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New Families of Special Polynomial Identities Based upon Combinatorial Sums Related to p-Adic Integrals

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Abstract: The aim of this paper is to study and investigate generating-type functions, which have been recently constructed by the author, with the aid of the Euler's identity, combinatorial sums, and p-adic integrals. Using these generating functions with their functional equation, we derive various interesting combinatorial sums and identities including new families of numbers and polynomials, the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Daehee numbers, the Changhee numbers, and other numbers and polynomials. Moreover, we present some revealing remarks and comments on the results of this paper.

Keywords: p-adic integrals; Volkenborn integral; generating function; special functions; Bernoulli numbers and polynomials; Euler numbers and polynomials; Stirling numbers; Daehee numbers; Changhee numbers; combinatorial numbers and sum

MSC: 11S80; 11B68; 05A15; 05A19; 30C15; 26C05; 12D10

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1. Introduction

In [1], we recently constructed generating-type functions for some new families of special polynomials and numbers via the umbral calculus convention method. We showed that these new families of special polynomials and numbers are associated with finite calculus, combinatorial numbers and polynomials, polynomial of the chordal graph, and special functions and their applications (cf. [1]).

The motivation of this paper is to give various combinatorial sums involving special numbers and polynomials via application of the p-adic integrals to functional equations of generating-type functions.

We [1] defined the following generating-type functions:

$$h\left(w; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \prod_{j=0}^{v-1} \left(f(w) - x_j\right)^{y_j}, \tag{1}$$

$$F_1\left(w; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \frac{w^v}{h\left(w; \overrightarrow{x_v}, \overrightarrow{y_v}\right)}, \tag{2}$$

and

$$F_2\left(w, z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = (1+w)^z F_1\left(w; \overrightarrow{x_v}, \overrightarrow{y_v}\right), \tag{3}$$

where $f(w)$ is an analytic function, $F_1\left(w; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ is a meromorphic function, $v \in \mathbb{N}$, v-tuples $\overrightarrow{x_v} = (x_0, x_1, \ldots, x_{v-1})$, $\overrightarrow{y_v} = (y_0, y_1, \ldots, y_{v-1})$, $x_j, y_j \in \mathbb{R}$ with $j = 0, 1, \ldots, v-1$, and $z, w \in \mathbb{R}$ (or $\mathbb{C}$).

The polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ of degree n and order y_j $(j = 0, 1, \ldots, v-1)$ are defined by means of the following generating function (cf. [1]):

$$F_2\left(w, z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \sum_{n=0}^{\infty} \theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) w^n. \tag{4}$$

Substituting $z = 0$ into (4), we [1] defined another new class of special numbers $\theta_n(\overrightarrow{x_v}, \overrightarrow{y_v}) := \theta_n(0; \overrightarrow{x_v}, \overrightarrow{y_v})$, which are defined by means of the following generating function:

$$F_1(w; \overrightarrow{x_v}, \overrightarrow{y_v}) = \sum_{n=0}^{\infty} \theta_n(\overrightarrow{x_v}, \overrightarrow{y_v}) w^n. \tag{5}$$

Combining (4) and (5) yields another relation between the polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$ and the numbers $\theta_n(\overrightarrow{x_v}, \overrightarrow{y_v})$, given as follows:

$$\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v}) = \sum_{j=0}^{n} \binom{z}{j} \theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v}) \tag{6}$$

where $n \in \mathbb{N}_0$ (cf. [1]).

We [1] defined the numbers $\ell_n(v) := \ell_n((0, 1, 2, \ldots, v-1))$ by means of the following generating function:

$$\begin{aligned} G_1(w; (0, 1, 2, \ldots, v-1)) &= \frac{w^v}{\prod_{j=0}^{v-1} (e^w - j)} \\ &= \sum_{n=0}^{\infty} \ell_n(v) \frac{w^n}{n!}, \end{aligned} \tag{7}$$

where

$$\ell_n(v) := \ell_n((0, 1, 2, \ldots, v-1)) = n! \theta_n((0, 1, 2, \ldots, v-1), (1, 1, \ldots, 1)),$$

for $m \in \mathbb{N}$,

$$\ell_0(v) = 0, \ell_0(m+1) = \ell_1(m+1) = \cdots = \ell_{m-1}(m+1) = 0$$

and

$$\ell_m(m+1) \neq 0$$

(cf. [1]).

