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Bipolar Picture Fuzzy Graphs with Application

Waheed Ahmad Khan [1,*], Babir Ali [1] and Abdelghani Taouti [2]

[1] Department of Mathematics, Attock Campus, University of Education, Attock 43600, Pakistan; msf1900184@ue.edu.pk
[2] ETS-Maths and NS Engineering Division, HCT, University City, Sharjah P.O. Box 7947, United Arab Emirates; gtaouti@hct.ac.ae or ganitaouti@yahoo.com.au
* Correspondence: wakhan@ue.edu.pk or sirwak2003@yahoo.com

Abstract: In this manuscript, we introduce and discuss the term bipolar picture fuzzy graphs along with some of its fundamental characteristics and applications. We also initiate the concepts of complete bipolar picture fuzzy graphs and strong bipolar picture fuzzy graphs. Firstly, we apply different types of operations to bipolar picture fuzzy graphs and then we introduce various products of bipolar picture fuzzy graphs. Several other terms such as order and size, path, neighbourhood degrees, busy values of vertices and edges of bipolar picture fuzzy graphs are also discussed. These terminologies also lay the foundations for the discussion about the regular bipolar picture fuzzy graphs. Moreover, we also discuss isomorphisms, weak and co-weak isomorphisms and automorphisms of bipolar picture fuzzy graphs. Finally, at the base of bipolar picture fuzzy graph we present the construction of a bipolar picture fuzzy acquaintanceship graph, which would be an important tool to measure the symmetry or asymmetry of acquaintanceship levels of social networks, computer networks etc.

Keywords: bipolar picture fuzzy graphs; ring sum of bipolar picture fuzzy graph; busy value of bipolar picture fuzzy edge; weak and co-weak isomorphisms of bipolar picture fuzzy graphs

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1. Introduction

In 1965, Zadeh [1] introduced the term fuzzy sets (FSs), which is extensively used in different fields such as life sciences, social sciences, engineering, theory of decision making, computer sciences etc. Subsequently, many generalizations of the fuzzy sets have been explored in the literature like interval-valued fuzzy sets (IVFSs), bipolar fuzzy sets (BPFs), intuitionistic fuzzy sets (IFSs), picture fuzzy sets (PFSs) and so on (see e.g., [2,3]). The term interval-valued fuzzy set (IVFS) was also introduced by Zadeh [4]. Another generalization of fuzzy sets termed bipolar fuzzy sets (BPFSs) was introduced in [5]. In bipolar fuzzy sets (BPFSs) the membership value was considered in the interval [-1, 1]. In continuation, recently, the term bipolar Pythagorean fuzzy sets along with its applications towards decision making theory is explored in [6]. Various types of relations on BFSs were introduced in [7]. Basically, the term bipolar fuzzy relations (BPFRs) is the direct extension of fuzzy relations. BPFRs were also given a name "bifuzzy relations". Some new types of bipolar fuzzy relations and bipolar fuzzy equivalence relation were discussed in [7]. Atanassov [8] introduced the notion of intuitionistic fuzzy sets which was another generalized form of the fuzzy sets. Similarly, the generalization of both the fuzzy sets and intuitionistics fuzzy sets termed picture fuzzy sets (PFSs) was initiated by Cuong [9]. He also studied several operations and characteristics of PFSs. PFS is described by assigning three memberships values to the object which are neutral, positive and negative. After this, Bo et al. [10] introduced few new operations and relations on PFSs. Cuong et al. [11] introduced various types of fuzzy logical operators in the setting of PFSs.

On the other hand, Rosenfeld [12] extended the scope of fuzzy sets towards graph theory by initiating the notion of fuzzy graphs(FGs). Later on, Bhattacharya [13] added several terms in the theory of fuzzy graphs. Different types of operations were introduced and applied on fuzzy graphs (FGs) in [14]. The term complement of fuzzy graphs (FGs) was introduced by Mordeson and Nair [15]. Generalization of fuzzy graphs named interval-valued fuzzy graphs (IVFGs) were initited in [16]. The concepts of intuitionistic fuzzy graphs (IFGs) were explored in [17]. Several operations were defined and applied to IFGs in [18]. The term complex Intuitionistic fuzzy graphs and its applications toward cellular networking were explored in [19].

The term bipolar fuzzy graphs (BPFGs) was introduced by Akram [20], he also studied several interesting properties of these graphs. Similarly, Yang et al. [21] presented different types of BPFGs. Talebi and Rashmanlou [22] introduced the terms complement and isomorphism on bipolar fuzzy graphs, Ghorai and Pal [23] defined generalized regular bipolar fuzzy graphs. Further to this, Poulik and Ghorai [24] explored different indices on bipolar fuzzy graphs. Several characterizations of bipolar fuzzy graphs were extensively explored in [25]. They also presented the adjacency sequence of a vertex and first and second fundamental sequences were described in a bipolar fuzzy graph illustrative example. They also demonstrated through examples that if G is a regular bipolar fuzzy graph (RBFG), then its underlying crisp graph need not be regular and they showed that all the vertices need not have the same adjacency sequence. Moreover, they verified that if G and its underlying crisp graph are regular, then all of the vertices need not have the same adjacency sequence. At the base of adjacency sequences, they also provided necessary and sufficient condition for a BFG to be a regular with at most four vertices.

Further to the above, Zuo et al. [26] initiated the notion of picture fuzzy graphs (PFGs). They applied several operations on PFGs and presented some applications of PFGs towards social networking. Afterwards, picture fuzzy multi-graph (PFMG) was introduced in [27]. Regular picture fuzzy graphs (RPFGs) along with its applications towards networking communications have been explored in [28]. Recently, Koczy et al. [29] more investigated the term PFGs and they added several significant graphical terms for PFGs and demonstrated them with examples. They also verified the superiority of PFGs over FGs and IFGs by providing suitable examples. Specifically, they described two real-life problems including a social network and a Wi-Fi-network through picture fuzzy graphs and showed that the picture fuzzy graphs are more feasible than any other existing fuzzy structures. Recently, Amanathulla et al. [30] initiated the concept of balanced picture fuzzy graphs (balanced PFGs). This is a special type of PFG through which one can (balanced PFGs) define the density of a PFG based on weight and size of the graph. They also provided an application of balanced PFG in business alliance.

In this paper, we initiate the concepts of bipolar picture fuzzy graphs, complete bipolar picture fuzzy graphs and strong bipolar picture fuzzy graphs. We introduce the terms size of bipolar picture fuzzy graphs, path of bipolar picture fuzzy graphs, busy value of vertices and edges of a bipolar picture fuzzy graphs. We also study isomorphisms, weak and co-weak isomorphisms and automorphism of bipolar picture fuzzy graphs. We deduce in Proposition 1 that isomorphism between two bipolar picture fuzzy graphs is an equivalence relation and hence we can study the symmetry between two social networks through it. Finally, we construct a bipolar picture fuzzy acquaintanceship graph, which is asymmetric.

2. Preliminaries

In this section, we present some basic concepts related to fuzzy graphs. One may consult [31] for the basics of classical graph theory.

Definition 1. *[1] A fuzzy set (FS) S defined on X is represented by the collection*

$$S = \{(x, \alpha_S(x)) : x \in X, \alpha_S(x) \in [0,1]\}$$

Definition 2. *[32] The Cartesian product of the FSs $S_1, ..., S_n$ on $X_1, ..., X_n$ is the FS on the product $X_1 \times ... \times X_n$ having a membership function*

$$\mu_{(S_1 \times ... \times S_n)}(x) = \{\min(\alpha_{S_i}(x_i)) : x = (x_1, ..., x_n), x_i \in X_i\}$$

Definition 3. *[32] The* mth *power of a fuzzy S on X has the membership function*

$$\alpha_{S^m}(x) = \{[\alpha_S(x)]^m : x \in X\}$$

Definition 4. *[33] A bipolar fuzzy set (BPFS) is the pair (α^P, α^N), where $\alpha^P : X \to [0,1]$ and $\alpha^N : X \to [-1,0]$ represent mappings.*

Definition 5. *[33] A set $0_S = (0_S^P, 0_S^N)$ (resp., $1_S = (1_S^P, 1_S^N)$) is termed bipolar fuzzy empty set (resp., the bipolar fuzzy whole set) on X and is described as*

$$0_S^P(x) = 0 = 0_S^N(x) \text{ (resp., } 1_S^P(x) = 1 \text{ and } 1_S^N(x) = -1)$$

for each $x \in X$.

Definition 6. *[34] For any two BPFs $S = (\alpha_S^P, \alpha_S^N)$ and $T = (\alpha_T^P, \alpha_T^N)$, we have*

$$(S \cap T)(x) = ((\alpha_S^P(x) \wedge \alpha_T^P(x)), (\alpha_S^N(x) \vee \alpha_T^N(x)))$$

$$(S \cup T)(x) = ((\alpha_S^P(x) \vee \alpha_T^P(x)), (\alpha_S^N(x) \wedge \alpha_T^N(x)))$$

Definition 7. *[33] A mapping $S = (\alpha_S^P, \alpha_S^N) : X \times X \to [-1,0] \times [0,1]$ is a bipolar fuzzy relation (BPFR) on X, where $\alpha_S^P(x,y) \in [0,1]$ and $\alpha_S^N(x,y) \in [-1,0]$.*

Definition 8. *[33] The empty BPFR (resp., the whole BPFR) on X may be described by*

$$\alpha_S^P(x,y) = 0 = \alpha_S^N(x,y) \text{ (resp., } \alpha S^P(x,y) = 1 \text{ and } \alpha_S^N(x,y) = -1)$$

for each $x, y \in X$.

Definition 9. *[8] An intuitionistic fuzzy set (IFS) S on X is the collection $S = \{(x, \alpha_S(x), \beta_S(x)) : x \in X\}$, where $\alpha_S : X \to [0,1]$ is a membership degree while $\beta_S : X \to [0,1]$ represents a non-membership degree of $x \in X$, also for each $x \in X$, $0 \le \alpha_S(x) + \beta_S(x) \le 1$.*

Definition 10. *[35] A bipolar intuitionistic fuzzy set (BPIFS) can be described as $S = \{x, \alpha^P(x), \alpha^N(x), \beta^P(x), \beta^N(x) : x \in X\}$, where $\alpha^P : X \to [0,1]$, $\alpha^N : X \to [-1,0]$, $\beta^P : X \to [0,1]$ and $\beta^N : X \to [-1,0]$ are the mappings satisfying*

$$0 \le \alpha^P(x) + \beta^P(x) \le 1$$

$$-1 \le \alpha^N(x) + \beta^N(x) \le 0$$

Definition 11. *[9] A picture fuzzy set (PFS) S on X is the collection $S = \{(x, \alpha_S(x), \gamma_S(x), \beta_S(x)) : x \in X\}$, where $\alpha_S(x) \in [0,1]$ is the positive membership degree of x in S, $\gamma_S(x) \in [0,1]$ represents the neutral membership degree of x in S and $\beta_S(x) \in [0,1]$ the negative membership degree of x in S, with α_S, γ_S and β_S satisfying $\alpha_S(x) + \gamma_S(x) + \beta_S(x) \le 1$, for all $x \in X$.*

Definition 12. *[20] A BPFG $S = \{u, \alpha^P(u), \alpha^N(u), \beta^P(u), \beta^N(u) : u \in U\}$, where $\alpha^P : U \to [0,1]$, $\alpha^N : U \to [-1,0]$, $\beta^P : U \to [0,1]$ and $\beta^N : U \to [-1,0]$ is said to be a bipolar fuzzy graph on underlying set U if, $\beta^P(u,v) \le min(\alpha^P(u), \alpha^P(v))$ and $\beta^N(u,v) \ge min(\alpha^N(u), \alpha^N(v))$, for all $u, v \in E = V \times V$.*

Definition 13. *[35] A bipolar intuitionistic fuzzy graph (BPIFG) on V is the pair* $G = (A, B)$*, where* $A = (\alpha_A^P(u),\ \alpha_A^N(u),\ \beta_A^P(u),\ \beta_A^N(u))$ *is a BPIFS on V and* $B = (\alpha_B^P(u),\ \alpha_B^N(u),\ \beta_B^P(u),\ \beta_B^N(u))$ *is a BPIFS on* $E \subseteq V \times V$ *satisfying*

$$\alpha_B^P(u,v) \leq min(\alpha_A^P(u), \alpha_A^P(v))$$
$$\alpha_B^N(u,v) \geq max(\alpha_A^N(u), \alpha_A^N(v))$$
$$\beta_B^P(u,v) \leq min(\beta_A^P(u), \beta_A^P(u))$$
$$\beta_B^N(u,v) \geq max(\beta_A^N(u), \beta_A^N(v))$$

for all $u, v \in E$*.*

Definition 14. *[35] A mapping* $S = (\alpha_S^P, \alpha_S^N, \beta_S^P, \beta_S^N) : X \times X \to [-1,0] \times [0,1] \times [-1,0] \times [0,1]$ *is a bipolar intuitionistic fuzzy relation (BPIFR) on X, where* $\alpha_S^P(x,y) \in [0,1]$*,* $\alpha_S^N(x,y) \in [-1,0]$*,* $\beta_S^P(x,y) \in [0,1]$ *and* $\beta_S^N(x,y) \in [-1,0]$*.*

Definition 15. *[9] A pair* $G = (A, B)$ *is said to be a picture fuzzy graph (PFG) on* $G^* = (V, E)$*, where* $A = (\alpha_A,\ \gamma_A\ ,\beta_A)$ *is a PFS on V and* $B = (\alpha_B,\ \gamma_B\ ,\beta_B)$ *is a PFS on* $E \subseteq V \in V$ *with*

$$\alpha_B(u,v) \leq min(\alpha_A(u), \alpha_A(v))$$
$$\gamma_B(u,v) \leq min(\gamma_A(u), \gamma_A(v))$$
$$\beta_B(u,v) \geq max(\beta_A(u), \beta_A(v))$$

3. Bipolar Picture Fuzzy Graphs (BPPFGs)

We begin this section with the definition of a bipolar picture fuzzy set (BPPFS) which is introduced by the first author (with Faiz and Taouti) in [36].

