

Article

Interactions between Homotopy and Topological Groups in Covering (C, R) Space Embeddings

Susmit Bagchi

Department of Aerospace and Software Engineering (Informatics), Gyeongsang National University, Jinju 660701, Korea; profsbagchi@gmail.com

Abstract: The interactions between topological covering spaces, homotopy and group structures in a fibered space exhibit an array of interesting properties. This paper proposes the formulation of finite covering space components of compact Lindelof variety in topological (C, R) spaces. The covering spaces form a Noetherian structure under topological injective embeddings. The locally path-connected components of covering spaces establish a set of finite topological groups, maintaining group homomorphism. The homeomorphic topological embedding of covering spaces and base space into a fibered non-compact topological (C, R) space generates two classes of fibers based on the location of identity elements of homomorphic groups. A compact general fiber gives rise to the discrete variety of fundamental groups in the embedded covering subspace. The path-homotopy equivalence is admitted by multiple identity fibers if, and only if, the group homomorphism is preserved in homeomorphic topological embeddings. A single identity fiber maintains the path-homotopy equivalence in the discrete fundamental group. If the fiber is an identity-rigid variety, then the fiber-restricted finite and symmetric translations within the embedded covering space successfully admits path-homotopy equivalence involving kernel. The topological projections on a component and formation of 2-simplex in fibered compact covering space embeddings generate a prime order cyclic group. Interestingly, the finite translations of the 2-simplexes in a dense covering subspace assist in determining the simple connectedness of the covering space components, and preserves cyclic group structure.

Keywords: topological spaces; covering spaces; fibers; path-homotopy; fundamental groups

MSC: 54B40; 54C25; 55P10; 55R15



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1. Introduction

The structures and properties of topological covering space C_o of any arbitrary base space X have been studied in detail by Lubkin [1]. The construction of covering space of an arbitrary base space is formulated without any requirement of additional local as well as global topological properties, such as path connectedness. In general, if a continuous surjection between two spaces is given by $p : C_o \rightarrow X$ then (C_o, p) is a *locally trivial sheaf* [1]. If the local connectedness of the topological space is further relaxed then $\forall B \subset X$ such that the covering space $p^{-1}(B)$ may not be uniquely determined, and as a result the suitable decomposition is necessary where $p^{-1}(B) = \bigcup_{i \in \Lambda} U_i$ condition is preserved (Note that Λ denotes an index set and each U_i is open). In another extreme let us consider the compact-open topology in the spaces of continuous functions represented by $C_f(X, Y)$ between the topological spaces X, Y . Suppose we consider the open subbases given by $O_B = \{f \in C_f(X, Y) : f(A) \subseteq U\}$ where A is compact and $U = U^o$. In this case the topological properties of Y are preserved by $C_f(X, Y)$. Interestingly, if the continuous surjection $p : (Y \sim) \rightarrow Y$ is a covering projection and the function $p_f : C_f(X, (Y \sim)) \rightarrow C_f(X, Y)$ is defined as $p_f((f \sim)) = p \circ (f \sim)$, then the function $p_f(\cdot)$ is also a covering map if, and only if, X is Hausdorff and contractible (i.e., compact CW-complex with finitely many

components) [2]. The characterizations of Hausdorff topological spaces can be made by employing the concept of compact-covering maps, which is analogous to the definition of sequence-covering maps [3]. Note that such covering projective spaces can successfully admit fibrations and injective embeddings. The preservation of fibrations in a covering space and projection requires specific topological properties. For example, if a topological space is locally compact then the preservation of covering fibration is maintained by the local homotopy of the respective topological space [4]. A topological space can admit a class of coverable topological groups by generalizing the concept of cover if the corresponding topological space is of a metrizable and connected variety [5].

This paper investigates the nature of interactions between homeomorphically embedded covering (C, R) spaces of Lindelof as well as Noetherian variety (under topological injective embeddings) and the finite topological groups within the covering spaces under fibration. The topological properties of corresponding structures and interactions are presented in detail, in view of algebraic as well as geometric topology. First we present the associated concepts and the resulting motivation as well as the summary of contributions made in this paper to address the wider audience. Note that in this paper Λ denotes an index set and $\text{hom}(A, B)$ represents that the structures A and B are homeomorphic to each other. Moreover, the sets of extended real numbers, complex numbers and positive integers are denoted as R, C and Z^+ respectively. The topological spaces under consideration in this paper are second-countable Hausdorff spaces. Furthermore, the topological space under a closed 2-simplex σ^2 is denoted by $|\sigma^2|$ whereas $\langle \sigma^2 \rangle$ denotes an open 2-simplex, and if p, q are two continuous paths, then $[p] * [q]$ represents homotopy class path products, where $p \neq q$ and the relation $\cong_H$ represents a path-homotopy equivalence.