We [1] also defined the polynomials $\ell_n(x; v)$ by means of the following generating function:

$$\frac{w^v}{\prod_{j=0}^{v-1} (e^w - j)} e^{zw} = \sum_{n=0}^{\infty} \ell_n(z; v) \frac{w^n}{n!}. \tag{8}$$

By using (8), we have

$$\ell_n(z; v) = \sum_{j=0}^{n} \binom{n}{j} z^j \ell_{n-j}(v), \tag{9}$$

where

$$\ell_{n-j}(v) := \ell_{n-j}(0, v)$$

(cf. [1]).

We also use the following standard notations throughout this paper:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of positive integers, the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $x \in \mathbb{R}$. The rising factorial polynomials $x^{(n)}$ and the falling factorial polynomials $x_{(n)}$ are given respectively by

$$\begin{aligned} x^{(n)} &= x(x+1)(x+2)\ldots(x+n-1), \\ x_{(n)} &= x(x-1)(x-2)\ldots(x-n+1), \end{aligned}$$

where $x^{(0)} = 1$, $x_{(0)} = 1$ and $n \in \mathbb{N}$, and also

$$(-1)^n(-x)_{(n)} = (x+n-1)_{(n)} = x^{(n)} \tag{10}$$

$n \in \mathbb{N}_0$ (cf. [1–22]).

The Bernoulli polynomials of order n and degree k are defined by means of the following generating function:

$$\left(\frac{w}{e^w - 1}\right)^n e^{wy} = \sum_{k=0}^{\infty} \frac{B_k^{(n)}(y)}{k!} w^k, \tag{11}$$

where $n \in \mathbb{Z}$. Setting $n = 0$, we have $B_k^{(0)}(y) = y^n$. Putting $y = 0$, we have the Bernoulli numbers of order n: $B_k^{(n)} = B_k^{(n)}(0)$. When $n = 1$, we have the Bernoulli numbers: $B_k = B_k^{(1)}$ (cf. [1,3,5,6,11,15,16,18–24]).

A relation between the numbers $\theta_n(\overrightarrow{x_v}, \overrightarrow{y_v})$ and the numbers $B_k^{(n)}$ is given as follows:

$$\theta_{n-y_0-y_1-\ldots-y_{v-1}}((1,1,1,\ldots,1),(y_0,y_1,\ldots,y_{v-1})) = \frac{n_{(v)}}{n!} B_{n-v}^{(y_0+y_1+\ldots+y_{v-1})}$$

where $n, y_0, y_1, \ldots, y_{v-1} \in \mathbb{N}_0$ with $n \geq y_0 + y_1 + \cdots + y_{v-1}$ (cf. [1]).

The Euler polynomials of order n and degree k are defined by means of the following generating function:

$$\left(\frac{2}{e^w + 1}\right)^n e^{wy} = \sum_{k=0}^{\infty} \frac{E_k^{(n)}(y)}{k!} w^k, \tag{12}$$

where $n \in \mathbb{Z}$. Setting $n = 0$, we have $E_k^{(0)}(y) = y^n$. Putting $y = 0$, we have the Euler numbers of order n: $E_k^{(n)} = E_k^{(n)}(0)$. When $n = 1$, we have the Euler numbers: $E_k = E_k^{(1)}$ (cf. [1,3,5,6,15,16,18–22,24,25]).

A relation between the numbers $\theta_n(\overrightarrow{x_v}, \overrightarrow{y_v})$ and the numbers $E_k^{(n)}$ is given as follows:

$$\theta_n((-1,-1,\ldots,-1),(y_0,y_1,\ldots,y_{v-1})) = \frac{n_{(v)}}{n! 2^{y_0+y_1+\ldots+y_{v-1}}} E_{n-v}^{(y_0+y_1+\ldots+y_{v-1})}$$

where $n, y_0, y_1, \ldots, y_{v-1} \in \mathbb{N}_0$ with $n \geq y_0 + y_1 + \cdots + y_{v-1}$ (cf. [1]).

