1}^{n} (1 + \alpha_k \alpha_k' + (\alpha_k + \alpha_k')\boldsymbol{e}_k) . \tag{39}$$

It is convenient to use a bijection:

$$\mathbb{R} \ni \vartheta_k \quad \longleftrightarrow \quad \alpha_k = \tanh \vartheta_k \in (-1, 1) \,. \tag{40}$$

Then:

$$\boldsymbol{a}\boldsymbol{a}' = a_0 a_0' \prod_{k=1}^{n} (1 + \alpha_k \alpha_k') \prod_{k=1}^{n} \left(1 + \frac{\alpha_k + \alpha_k'}{1 + \alpha_k \alpha_k'} \boldsymbol{e}_k\right), \tag{41}$$

where:

$$\frac{\alpha_k + \alpha_k'}{1 + \alpha_k \alpha_k'} = \frac{\tanh \vartheta_k + \tanh \vartheta_k'}{1 + \tanh \vartheta_k \, \tanh \vartheta_k'} = \tanh(\vartheta_k + \vartheta_k') \,. \tag{42}$$

Taking into account that:

$$a_0 a_0' \prod_{k=1}^{n} (1 + \alpha_k \alpha_k') > 0 \quad \text{and} \quad |\tanh(\vartheta_k + \vartheta_k')| < 1 \,, \tag{43}$$

we conclude that $\boldsymbol{a}\boldsymbol{a}'$ is of the form of (38).

The inverse element always exists and is computed as a special case of (36):

$$\boldsymbol{a}^{-1} = \frac{\sum_{k=1}^{n} (1 - \alpha_k \boldsymbol{e}_k)}{a_0 \prod_{k=1}^{n} \left(1 - \alpha_k^2\right)} . \tag{44}$$

Simple connectedness follows immediately if we consider the following homotopy:

$$\boldsymbol{a}(t) = (1 - t + t a_0) \prod_{k=1}^{n} (1 + t \alpha_k \boldsymbol{e}_k) \,, \tag{45}$$

where $t \in [0, 1]$. □

Remark 3. *One can easily see that:*

$$\tilde{\mathbb{S}}^{1+n}_{+} = F(\mathbb{S}^{1+n}_{+}) \,, \tag{46}$$

where:

$$\mathbb{S}^{1+n}_{+} := \{\overset{0}{a} : \; a_0 > 0 \,, \; |a_k| < |a_0| \quad \text{for} \quad k = 1, \ldots, n\} \tag{47}$$

is closely related to the "restricted space subset" [11] *(scalar components of scators from the restricted subset can be both positive and negative).*

The bijection (40) suggests a convenient parameterization of the group $\tilde{\mathbb{S}}_+^{1+n}$ using the exponential representation. Indeed, taking into account (11), we compute:

$$\exp(\vartheta_k \boldsymbol{e}_k) = \sum_{j=0}^{\infty} \frac{(\vartheta_k)^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{(\vartheta_k)^{2j+1}}{(2j+1)!} \boldsymbol{e}_k = \cosh \vartheta_k + \boldsymbol{e}_k \sinh \vartheta_k \,. \tag{48}$$

In other words,

$$\exp(\vartheta_k \boldsymbol{e}_k) = \cosh \vartheta_k (1 + \tanh \vartheta_k \boldsymbol{e}_k) \,, \tag{49}$$

and finally,

$$1 + \alpha_k \boldsymbol{e}_k \equiv \frac{\exp(\vartheta_k \boldsymbol{e}_k)}{\cosh \vartheta_k} \qquad (\text{where} \quad \alpha_k = \tanh \vartheta_k). \tag{50}$$

Therefore,

$$\exp\left(\sum_{k=1}^{n} \vartheta_k \boldsymbol{e}_k \right) = \prod_{k=1}^{n} (\cosh \vartheta_k + \boldsymbol{e}_k \sinh \vartheta_k) = \left(\prod_{k=1}^{n} \cosh \vartheta_k \right) \prod_{k=1}^{n} (1 + \boldsymbol{e}_k \tanh \vartheta_k) \,. \tag{51}$$