Definition 16. *[36] Let X be a nonempty set. A bipolar picture fuzzy set (BPPFS) on X is the collection* $S = \{x, \alpha^P(x), \alpha^N(x), \gamma^P(x), \gamma^N(x), \beta^P(x), \beta^N(x) : x \in X\}$*, where* $\alpha^P : X \to [0,1]$*,* $\alpha^N : X \to [-1,0]$*,* $\gamma^P : X \to [0,1]$*,* $\gamma^N : X \to [-1,0]$*,* $\beta^P : X \to [0,1]$ *and* $\beta^N : X \to [-1,0]$ *are the mappings with* $0 \leq \alpha^P(x) + \gamma^P(x) + \beta^P(x) \leq 1$*,* $-1 \leq \alpha^N(x) + \gamma^N(x) + \beta^N(x) \leq 0$*.*

Following [36], for each x in X, $\alpha^P(x)$ stands for the positive membership degree, $\beta^P(x)$ for the positive non-membership degree and $\gamma^P(x)$ for the positive neutral degree. Alternatively, $\alpha^N(x)$ represents the negative membership degree, $\beta^N(x)$ is the negative non-membership degree and $\gamma^N(x)$ is a negative neutral degree. On the other hand, if $\alpha^P(x) \neq 0$ while all other mappings are mapped to zero then it means that x has only a positive membership property of the bipolar picture fuzzy set. Similarly, if $\alpha^N(x) \neq 0$ while all other mappings matched to zero (or equal to zero) then it reflects that x has only the negative membership property of a BPPFS. Additionally, if $\gamma^P(x) \neq 0$ and remaining mappings are mapped to zero then it reflects that x has only the positive neutral property of a BPPFS. By $\gamma^N(x) \neq 0$ and the other mapping goes to zero then we mean that x has only the negative neutral property of a BPPFS. However, if $\beta^P(x) \neq 0$ while all other mapping matched to zero then it implies that x has only the positive nonmembership property of a BPPFS. Finally, if $\beta^N(x) \neq 0$ while remaining are zero then it implies that x has only the negative nonmembership property in a BPPFS.

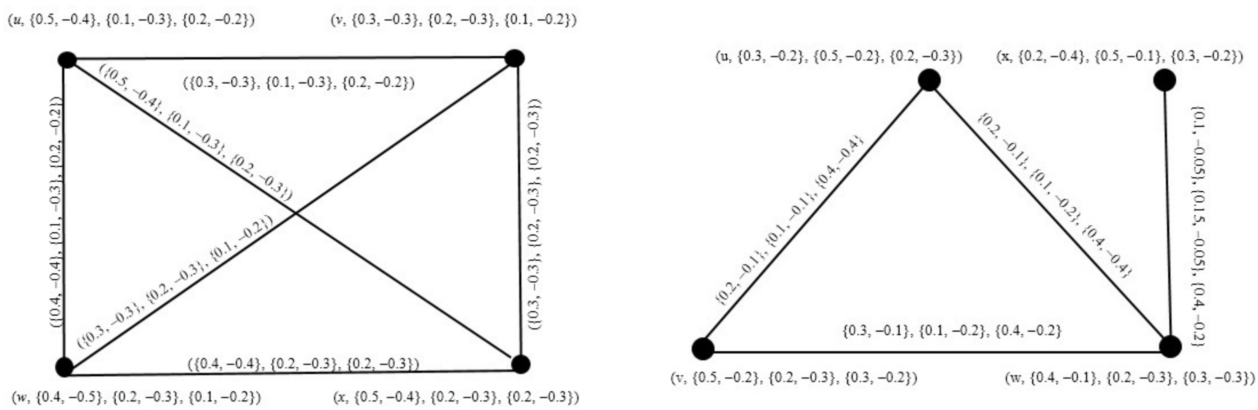
Definition 17. *Let* $G^* = (V, E)$ *be a graph. A pair* $G = (C, D)$ *is said to be a bipolar picture fuzzy graph (BPPFG) on* G^**, where* $C = \{\alpha_C^P(u),\ \alpha_C^N(u),\ \gamma_C^P(u),\ \gamma_C^N(u),\ \beta_C^P(u),\ \beta_C^N(u)\}$ *is a bipolar picture fuzzy set on V and* $D = \{\alpha_D^P(u,v),\ \alpha_D^N(u,v),\ \gamma_D^P(u,v),\ \gamma_D^N(u,v),\ \beta_D^P(u,v),\ \beta_D^N(u,v)\}$ *is a bipolar picture fuzzy set on* $E \subseteq V \times V$ *such that for every edge* $uv \in E$*,*

$$\alpha_D^P(uv) \le min(\alpha_C^P(u), \alpha_C^P(v)), \qquad \alpha_D^N(uv) \ge max(\alpha_C^N(u), \alpha_C^N(v))$$
$$\gamma_D^P(uv) \le min(\gamma_C^P(u), \gamma_C^P(u)), \qquad \gamma_D^N(uv) \ge max(\gamma_C^N(u), \gamma_C^N(v))$$
$$\beta_D^P(uv) \ge max(\beta_C^P(u), \beta_C^P(v)), \qquad \beta_D^N(uv) \le min(\beta_C^N(u), \beta_C^N(v))$$

satisfying

$$0 \le \alpha_D^P(uv) + \gamma_D^P(uv) + \beta_D^P(uv) \le 1$$
$$-1 \le \alpha_D^P(uv) + \gamma_D^P(uv) + \beta_D^P(uv) \le 0$$

Example 1. *One can easily verify that the graphs shown in Figure 1a,b are BPPFGs.*



(a) Complete bipolar picture fuzzy graph

(b) (Simple) Bipolar picture fuzzy graph

Figure 1. Bipolar picture fuzzy graph.

Definition 18. *The order $O(G)$ of a BPPFG $G = (C, D)$ is defined by $O(G) = (O_\alpha(G), O_\gamma(G), O_\beta(G))$, where*

$$O_\alpha(G) = (\sum_{u_i \in V} O_{\alpha^P}(G), \sum_{u_i \in V} O_{\alpha^N}(G))$$
$$O_\gamma(G) = (\sum_{u_i \in V} O_{\gamma^P}(G), \sum_{u_i \in V} O_{\gamma^N}(G)) \quad and$$
$$O_\beta(G) = (\sum_{u_i \in V} O_{\beta^P}(G), \sum_{u_i \in V} O_{\beta^N}(G))$$

Definition 19. *The size $S(G)$ of a BPPFG $G = (C, D)$ is denoted and defined by $S(G) = (S_\alpha(G), S_\gamma(G), S_\beta(G))$, where*

$$S_\alpha(G) = (\sum_{u_i \in V} S_{\alpha^P}(G), \sum_{u_i \in V} S_{\alpha^N}(G))$$
$$S_\gamma(G) = (\sum_{u_i \in V} S_{\gamma^P}(G), \sum_{u_i \in V} S_{\gamma^N}(G)) \quad and$$
$$S_\beta(G) = (\sum_{u_i \in V} S_{\beta^P}(G), \sum_{u_i \in V} S_{\beta^N}(G))$$

Definition 20. *Let $J^* = (V_1, E_1)$ and $K^* = (V_2, E_2)$ be two graphs. Let $J = (C_1, D_1)$ be a BPPFG on $J^* = (V_1, E_1)$, where $C_1 = \{\alpha_{C_1}^P(u), \alpha_{C_1}^N(u), \gamma_{C_1}^P(u), \gamma_{C_1}^N(u), \beta_{C_1}^P(u), \beta_{C_1}^N(u)\}$ is a BPPFS on V_1 and $D_1 = \{\alpha_{D_1}^P(u), \alpha_{D_1}^N(u), \gamma_{D_1}^P(u), \gamma_{D_1}^N(u), \beta_{D_1}^P(u), \beta_{D_1}^N(u)\}$ is a BPPFS on E_1, respectively. Let $K = (C_2, D_2)$ be a BPPFG on $K^* = (V_2, E_2)$, where $C_2 = \{\alpha_{C_2}^P(u), \alpha_{C_2}^N(u), \gamma_{C_2}^P(u), \gamma_{C_2}^N(u), \beta_{C_2}^P(u), \beta_{C_2}^N(u)\}$ is a BPPFS on V_2 and $D_2 = \{\alpha_{D_2}^P(u), \alpha_{D_2}^N(u), \gamma_{D_2}^P(u), \gamma_{D_2}^N(u),$*

$\beta^P_{D_2}(u)$, $\beta^N_{D_2}(u)\}$ is a BPPFS on E_2 be the two BPPFGs. Then the operations union and intersection between J and K can be defined as

$$J \cup K = (C_1 \cup C_2, D_1 \cup D_2) \tag{1}$$

For any vertex u:
Case (i):
$C_1 \cup C_2 = \{u, max(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), max(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), min(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), max(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), min(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), max(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_1 - V_2\}$
Case (ii):
$C_1 \cup C_2 = \{u, max(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), max(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), min(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), max(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), min(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), max(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_2 - V_1\}$
Case (iii):
$C_1 \cup C_2 = \{u, max(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), max(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), min(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), max(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), min(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), max(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_1 \cap V_2\}$
Similarly, for any edge uv:
Case (i):
$D_1 \cup D_2 = \{uv, max(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), max(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), min(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), max(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), min(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), max(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_1 - E_2\}$
Case (ii):
$D_1 \cup D_2 = \{uv, max(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), max(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), min(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), max(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), min(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), max(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_2 - E_1\}$
Case (iii):
$D_1 \cup D_2 = \{uv, max(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), max(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), min(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), max(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), min(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), max(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_1 \cap E_2\}$

$$J \cap K = (C_1 \cap C_2, D_1 \cap D_2) \tag{2}$$

For any vertex u:
Case (i):
$C_1 \cap C_2 = \{u, min(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), min(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), max(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), min(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), max(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), min(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_1 - V_2\}$
Case (ii):
$C_1 \cap C_2 = \{u, min(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), min(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), max(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), min(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), max(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), min(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_2 - V_1\}$
Case (iii):
$C_1 \cap C_2 = \{u, min(\alpha^P_{C_1}(u), \alpha^P_{C_2}(u)), min(\gamma^P_{C_1}(u), \gamma^P_{C_2}(u)), max(\beta^P_{C_1}(u), \beta^P_{C_2}(u)), min(\alpha^N_{C_1}(u), \alpha^N_{C_2}(u)), max(\gamma^N_{C_1}(u), \gamma^N_{C_2}(u)), min(\beta^N_{C_1}(u), \beta^N_{C_2}(u)) : u \in V_1 \cap V_2\}$
Similarly, for any edge uv:
Case (i):
$D_1 \cap D_2 = \{uv, min(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), min(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), max(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), min(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), max(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), min(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_1 - E_2\}$.
Case (ii):
$D_1 \cap D_2 = \{uv, min(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), min(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), max(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), min(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), max(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), min(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_2 - E_1\}$.
Case (iii):
$D_1 \cap D_2 = \{uv, min(\alpha^P_{D_1}(uv), \alpha^P_{D_2}(uv)), min(\gamma^P_{D_1}(uv), \gamma^P_{D_2}(uv)), max(\beta^P_{D_1}(uv), \beta^P_{D_2}(uv)), min(\alpha^N_{D_1}(uv), \alpha^N_{D_2}(uv)), max(\gamma^N_{D_1}(uv), \gamma^N_{D_2}(uv)), min(\beta^N_{D_1}(uv), \beta^N_{D_2}(uv)) : uv \in E_1 \cap E_2\}$.

Definition 21. *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be the two BPPFGs on $G^* = (V_1, E_1)$ and $G^{**} = (V_2, E_2)$, respectively. Then the ring sum $G_1 \bigoplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$ of BPPFGs of G_1 and G_2 is the graph $G = (C, D)$, where $C = (\alpha^P_C, \alpha^N_C, \gamma^P_C, \gamma^N_C, \beta^P_C, \beta^N_C)$ is bipolar picture fuzzy set on $V = V_1 \cup V_2$ and $D = (\alpha^P_D, \alpha^N_D, \gamma^P_D, \gamma^N_D, \beta^P_D, \beta^N_D)$ is a bipolar picture fuzzy*

set on $E = E_1 \cup E_2 - (E_1 \cap E_2)$ satisfying the following conditions.
(A)

$$\alpha_C^P(u) = \begin{cases} \alpha_{C_1}^P(u) & if \quad u \in V_1 \\ \alpha_{C_2}^P(u) & if \quad u \in V_2 \\ \alpha_{C_1}^P(u) \wedge \alpha_{C_2}^P(u) & if \quad u \in V_1 \cap V_2 \end{cases}$$

(B)

$$\alpha_C^P(u,v) = \begin{cases} \alpha_{C_1}^P(u,v) & if \quad u,v \in E_1 - E_2 \\ \alpha_{C_2}^P(u,v) & if \quad u,v \in E_2 - E_1 \\ \alpha_{C_1}^P(u,v) \wedge \alpha_{C_2}^P(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

(C)

$$\alpha_C^N(u) = \begin{cases} \alpha_{C_1}^N(u) & if \quad u \in V_1 \\ \alpha_{C_2}^N(u) & if \quad u \in V_2 \\ \alpha_{C_1}^N(u) \vee \alpha_{C_2}^P(u) & if \quad u \in V_1 \cap V_2 \end{cases}$$

(D)

$$\alpha_C^N(u,v) = \begin{cases} \alpha_{C_1}^N(u,v) & if \quad u,v \in E_1 - E_2 \\ \alpha_{C_2}^N(u,v) & if \quad u,v \in E_2 - E_1 \\ \alpha_{C_1}^N(u,v) \vee \alpha_{C_2}^P(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