The Apostol–Bernoulli numbers, $\mathcal{B}_n(\lambda)$, are defined by

$$f_b(w;\lambda) = \frac{w}{\lambda e^w - 1} = \sum_{n=0}^{\infty} \frac{\mathcal{B}_n(\lambda)}{n!} w^n, \tag{13}$$

(cf. [2]; for detail, see also [1,8,16,18–22]).

A relation between the numbers $\ell_n(v)$ and the Apostol–Bernoulli numbers $\mathcal{B}_n(\lambda)$ is given by the following theorem:

Theorem 1 (*cf.* [1])**.** *Let $n \in \mathbb{N}_0$ and $v \in \mathbb{N}$ with $v > 1$. Then, we have*

$$\ell_n(v) = \frac{1}{v-1} \sum_{j=0}^{n} \binom{n}{j} \ell_{n-j}(v-1) \mathcal{B}_j\left(\frac{1}{v-1}\right). \tag{14}$$

The Apostol–Euler polynomials, $\mathcal{E}_n(\lambda)$, are defined by

$$f_e(w;\lambda) = \frac{2}{\lambda e^w + 1} = \sum_{n=0}^{\infty} \frac{\mathcal{E}_n(\lambda)}{n!} w^n, \tag{15}$$

(cf. [1,8,15,16,18–22]).

Using generating functions $f_b(w;\lambda)$ and $f_e(w;\lambda)$ yields the following well-known important relation:

$$(n+1)\mathcal{E}_n(\lambda) = -2\mathcal{B}_{n+1}(-\lambda) \tag{16}$$

(cf. [22]).

A relation between the numbers $\ell_n(v)$ and the Apostol–Euler numbers $\mathcal{E}_n(\lambda)$ is given by the following theorem:

Theorem 2 (*cf.* [1])**.** *Let $n \in \mathbb{N}_0$ and $v \in \mathbb{N}$ with $v > 1$. Then, we have*

$$\ell_n(v) = \sum_{j=0}^{n} \binom{n}{j} \frac{(n-j)\ell_j(v-1)}{2(1-v)} \mathcal{E}_{n-j-1}\left(\frac{1}{1-v}\right). \tag{17}$$

The Stirling numbers of the first kind, $S_1(v,d)$, are defined by means of the following generating function:

$$(\log(1+w))^d = \sum_{v=0}^{\infty} \frac{d! S_1(v,d)}{v!} w^v. \tag{18}$$

Using (18), we have

$$S_1(v,d) = 0$$

if $d > v$ (cf. [3,14,22]; see also the references cited in each of these earlier works).

The Stirling numbers of the first kind are also given by

$$y_{(v)} = \sum_{d=0}^{v} S_1(v,d) y^d, \tag{19}$$

(cf. [3,14,22,25]).

A relation between the polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ and the numbers $S_1(j,m)$ is given as follows:

$$\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \sum_{j=0}^{n} \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) \sum_{m=0}^{j} \frac{z^m S_1(j,m)}{j!} \tag{20}$$

where $n \in \mathbb{N}_0$ (cf. [1]).

The Bernoulli polynomials of the second kind, $b_m(y)$, are defined by

$$\frac{w(1+w)^y}{\log(1+w)} = \sum_{m=0}^{\infty} \frac{b_m(y)}{m!} w^m \tag{21}$$

such that for $y = 0$, we have the Bernoulli numbers of the second kind (Cauchy numbers of the first kind), which is denoted by $b_m(0)$ (cf. [3,25]).

The Stirling numbers of the second kind, $S_2(v,d)$, are defined by

$$(e^w - 1)^d = \sum_{v=0}^{\infty} \frac{d! S_2(v,d)}{v!} w^v, \tag{22}$$

where $d \in \mathbb{N}_0$ (cf. [1–22,25]).

The Stirling numbers of the second kind are also given by

$$y^v = \sum_{d=0}^{v} S_2(v,d) y_{(d)}, \tag{23}$$

(cf. [1–26]; see also the references cited in each of these earlier works).