Thus, any scator from $\tilde{\mathbb{S}}_+^{1+n}$ can be represented as:

$$\boldsymbol{a} = \frac{a_0}{\prod_{k=1}^{n} \cosh \vartheta_k} \exp\left(\sum_{k=1}^{n} \vartheta_k \boldsymbol{e}_k \right). \tag{52}$$

Theorem 5. *Any element $\boldsymbol{a} \in \tilde{\mathbb{S}}_{inv}^{1+n}$ can be represented as:*

$$\boldsymbol{a} = \pm \boldsymbol{e}_J \boldsymbol{a}_+ \tag{53}$$

where $\boldsymbol{a}_+ \in \tilde{\mathbb{S}}_+^{1+n}$ and J is a multi-index.

Proof. Given an element $\boldsymbol{a}$ of $\tilde{\mathbb{S}}_{inv}^{1+n}$, we use the following identity,

$$\boldsymbol{e}_j (1 + \alpha_j \boldsymbol{e}_j) = \alpha_j (1 + \alpha_j^{-1} \boldsymbol{e}_j) \quad (\text{for} \ \ \alpha_j \neq 0), \tag{54}$$

wherever the coefficient $\alpha_j > 1$. Thus, the element $\boldsymbol{a}$ can be expressed, up to the sign, as a product of some number of basis vectors (i.e., shortly, $\boldsymbol{e}_J$) multiplied by an element of $\tilde{\mathbb{S}}_+^{1+n}$. Hence, we obtain (53). □

4. Embedded Scator Space as an Intersection of Quadrics in a Higher-Dimensional Space

We showed that the set of scators, $\mathbb{S}^{1+n}$, is embedded in the space $A^{1,n}$ of dimension 2^n. In this section, we study the geometry of the embedding $\tilde{\mathbb{S}}^{1+n} \subset A^{1,n}$, assuming tacitly that the basis (12) is orthonormal (i.e., $A^{1,n} \simeq R^{2^n}$). The coordinates in the space R^{2^n} are denoted by x_J, where J is a multi-index.

The embedded scator space seems to consist of two parts. The first part, defined by the condition $x_0 \neq 0$, contains scators parameterized by $x_0, x_1, \ldots, x_n$ in the following way:

$$\boldsymbol{x} = x_0 \prod_{k=1}^{n} \left(1 + \boldsymbol{e}_k \frac{x_k}{x_0} \right) = x_0 + \sum_{i=1}^{n} x_i \boldsymbol{e}_i + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{x_i x_j}{x_0} \boldsymbol{e}_i \boldsymbol{e}_j + \sum_{i=1}^{k} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \frac{x_i x_j x_k}{x_0^2} \boldsymbol{e}_i \boldsymbol{e}_j \boldsymbol{e}_k + \ldots \,. \tag{55}$$

We denote by x_{ij} the coefficient of $\boldsymbol{x}$ by $\boldsymbol{e}_{ij} \equiv \boldsymbol{e}_i \boldsymbol{e}_j$, and in general, the coefficient of $\boldsymbol{x}$ by $\boldsymbol{e}_J$ is denoted by x_J. Thus:

$$x_{ij} = \frac{x_i x_j}{x_0} \,, \qquad x_{i_1 \ldots i_k} = \frac{x_{i_1} \ldots x_{i_k}}{x_0^{k-1}} \quad \text{for } k = 2, \ldots, n \,, \tag{56}$$

and we may shortly write down:

$$\boldsymbol{x} = \sum_J x_J \boldsymbol{e}_J \,. \tag{57}$$

The norm of $\boldsymbol{x}$ is identified with the norm of the corresponding scator (i.e., $\|\boldsymbol{x}\|^2 := \|\pi(\boldsymbol{x})\|^2$), but the formula $\|\boldsymbol{x}\|^2 = \boldsymbol{x}\boldsymbol{x}^*$ holds, as well:

$$\|\boldsymbol{x}\|^2 = \boldsymbol{x}\boldsymbol{x}^* = x_0^2 \prod_{k=1}^{n} \left(1 + \boldsymbol{e}_k \frac{x_k}{x_0}\right) \prod_{k=1}^{n} \left(1 - \boldsymbol{e}_k \frac{x_k}{x_0}\right) = x_0^2 \prod_{k=1}^{n} \left(1 - \frac{x_k^2}{x_0^2}\right). \tag{58}$$