(E)

$$\gamma_C^P(u) = \begin{cases} \gamma_{C_1}^P(u) & if \quad u \in V_1 \\ \gamma_{C_2}^P(u) & if \quad u \in V_2 \\ \gamma_{C_1}^P(u) \wedge \gamma_{C_2}^P(u) & if \quad u \in V_1 \cap V_2 \end{cases}$$

(F)

$$\gamma_C^P(u,v) = \begin{cases} \gamma_{C_1}^P(u,v) & if \quad u,v \in E_1 - E_2 \\ \gamma_{C_2}^P(u,v) & if \quad u,v \in E_2 - E_1 \\ \gamma_{C_1}^P(u,v) \wedge \gamma_{C_2}^P(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

(G)

$$\gamma_C^N(u) = \begin{cases} \gamma_{C_1}^N(u) & if \quad u \in V_1 \\ \gamma_{C_2}^N(u) & ifu \quad \in V_2 \\ \gamma_{C_1}^N(u) \vee \gamma_{C_2}^N(u) & if \quad u \in V_1 \cap V_2 \end{cases}$$

(H)

$$\gamma_C^N(u,v) = \begin{cases} \gamma_{C_1}^N(u,v) & if \quad u,v \in E_1 - E_2 \\ \gamma_{C_2}^N(u,v) & if \quad u,v \in E_2 - E_1 \\ \gamma_{C_1}^N(u,v) \vee \gamma_{C_2}^N(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

(I)

$$\beta_C^P(u) = \begin{cases} \beta_{C_1}^P(u) & if \quad u \in V_1 \\ \beta_{C_2}^P(u) & if \quad u \in V_2 \\ \beta_{C_1}^P(u) \vee \beta_{C_2}^P(u) & if \quad u \in V_1 \cap V_2 \end{cases}$$

(J)

$$\beta_C^P(u,v) = \begin{cases} \beta_{C_1}^P(u,v) & if \quad u,v \in E_1 - E_2 \\ \beta_{C_2}^P(u,v) & if \quad u,v \in E_2 - E_1 \\ \beta_{C_1}^P(u,v) \vee \beta_{C_2}^P(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

(K)

$$\beta_C^N(u) = \begin{cases} \beta_{C_1}^N(u) & if \quad u \in V_1 \\ \beta_{C_2}^N(u) & if \quad u \in V_2 \\ \beta_{C_1}^N(u) \wedge \beta_{C_2}^N(u) & if u \in V_1 \cap V_2 \end{cases}$$

(L)

$$\gamma_C^N(u,v) = \begin{cases} \beta_{C_1}^N(u,v) & if \quad u,v \in E_1 - E_2 \\ \beta_{C_2}^N(u,v) & if \quad u,v \in E_2 - E_1 \\ \beta_{C_1}^N(u,v) \wedge \beta_{C_2}^N(u,v) & if \quad u,v \in E_1 \cap E_2 \end{cases}$$

where uv represents an edge between the two vertices u, v while E_1, E_2 represent edges sets in G_1 and G_2, respectively.

Theorem 1. *Ring sum of two BPPFGs is a BPPFG.*

Proof. Let us consider two BPPFGs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ defined on crisp graphs $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$. Then, their ring sum $G_1 \oplus G_2 = G = (C, D)$ is BPPFG. Where $C = (\alpha_C^P(u), \alpha_C^N(u), \gamma_C^P(u), \gamma_C^N(u), \beta_C^P(u), \beta_C^N(u))$ and $D = (\alpha_D^P(u,v), \alpha_D^N(u,v), \gamma_D^P(u,v), \gamma_D^N(u,v), \beta_D^P(u,v), \beta_D^N(u,v))$. Then we have the following cases.
Case 1:
If $u \in V_1$, then $\alpha_C^P(u) = \alpha_{C_1}^P(u) \in V_1$, which is a BPPFS on V_1. Additionally, if $u, v \in V_1$, then $\alpha_C^P(u,v) = \alpha_{C_1}^P(u,v) \in E_1$, which is a BPPFS on E_1.
Case 2:
If $u \in V_2$, then $\alpha_C^P(u) = \alpha_{C_2}^P(u) \in V_2$, which is a BPPFS on V_2. Additionally, if $u, v \in V_2$, then $\alpha_C^P(u,v) = \alpha_{C_2}^P(u,v) \in E_2$, which is a BPPFS on E_2.
Case 3:
If $u \in V_1 \cap V_2$, then $\alpha_C^P(u) = \alpha_{C_1}^P(u) \wedge \alpha_{C_2}^P(u) \in V_1 \cap V_2$, which is a BPPFS. Additionally, if $u, v \in V_1 \cap V_2$, then $\alpha_C^P(u,v) = \alpha_{C_1}^P(u,v) \wedge \alpha_{C_2}^P(u,v) \in E_1 \cap E_2$, which is BPPFR on $V_1 \cap V_2 \times V_1 \cap V_2$.
Similarly, we can show for all $\alpha_C^N(u)$, $\gamma_C^P(u)$, $\gamma_C^N(u)$, $\beta_C^P(u)$, $\beta_C^N(u) \in C$ and $\alpha_D^N(u,v)$, $\gamma_D^P(u,v)$, $\gamma_D^N(u,v)$, $\beta_D^P(u,v)$, $\beta_D^N(u,v) \in D$. Since, $V_1, E_1 \in G_1$, $V_2, E_2 \in G_2$ and G_1, G_2 are BPPFGs. Hence $G_1 \oplus G_2 = G$ is a BPPFG. □

Proposition 1. *Let $H = (C, D)$ be a BPPFG on $G = (V, E)$. Then $H \cup H = H \cap H = H$ and $H \oplus H = \emptyset$ are BPPFGs.*

Proof. Let $H = (C, D)$ be a BPPFG on $H^* = (V, E)$, where $C = \{\alpha_C^P(u), \alpha_C^N(u), \gamma_C^P(u), \gamma_C^N(u), \beta_C^P(u), \beta_C^N(u)\}$ is a BPPFS on V and $D = \{\alpha_D^P(u), \alpha_D^N(u), \gamma_D^P(u), \gamma_D^N(u), \beta_D^P(u), \beta_D^N(u)\}$ is a BPPFS on E, respectively. For $H \cup H = (C \cup C, D \cup D)$, by Definition 20(1), we have
$C \cup C = \{u, \max(\alpha_C^P(u), \alpha_C^P(u)), \max(\gamma_C^P(u), \gamma_C^P(u)), \min(\beta_C^P(u), \beta_C^P(u)), \max(\alpha_C^N(u), \alpha_C^N(u)), \min(\gamma_C^N(u), \gamma_C^N(u)), \max(\beta_C^N(u), \beta_C^N(u)) : u \in V\}$ and
$D \cup D = \{uv, \max(\alpha_D^P(uv), \alpha_D^P(uv)), \max(\gamma_D^P(uv), \gamma_D^P(uv)), \min(\beta_D^P(uv), \beta_D^P(uv)), \max(\alpha_D^N(uv), \alpha_D^N(uv)), \min(\gamma_D^N(uv), \gamma_D^N(uv)), \max(\beta_D^N(uv), \beta_D^N(uv)) : u, v \in E\}$.
Thus, we have $C \cup C = C$ and $D \cup D = D$. Hence $H \cup H = H$.
Similarly, for $H \cap H = (C \cap C, D \cap D)$, by Definition 20(2), we have
$C \cap C = \{u, min(\alpha_C^P(u), \alpha_C^P(u)), \min(\gamma_C^P(u), \gamma_C^P(u)), \max(\beta_C^P(u), \beta_C^P(u)), \min(\alpha_C^N(u), \alpha_C^N(u)), \max(\gamma_C^N(u), \gamma_C^N(u)), \min(\beta_C^N(u), \beta_C^N(u)) : u \in V\}$ and
$D \cap D = \{uv, min(\alpha_D^P(uv), \alpha_D^P(uv)), \min(\gamma_D^P(uv), \gamma_D^P(uv)), \max(\beta_D^P(uv), \beta_D^P(uv)), \min(\alpha_D^N(uv), \alpha_D^N(uv)), \max(\gamma_D^N(uv), \gamma_D^N(uv)), \min(\beta_D^N(uv), \beta_D^N(uv)) : uv \in E\}$.
Thus, $C \cap C = C$ and $D \cap D = D$ implies $H \cap H = H$.

Finally, to prove $H \oplus H = \emptyset$. Let $u \in V$ be any vertex, then by Definition (21) we have

$$\alpha_C^P(u) = \begin{cases} \alpha_C^P(u) & if \quad u \in V \\ \alpha_C^P(u) & if \quad u \in V \\ \alpha_C^P(u) \wedge \alpha_C^P(u) & if \quad u \in V \cap V \end{cases}$$

Hence, $\alpha_C^P(u) = \alpha_C^P(u), \forall\, u \in V$. Similarly, for any edge $(u,v) \in E$. Following Definition 21, we have

$$\alpha_C^P(u,v) = \begin{cases} \alpha_C^P(u,v) & if \quad u,v \in E - E \\ \alpha_C^P(u,v) & if \quad u,v \in E - E \\ \alpha_C^P(u,v) \wedge \alpha_C^P(u,v) & if \quad u,v \in E \cap E \end{cases}$$

It implies $\alpha_C^P(u,v) = \emptyset, \forall\, uv \in E - E$. Thus, $H \oplus H = \emptyset$, which completes the proof. □

Definition 22. *The open neighbourhood degree of a vertex m of a BPPFG $H = (C, D)$ is deg(m) = $(d(\alpha_C^P(u)), d(\alpha_C^N(u)), d(\gamma_C^P(u)), d(\gamma_C^N(u)), d(\beta_C^P(u)), d(\beta_C^N(u)))$, where*

$$\begin{aligned} d(\alpha_C^P(m)) &= \sum_{n \in N(x)} \alpha_C^P(n), \qquad & d(\alpha_C^N(m)) &= \sum_{n \in N(x)} \alpha_C^N(n) \\ d(\gamma_C^P(m)) &= \sum_{n \in N(x)} \gamma_C^P(n), \qquad & d(\gamma_C^N(m)) &= \sum_{n \in N(x)} \gamma_C^N(n) \\ d(\beta_C^P(m)) &= \sum_{n \in N(x)} \beta_C^P(n), \qquad & d(\beta_C^N(m)) &= \sum_{n \in N(x)} \beta_C^N(n) \end{aligned}$$

Definition 23. *A vertex u in a BPPFG $H = (C, D)$ is said to be a busy vertex, if*

$$\begin{aligned} \alpha_C^P(u) &\le d(\alpha_C^P(u)), \qquad & \alpha_C^N(u) &\ge d(\alpha_C^N(u)) \\ \gamma_C^P(u) &\le d(\gamma_C^P(u)), \qquad & \gamma_C^N(u) &\ge d(\gamma_C^N(u)) \quad and \\ \beta_C^P(u) &\ge d(\beta_C^P(u)), \qquad & \beta_C^N(u) &\le d(\beta_C^N(u)) \end{aligned}$$

Otherwise, it is a free vertex.

Definition 24. *The busy value of a vertex u of a BPPFG $H = (C, D)$ is defined by $J(u) = (J(\alpha_C^P)(u), J(\alpha_C^N)(u), J(\gamma_C^P)(u), J(\gamma_C^N)(u), J(\beta_C^P)(u), J(\beta_C^N)(u))$, where*

$$\begin{aligned} J(\alpha_C^P)(u) &= \sum \alpha_C^P(u) \wedge \alpha_C^P(u_i), \qquad & J(\alpha_C^N)(u) &= \sum \alpha_C^N(u) \vee \alpha_C^N(u_i) \\ J(\gamma_C^P)(u) &= \sum \gamma_C^P(u) \wedge \gamma_C^P(u_i), \qquad & J(\gamma_C^N)(u) &= \sum \gamma_C^N(u) \vee \gamma_C^N(u_i) \\ J(\beta_C^P)(u) &= \sum \beta_C^P(u) \vee \beta_C^P(u_i), \qquad & J(\beta_C^N)(u) &= \sum \beta_C^N(u) \wedge \beta_C^N(u_i) \end{aligned}$$

$u_{i,s}$ represent the neighbors of u, the sum of the busy values of all vertices of H i.e., $J(H) = \sum J(u_i)$ is said to be a busy value of a BPPFG H.