The Daehee numbers, D_n, are defined by means of the following generating function:

$$\frac{1}{w}\log(1+w) = \sum_{n=0}^{\infty} \frac{D_n}{n!} w^n \tag{24}$$

(cf. [16] (p. 45), [11,18,20]). Using (24), we have

$$D_n = \frac{(-1)^n n!}{n+1} \tag{25}$$

(cf. [11]; see also [14,18,20]).

The Changhee numbers of the first kind, Ch_n, are defined by

$$\frac{2}{w+2} = \sum_{n=0}^{\infty} \frac{Ch_n}{n!} w^n \tag{26}$$

(cf. [12,18,20]). Using (26), we have

$$Ch_n = \frac{(-1)^n n!}{2^n} = \sum_{k=0}^{n} S_1(n,k) E_k \tag{27}$$

(cf. [12]; see also [14,18,20]).

Kucukoglu and Simsek [14] defined a new sequence of special numbers $\beta_n(k)$ by means of the following generating function:

$$\left(1 - \frac{z}{2}\right)^k = \sum_{n=0}^{\infty} \beta_n(k) \frac{z^n}{n!}, \tag{28}$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $|z| < 2$.

By using (28), we have

$$\beta_n(k) = \frac{(-1)^n n!}{2^n} \binom{k}{n}, \tag{29}$$

where $n, k \in \mathbb{N}_0$ (cf. [14] (Equation (4.9))).

Therefore, we summarize the content of the paper as follows:

In Section 2, we give identities and combinatorial sums including the polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$, the numbers $\ell_n(v)$, the Stirling numbers, the Daehee numbers, and the Bernoulli numbers of the second kind.

In Section 3, by using p-adic integrals on the set of p-adic integers, we give p-adic integral formulas for the polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$.

In Section 4, making use of these p-adic integral formulas for the polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$, we give some combinatorial sums including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Daehee numbers, and the Changhee numbers.

In Section 5, we give the conclusion section of this paper.

2. Identities, Relations, and Combinatorial Sums Derived from Generating Functions

In this section, we give some identities, relations, and combinatorial sums involving the new families of polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$, the numbers $\ell_n(v)$, the Stirling numbers, the Daehee numbers, and the Bernoulli numbers of the second kind.

Using (4), (5), (21), and (24), we obtain

$$\sum_{n=0}^{\infty} \theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v}) w^n = \sum_{n=0}^{\infty} \frac{b_n(z)}{n!} w^n \sum_{n=0}^{\infty} \frac{D_n}{n!} w^n \sum_{n=0}^{\infty} \theta_n(\overrightarrow{x_v}, \overrightarrow{y_v}) w^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v}) w^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \frac{D_{k-j}}{k!} \theta_{n-k}(\overrightarrow{x_v}, \overrightarrow{y_v}) b_j(z) w^n.$$

Equating coefficients w^n on both sides of the above equation, we arrive at the following theorem:

Theorem 3. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{k}{j} \frac{D_{k-j}}{k!} \theta_{n-k}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) b_j(z). \tag{30}$$

Combining (25) with (30), we arrive at the following corollary:

Corollary 1. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right) = \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(k-j)!\theta_{n-k}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{(k-j+1)k!} b_j(z).$$

Combining (14) with the following well-known formula

$$\mathcal{B}_n(\alpha) = \frac{n\alpha}{(\alpha-1)^n} \sum_{d=1}^{n-1} (-1)^d d! \alpha^{d-1} (\alpha-1)^{n-d-1} S_2(n-1, d) \tag{31}$$

(cf. [2]), we arrive at the following theorem:

Theorem 4. *Let $n \in \mathbb{N}$ with $n > 1$. Then, we have*

$$\ell_n(v) + \frac{n}{v-2} \ell_{n-1}(v-1) = -\sum_{j=2}^{n} \sum_{d=1}^{j-1} \binom{n}{j} \frac{jd!}{(v-2)^{d+1}} S_2(j-1, d) \ell_{n-j}(v-1).$$

By using (8), we obtain

$$w^v e^{xw} = \prod_{j=0}^{v-1} (e^w - j) \sum_{n=0}^{\infty} \ell_n(x; v) \frac{w^n}{n!}.$$