Therefore,

$$\|\boldsymbol{x}\|^2 = x_0^2 - \sum_{k=1}^{n} x_k^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{x_i^2 x_j^2}{x_0^2} - \sum_{i=1}^{k} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} \frac{x_i^2 x_j^2 x_k^2}{x_0^4} + \ldots \,, \tag{59}$$

and taking into account (56),

$$\|\boldsymbol{x}\|^2 = x_0^2 - \sum_{k=1}^{n} x_k^2 + \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_{ij}^2 - \sum_{i=1}^{k} \sum_{j=i+1}^{n} \sum_{k=j+1}^{n} x_{ijk}^2 + \ldots = \sum_J (-1)^{|J|} x_J^2 \,, \tag{60}$$

where $|J|$ denotes the cardinality (number of elements) of the multi-index J.

Corollary 4. *The scator metric in S_*^{1+n} coincides with the pseudo-Euclidean metric (60) in the 2^n-dimensional space $A^{1,n}$. The metric (60) has signature zero.*

The second part of the embedded scator space, corresponding to $x_0 = 0$, apparently consists of n coordinate axes (lines along $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$). However, we suggest another interpretation (planes instead of lines), motivated by the following low-dimensional cases.

4.1. Scator Transformations for N = 2 as Isometries in a Four-Dimensional Space of Zero Signature

Let us consider the case $n = 2$. An element $\boldsymbol{x}$ of $A^{1,2}$, given by:

$$\boldsymbol{x} = x_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_{12}\boldsymbol{e}_1\boldsymbol{e}_2 \,, \tag{61}$$

is an embedded scator if and only if one of the following possibilities hold:

$$\begin{aligned} & x_{12} = \frac{x_1 x_2}{x_0} \,, \qquad x_0 \neq 0 \,, \\ & x_0 = x_1 = x_{12} = 0 \,, \\ & x_0 = x_2 = x_{12} = 0 \,. \end{aligned} \tag{62}$$

It is tempting to replace conditions (62) by one equation $x_1 x_2 = x_0 x_{12}$, defining a quadric in $A^{1,2}$:

$$\boldsymbol{x} = x_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_{12}\boldsymbol{e}_1\boldsymbol{e}_2 \,, \qquad x_1 x_2 = x_0 x_{12} \,. \tag{63}$$

The equation $x_1 x_2 = x_0 x_{12}$ is not equivalent to (62), because the constraint $x_{12} = 0$ is not its necessary consequence. However, we conjecture that the quadric (63) can be more fundamental than (62). Therefore, we find transformations preserving the following quadratic constraints:

$$\begin{aligned} & x_0^2 - x_1^2 - x_2^2 + x_{12}^2 = C \,, \\ & x_1 x_2 - x_0 x_{12} = 0 \,, \end{aligned} \tag{64}$$

where C is a constant. This system of two equations can be rewritten in the following, equivalent, form:

$$\begin{aligned} &(x_0 + x_{12})^2 - (x_1 + x_2)^2 = C \,, \\ &(x_0 - x_{12})^2 - (x_1 - x_2)^2 = C \,, \end{aligned} \tag{65}$$

which means that this is an intersection of two hyperbolic cylinders. The most general linear transformation (modulo reflections) preserving these two quadrics is a system of two hyperbolic "rotations" (boosts):

$$\begin{aligned} &\begin{pmatrix} \tilde{x}_0 + \tilde{x}_{12} \\ \tilde{x}_1 + \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} x_0 + x_{12} \\ x_1 + x_2 \end{pmatrix}, \\ &\begin{pmatrix} \tilde{x}_0 - \tilde{x}_{12} \\ \tilde{x}_1 - \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} x_0 - x_{12} \\ x_1 - x_2 \end{pmatrix}, \end{aligned} \tag{66}$$

where φ and ψ are constant parameters. It corresponds to the following linear transformation in the space $A^{1,2}$:

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{12} \\ \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a+c & a-c & b+d & b-d \\ a-c & a+c & b-d & b+d \\ b+d & b-d & a+c & a-c \\ b-d & b+d & a-c & a+c \end{pmatrix} \begin{pmatrix} x_0 \\ x_{12} \\ x_1 \\ x_2 \end{pmatrix}. \tag{67}$$

where:

$$a = \cosh\varphi \,, \quad b = \sinh\varphi \,, \quad c = \cosh\psi, \quad d = \sinh\psi \,. \tag{68}$$

Equation (67) can be shortly written as $\tilde{x} = Ax$. Note that A is symmetric ($A^T = A$). Moreover,

$$A^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{69}$$

which means that $A \in O(2,2)$. We can verify in a straightforward way that $\det A = 1$, which means that $A \in SO(2,2)$.

Theorem 6. *The orthogonal transformation* (67) *can be realized as a multiplication by the unit scator* $\exp(\vartheta_1 \boldsymbol{e}_1 + \vartheta_2 \boldsymbol{e}_2)$, *where* $\tanh\vartheta_k = \beta_k$ *($k = 1,2$):*

$$\tilde{\boldsymbol{x}} = \exp(\vartheta_1 \boldsymbol{e}_1 + \vartheta_2 \boldsymbol{e}_2)\, \boldsymbol{x} \equiv \gamma_1\gamma_2 (1 - \beta_1 \boldsymbol{e}_1)(1 - \beta_2 \boldsymbol{e}_2)\, \boldsymbol{x} \,, \tag{70}$$

where $\gamma_k = \dfrac{1}{\sqrt{1-\beta_k^2}}$ *($k = 1,2$).*

Proof. Taking into account (61) and performing the multiplication on the right-hand side of (70), we obtain:

$$\begin{aligned} \tilde{x}_0 &= \gamma_1\gamma_2 (x_0 - \beta_1 x_1 - \beta_2 x_2 + \beta_1\beta_2 x_{12}) \,, \\ \tilde{x}_1 &= \gamma_1\gamma_2 (-\beta_1 x_0 + x_1 + \beta_1\beta_2 x_2 - \beta_2 x_{12}) \,, \\ \tilde{x}_2 &= \gamma_1\gamma_2 (-\beta_2 x_0 + \beta_1\beta_2 x_1 + x_2 - \beta_1 x_{12}) \,, \\ \tilde{x}_{12} &= \gamma_1\gamma_2 (\beta_1\beta_2 x_0 - \beta_2 x_1 - \beta_1 x_2 + x_{12}) \,, \end{aligned} \tag{71}$$

or in the matrix form:

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_{12} \\ \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \gamma_1\gamma_2 \begin{pmatrix} 1 & \beta_1\beta_2 & -\beta_1 & -\beta_2 \\ \beta_1\beta_2 & 1 & -\beta_2 & -\beta_1 \\ -\beta_1 & -\beta_2 & 1 & \beta_1\beta_2 \\ -\beta_2 & -\beta_1 & \beta_1\beta_2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_{12} \\ x_1 \\ x_2 \end{pmatrix} . \tag{72}$$

Matrices (67) and (72) are identical if:

$$\begin{aligned} a &= \gamma_1\gamma_2(1+\beta_1\beta_2)\,, \\ b &= -\gamma_1\gamma_2(\beta_1+\beta_2)\,, \\ c &= \gamma_1\gamma_2(1-\beta_1\beta_2)\,, \\ d &= \gamma_1\gamma_2(\beta_2-\beta_1)\,. \end{aligned} \tag{73}$$

Computing β_1 and β_2 from (73), we obtain the inverse transformation:

$$\beta_1 = -\frac{b+d}{a+c} = -\frac{\sinh\varphi + \sinh\psi}{\cosh\varphi + \cosh\psi}\,, \qquad \beta_2 = -\frac{b-d}{a+c} = -\frac{\sinh\varphi - \sinh\psi}{\cosh\varphi + \cosh\psi}\,. \tag{74}$$