Definition 25. *The busy value of an edge uv of a BPPFG $H = (C, D)$ is defined by $J(uv) = (J(\alpha_D^P)(uv), J(\alpha_D^N)(uv), J(\gamma_D^P)(uv), J(\gamma_D^N)(uv), J(\beta_D^P)(uv), J(\beta_D^N)(uv))$ such that*

$$\begin{aligned} J(\alpha_D^P(uv)) &\le min(J(\alpha_C^P(u)), J(\alpha_C^P(v))), \qquad & J(\alpha_D^N(uv)) &\ge max(J(\alpha_C^N(u)), J(\alpha_C^N(v))) \\ J(\gamma_D^P(uv)) &\le min(J(\gamma_C^P(u), \gamma_C^P(u))), \qquad & J(\gamma_D^N(uv)) &\ge max(J(\gamma_C^N(u)), J(\gamma_C^N(v))) \\ J(\beta_D^P(uv)) &\ge max(J(\beta_C^P(u)), J(\beta_C^P(v))), \qquad & J(\beta_D^N(uv)) &\le min(J(\beta_C^N(u)), J(\beta_C^N(v))) \end{aligned}$$

Definition 26. *The set of sequence of different vertices $v_0, v_1, v_2, \ldots, v_k$ is the path p in a BPPFG $H = (C, D)$ such that $(\alpha^P(v_{i-1}, v_i), \gamma^P(v_{i-1}, v_i), \beta^P(v_{i-1}, v_i)) \ge 0$ and $(\alpha^N(v_{i-1}, v_i), \gamma^N(v_{i-1}, v_i), \beta^N(v_{i-1}, v_i)) \le 0$; $i = 1, 2, 3, \ldots, k$.*

Definition 27. *Two vertices u and v are connected by a path p i.e., $p: u_0, u_1, u_2, ...u_{k-1}, u_k$ of length l in a BPPFG $H = (C, D)$. Then, $\alpha^P(u,v), \gamma^P(u,v), \beta^P(u,v)$, $\alpha^N(u,v), \gamma^N(u,v)$ and $\beta^N(u,v)$ are illustrated as follows.*

$$
\begin{aligned}
\alpha^P(u,v) =& \alpha^P(u,u_1) \wedge \alpha^P(u_1,u_2) \wedge \alpha^P(u_2,u_3) \wedge ... \wedge \alpha^P(u_{k-1},v)\\
\gamma^P(u,v) =& \gamma^P(u,u_1) \wedge \gamma^P(u_1,u_2) \wedge \gamma^P(u_2,u_3) \wedge ... \wedge \gamma^P(u_{k-1},v)\\
\beta^P(u,v) =& \beta^P(u,u_1) \vee \beta^P(u_1,u_2) \vee \beta^P(u_2,u_3) \vee ... \vee \beta^P(u_{k-1},v)\\
\alpha^N(u,v) =& \alpha^N(u,u_1) \vee \alpha^N(u_1,u_2) \vee \alpha^N(u_2,u_3) \vee ... \vee \alpha^N(u_{k-1},v)\\
\gamma^N(u,v) =& \gamma^N(u,u_1) \vee \gamma^N(u_1,u_2) \vee \gamma^N(u_2,u_3) \vee ... \vee \gamma^N(u_{k-1},v) \quad and\\
\beta^N(u,v) =& \beta^N(u,u_1) \wedge \beta^N(u_1,u_2) \wedge \beta^N(u_2,u_3) \wedge ... \wedge \beta^N(u_{k-1},v)
\end{aligned}
$$

Theorem 2. *Let $H = (C, D)$ be a BPPFG. If H contains a "$x - y$" walk of length k, then H contains a "$x - y$" path of length k.*

3.1. Different Types of Products of Bipolar Picture Fuzzy Graphs

Definition 28. *The strong product of two BPPFGs $H_1 = (C_1, D_1)$, where $C_1 = (\alpha^P_{C_1}, \alpha^N_{C_1}, \gamma^P_{C_1}, \gamma^N_{C_1}, \beta^P_{C_1}, \beta^N_{C_1})$, $D_1 = (\alpha^P_{D_1}, \alpha^N_{D_1}, \gamma^P_{D_1}, \gamma^N_{D_1}, \beta^P_{D_1}, \beta^N_{D_1})$ and $H_2 = (C_2, D_2)$, where $C_2 = (\alpha^P_{C_2}, \alpha^N_{C_2}, \gamma^P_{C_2}, \gamma^N_{C_2}, \beta^P_{C_2}, \beta^N_{C_2})$, $D_2 = (\alpha^P_{D_2}, \alpha^N_{D_2}, \gamma^P_{D_2}, \gamma^N_{D_2}, \beta^P_{D_2}, \beta^N_{D_2})$, where we take $V_1 \cap V_2 = \emptyset$, is defined as*
$H_1 \otimes H_2 = (\alpha^P_{C_1} \otimes \alpha^P_{C_2}, \gamma^P_{C_1} \otimes \gamma^P_{C_2}, \beta^P_{C_1} \otimes \beta^P_{C_2}, \alpha^N_{C_1} \otimes \alpha^N_{C_2}, \gamma^N_{C_1} \otimes \gamma^N_{C_2}, \beta^N_{C_1} \otimes \beta^N_{C_2}, \alpha^P_{D_1} \otimes \alpha^P_{D_2}, \gamma^P_{D_1} \otimes \gamma^P_{D_2}, \beta^P_{D_1} \otimes \beta^P_{D_2}, \alpha^N_{D_1} \otimes \alpha^N_{D_2}, \gamma^N_{D_1} \otimes \gamma^N_{D_2}, \beta^N_{D_1} \otimes \beta^N_{D_2})$ of $H^* = (V_1 \times V_2, E)$. Where E $= \{(m,x_1)(m,x_2) : m \in V_1, (x_1,x_2) \in E_2\} \cup \{(m_1,z)(m_2,z) : z \in V_2, (m_1,m_2) \in E_1\} \cup \{(m_1,y_1)(m_2,y_2) : (m_1,m_2) \in E_1, (y_1,y_2) \in E_2\}$
and
$\alpha^P_{C_1} \otimes \alpha^P_{C_2}(m,n) = \alpha^P_{C_1}(m) \vee \alpha^P_{C_2}(n)$, $\gamma^P_{C_1} \otimes \gamma^P_{C_2}(m,n) = \gamma^P_{C_1}(m) \vee \gamma^P_{C_2}(n)$, $\beta^P_{C_1} \otimes \beta^P_{C_2}(m,n) = \beta^P_{C_1}(m) \wedge \beta^P_{C_2}(n)$,
$\alpha^N_{C_1} \otimes \alpha^N_{C_2}(m,n) = \alpha^N_{C_1}(m) \vee \alpha^N_{C_2}(n)$, $\gamma^N_{C_1} \otimes \gamma^N_{C_2}(m,n) = \gamma^N_{C_1}(m) \vee \gamma^N_{C_2}(n)$, $\beta^N_{C_1} \otimes \beta^N_{C_2}(m,n) = \beta^N_{C_1}(m) \wedge \beta^N_{C_2}(n)$,
for all $(m, m_1, m_2, x_1, x_2, y_1, y_2) \in V_1 \times V_2$. Similarly,
$\alpha^P_{D_1} \otimes \alpha^P_{D_2}(m,n) = \alpha^P_{D_1}(m) \vee \alpha^P_{D_2}(n)$, $\gamma^P_{D_1} \otimes \gamma^P_{D_2}(m,n) = \gamma^P_{D_1}(m) \vee \gamma^P_{D_2}(n)$, $\beta^P_{D_1} \otimes \beta^P_{D_2}(m,n) = \beta^P_{D_1}(m) \wedge \beta^P_{D_2}(n)$,
$\alpha^P_{D_1} \otimes \alpha^P_{D_2}(m_1,x_1)(m_2,x_2) = \alpha^P_{D_1}(m_1,m_2) \vee \alpha^P_{D_2}(x_1,x_2)$, $\gamma^P_{D_1} \otimes \gamma^P_{D_2}(m_1,x_1)(m_2,x_2) = \gamma^P_{D_1}(m_1,m_2) \vee \gamma^P_{D_2}(x_1,x_2)$,
$\beta^P_{D_1} \otimes \beta^P_{D_2}(m_1,x_1)(m_2,x_2) = \beta^P_{D_1}(m_1,m_2) \wedge \beta^P_{D_2}(x_1,x_2)$, $\alpha^N_{D_1} \otimes \alpha^N_{D_2}(m_1,x_1)(m_2,x_2) = \alpha^N_{D_1}(m_1,m_2) \vee \alpha^N_{D_2}(x_1,x_2)$,
$\gamma^N_{D_1} \otimes \gamma^N_{D_2}(m_1,x_1)(m_2,x_2) = \gamma^N_{D_1}(m_1,m_2) \vee \gamma^N_{D_2}(x_1,x_2)$, $\beta^N_{D_1} \otimes \beta^N_{D_2}(m_1,x_1)(m_2,x_2) = \beta^N_{D_1}(m_1,m_2) \wedge \beta^N_{D_2}(x_1,x_2)$.

Remark 1. *The strong product of two BPPFGs is always a BPPFG.*

Definition 29. *The semi-strong product of two BPPFGs $G_1 = (C_1, D_1)$, where $C_1 = (\alpha^P_{C_1}, \alpha^N_{C_1}, \gamma^P_{C_1}, \gamma^N_{C_1}, \beta^P_{C_1}, \beta^N_{C_1})$, $D_1 = (\alpha^P_{D_1}, \alpha^N_{D_1}, \gamma^P_{D_1}, \gamma^N_{D_1}, \beta^P_{D_1}, \beta^N_{D_1})$ with crisp graphs $G_1^* = (V_1, E_1)$ and $G_2 = (C_2, D_2)$, where $C_2 = (\alpha^P_{C_2}, \alpha^N_{C_2}, \gamma^P_{C_2}, \gamma^N_{C_2}, \beta^P_{C_2}, \beta^N_{C_2})$, $D_2 = (\alpha^P_{D_2}, \alpha^N_{D_2}, \gamma^P_{D_2}, \gamma^N_{D_2}, \beta^P_{D_2}, \beta^N_{D_2})$ with crisp graph $G_2^* = (V_2, E_2)$, where we assume that $V_1 \cup V_2 = \emptyset$, is defined to be the BPPFG $G_1 \circ G_2 = (\alpha_1 \circ \alpha_2, \beta_1 \circ \beta_2, \gamma_1 \circ \gamma_2)$ with crisp graph $G^* = (V_1 \times V_2, E)$ such that $E = \{(x,y_1)(x,y_2) : x \in V_1, (y_1,y_2) \in E_2\} \cup \{(x_1,y_1)(x_2,y_2) : (x_1,x_2) \in E_1, (y_1,y_2) \in E_2\}$. Then*
(i)
$(\alpha^p_{C_1} \circ \alpha^P_{C_2})(x,y) = min(\alpha^P_{C_1}(x), \alpha^P_{C_2}(y))$, $(\alpha^N_{C_1} \circ \alpha^N_{C_2})(x,y) = max(\alpha^N_{C_1}(x), \alpha^N_{C_2}(y))$ for all $(x,y) \in V_1 \times V_2$

(ii)
$(\gamma^{p}_{C_1} \circ \gamma^{P}_{C_2})(x,y) = min(\gamma^{P}_{C_1}(x), \gamma^{P}_{C_2}(y))$, $(\gamma^{N}_{C_1} \circ \gamma^{N}_{C_2})(x,y) = max(\gamma^{N}_{C_1}(x), \gamma^{N}_{C_2}(y))$ *for all* $(x,y) \in V_1 \times V_2$
(iii)
$(\beta^{p}_{C_1} \circ \beta^{P}_{C_2})(x,y) = max(\beta^{P}_{C_1}(x), \beta^{P}_{C_2}(y))$, $(\beta^{N}_{C_1} \circ \beta^{N}_{C_2})(x,y) = min(\beta^{N}_{C_1}(x), \beta^{N}_{C_2}(y))$ *for all* $(x,y) \in V_1 \times V_2$
(iv)
$(\alpha^{P}_{D_1} \circ \alpha^{P}_{D_2})((x,y_1)(x,y_2)) = min(\alpha^{P}_{C_1}(x), \alpha^{P}_{D_2}(y_1,y_2))$ *and* $(\alpha^{P}_{D_1} \circ \alpha^{P}_{D_2})((x_1,y_1)(x_2,y_2)) = min(\alpha^{P}_{D_1}(x_1,x_2), \alpha^{P}_{D_2}(y_1,y_2))$
$(\alpha^{N}_{D_1} \circ \alpha^{N}_{D_2})((x,y_1)(x,y_2)) = max(\alpha^{N}_{C_1}(x), \alpha^{N}_{D_2}(y_1,y_2))$ *and* $(\alpha^{N}_{D_1} \circ \alpha^{N}_{D_2})((x_1,y_1)(x_2,y_2)) = max(\alpha^{N}_{D_1}(x_1,x_2), \alpha^{N}_{D_2}(y_1,y_2))$
(v)
$(\gamma^{P}_{D_1} \circ \gamma^{P}_{D_2})((x,y_1)(x,y_2)) = min(\gamma^{P}_{C_1}(x), \gamma^{P}_{D_2}(y_1,y_2))$ *and* $(\gamma^{P}_{D_1} \circ \gamma^{P}_{D_2})((x_1,y_1)(x_2,y_2)) = min(\gamma^{P}_{D_1}(x_1,x_2), \gamma^{P}_{D_2}(y_1,y_2))$
$(\gamma^{N}_{D_1} \circ \gamma^{N}_{D_2})((x,y_1)(x,y_2)) = max(\gamma^{N}_{C_1}(x), \gamma^{N}_{D_2}(y_1,y_2))$ *and* $(\gamma^{N}_{D_1} \circ \gamma^{N}_{D_2})((x_1,y_1)(x_2,y_2)) = max(\gamma^{N}_{D_1}(x_1,x_2), \gamma^{N}_{D_2}(y_1,y_2))$
(vi)
$(\beta^{P}_{D_1} \circ \beta^{P}_{D_2})((x,y_1)(x,y_2)) = max(\beta^{P}_{C_1}(x), \beta^{P}_{D_2}(y_1,y_2))$ *and* $(\beta^{P}_{D_1} \circ \beta^{P}_{D_2})((x_1,y_1)(x_2,y_2)) = max(\beta^{P}_{D_1}(x_1,x_2), \beta^{P}_{D_2}(y_1,y_2))$
$(\beta^{N}_{D_1} \circ \beta^{N}_{D_2})((x,y_1)(x,y_2)) = min(\beta^{N}_{C_1}(x), \beta^{N}_{D_2}(y_1,y_2))$ *and* $(\beta^{N}_{D_1} \circ \beta^{N}_{D_2})((x_1,y_1)(x_2,y_2)) = min(\beta^{N}_{D_1}(x_1,x_2), \beta^{N}_{D_2}(y_1,y_2))$.