By combining the above equation with (19), we obtain

$$w^v \sum_{n=0}^{\infty} \frac{x^n w^n}{n!} = \sum_{d=0}^{v} S_1(v, d) \sum_{n=0}^{\infty} \frac{d^n w^n}{n!} \sum_{n=0}^{\infty} \ell_n(x; v) \frac{w^n}{n!}.$$

Therefore,

$$\sum_{n=0}^{\infty} n_{(v)} \frac{x^{n-v} w^n}{n!} = \sum_{d=0}^{v} S_1(v, d) \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} d^{n-j} \ell_j(x; v) \frac{w^n}{n!}.$$

Equating coefficients $\frac{w^n}{n!}$ on both sides of the above equation yields the following theorem:

Theorem 5. *Let $n, v \in \mathbb{N}$. Then, we have*

$$\sum_{d=0}^{v} \sum_{j=0}^{n} \binom{n}{j} d^{n-j} S_1(v, d) \ell_j(x; v) = n_{(v)} x^{n-v}. \tag{32}$$

Theorem 6. *Let $m, v \in \mathbb{N}$. Then, we have*

$$\begin{aligned} \ell_m(x+iy;v) &= \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j y^{2j} \binom{m}{2j} \ell_{m-2j}(x;v) \\ &\quad + i \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1} y^{2j+1} \ell_{m-2j-1}(x;v). \end{aligned} \tag{33}$$

Proof. Substituting $z = x + iy$ ($i^2 = -1$) into (8) and using (7), with the aid of the well-known Euler identity $e^{ywi} = \cos(yw) + i\sin(yw)$, we obtain

$$\frac{w^v e^{xw}}{\prod_{j=0}^{v-1}(e^w - j)} (\cos(yw) + i\sin(yw)) = \sum_{n=0}^{\infty} \ell_n(z;v) \frac{w^n}{n!}.$$

Combining the above equation with the MacLaurin Series for the trigonometric functions $\cos(yw)$ and $\sin(yw)$ yields

$$\begin{aligned} \sum_{m=0}^{\infty} \ell_m(x+iy;v) \frac{w^m}{m!} &= \sum_{m=0}^{\infty} (-1)^m \frac{(yw)^{2m}}{(2m)!} \sum_{m=0}^{\infty} \ell_m(x;v) \frac{w^m}{m!} \\ &\quad + i \sum_{m=0}^{\infty} (-1)^m \frac{(yw)^{2m+1}}{(2m+1)!} \sum_{m=0}^{\infty} \ell_m(x;v) \frac{w^m}{m!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{m=0}^{\infty} \ell_m(x+iy;v) \frac{w^m}{m!} &= \sum_{m=0}^{\infty} \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j y^{2j} \binom{m}{2j} \ell_{m-2j}(x;v) \\ &\quad + i \sum_{m=0}^{\infty} \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1} y^{2j+1} \ell_{m-2j-1}(x;v). \end{aligned}$$

Comparing the coefficients of $\frac{w^m}{m!}$ on both sides of the above equation, we obtain the desired result. □

Putting $x = y$ in (33), we obtain the following corollary:

Corollary 2. *Let $m, v \in \mathbb{N}$. Then, we have*

$$\begin{aligned} \sum_{j=0}^{m} \binom{m}{j} x^j (1+i)^j \ell_{m-j}(v) &= \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j x^{2j} \binom{m}{2j} \ell_{m-2j}(x;v) \\ &\quad + i \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1} x^{2j+1} \ell_{m-2j-1}(x;v). \end{aligned} \tag{34}$$

After the necessary algebraic calculations in Equation (34), the following combinatorial finite sum is obtained:

$$(1+i)^m = \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j \binom{m}{2j} + i \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1}. \tag{35}$$

By using (35), we arrive at the following well-known formulas:

$$\operatorname{Re}(1+i)^m = \sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j \binom{m}{2j}$$

and

$$\operatorname{Im}(1+i)^m = \sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1},$$

where $\operatorname{Re}(x+iy) = x$ and $\operatorname{Im}(x+iy) = y$.