One can check by straightforward computation that the other two equations resulting from (73), namely:

$$\gamma_1\gamma_2 = \frac{1}{2}(a+c)\,, \qquad \gamma_1\gamma_2\beta_1\beta_2 = \frac{1}{2}(a-c)\,, \tag{75}$$

are then identically satisfied (taking into account $a^2 - b^2 = 1$ and $c^2 - d^2 = 1$). Therefore, substituting (74) into (70), we obtain the matrix A given by (67). □

4.2. Embedded Scators as the Intersection of Quadrics in the Case N = 3

If $x_0 \neq 0$, then the general element of $A^{1,3}$, given by:

$$\boldsymbol{x} = x_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_{12}\boldsymbol{e}_1\boldsymbol{e}_2 + x_{13}\boldsymbol{e}_1\boldsymbol{e}_3 + x_{23}\boldsymbol{e}_2\boldsymbol{e}_3 + x_{123}\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\,, \tag{76}$$

is an embedded scator if the last four coordinates are parameterized by the four first coordinates as follows:

$$x_{12} = \frac{x_1x_2}{x_0}\,, \qquad x_{23} = \frac{x_2x_3}{x_0}\,, \qquad x_{13} = \frac{x_1x_3}{x_0}\,, \qquad x_{123} = \frac{x_1x_2x_3}{x_0^2}\,. \tag{77}$$

Equation (77) implies that the scator norm is a pseudo-Euclidean norm in eight-dimensional space $A^{1,3}$:

$$\|\boldsymbol{x}\|^2 = x_0^2\left(1-\frac{x_1^2}{x_0^2}\right)\left(1-\frac{x_2^2}{x_0^2}\right)\left(1-\frac{x_3^2}{x_0^2}\right) = x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 - x_{123}^2\,. \tag{78}$$

The system (77) can be rewritten as the intersection of nine quadrics:

$$\begin{aligned} x_1x_2 &= x_0x_{12}\,, & x_{12}x_3 &= x_0x_{123}\,, & x_{13}x_{23} &= x_3x_{123}\,, \\ x_1x_3 &= x_0x_{13}\,, & x_{13}x_2 &= x_0x_{123}\,, & x_{12}x_{23} &= x_2x_{123}\,, \\ x_2x_3 &= x_0x_{23}\,, & x_{23}x_1 &= x_0x_{123}\,, & x_{12}x_{13} &= x_1x_{123}\,. \end{aligned} \tag{79}$$

These equations are not independent, of course. Now, we can ask about the consequences of (79) in the case of $x_0 = 0$. First, it follows from the three equations on the left that at least two of the three coordinates x_1, x_2, x_3 vanish. Suppose that $x_1 = x_2 = 0$. Then:

$$x_3x_{12} = 0\,, \qquad x_{12}x_{13} = 0\,, \qquad x_{12}x_{23} = 0\,, \qquad x_{13}x_{23} = x_3x_{123}\,. \tag{80}$$

Hence, either $x_{12} = 0$ and $x_{13}x_{23} = x_3x_{123}$ or $x_3 = x_{13} = x_{23} = 0$. Analogous results follows if we take $x_1 = x_3 = 0$ or $x_2 = x_3 = 0$. Thus, we arrive at the following set of general solutions to the system (79):

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{e}_3(x_3 + x_{13}\boldsymbol{e}_1 + x_{23}\boldsymbol{e}_2 + x_{123}\boldsymbol{e}_1\boldsymbol{e}_2), \qquad & x_{13}x_{23} = x_3x_{123}\,, \\ \boldsymbol{x} &= \boldsymbol{e}_2(x_2 + x_{12}\boldsymbol{e}_1 + x_{23}\boldsymbol{e}_3 + x_{123}\boldsymbol{e}_1\boldsymbol{e}_3), \qquad & x_{12}x_{23} = x_2x_{123}\,, \\ \boldsymbol{x} &= \boldsymbol{e}_1(x_1 + x_{12}\boldsymbol{e}_2 + x_{13}\boldsymbol{e}_3 + x_{123}\boldsymbol{e}_2\boldsymbol{e}_3), \qquad & x_{12}x_{13} = x_1x_{123}\,. \end{aligned} \tag{81}$$

Note that the solution $x_3 = x_{13} = x_{23} = 0$ (when $x_1 = x_2 = 0$) is included as a special case of the second equation of (81). Therefore, the subset $x_0 = 0$ reduces to the union of three two-dimensional quadrics; compare (63). We conjecture that a similar property holds in higher dimensions, as well.