Example 2. *Let us consider two BPPFGs graphs given in Figure 1a,b. Then their semi-strong product is as follows.*
(i) $(\alpha^{p}_{C_1} \circ \alpha^{P}_{C_2})(x,y) = min(\alpha^{P}_{C_1}(x), \alpha^{P}_{C_2}(y))$, $(\alpha^{N}_{C_1} \circ \alpha^{N}_{C_2})(x,y) = max(\alpha^{N}_{C_1}(x), \alpha^{N}_{C_2}(y))$ *for all* $(x,y) \in V_1 \times V_2$
(ii) $(\gamma^{p}_{C_1} \circ \gamma^{P}_{C_2})(x,y) = min(\gamma^{P}_{C_1}(x), \gamma^{P}_{C_2}(y))$, $(\gamma^{N}_{C_1} \circ \gamma^{N}_{C_2})(x,y) = max(\gamma^{N}_{C_1}(x), \gamma^{N}_{C_2}(y))$ *for all* $(x,y) \in V_1 \times V_2$
(iii) $(\beta^{p}_{C_1} \circ \beta^{P}_{C_2})(x,y) = max(\beta^{P}_{C_1}(x), \beta^{P}_{C_2}(y))$, $(\beta^{N}_{C_1} \circ \beta^{N}_{C_2})(x,y) = min(\beta^{N}_{C_1}(x), \beta^{N}_{C_2}(y))$ *for all* $(x,y) \in V_1 \times V_2$.
Consequently, for vertex u:
$(\alpha^{p}_{C_1} \circ \alpha^{P}_{C_2})(x_1,y_2) = min(0.6, 0.3) = 0.3$, $(\alpha^{N}_{C_1} \circ \alpha^{N}_{C_2})(x_1,y_2) = max(-0.4, -0.5) = -0.4$
$(\gamma^{p}_{C_1} \circ \gamma^{P}_{C_2})(x_1,y_2) = min(0.1, 0.5) = 0.1$, $(\gamma^{N}_{C_1} \circ \gamma^{N}_{C_2})(x_1,y_2) = max(-0.3, -0.2) = -0.2$
$(\beta^{p}_{C_1} \circ \beta^{P}_{C_2})(x_1,y_2) = max(0.2, 0.2) = 0.2$, $(\beta^{N}_{C_1} \circ \beta^{N}_{C_2})(x_1,y_2) = min(-0.2, -0.3) = -0.3$
(u, 0.3, −0.4, 0.1, −0.2, 0.2, −0.3)
Similarly, for vertex v, w and x:
(v, 0.3, −0.2, 0.2, −0.3, 0.3, −0.3), (w, 0.2, −0.1, 0.2, −0.3, 0.3, −0.2), (x, 0.2, −0.4, 0.4, −0.1, 0.3, −0.3)
Now edges of the semi−strong product of two graphs can be obtained by using (iv), (v) and (vi) of Definition 28
For an edge uv: (0.2, −0.1, 0.1, −0.1, 0.4, −0.2) For an edge wx: (0.1, −0.01, 0.15, −0.05, 0.5, −0.3)
For an edge vw: (0.3, −0.01, 0.1, −0.3, 0.4, −0.3) For an edge vx: (0.1, −0.2, 0.15, −0.3, 0.4, −0.3).
Graph shown in Figure 2 is the semi-strong product of the graphs of Figure 1a,b.

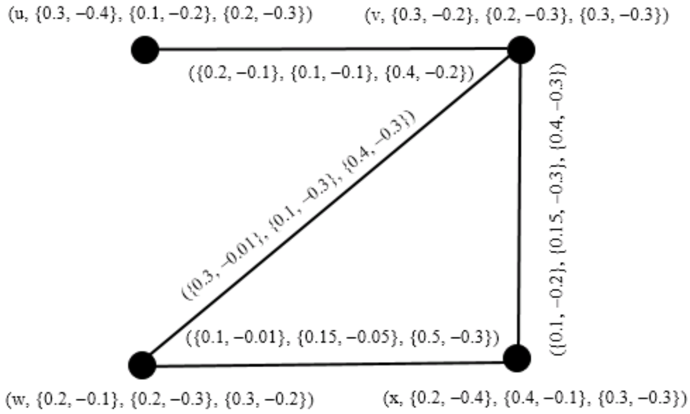


Figure 2. Semi-strong product of bipolar picture fuzzy graphs shown in Figure 1a,b.

Definition 30. *The normal product of two BPPFGs* $H_1 = (V_1, C_1, D_1)$ *and* $H_2 = (V_2, C_2, D_2)$ *with underlying crisp graphs* $H_1^* = (V_1, E_1)$ *and* $H_2^* = (V_2, E_2)$*, respectively, is defined as a BPPFG* $G = G_1 \bullet G_2 = (A_1 \bullet A_2, B_1 \bullet B_2)$ *with underline crisp graph* $H^* = (V, E)$*, where* $V = V_1 \times V_2$ *and* $E = \{(u,v)(w,x) : u = w, vx \in E_2 \text{ or } v = x, uw \in E_1\} \cup E = \{(u,w)(v,x) : uw \in E_1, vx \in E_2\}$ *with*

(i)

$\alpha^P_{C_1 \bullet C_2}(u,v) = (\alpha^P_{C_1}(u) \wedge \alpha^P_{C_2}(v))$, $\alpha^N_{C_1 \bullet C_2}(u,v) = (\alpha^N_{C_1}(u) \vee \alpha^N_{C_2}(v))$

$\gamma^P_{C_1 \bullet C_2}(u,v) = (\gamma^P_{C_1}(u) \wedge \gamma^P_{C_2}(v))$, $\gamma^N_{C_1 \bullet C_2}(u,v) = (\gamma^N_{C_1}(u) \vee \gamma^N_{C_2}(v))$

$\beta^P_{C_1 \bullet C_2}(u,v) = (\beta^P_{C_1}(u) \vee \beta^P_{C_2}(v))$, $\beta^N_{C_1 \bullet C_2}(u,v) = (\beta^N_{C_1}(u) \wedge \beta^N_{C_2}(v))$

for all $u, v \in V$

(ii)

$\alpha^P_{D_1 \bullet D_2}((u,v)(u,w)) = (\alpha^P_{C_1}(u) \wedge \alpha^P_{D_2}(u,w))$, $\alpha^N_{D_1 \bullet D_2}((u,v)(u,w)) = (\alpha^N_{C_1}(u) \vee \alpha^N_{D_2}(u,w))$

$\gamma^P_{D_1 \bullet D_2}((u,v)(u,w)) = (\gamma^P_{C_1}(u) \wedge \gamma^P_{D_2}(u,w))$, $\gamma^N_{D_1 \bullet D_2}((u,v)(u,w)) = (\gamma^N_{C_1}(u) \vee \gamma^N_{D_2}(u,w))$

$\beta^P_{D_1 \bullet D_2}((u,v)(u,w)) = (\beta^P_{C_1}(u) \vee \beta^P_{D_2}(u,w))$, $\beta^N_{D_1 \bullet D_2}((u,v)(u,w)) = (\beta^N_{C_1}(u) \wedge \beta^N_{D_2}(u,w))$

for all $u \in V_1$ *and* $vw \in E_2$

(iii)

$\alpha^P_{D_1 \bullet D_2}((u,w)(v,w)) = (\alpha^P_{C_1}(w) \wedge \alpha^P_{D_2}(u,v))$, $\alpha^N_{D_1 \bullet D_2}((u,w)(v,w)) = (\alpha^N_{C_1}(w) \vee \alpha^N_{D_2}(u,v))$

$\gamma^P_{D_1 \bullet D_2}((u,w)(v,w)) = (\gamma^P_{C_1}(w) \wedge \gamma^P_{D_2}(u,v))$, $\gamma^N_{D_1 \bullet D_2}((u,w)(v,w)) = (\gamma^N_{C_1}(w) \vee \gamma^N_{D_2}(u,v))$

$\beta^P_{D_1 \bullet D_2}((u,w)(v,w)) = (\beta^P_{C_1}(w) \vee \beta^P_{D_2}(u,v))$, $\beta^N_{D_1 \bullet D_2}((u,w)(v,w)) = (\beta^N_{C_1}(w) \wedge \beta^N_{D_2}(u,v))$

for all $w \in V_1$ *and* $uv \in E_1$

(iv)

$\alpha^P_{D_1 \bullet D_2}((u,v)(w,x)) = (\alpha^P_{C_1}(u,w) \wedge \alpha^P_{D_2}(v,x))$, $\alpha^N_{D_1 \bullet D_2}((u,v)(w,x)) = (\alpha^N_{C_1}(u,w) \vee \alpha^N_{D_2}(v,x))$

$\gamma^P_{D_1 \bullet D_2}((u,v)(w,x)) = (\gamma^P_{C_1}(u,w) \wedge \gamma^P_{D_2}(v,x))$, $\gamma^N_{D_1 \bullet D_2}((u,v)(w,x)) = (\gamma^N_{C_1}(u,w) \vee \gamma^N_{D_2}(v,x))$

$\beta^P_{D_1 \bullet D_2}((u,v)(w,x)) = (\beta^P_{C_1}(u,w) \vee \beta^P_{D_2}(v,x))$, $\beta^N_{D_1 \bullet D_2}((u,v)(w,x)) = (\beta^N_{C_1}(u,w) \wedge \beta^N_{D_2}(v,x))$

for all $uw \in E_1$ *and* $vx \in E_2$.

Definition 31. *Let* $H = H_1 \bullet H_2$ *with underlying crisp graph* $G^* = (V, E)$*, where* $V = V_1 \times V_2$, $E = E_1 \times E_2$ *be the normal product of two BPPFGs* $H_1 = (C_1, D_1)$ *and* $H_2 = (C_2, D_2)$ *with crisp graphs* $G_1^* = (V_1, E_1)$ *and* $G_2^* = (V_2, E_2)$*, respectively. Then the degree of the vertex* (u_1, u_2) *in* V *is denoted by* $d(H_1 \bullet H_2(u,v)) = d(\alpha^P_{H_1} \bullet \alpha^P_{H_2})(u,v)$, $d(\alpha^N_{H_1} \bullet \alpha^N_{H_2})$, $d(\gamma^P_{H_1} \bullet \gamma^P_{H_2})(u,v)$, $d(\gamma^N_{H_1} \bullet \gamma^N_{H_2})(u,v)$, $d(\beta^P_{H_1} \bullet \beta^P_{H_2})(u,v)$, $d(\beta^N_{H_1} \bullet \beta^N_{H_2})(u,v)$ *and is defined by*

(i)

$$d(\alpha^P_{H_1} \bullet \alpha^P_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\alpha^P_{C_1}(v) \wedge \alpha^P_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\alpha^P_{D_1}(u,w) \wedge \alpha^P_{C_2}(v)) =$$

$$\sum_{(u,w)\in E_1} (\alpha^P_{D_1}(u,w) \wedge \alpha^P_{D_2}(v,x))$$

(ii)

$$d(\alpha^N_{H_1} \bullet \alpha^N_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\alpha^N_{C_1}(v) \vee \alpha^N_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\alpha^N_{D_1}(u,w) \vee \alpha^N_{C_2}(v)) = \sum_{(u,w)\in E_1} (\alpha^N_{D_1}(u,w) \vee \alpha^N_{D_2}(v,x))$$

(iii)

$$d(\gamma^P_{H_1} \bullet \gamma^P_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\gamma^P_{C_1}(v) \wedge \gamma^P_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\gamma^P_{D_1}(u,w) \wedge \gamma^P_{C_2}(v)) = \sum_{(u,w)\in E_1} (\gamma^P_{D_1}(u,w) \wedge \gamma^P_{D_2}(v,x))$$

(iv)

$$d(\gamma^N_{H_1} \bullet \gamma^N_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\gamma^N_{C_1}(v) \vee \gamma^N_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\gamma^N_{D_1}(u,w) \vee \gamma^N_{C_2}(v)) = \sum_{(u,w)\in E_1} (\gamma^N_{D_1}(u,w) \vee \gamma^N_{D_2}(v,x))$$

(v)

$$d(\beta^P_{H_1} \bullet \beta^P_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\beta^P_{C_1}(v) \vee \beta^P_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\beta^P_{D_1}(u,w) \vee \beta^P_{C_2}(v)) = \sum_{(u,w)\in E_1} (\beta^P_{D_1}(u,w) \vee \beta^P_{D_2}(v,x))$$

(vi)

$$d(\beta^N_{H_1} \bullet \beta^N_{H_2})(u,v) = \sum_{v=x,(u,w)\in E_2} (\beta^N_{C_1}(v) \wedge \beta^N_{D_2}(u,w)) + \sum_{v=x,(u,w)\in E_1} (\beta^N_{D_1}(u,w) \wedge \beta^N_{C_2}(v)) = \sum_{(u,w)\in E_1} (\beta^N_{D_1}(u,w) \wedge \beta^N_{D_2}(v,x)).$$

Theorem 3. *Let $H_1 = (V_1, C_1, D_1)$ and $H_2 = (V_2, C_2, D_2)$ be two BPPFGs. If $\alpha^P_{C_1} \geq \alpha^P_{D_2}$, $\alpha^N_{C_1} \leq \alpha^N_{D_2}$, $\gamma^P_{C_1} \geq \gamma^P_{D_2}$, $\gamma^N_{C_1} \leq \gamma^N_{D_2}$, $\beta^P_{C_1} \leq \beta^P_{D_2}$, $\beta^N_{C_1} \geq \beta^N_{D_2}$ $\alpha^P_{C_2} \geq \alpha^P_{D_1}$, $\alpha^N_{C_2} \leq \alpha^N_{D_1}$, $\gamma^P_{C_2} \geq \gamma^P_{D_1}$, $\gamma^N_{C_2} \leq \gamma^N_{D_1}$, $\beta^P_{C_2} \leq \beta^P_{D_1}$, $\beta^N_{C_2} \geq \beta^N_{D_1}$ and $\alpha^P_{D_2} \geq \alpha^P_{D_1}$, $\alpha^N_{D_2} \leq \alpha^N_{D_1}$, $\gamma^P_{D_2} \geq \gamma^P_{D_1}$, $\gamma^N_{D_2} \leq \gamma^N_{D_1}$, $\beta^P_{D_2} \leq \beta^P_{D_1}$, $\beta^N_{D_2} \geq \beta^N_{D_1}$, then $d_{H_1 \bullet H_2}(u_1, u_2) = |V_2|\, d_{H_1}(u_1) + d_{H_2}(u_2)$.*

3.2. Homomorphism of Bipolar Picture Fuzzy Graphs

Definition 32. *Let H_1 and H_2 be the two BPPFGs. A homomorphism $f : H_1 \to H_2$ is the map $f : V_1 \to V_2$ satisfying*

(a) $\alpha^P_{C_1}(u) \leq \alpha^P_{C_2}(f(u))$, $\alpha^N_{C_1}(u) \geq \alpha^N_{C_2}(f(u))$

(b) $\gamma^P_{C_1}(u) \leq \gamma^P_{C_2}(f(u))$, $\gamma^N_{C_1}(u) \geq \gamma^N_{C_2}(f(u))$

(c) $\beta^P_{C_1}(u) \geq \beta^P_{C_2}(f(u))$, $\beta^N_{C_1}(u) \leq \beta^N_{C_2}(f(u))$

(d) $\alpha^P_{D_1}(uv) \leq \alpha^P_{D_2}(f(u)f(v))$, $\alpha^N_{D_1}(uv) \geq \alpha^N_{D_2}(f(u)f(v))$

(e) $\gamma^P_{D_1}(uv) \leq \gamma^P_{D_2}(f(u)f(v))$, $\gamma^N_{D_1}(uv) \geq \gamma^N_{D_2}(f(u)f(v))$

(f) $\beta^P_{D_1}(uv) \geq \beta^P_{D_2}(f(u)f(v))$, $\beta^N_{D_1}(uv) \leq \beta^N_{D_2}(f(u)f(v))$

for all $u \in V_1$, $uv \in E_1$.