Therefore, we obtain the following well-known formulas:

$$\sum_{j=0}^{\left[\frac{m}{2}\right]} (-1)^j \binom{m}{2j} = \frac{(1+i)^m + (1-i)^m}{2}$$

and

$$\sum_{j=0}^{\left[\frac{m-1}{2}\right]} (-1)^j \binom{m}{2j+1} = \frac{(1+i)^m + (1-i)^m}{2i}$$

(cf. [4] (Equations (2.26) and (2.30)); see also [6]).

Remark 1. *Notice that the results introduced in this paper would generalize and improve many works on the subject. For example, in the paper* [23] (Theorem 4, Equation (7), p. 5)*, Iordanescu et al. gave a generalized Euler formula. In future studies, with the help of the aforementioned generalized Euler formula, Formulas* (33)–(35) *of the current paper can be further generalized, and researchers who work on the generalization and unification of the Euler formula may obtain interesting results reducible to Formulas* (33)–(35) *of the present paper.*

3. p-Adic Integrals of the Polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ on the Set of p-Adic Integers

In this section, we give p-adic integrals of the polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ on $\mathbb{Z}_p$, which denotes the set of p-adic integers. These formulas include the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Daehee numbers, and the Changhee numbers.

In order to give p-adic integrals of the polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$ on $\mathbb{Z}_p$, we need some fundamental properties of the p-adic distributions and p-adic integrals. Assuming that p is an odd prime number, the $|.|_p$ map on $\mathbb{Q}$ is defined by

$$|x|_p = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where for $m \in \mathbb{N}$, $ord_p(m)$ denotes the greatest integer k ($k \in \mathbb{N}_0$) such that p^k divides m in $\mathbb{Z}$. If $m = 0$, then $ord_p(m) = \infty$. Let $\mathbb{Q}_p$ denote the set of p-adic rational numbers (cf. [17,20]).

The Haar distribution is defined by

$$\mu_1\left(x + p^N\mathbb{Z}_p\right) = \mu_1(x) = \frac{1}{p^N} \tag{36}$$

on $\mathbb{Z}_p$ (cf. [7,17,20]), and the distribution $\mu_{-1}\left(x + p^N\mathbb{Z}_p\right)$ on $\mathbb{Z}_p$ is defined by

$$\mu_{-1}\left(x + p^N\mathbb{Z}_p\right) = \mu_{-1}(x) = (-1)^x \tag{37}$$

(cf. [8,10,20]).

The Volkenborn integral (or the p-adic bosonic integral) of a uniformly differential function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty} p^{-N} \sum_{d=0}^{p^N-1} f(d), \tag{38}$$

(cf. [7,17,20]).

The p-adic fermionic integral of a uniformly differential function f on $\mathbb{Z}_p$ is given by

$$\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \tag{39}$$

(cf. [8]; see also [10,20]).

Note that the Volkenborn integral (or the bosonic integral) and the p-adic fermionic integral have various different applications in mathematics, in mathematical physics, and in other areas. Using the Volkenborn integral, generating functions for Bernoulli-type numbers and polynomials and combinatorial numbers and polynomials are constructed and investigated (cf. [7,11,13,18–21,26]). On the other hand, using the p-adic fermionic integral, generating functions for Euler-type numbers and polynomials and Genocchi-type numbers and polynomials are constructed and investigated (cf. [8,10,12,13,18–21,26]). By using p-adic integrals, the theory of the generating functions, ultrametric calculus, the quantum groups, cohomology groups, q-deformed oscillator, and p-adic models have been studied (cf. [17,20]).

Some well-known formulas for the Volkenborn integral are given as follows:

$$\int_{\mathbb{Z}_p} \binom{y}{n} d\mu_1(y) = \frac{(-1)^n}{n+1}, \tag{40}$$

where $n \in \mathbb{N}_0$ (cf. [17]).

The p-adic integral representation of the Bernoulli numbers B_n is given by

$$B_n = \int_{\mathbb{Z}_p} y^n d\mu_1(y), \tag{41}$$

where $n \in \mathbb{N}_0$ (cf. [17]; see also [7,20] and the references cited in each of these earlier works).