5. Lorentz Transformation vs. Scator Transformation

Lorentz transformations form the well-known group of symmetries of the $1+3$-dimensional Minkowski space. In this section, we compare the Lorentz group with the group of symmetries of the embedded scator space introduced in Section 3.

5.1. Lorentz Transformation in the Matrix Form

The Lorentz transformation can be represented in the following matrix form [20]:

$$\tilde{x} = Lx\,, \tag{82}$$

where $x = (x_0, x_1, x_2, x_3)^T$ and:

$$L = \begin{pmatrix} \gamma & -\gamma\beta_1 & -\gamma\beta_2 & -\gamma\beta_3 \\ -\gamma\beta_1 & 1 + \frac{\gamma^2}{\gamma+1}\beta_1^2 & \frac{\gamma^2}{\gamma+1}\beta_1\beta_2 & \frac{\gamma^2}{\gamma+1}\beta_1\beta_3 \\ -\gamma\beta_2 & \frac{\gamma^2}{\gamma+1}\beta_2\beta_1 & 1 + \frac{\gamma^2}{\gamma+1}\beta_2^2 & \frac{\gamma^2}{\gamma+1}\beta_2\beta_3 \\ -\gamma\beta_3 & \frac{\gamma^2}{\gamma+1}\beta_3\beta_1 & \frac{\gamma^2}{\gamma+1}\beta_3\beta_2 & 1 + \frac{\gamma^2}{\gamma+1}\beta_3^2 \end{pmatrix} \tag{83}$$

where:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \ldots \tag{84}$$

and $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$. The Lorentz factor γ is defined for $|\beta| < 1$. We can decompose L as follows:

$$L = \gamma \begin{pmatrix} 1 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 1 & 0 & 0 \\ -\beta_2 & 0 & 1 & 0 \\ -\beta_3 & 0 & 0 & 1 \end{pmatrix} + \frac{\gamma - 1}{\beta^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_2^2 - \beta_3^2 & \beta_1\beta_2 & \beta_1\beta_3 \\ 0 & \beta_2\beta_1 & -\beta_1^2 - \beta_3^2 & \beta_2\beta_3 \\ 0 & \beta_3\beta_1 & \beta_3\beta_2 & -\beta_1^2 - \beta_2^2 \end{pmatrix}, \tag{85}$$

where we applied the identity:

$$\frac{\gamma - 1}{\beta^2} \equiv \frac{\gamma^2}{\gamma + 1} = \frac{1}{2} + \frac{3}{8}\beta^2 + \ldots \tag{86}$$

Thus, taking into account (84) and (86), we can easily expand L in the Taylor series with respect to β_1, β_2 and β_3. The series is convergent in the open ball $|\beta| < 1$.

5.2. Scator Transformation

As a natural analogue of Lorentz transformations in the scator space, one takes multiplications by unit scators [5,6], because, preserving the scator norm, they are isometries of the scator space $\mathbb{S}^{1+3}$:

$$(\tilde{x}_0 + \tilde{x}_1 \overset{o}{\boldsymbol{e}}_1 + \tilde{x}_2 \overset{o}{\boldsymbol{e}}_2 + \tilde{x}_3 \overset{o}{\boldsymbol{e}}_3) = \pi\Big(\gamma_1\gamma_2\gamma_3(1-\beta_1\boldsymbol{e}_1)(1-\beta_2\boldsymbol{e}_2)(1-\beta_3\boldsymbol{e}_3)F(\overset{o}{x})\Big) \tag{87}$$

where we use the fundamental embedding F and the projection π: $\mathbb{A}^{1,3} \to \mathbb{S}^{1+3}$.