Definition 33. *Let H_1 and H_2 be the two BPPFGs. An isomorphism $f : H_1 \to H_2$ is a bijective mapping $f : V_1 \to V_2$ which satisfies*

(a) $\alpha^P_{C_1}(u) = \alpha^P_{C_2} f(u)$, $\alpha^N_{C_1}(u) = \alpha^N_{C_2} f(u)$

(b) $\gamma^P_{C_1}(u) = \gamma^P_{C_2} f(u)$, $\gamma^N_{C_1}(u) = \gamma^N_{C_2} f(u)$

(c) $\beta^P_{C_1}(u) = \beta^P_{C_2} f(u)$, $\beta^N_{C_1}(u) = \beta^N_{C_2} f(u)$

(d) $\alpha^P_{D_1}(u,v) = \alpha^P_{D_2}(f(u)f(v))$, $\alpha^N_{D_1}(u,v) = \alpha^N_{D_2}(f(u)f(v))$

(e) $\gamma^P_{D_1}(u,v) = \gamma^P_{D_2}(f(u)f(v))$, $\gamma^N_{D_1}(u,v) = \gamma^N_{D_2}(f(u)f(v))$

(f) $\beta^P_{D_1}(u,v) = \beta^P_{D_2}(f(u)f(v))$, $\beta^N_{D_1}(u,v) = \beta^N_{D_2}(f(u)f(v))$

for all $x_1 \in V_1$, $x_1y_1 \in E_1$.

Proposition 2. *The isomorphism between BPPFGs is an equivalence relation.*

Definition 34. *Let H_1 and H_2 be the two BPPFGs. Then a weak isomorphism $h : G_1 \to G_2$ is a bijective map $h : V_1 \to V_2$ satifying*
(a) *h is a homomorphism.*
(b) $\alpha^P_{C_1}(u) = \alpha^P_{C_2} f(u)$, $\alpha^N_{C_1}(u) = \alpha^N_{C_2} f(u)$
(c) $\gamma^P_{C_1}(u) = \gamma^P_{C_2} f(u)$, $\gamma^N_{C_1}(u) = \gamma^N_{C_2} f(u)$
(d) $\beta^P_{C_1}(u) = \beta^P_{C_2} f(u)$, $\beta^N_{C_1}(u) = \beta^N_{C_2} f(u)$
for all $u \in V$. Evidently, the co-weak isomorphism fixes only the weights of the vertices.

Definition 35. *Let G_1, G_2 be the two BPPFGs. The co-weak isomorphism $h : G_1 \to G_2$ is the bijective map $h : V_1 \to V_2$ which satisfies*
(a) *h is a homomorphism*
(b) $\alpha^P_{D_1}(u,v) = \alpha^P_{D_2}(f(u)f(v))$, $\alpha^N_{D_1}(u,v) = \alpha^N_{D_2}(f(u)f(v))$
(c) $\gamma^P_{D_1}(u,v) = \gamma^P_{D_2}(f(u)f(v))$, $\gamma^N_{D_1}(u,v) = \gamma^N_{D_2}(f(u)f(v))$
(d) $\beta^P_{D_1}(u,v) = \beta^P_{D_2}(f(u)f(v))$, $\beta^N_{D_1}(u,v) = \beta^N_{D_2}(f(u)f(v))$
for all $uv \in E_1$. Evidently, the co-weak isomorphism fixes only the weights of the edges.

Proposition 3. *Weak isomorphism between BPPFGs always induces a partial order relation.*

Theorem 4. *Let $G = (A, B)$ be a BPPFG and Aut(G) be the set of all automorphisms of G. Then (Aut(G), $\circ$) forms a group.*

Proof. Let ρ, τ, $\varrho \in Aut(G)$ and let $u, v \in V$. Then

$$\alpha^P_C((\rho \circ \tau)(u)) = \alpha^P_C(\rho(\tau(u))) \geq \alpha^P_C(\rho(u)) \geq \alpha^P_C(u)$$
$$\alpha^N_C((\rho \circ \tau)(u)) = \alpha^N_C(\rho(\tau(u))) \leq \alpha^N_C(\rho(u)) \leq \alpha^N_C(u)$$
$$\gamma^P_C((\rho \circ \tau)(u)) = \gamma^P_C(\rho(\tau(u))) \geq \gamma^P_C(\rho(u)) \geq \gamma^P_C(u)$$
$$\gamma^N_C((\rho \circ \tau)(u)) = \gamma^N_C(\rho(\tau(u))) \leq \gamma^N_C(\rho(u)) \leq \gamma^N_C(u)$$
$$\beta^P_C((\rho \circ \tau)(u)) = \beta^P_C(\rho(\tau(u))) \leq \beta^P_C(\rho(u)) \leq \beta^P_C(u)$$
$$\beta^N_C((\rho \circ \tau)(u)) = \beta^N_C(\rho(\tau(u))) \geq \beta^N_C(\rho(u)) \geq \beta^N_C(u)$$
$$\alpha^P_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \alpha^P_D(\rho(\tau(u)))(\rho(\tau(v))) \geq \alpha^P_D((\rho(u))(\rho(v))) \geq \alpha^P_D(uv)$$
$$\gamma^P_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \gamma^P_D((\rho(\tau(u)))(\rho(\tau(v)))) \geq \gamma^P_D((\rho(u))(\rho(v))) \geq \gamma^P_D(uv)$$
$$\beta^P_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \beta^P_D((\rho(\tau(u)))(\rho(\tau(v)))) \leq \beta^P_D((\rho(u))(\rho(v))) \leq \beta^P_D(uv)$$
$$\alpha^N_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \alpha^N_D((\rho(\tau(u)))(\rho(\tau(v)))) \leq \alpha^N_D((\rho(u))(\rho(v))) \leq \alpha^N_D(uv)$$
$$\gamma^N_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \gamma^N_D((\rho(\tau(u)))(\rho(\tau(v)))) \leq \gamma^N_D((\rho(u))(\rho(v))) \leq \gamma^N_D(uv)$$
$$\beta^N_D((\rho \circ \tau)(u)(\rho \circ \tau)(v)) = \beta^N_D((\rho(\tau(u)))(\rho(\tau(v)))) \geq \beta^N_D((\rho(u))(\rho(v))) \geq \beta^N_D(uv)$$

Thus, $\rho \circ \tau \in$ Aut(G). Similarly, one can easily prove that $(\rho \circ \tau) \circ \varrho = \rho \circ (\tau \circ \varrho)$, where ρ, τ, $\varrho \in$ Aut(G). Additionally, we have the inverses for each $\rho \in$ Aut(G) defined as $\alpha^P_C(\rho^{-1}) = \alpha^P_C(\rho)$, $\alpha^N_C(\rho^{-1}) = \alpha^N_C(\rho)$, $\gamma^P_C(\rho^{-1}) = \gamma^P_C(\rho)$, $\gamma^N_C(\rho^{-1}) = \gamma^N_C(\rho)$, $\beta^P_C(\rho^{-1}) = \beta^P_C(\rho)$, $\beta^N_C(\rho^{-1}) = \beta^N_C(\rho)$. Similarly, there exists $e \in$ Aut(G). Let $\rho \circ e = \rho = e \circ \rho$. $\alpha^P_C((\rho \circ e)(u)) = \alpha^P_C(\rho(u))$ $\forall e, \rho \in$ Aut(G) is the identity element. Hence (Aut(G), $\circ$) forms a group. $\square$

Proposition 4. *Let $H = (C, D)$ be a BPPFG and Aut(H) be the set of all automorphisms of H. Let $h = (\alpha_h^P, \gamma_h^P, \beta_h^P, \alpha_h^N, \gamma_h^N, \beta_h^N)$ be a BPPFS in Aut(H) defined by*

$$\alpha_h^P(\rho) = sup\{\alpha_D^P(\rho(u), \rho(v))\}, \qquad \alpha_h^N(\rho) = inf\{\alpha_D^N(\rho(u), \rho(v))\}$$
$$\gamma_h^P(\rho) = sup\{\gamma_D^P(\rho(u), \rho(v))\}, \qquad \gamma_h^N(\rho) = inf\{\gamma_D^N(\rho(u), \rho(v))\}$$
$$\beta_h^P(\rho) = inf\{\beta_D^P(\rho(u), \rho(v))\}, \qquad \beta_h^N(\rho) = sup\{\beta_D^N(\rho(u), \rho(v))\}$$

for all $(u, v) \in V \times V$, $\rho \in Aut(H)$. Then, $h = (\alpha_h^P, \gamma_h^P, \beta_h^P, \alpha_h^N, \gamma_h^N, \beta_h^N)$ is a bipolar picture fuzzy group on Aut(H).

Proof. Follows from Theorem 3. □

3.3. Complete and Strong Bipolar Picture Fuzzy Graphs

Definition 36. *A BPPFG $G = (C, D)$ of a graph $G^* = (V, E)$, where $C = \{\alpha_C^P(u), \alpha_C^N(u), \gamma_C^P(u), \gamma_C^N(u), \beta_C^P(u), \beta_C^N(u)\}$ and $D = \{\alpha_D^P(u), \alpha_D^N(u), \gamma_D^P(u), \gamma_D^N(u), \beta_D^P(u), \beta_D^N(u)\}$ is called a complete bipolar picture fuzzy graph (complete BPPFG) if*

$$\alpha_D^P(uv) = min(\alpha_C^P(u), \alpha_C^P(v)), \qquad \alpha_D^N(uv) = max(\alpha_C^N(u), \alpha_C^N(v))$$
$$\gamma_D^P(uv) = min(\gamma_C^P(u), \gamma_C^P(v)), \qquad \gamma_D^N(uv) = max(\gamma_c^N(u), \gamma_c^N(v))$$
$$\beta_D^P(uv) = max(\beta_C^P(u), \beta_C^P(v)), \qquad \beta_D^N(uv) = min(\beta_C^N(u), \beta_C^N(v))$$

for all $u, v \in V$.

Example 3. *One can easily verify that the graph shown in Figure 1a is a complete BPPFG.*

Theorem 5. *Let $H_1 = (C_1, D_1)$ and $H_2 = (C_2, D_2)$ be two complete BPPFGs. Then their direct product $H_1 \otimes H_2$ is also a complete BPPFG.*

Proof. As we know that the strong product of BPPFGs is a BPPFG and each pair of vertices are adjacent, $E \subseteq V_1 \times V_2$. Now, for all $(u, v_1)(u, v_2) \in E$, since H_2 is complete
$(\alpha_{D_1}^P \otimes \alpha_{D_2}^P)((u, v_1), (u, v_2)) = \alpha_{C_1}^P(u) \wedge \alpha_{D_2}^P(v_1 v_2) = \alpha_{C_1}^P(u) \wedge \alpha_{C_2}^P(v_1) \wedge \alpha_{C_2}^P(v_2) = (\alpha_{C_1}^P \otimes \alpha_{C_2}^P)((u)) \wedge (\alpha_{C_1}^P \otimes \alpha_{C_2}^P)((v_1, v_2))$
$(\alpha_{D_1}^N \otimes \alpha_{D_2}^N)((u, v_1), (u, v_2)) = \alpha_{C_1}^N(u) \vee \alpha_{D_2}^N(v_1 v_2) = \alpha_{C_1}^N(u) \vee \alpha_{C_2}^N(v_1) \vee \alpha_{C_2}^N(v_2) = (\alpha_{C_1}^N \otimes \alpha_{C_2}^N)((u)) \vee (\alpha_{C_1}^N \otimes \alpha_{C_2}^N)((v_1, v_2))$
$(\gamma_{D_1}^P \otimes \gamma_{D_2}^P)((u, v_1), (u, v_2)) = \gamma_{C_1}^P(u) \wedge \gamma_{D_2}^P(v_1 v_2) = \gamma_{C_1}^P(u) \wedge \gamma_{C_2}^P(v_1) \wedge \gamma_{C_2}^P(v_2) = (\gamma_{C_1}^P \otimes \gamma_{C_2}^P)((u)) \wedge (\gamma_{C_1}^P \otimes \gamma_{C_2}^P)((v_1, v_2))$
$(\gamma_{D_1}^N \otimes \gamma_{D_2}^N)((u, v_1), (u, v_2)) = \gamma_{C_1}^N(u) \vee \gamma_{D_2}^N(v_1 v_2) = \gamma_{C_1}^N(u) \vee \gamma_{C_2}^N(v_1) \vee \gamma_{C_2}^N(v_2) = (\gamma_{C_1}^N \otimes \gamma_{C_2}^N)((u)) \vee (\gamma_{C_1}^N \otimes \gamma_{C_2}^N)((v_1, v_2))$
$(\beta_{D_1}^P \otimes \beta_{D_2}^P)((u, v_1), (u, v_2)) = \beta_{C_1}^P(u) \vee \beta_{D_2}^P(v_1 v_2) = \beta_{C_1}^P(u) \vee \beta_{C_2}^P(v_1) \vee \beta_{C_2}^P(v_2) = (\beta_{C_1}^P \otimes \beta_{C_2}^P)((u)) \vee (\beta_{C_1}^P \otimes \beta_{C_2}^P)((v_1, v_2))$
$(\beta_{D_1}^N \otimes \beta_{D_2}^N)((u, v_1), (u, v_2)) = \beta_{C_1}^N(u) \wedge \beta_{D_2}^N(v_1 v_2) = \beta_{C_1}^N(u) \wedge \beta_{C_2}^N(v_1) \wedge \beta_{C_2}^N(v_2) = (\beta_{C_1}^N \otimes \beta_{C_2}^N)((u)) \wedge (\beta_{C_1}^N \otimes \beta_{C_2}^N)((v_1, v_2))$
If $((u_1, w)(u_2, w)) \in E$, then
$(\alpha_{D_1}^P \otimes \alpha_{D_2}^P)((u_1, w), (u_2, w)) = \alpha_{D_1}^P(u_1 u_2) \wedge \alpha_{C_2}^P(w) = \alpha_{C_1}^P(u_1) \wedge \alpha_{C_1}^P(u_2) \wedge \alpha_{C_2}^P(w) = (\alpha_{C_1}^P \otimes$