The p-adic integral representations of the Daehee numbers D_n are given by

$$D_n = \int_{\mathbb{Z}_p} y_{(n)} d\mu_1(y), \tag{42}$$

(cf. [11]) and

$$\int_{\mathbb{Z}_p} y_{(n)} d\mu_1(y) = \sum_{l=0}^{n} S_1(n,l) B_l, \tag{43}$$

where $n \in \mathbb{N}_0$ (cf. [11]).

Combining (25) with (42), we have the following integral formula:

$$\int_{\mathbb{Z}_p} y_{(n)} d\mu_1(y) = \frac{(-1)^n n!}{n+1}, \tag{44}$$

where $n \in \mathbb{N}_0$ (cf. [11,17,20]).

The p-adic fermionic integral representation of the Euler numbers E_n is given by

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x), \tag{45}$$

where $n \in \mathbb{N}_0$ (cf. [8,10,13,20]; see also the references cited in each of these earlier works).

We also need the following interesting well-known formulas:

$$\int_{\mathbb{Z}_p} \binom{y}{n} d\mu_{-1}(y) = (-1)^n 2^{-n} \tag{46}$$

(cf. [12] (Theorem 2.3)) and

$$\int_{\mathbb{Z}_p} y_{(n)} d\mu_{-1}(y) = (-1)^n 2^{-n} n!, \tag{47}$$

where $n \in \mathbb{N}_0$ (cf. [12]).

Combining (27) with (47), we have the following p-adic fermionic integral representation of the Changhee numbers Ch_n:

$$Ch_n = \int_{\mathbb{Z}_p} y_{(n)} d\mu_{-1}(y), \tag{48}$$

where $n \in \mathbb{N}_0$ (*cf.* [12]).

3.1. Volkenborn Integral of the Polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$ on $\mathbb{Z}_p$

Here, using p-adic integrals of the polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$, we give combinatorial sums involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Daehee numbers, and the Changhee numbers.

Applying the Volkenborn integral to Equations (6) and (20) on $\mathbb{Z}_p$, and using (41), (43), and (44), after some elementary calculations, we obtain the following Volkenborn integral of the polynomials of $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$:

Theorem 7. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\int_{\mathbb{Z}_p} \theta_n(x; \overrightarrow{x_v}, \overrightarrow{y_v}) d\mu_1(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} \frac{B_m S_1(j,m) \theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v})}{j!}, \tag{49}$$

$$\int_{\mathbb{Z}_p} \theta_n(x; \overrightarrow{x_v}, \overrightarrow{y_v}) d\mu_1(x) = \sum_{j=0}^{n} (-1)^j \frac{1}{j+1} \theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v}), \tag{50}$$

and

$$\int_{\mathbb{Z}_p} \theta_n(x; \overrightarrow{x_v}, \overrightarrow{y_v}) d\mu_1(x) = \sum_{j=0}^{n} \frac{\theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v}) D_j}{j!}. \tag{51}$$

3.2. p-Adic Fermionic Integrals of the Polynomials $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$ on $\mathbb{Z}_p$

Applying the p-adic fermionic integral to Equation (6) and Equation (20) on $\mathbb{Z}_p$, and using (45), (46), (48), and (29), after some elementary calculations, we obtain the following p-adic fermionic integral of the polynomials of $\theta_n(z; \overrightarrow{x_v}, \overrightarrow{y_v})$:

Theorem 8. *Let $n, k \in \mathbb{N}_0$. Then, we have*

$$\int_{\mathbb{Z}_p} \theta_n\left(x; \overrightarrow{x_v}, \overrightarrow{y_v}\right) d\mu_{-1}(x) = \sum_{j=0}^{n} \sum_{m=0}^{j} \frac{E_m S_1(j,m) \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j!}, \tag{52}$$

$$\int_{\mathbb{Z}_p} \theta_n\left(x; \overrightarrow{x_v}, \overrightarrow{y_v}\right) d\mu_{-1}(x) = \sum_{j=0}^{n} (-1)^j \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{2^j}, \tag{53}$$

$$\int_{\mathbb{Z}_p} \theta_n\left(x; \overrightarrow{x_v}, \overrightarrow{y_v}\right) d\mu_{-1}(x) = \sum_{j=0}^{n} \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) Ch_j}{j!}, \tag{54}$$

and

$$\int_{\mathbb{Z}_p} \theta_n\left(x; \overrightarrow{x_v}, \overrightarrow{y_v}\right) d\mu_{-1}(x) = \sum_{j=0}^{n} \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) \beta_j(k)}{j! \binom{k}{j}}. \tag{55}$$