The scator transformation (87) preserves the scator norm (78) and, considered as a transformation in the space $\mathbb{R}^{1,3}$, is nonlinear:

$$\begin{aligned} \tilde{x}_0 &= x_0\gamma_1\gamma_2\gamma_3\left(1-\beta_1\frac{x_1}{x_0}\right)\left(1-\beta_2\frac{x_2}{x_0}\right)\left(1-\beta_3\frac{x_3}{x_0}\right), \\ \tilde{x}_k &= \tilde{x}_0\left(\frac{\frac{x_k}{x_0}-\beta_k}{1-\beta_k\frac{x_k}{x_0}}\right) \qquad (k=1,2,3)\,. \end{aligned} \tag{88}$$

It becomes linear when considered in the larger space $\mathbb{R}^{2^n}$, as described in Section 4.

We propose here also another approach, which consists of linearizing the nonlinear transformation (87) with respect to variables $x_0, x_1, \ldots, x_n$. Then:

$$(\tilde{x}_0 + \tilde{x}_1\boldsymbol{e}_1 + \tilde{x}_2\boldsymbol{e}_2 + \tilde{x}_3\boldsymbol{e}_3) = \pi(\gamma_1\gamma_2\gamma_3(1-\beta_1\boldsymbol{e}_1)(1-\beta_2\boldsymbol{e}_2)(1-\beta_3\boldsymbol{e}_3)(x_0 + x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3)). \tag{89}$$

Rewriting it in the matrix form, we obtain:

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} = \gamma_1\gamma_2\gamma_3 \begin{pmatrix} 1 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & 1 & \beta_1\beta_2 & \beta_1\beta_3 \\ -\beta_2 & \beta_2\beta_1 & 1 & \beta_2\beta_3 \\ -\beta_3 & \beta_3\beta_1 & \beta_3\beta_2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{90}$$

This linearized approach can be also be viewed at as identifying the Minkowski space with scators in the "additive representation" $\mathbb{R}^{1+n}$, while the elements of the transformation group are identified with scators in the "multiplicative representation" [11]. It is worth noting that the scator transformation (90) is defined on a larger domain (the open cube $|\beta_k| < 1$ for $k = 1, 2, 3$) than the Lorentz transformation.

The components of Formula (88) are reminiscent of the Lorentz rule for the relativistic sum of velocities (in the one-dimensional case), which has motivated some physical applications [5,6]. Certainly, the commutative properties of the scator product look promising in comparison to Einstein's addition of vector velocities, which is neither associative nor commutative, and an extra rotation is necessary to satisfy gyro-group properties [21,22]. However, comparing the matrices (83) and (90), we see that the scator approach yields results that, in general, are quite different from the classical special relativity. In order to obtain physically plausible conclusions, one should apply scators in a very special range of variables and parameters.

6. Conclusions

In the generic case, the scator product, proposed by Fernández-Guasti and Zaldívar [1], is induced by another product (in another space, namely $A^{1,n}$), which is commutative, associative, and distributive over addition. The space $\mathbb{A}^{1,n}$, spanned by vectors $\boldsymbol{e}_k$ ($k = 1, \ldots, n$)) and their products, may be understood as a commutative analogue of the geometric algebra or the Clifford algebra [14,15]. In this context, we can interpret $\boldsymbol{e}_{jk}$ as bivectors and $\boldsymbol{e}_J$ (where J is a multi-index) as multivectors. The set $F(\mathbb{S}_*^{1+n})$ (the fundamental embedding of scators with nonvanishing scalar component) has a natural group structure reminiscent of a commutative analogue of the Clifford (or Lipschitz) group [23]. Theorems 1 and 2 show that this group is, indeed, a natural model of the scator space. Another interesting feature of the space $A^{1,n}$ is the metric structure induced by the scator metric. It turns out that this is a pseudo-Cartesian metric (the squared norm of basis multivectors $\boldsymbol{e}_J$ is one for even multiple indices and -1 for odd multiple indices). Section 4 presents a new interpretation of the scator product as an isometry (orthogonal transformation) in this space.