$\alpha^P_{C_2})((u_1u_2)) \wedge (\alpha^P_{C_1} \otimes \alpha^P_{C_2})(w)$.
Similarly, one can easily verify that

$$
\begin{aligned}
(\alpha^N_{D_1} \otimes \alpha^N_{D_2})((u_1,w)(u_2,w)) &= (\alpha^N_{C_1} \otimes \alpha^N_{C_2})(u_1,w) \vee (\alpha^N_{C_1} \otimes \alpha^N_{C_2})(u_2,w)\\
(\gamma^P_{D_1} \otimes \gamma^P_{D_2})((u_1,w)(u_2,w)) &= (\gamma^P_{C_1} \otimes \gamma^P_{C_2})(u_1,w) \wedge (\gamma^P_{C_1} \otimes \gamma^P_{C_2})(u_2,w)\\
(\gamma^N_{D_1} \otimes \gamma^N_{D_2})((u_1,w)(u_2,w)) &= (\gamma^N_{C_1} \otimes \gamma^N_{C_2})(u_1,w) \vee (\gamma^N_{C_1} \otimes \gamma^N_{C_2})(u_2,w)\\
(\beta^P_{D_1} \otimes \beta^P_{D_2})((u_1,w)(u_2,w)) &= (\beta^P_{C_1} \otimes \beta^P_{C_2})(u_1,w) \vee (\beta^P_{C_1} \otimes \beta^P_{C_2})(u_2,w)\\
(\beta^N_{D_1} \otimes \beta^N_{D_2})((u_1,w)(u_2,w)) &= (\beta^N_{C_1} \otimes \beta^N_{C_2})(u_1,w) \wedge (\beta^N_{C_1} \otimes \beta^N_{C_2})(u_2,w)
\end{aligned}
$$

If $(u_1,v_1)(u_2,v_2) \in E$, then as H_1 and H_2 are complete
$(\alpha^P_{D_1} \otimes \alpha^P_{D_2})((u_1,v_1)(u_2,v_2)) = \alpha^P_{D_1}(u_1u_2) \wedge \alpha^P_{D_2}(v_1,v_2) = \alpha^P_{C_1}(u_1) \wedge \alpha^P_{C_1}(v_1) \wedge \alpha^P_{C_2}(u_1) \wedge \alpha^P_{C_2}(v_2)$.
Similarly, we can show that

$$
\begin{aligned}
(\alpha^N_{D_1} \otimes \alpha^N_{D_2})((u_1,v_1)(u_2,v_2)) &= \alpha^N_{C_1}(u_1) \vee \alpha^N_{C_1}(v_1) \vee \alpha^N_{C_2}(u_1) \vee \alpha^N_{C_2}(v_2)\\
(\gamma^P_{D_1} \otimes \gamma^P_{D_2})((u_1,v_1)(u_2,v_2)) &= \gamma^P_{C_1}(u_1) \wedge \gamma^P_{C_1}(v_1) \wedge \gamma^P_{C_2}(u_1) \wedge \gamma^P_{C_2}(v_2)\\
(\gamma^N_{D_1} \otimes \gamma^N_{D_2})((u_1,v_1)(u_2,v_2)) &= \gamma^N_{C_1}(u_1) \vee \gamma^N_{C_1}(v_1) \vee \gamma^N_{C_2}(u_1) \vee \gamma^N_{C_2}(v_2)\\
(\beta^P_{D_1} \otimes \beta^P_{D_2})((u_1,v_1)(u_2,v_2)) &= \beta^P_{C_1}(u_1) \vee \beta^P_{C_1}(v_1) \vee \beta^P_{C_2}(u_1) \vee \beta^P_{C_2}(v_2)\\
(\beta^P_{D_1} \otimes \beta^P_{D_2})((u_1,v_1)(u_2,v_2)) &= \beta^P_{C_1}(u_1) \wedge \beta^P_{C_1}(v_1) \wedge \beta^P_{C_2}(u_1) \wedge \beta^P_{C_2}(v_2).
\end{aligned}
$$

Hence, $H_1 \otimes H_2$ is a complete BPPFG. □

Definition 37. *A BPPFG $G = (C,D)$ on a graph $G^* = (V,E)$, where $C = \{\alpha^P_C(u), \alpha^N_C(u), \gamma^P_C(u), \gamma^N_C(u), \beta^P_C(u), \beta^N_C(u)\}$ and $D = \{\alpha^P_D(u), \alpha^N_D(u), \gamma^P_D(u), \gamma^N_D(u), \beta^P_D(u), \beta^N_D(u)\}$ is said to be a strong bipolar picture fuzzy graph (in short, BPPFG) if*

$$
\begin{aligned}
\alpha^P_D(uv) &= min(\alpha^P_C(u), \alpha^P_C(v)), & \alpha^N_D(uv) &= max(\alpha^N_C(u), \alpha^N_C(v))\\
\gamma^P_D(uv) &= min(\gamma^P_C(u), \gamma^P_C(v)), & \gamma^N_D(uv) &= max(\gamma^N_c(u), \gamma^N_c(v))\\
\beta^P_D(uv) &= max(\beta^P_C(u), \beta^P_C(v)), & \beta^N_D(uv) &= min(\beta^N_C(u), \beta^N_C(v))
\end{aligned}
$$

for all $u, v \in E$.

Example 4. *The graph shown in Figure 3 is a strong BPPFG.*

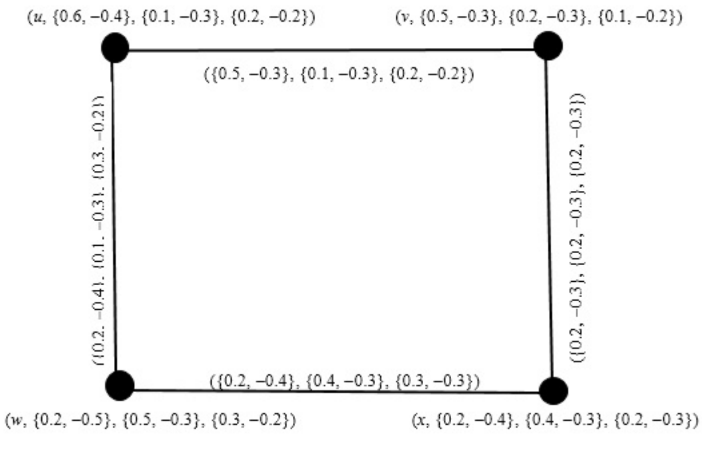


Figure 3. Strong bipolar picture fuzzy graph.

Remark 2. *Every complete BPPFG implies a strong BPPFG but the converse does not exist.*

Definition 38. *The complement of a strong BPPFG* $G = (C, D)$ *of a graph* $G^* = (V, E)$*, where* $C = \{\alpha_C^P(u), \alpha_C^N(u), \gamma_C^P(u), \gamma_C^N(u), \beta_C^P(u), \beta_C^N(u)\}$ *and* $D = \{\alpha_D^P(u), \alpha_D^N(u), \gamma_D^P(u), \gamma_D^N(u), \beta_D^P(u), \beta_D^N(u)\}$ *is a BPPFG* $\overline{G} = (\overline{C}, \overline{D})$ *of* $\overline{G^*} = (V, V \times V)$*, where* $\overline{C} = C = \{\alpha_C^P(u), \alpha_C^N(u), \gamma_C^P(u), \gamma_C^N(u), \beta_C^P(u), \beta_C^N(u)\}$ *and* $\overline{D} = D = \{\overline{\alpha_D^P}(uv), \overline{\alpha_D^N}(uv), \overline{\gamma_D^P}(uv), \overline{\gamma_D^N}(uv), \overline{\beta_D^P}(uv), \overline{\beta_D^N}(uv)\}$ *is defined by*

$$\overline{\alpha_D^P}(uv) = min(\alpha_C^P(u), \alpha_C^P(v)) - \alpha_D^P(uv), \qquad \overline{\alpha_D^N}(uv) = max(\alpha_C^N(u), \alpha_C^N(v)) - \alpha_D^N(uv)$$
$$\overline{\gamma_D^P}(uv) = min(\gamma_C^P(u), \gamma_C^P(v)) - \gamma_D^P(uv), \qquad \overline{\gamma_D^N}(uv) = max(\gamma_c^N(u), \gamma_c^N(v)) - \gamma_D^N(uv)$$
$$\overline{\beta_D^P}(uv) = max(\beta_C^P(u), \beta_C^P(v)) - \beta_D^P(uv), \qquad \overline{\beta_D^N}(uv) = min(\beta_C^N(u), \beta_C^N(v)) - \beta_D^N(uv)$$

for all $u, v \in V$, $uv \in \overline{V^2}$.

Example 5. *Graph in Figure 4 is the complement of a strong BPPFG shown in Figure 3.*

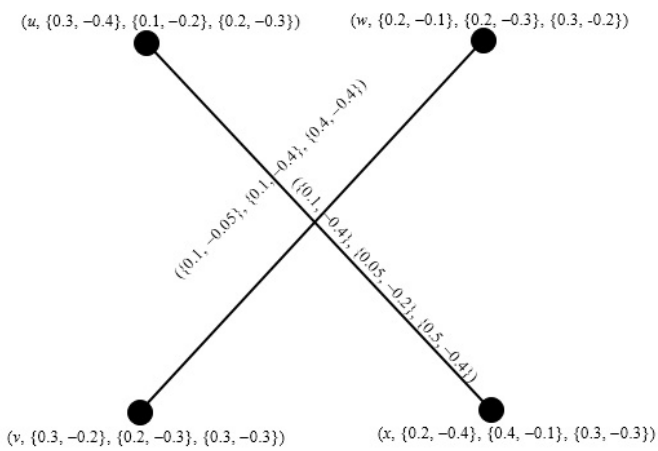


Figure 4. Complement of a strong bipolar picture fuzzy graph given in Figure 3.

Theorem 6. *Let* $G_1 = (C_1, D_1)$ *and* $G_2 = (C_2, D_2)$ *be the two strong BPPFGs. Then* $G_1 \sqcap G_2$ *is strong BPPFG.*

Proof. Let $(u_1, v_1)(u_2, v_2) \in E$. Since G_1 and G_2 are strong BPPFGs, we have
$(\alpha_{D_1}^P \sqcap \alpha_{D_2}^P)((u_1, v_1)(u_2, v_2)) = \alpha_{D_1}^P(u_1, v_1) \wedge \alpha_{D_2}^P(u_2, v_2) = \alpha_{C_1}^P(u_1) \wedge \alpha_{C_2}^P(u_2) \wedge \alpha_{C_1}^P(v_1) \wedge \alpha_{C_2}^P(v_2) = (\alpha_{C_1}^P \sqcap \alpha_{C_2}^P)(u_1, v_1) \wedge (\alpha_{C_1}^P \sqcap \alpha_{C_2}^P)(u_2, v_2)$
$(\alpha_{D_1}^N \sqcap \alpha_{D_2}^N)((u_1, v_1)(u_2, v_2)) = \alpha_{D_1}^N(u_1, v_1) \vee \alpha_{D_2}^N(u_2, v_2) = \alpha_{C_1}^N(u_1) \vee \alpha_{C_2}^N(u_2) \vee \alpha_{C_1}^N(v_1) \vee \alpha_{C_2}^N(v_2) = (\alpha_{C_1}^N \sqcap \alpha_{C_2}^N)(u_1, v_1) \wedge (\alpha_{C_1}^N \sqcap \alpha_{C_2}^N)(u_2, v_2)$
$(\gamma_{D_1}^P \sqcap \gamma_{D_2}^P)((u_1, v_1)(u_2, v_2)) = \gamma_{D_1}^P(u_1, v_1) \wedge \gamma_{D_2}^P(u_2, v_2) = \gamma_{C_1}^P(u_1) \wedge \gamma_{C_2}^P(u_2) \wedge \gamma_{C_1}^P(v_1) \wedge \gamma_{C_2}^P(v_2) = (\gamma_{C_1}^P \sqcap \gamma_{C_2}^P)(u_1, v_1) \wedge (\gamma_{C_1}^P \sqcap \gamma_{C_2}^P)(u_2, v_2)$
$(\gamma_{D_1}^N \sqcap \gamma_{D_2}^N)((u_1, v_1)(u_2, v_2)) = \gamma_{D_1}^N(u_1, v_1) \vee \gamma_{D_2}^N(u_2, v_2) = \gamma_{C_1}^N(u_1) \vee \gamma_{C_2}^N(u_2) \vee \gamma_{C_1}^N(v_1) \vee \gamma_{C_2}^N(v_2) = (\gamma_{C_1}^N \sqcap \gamma_{C_2}^N)(u_1, v_1) \wedge (\gamma_{C_1}^N \sqcap \gamma_{C_2}^N)(u_2, v_2)$
$(\beta_{D_1}^P \sqcap \beta_{D_2}^P)((u_1, v_1)(u_2, v_2)) = \beta_{D_1}^P(u_1, v_1) \vee \beta_{D_2}^P(u_2, v_2) = \beta_{C_1}^P(u_1) \wedge \beta_{C_2}^P(u_2) \wedge \beta_{C_1}^P(v_1) \wedge \beta_{C_2}^P(v_2) = (\beta_{C_1}^P \sqcap \beta_{C_2}^P)(u_1, v_1) \vee (\beta_{C_1}^P \sqcap \beta_{C_2}^P)(u_2, v_2)$
$(\beta_{D_1}^N \sqcap \beta_{D_2}^N)((u_1, v_1)(u_2, v_2)) = \beta_{D_1}^N(u_1, v_1) \wedge \beta_{D_2}^N(u_2, v_2) = \beta_{C_1}^N(u_1) \wedge \beta_{C_2}^N(u_2) \wedge \beta_{C_1}^N(v_1) \wedge \beta_{C_2}^N(v_2) = (\beta_{C_1}^N \sqcap \beta_{C_2}^N)(u_1, v_1) \wedge (\beta_{C_1}^N \sqcap \beta_{C_2}^N)(u_2, v_2)$.
□