4. Combinatorial Sums and Identities Derived from *p*-Adic Integrals

In this section, by making use of these p-adic integral formulas for the polynomials $\theta_n\left(z; \overrightarrow{x_v}, \overrightarrow{y_v}\right)$, we give some combinatorial sums including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Daehee numbers, and the Changhee numbers.

Combining (49) and (50) with (51), we obtain the following combinatorial sums, respectively:

Theorem 9. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n} \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) D_j}{j!} = \sum_{j=0}^{n} (-1)^j \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j+1}.$$

Theorem 10. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n} \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) D_j}{j!} = \sum_{j=0}^{n} \sum_{m=0}^{j} \frac{B_m S_1(j,m) \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j!}.$$

Theorem 11. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n} (-1)^j \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j+1} = \sum_{j=0}^{n} \sum_{m=0}^{j} \frac{B_m S_1(j,m) \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j!}.$$

Combining (52) and (53) with (54), we also obtain the following combinatorial sums, respectively:

Theorem 12. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n} \sum_{m=0}^{j} \frac{E_m S_1(j,m) \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j!} = \sum_{j=0}^{n} (-1)^j \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{2^j}.$$

Theorem 13. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n} \sum_{m=0}^{j} \frac{E_m S_1(j,m) \theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right)}{j!} = \sum_{j=0}^{n} \frac{\theta_{n-j}\left(\overrightarrow{x_v}, \overrightarrow{y_v}\right) Ch_j}{j!}.$$

Theorem 14. *Let $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{j=0}^{n}(-1)^j \frac{\theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v})}{2^j} = \sum_{j=0}^{n} \frac{\theta_{n-j}(\overrightarrow{x_v}, \overrightarrow{y_v}) Ch_j}{j!}.$$

Applying the Volkenborn integral to Equation (32) on $\mathbb{Z}_p$, and using Equation (41), after some elementary calculations, we arrive at the following theorem:

Theorem 15. *Let $n, v \in \mathbb{N}$ with $n \geq v$. Then, we have*

$$B_{n-v} = \frac{1}{n_{(v)}} \sum_{d=0}^{v} \sum_{j=0}^{n} \sum_{m=0}^{j} \binom{n}{j} \binom{j}{m} d^{n-j} S_1(v,d) \ell_{j-m}(v) B_m.$$

Applying the p-adic fermionic integral to Equation (32) on $\mathbb{Z}_p$, and using (45), after some elementary calculations, we arrive at the following theorem:

Theorem 16. *Let $n, v \in \mathbb{N}$ with $n \geq v$. Then, we have*

$$E_{n-v} = \frac{1}{n_{(v)}} \sum_{d=0}^{v} \sum_{j=0}^{n} \sum_{m=0}^{j} \binom{n}{j} \binom{j}{m} d^{n-j} S_1(v,d) \ell_{j-m}(v) E_m.$$

5. Conclusions

In [1], we gave applications of new constructed families of generating-type functions interpolating new and known classes of polynomials and numbers. In this paper, we studied these generating functions with their functional equations. By applying p-adic integrals to these generating functions and their functional equations, we gave p-adic integral formulas for these new classes of polynomials and numbers. Using these generating functions with their functional equations, we also derived many novel combinatorial sums and identities involving these polynomials and numbers and also the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Daehee numbers, and the Changhee numbers.

In addition, we have given some clarifying explanations and comments on the results of this article. The results, including the special classes of polynomials and numbers presented in this article and combinatorial sums derived from them, have the potential to be used by researchers in similar areas.

The applications of the special numbers and polynomials produced by the generating functions in this paper are planned to be studied and investigated in the near future.

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