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References

1. Fernández-Guasti, M.; Zaldívar, F. A Hyperbolic Non-Distributive Algebra in 1 + 2 dimensions. *Adv. Appl. Clifford Algebr.* **2013**, *23*, 639–656. [CrossRef]
2. Fernández-Guasti, M. A Non-distributive Extension of Complex Numbers to Higher Dimensions. *Adv. Appl. Clifford Algebr.* **2015**, *25*, 829–849. [CrossRef]
3. Fernández-Guasti, M. Components exponential scator holomorphic function. *Math. Meth. Appl. Sci.* **2020**, *43*, 1017–1034. [CrossRef]
4. Cieśliński, J.L.; Zhalukevich, D. Explicit Formulas for All Scator Holomorphic Functions in the (1 + 2)-Dimensional Case. *Symmetry* **2020**, *12*, 1550. [CrossRef]
5. Fernández-Guasti, M. Time and space transformations in a scator deformed Lorentz metric. *Eur. Phys. J. Plus* **2014**, *129*, 195. [CrossRef]
6. Fernández-Guasti, M. Composition of velocities and momentum transformations in a scator-deformed Lorentz metric. *Eur. Phys. J. Plus* **2020**, *135*, 542. [CrossRef]
7. Magueijo, J.; Smolin, L. Generalized Lorentz invariance with an invariant energy scale. *Phys. Rev. D* **2003**, *67*, 044017. [CrossRef]
8. Amelino-Camelia, G. Doubly-Special Relativity: Facts, Myths and Some Key Open Questions. *Symmetry* **2010**, *2*, 230–271. [CrossRef]
9. Kobus, A.; Cieśliński, J.L. On the Geometry of the Hyperbolic Scator Space in $1+2$ Dimensions. *Adv. Appl. Clifford Algebr.* **2017**, *27*, 1369–1386. [CrossRef]
10. Kobus, A.; Cieśliński, J.L. Geometric and Differential Features of Scators as Induced by Fundamental Embedding. *Symmetry* **2020**, *12*, 1880. [CrossRef]
11. Fernández-Guasti, M.; Zaldívar, F. Multiplicative representation of a hyperbolic non distributive algebra. *Adv. Appl. Clifford Algebr.* **2014**, *24*, 661–674. [CrossRef]
12. Fernández-Guasti, M. Differential quotients in elliptic scator algebra. *Math. Meth. Appl. Sci.* **2018**, *41*, 4827–4840. [CrossRef]
13. Cieśliński, J.L.; Kobus, A. On the Product Rule for the Hyperbolic Scator Algebra. *Axioms* **2020**, *9*, 55. [CrossRef]
14. Hestenes, D.; Sobczyk, G. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*; D. Reidel Publ. Co.: Dordrecht, The Netherlands, 1984.
15. Vaz, J.; da Rocha, R. *An Introduction to Clifford Algebras and Spinors*; Oxford University Press: Oxford, UK, 2016.
16. Okubo, S. *Introduction to Octonion and Other Non-Associative Algebras in Physics*; Cambridge University Press: Cambridge, UK, 1995.
17. Gogberashvili, M. Octonionic geometry. *Adv. Appl. Clifford Algebras* **2005**, *15*, 55–66.
18. Dzhunushaliev, V. Colorless operators in a non-associative quantum theory. *Phys. Lett. A* **2006**, *355*, 298–302.
19. Fernández-Guasti, M. Associativity in scator algebra and the quantum wavefunction collapse. *Univers J. Math. Appl.* **2018**, *1*, 80–88.
20. Furry, W.H. Lorentz Transformation and the Thomas Precession. *Am. J. Phys.* **1955**, *23*, 517.
21. Ungar, A.A. Thomas rotation and the parameterization of the Lorentz transformation group. *Found. Phys. Lett.* **1988**, *1*, 57–89.
22. Ungar, A.A. *Beyond the Einstein Addition Law and Its Gyroscopic Thomas Precession*; Springer: Berlin/Heidelberg, Germany, 2002. [CrossRef]
23. Cieśliński, J.L. Divisors of zero in the Lipschitz semigroup. *Adv. Appl. Clifford Algebr.* **2007**, *17*, 153–157. [CrossRef]