4. Application

Modelling by using graphs has vast applications in various fields of computer science, mathematics, chemistry, physics, social sciences etc. Usually such types of models require more arrangements than merely the adjacencies among the vertices. In the study of social circuits, it is found that two people know each other i.e., if they are familiar (acquainted), or whether they are friends of each others (in the real world or in the virtual world such as Instagram) and so on. We can label each person in a particular group of people by a vertex u. There is an undirected edge between a vertex u and v if two people has a relationship with each other. In such type of graphs no multiple edges and usually no loops are needed. There is an edge between the vertices u and v when there is any acquaintanceship exists between them. In such graphs there does not exist any loop or multiple edges. In acquaintanceship graphs, the vertex (node) represents the level of acquaintanceship (how much a person is socialized or familiar/friendly) of a person while the the edge is the acquaintanceship between two persons in the social network. Since each vertex has equal importance in the classical graphs, it is not possible to graph the social networks model properly through them. In addition, all social units (individual or organization) present in social groups must be considered with equal importance in the classical graph theory. However, in the real life, the situation is different. Similarly, every edge (relationship) has an equal strength in the classical graphs. Moreover, in classical graphs it is assumed that the relationship between two social units are of equal strength, however, in real life it is not possible. Thus the acquaintance of the person has fuzzy boundary and hence can be better represented through the fuzzy graphs. In fuzzy acquaintanceship graph, each vertex represents the person and its membership value which reflects the strength of acquaintance of the person within the social group. Hence we present a fuzzy acquaintanceship graph, a bipolar fuzzy acquaintanceship and consequently a bipolar picture fuzzy acquaintanceship graph models to find out that how much the person is acquainted (social) within a group. Bipolar picture fuzzy acquaintanceship graph models would be more efficient to detect the symmetry or asymmetry existing between entities through the levels of acquaintanceships in social networks, computer networks etc.

4.1. Fuzzy Acquaintanceship Graph

We take a fuzzy acquaintanceship graph of a social network which is shown in Figure 5. In which the nodes represent the degree of the level of acquaintance of a person within the social group. The degree of the level of acquaintance is expressed in its membership value. Degree of membership states that how much a person is acquainted e.g., X is 60% acquainted within the group. The edges of a graph describe the acquaintanceship level of one person with the other person. The membership degree of edges can be considered in terms of positive percentage e.g., Y has 40% acquaintanceship level with X and so.

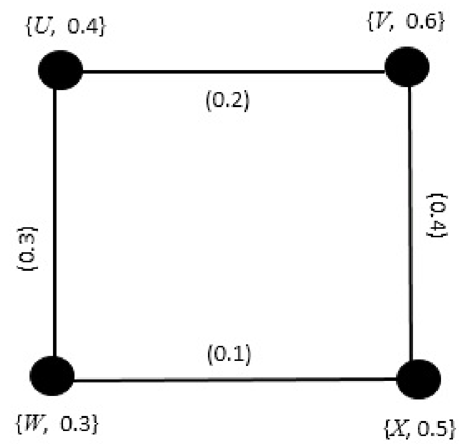


Figure 5. Fuzzy acquaintanceship graph.

4.2. Bipolar Fuzzy Acquaintanceship Graph

The acquaintanceship of a person may be positive or negative. Suppose if a person A and B belong to a social network but having not a good relationships between them then the acquaintanceship between them is negative. We can depict such circumstances through the bipolar fuzzy acquaintanceship graph. Consider a bipolar fuzzy acquaintanceship graph of a social group shown in Figure 6. In which the nodes are reflecting the degree of the level of acquaintanceship of a person belongs to a social group and the edges represent the degree of acquaintanceship levels among the persons. Degree of positive membership can be interpreted as how much a person acquainted and negative membership tells us that how much a person losses the the level of acquaintance, X has 50% level of acquaintance within the group but it loses 20% level in the same group. Edges of the graph reflect the acquaintance of one person with the other persons in the group. The positive and negative memberships degrees of edges describes the percentage of positive and negative acquaintance,for instance e.g., X is acquainted 10% with W and W is not acquainted 10% with X.

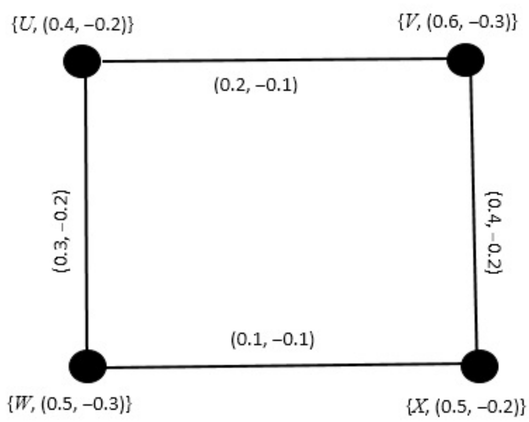


Figure 6. Bipolar fuzzy acquaintanceship graph.

4.3. Bipolar Picture Fuzzy Acquaintanceship Graph

The degree of the acquaintanceship of a person is defined in terms of its membership (positive, negative), non-membership (positive, negative) and neutral membership (positive, negative) values. The degree of the membership (positive, negative) can be interpreted as a good acquaintanceship (gaining, losing). By a good acquaintanceship, we mean the acquaintance with intimacy. The degree of non-membership (positive, negative) can be interpreted as a bad acquaintanceship (gaining, losing). Bad acquaintanceship means acquaintance with ill-famed. The degree of neutral membership (positive, negative) represents that the person having a loose acquaintanceship (gaining, losing). By a loose acquaintanceship, we mean someone we do not know well enough but we probably see them around occasionally. In Figure 7, X gains (resp., loses) 30% (resp., 50%) good acquaintanceship, he gains 20% (resp., loses 10%) bad acquaintanceship but he gains (resp., loses) 30% (resp., loses 20%) loose acquaintanceship within the social group. On the other hands, the edges of a graph (Figure 7) reflect the acquaintanceship of one person with another person. The degree of a membership (positive and negative), non-membership (positive and negative) and neutral membership(positive and negative) of the edges can be interpreted as the percentage of good acquaintanceship (gaining, losing), bad acquaintanceship (gaining, losing) and non-acquaintanceships (gaining, losing). Furthermore, it is easy to verify that the values of the edges of a graph in Figure 7 are satisfying the below conditions.

$$\begin{aligned} \alpha_D^P(uv) &\leq min(\alpha_C^P(u), \alpha_C^P(v)), & \alpha_D^N(uv) &\geq max(\alpha_C^N(u), \alpha_C^N(v)) \\ \gamma_D^P(uv) &\leq min(\gamma_C^P(u), \gamma_C^P(u)), & \gamma_D^N(uv) &\geq max(\gamma_C^N(u), \gamma_C^N(v)) \\ \beta_D^P(uv) &\geq max(\beta_C^P(u), \beta_C^P(v)), & \beta_D^N(uv) &\leq min(\beta_C^N(u), \beta_C^N(v)). \end{aligned}$$

Refer to the graph shown in Figure 7, we have
$\alpha_D^P(UV) \leq \min\,(\alpha_C^P(U), \alpha_C^P(V)) \quad \Rightarrow \quad \alpha_D^P(UV) \leq \min\,(0.4, 0.2) \quad \Rightarrow \quad 0.1 \leq 0.2$
$\alpha_D^N(UV) \geq \max\,(\alpha_C^N(U), \alpha_C^N(V)) \quad \Rightarrow \quad \alpha_D^N(UV) \geq \max\,(\text{-}0.2, \text{-}0.1) \quad \Rightarrow \quad -0.05 \geq -0.1$
$\gamma_D^P(UV) \leq \min\,(\gamma_C^P(U), \gamma_C^P(V)) \quad \Rightarrow \gamma_D^P(UV) \leq \min\,(0.1, 0.3) \quad \Rightarrow \quad 0.1 \leq 0.1$
$\gamma_D^N(UV) \geq \max\,(\gamma_C^N(U), \gamma_C^N(V)) \quad \Rightarrow \gamma_D^N(UV) \geq \max\,(\text{-}0.3, \text{-}0.3) \quad \Rightarrow \quad -0.2 \geq -0.3$
$\beta_D^P(UV) \geq \max\,(\beta_C^P(U), \beta_C^P(V)) \Rightarrow \beta_D^P(UV) \geq \max\,(0.2, 0.5) \quad \Rightarrow \quad 0.6 \geq 0.5$
$\beta_D^N(UV) \leq \min\,(\beta_C^N(U), \beta_C^N(V)) \Rightarrow \beta_D^N(UV) \leq \min\,(-0.4, -0.4) \quad \Rightarrow \quad -0.5 \leq -0.4.$

Hence by doing same calculations for the other vertices and edges of the graph shown in Figure 7, it is easy to verify that the graph given in Figure 7 is a bipolar picture fuzzy acquaintanceship graph. Similarly, by the values of vertices and edges, one can easily deduce that the graph in Figure 7 is asymmetric.

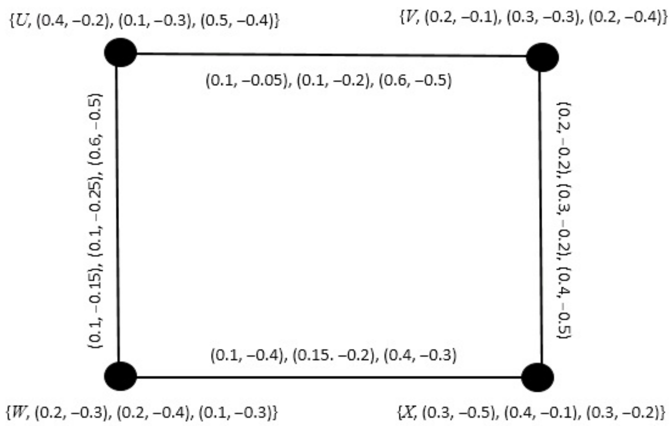


Figure 7. Bipolar picture fuzzy acquaintanceship graph.

5. Conclusions

Fuzzy graphs theory plays a significant role in modeling many real world problems containing uncertainties in different fields such as decision making theory, computer science, optimization problems, data analysis, networking etc. In this perspective, a number of generalizations of fuzzy graph have been introduced to deal with the difficult and complex real life problems. The picture fuzzy set is a direct extension of both the fuzzy sets and intuitonistic fuzzy sets. Bipolar fuzzy set is another generalized form of fuzzy set which is also an effective tool for the multiagent decision analysis. The main goal of this manuscript is to initiate the concepts of bipolar picture fuzzy graph and its different characterizations. In this article, first we propose the definition of bipolar picture fuzzy graphs based on the bipolar picture fuzzy relation. In this article, we have introduced the terms bipolar picture fuzzy graphs, complete bipolar picture fuzzy graphs and strong bipolar picture fuzzy graphs along with their several fundamental properties. For the sake of investigations, we have introduced and applied numerous operations like union, intersection, complement, ring sum etc. on bipolar picture fuzzy graphs. We also introduce different types of products of bipolar picture fuzzy graphs like semi-strong product, direct product, normal products etc. Several other terms such as order and size, path neighborhood degrees, busy values of vertices and edges of bipolar picture fuzzy graphs are also studied. These terminologies also laid the foundation for the discussion of regular bipolar picture fuzzy graphs. Furthermore, we also discuss isomorphisms, weak and co-weak isomorphisms and automorphisms of bipolar picture fuzzy graphs. During this, we have proved that the set of all automorphisms of a bipolar picture fuzzy graph forms a group. Finally, we construct a bipolar picture fuzzy acquaintanceship graph which reflects the importance of our theoretical results produced in this article. Evidently, the network modelled through a bipolar picture fuzzy acquaintanceship graph shown in Figure 7 has no any symmetry. However, we can also model a symmetric relation through the bipolar picture fuzzy acquaintanceship graph. On the same patterns, one could express collaboration graph, computer networking, social networking, web graphs in the frame of bipolar picture fuzzy graphs. In general, numbers of applications of bipolar fuzzy graphs and picture fuzzy graphs have been explored in different fields of social, natural and computer sciences. Evidently, bipolar picture fuzzy graphs would be an important tool to deal with real world problems containing uncertainties. Finally, one can extend this work by introducing bipolar interval-valued picture fuzzy graphs.

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