Motivation and Contributions

It is known that the Poincare group displays a set of interactive properties in the topological covering spaces. It is noted that every filtered Galois group is isomorphic to the corresponding Poincare filtered group of regular covering space where the topological space is a connected variety [1]. Moreover in a covering projection $u : Y \rightarrow X$ if a connected topological group G exists in the simply connected base space then there is a connected covering space group G_c such that $u : G_c \rightarrow G$ preserves respective covering projection properties while incorporating group homomorphism [6]. Interestingly, the varieties of Hausdorff and contractible topological spaces of continuous functions admit fibrations and topological embeddings under injective inclusions [2]. Recall that the covering spaces are generally considered to be path-connected topological spaces. There is a natural way to establish fibrations in a path-connected space X . If $X^{[0,1]}$ denotes the space of every path in a path-connected topological space X then there is a natural fibration given by $\pi : X^{[0,1]} \rightarrow X^2$ such that $\pi(f \in X^{[0,1]}) = (f(0), f(1))$ where $f : [0, 1] \rightarrow X$ is continuous [7]. However a *discretely* fibered covering space can also be formulated in a covering projection [8]. These observations suggest the importance of investigating the properties of interactions of topological groups and covering spaces under embeddings in a fibered topological (C, R) space which admits Noetherian convex P-separations [9,10]. Hence, the motivating questions can be summarized as: (1) what are the properties of interactions between the topological groups in homeomorphically embedded compact Lindelof–Noetherian planar covering spaces and the fibers in a path connected (C, R) subspace, (2) is it possible to categorize the fibers in such covering spaces to establish a topological structure in path-connected embeddings in a (C, R) space, and (3) what are the properties of generated planar simplicial complexes in view of geometric topology within the respective embedded covering (C, R) spaces under fibration? These questions are addressed in this paper in relative detail.

The main contributions made in this paper can be summarized as follows. The concepts related to compact Lindelof as well as finite Noetherian (under topological injective embeddings) covering space components are introduced and the formulation of homomorphic topological groups within such covering spaces is presented. Next, the topological

properties of embeddings of such covering spaces in a fibered topological (C, R) space are investigated in detail, involving finite as well as symmetric translations restricted on the identity-rigid fibers. It is shown that the embeddings give rise to two varieties of fibers and the path-homotopy equivalence is preserved by different topological structures within the embedded subspace. Moreover a discrete-loop variety of fundamental groups is generated within the embedded subspace under fibration.

The rest of the paper is organized as follows. The preliminary concepts are presented in Section 2 as a set of existing definitions and theorems. The definitions related to proposed topological structures are presented in Section 3. The main results are presented in Section 4. Finally, Section 5 concludes the paper.

2. Preliminary Concepts

In algebraic topology, the structure and properties of topological spaces are investigated by studying the behaviors of open as well as closed continuous functions within the space along with their continuous deformations. In this section, a set of classical as well as contemporary results are presented in relation to the covering spaces, fibrations, covering homotopy varieties and associated isomorphism of fundamental groups. Let two topological spaces be denoted as (X, τ_X) and (Y, τ_Y) such that $X = A \cup B$ where $A = \overline{A}$ and $B = \overline{B}$. If we consider two continuous functions $f_1 : A \rightarrow Y$ and $f_2 : B \rightarrow Y$ then the continuity of $\langle f_1, f_2 \rangle$ is preserved if, and only if, the following conditions are satisfied.

$$\begin{aligned} E &= (A \cap B) \neq \phi, \\ g &: X \rightarrow Y, \\ \forall x \in E, f_1(x) &= f_2(x), \\ (x \in A) &\Rightarrow (g(x) = f_1(x)), \\ (x \in B) &\Rightarrow (g(x) = f_2(x)). \end{aligned} \quad (1)$$

The concept of covering space is central to the algebraic topology. An elementary neighborhood of a topological space (X, τ_X) is a subspace of X which can be surjectively mapped under additional conditions leading to the concept of covering space. The definition of covering space and projection is presented as follows.

2.1. Definition: Covering Spaces

Let a continuous surjective function be given as $u : Y \rightarrow X$ where the topological space X is called a base space. If it is true that $\forall B \subset X$, the function $u^{-1}(B)$ generates covers which are homeomorphically mapped onto B , then the function $u : Y \rightarrow X$ is called a covering projection or covering map.

It is important to note that base subspace B is considered to be path-connected topological subspace. If we consider $S^1 \subset C$ on a complex plane then it results in the formulation of covering path theorem which is presented as follows [11,12].

Theorem 1. If $\theta : [0, 1] \rightarrow S^1$ is a path such that $\theta(0) = 1$ then there is a unique covering path $\bar{\theta} : [0, 1] \rightarrow S^1$ such that $\bar{\theta}(0) = 0$.

The above theorem can be further generalized for any $r \in R$. Suppose a continuous function is defined as $f : R \rightarrow S^1$ such that $f(r) = e^{2\pi r i}$. If it is considered that $\exists r, f(r) = \theta(0)$ then the corresponding unique covering path $\bar{\theta}(\cdot)$ can also be formulated. The respective covering homotopy is a related concept which can be presented in the following theorem [11,13,14].

Theorem 2. If the function $H : [0, 1]^2 \rightarrow S^1$ is a homotopy such that $H(0, 0) = 1$ then the function $\bar{H} : [0, 1]^2 \rightarrow R$ is a covering homotopy where $\bar{H}(0, 0) = 0$.

Interestingly, the covering homotopy property needs a generalized reconstruction in the covering spaces by considering the uniqueness of a covering path. If there is a

continuous function $q : [0, 1] \rightarrow B$ and in the corresponding covering space $\exists y_0 \in Y$ such that $u(y_0) = q(0)$ then the *covering homotopy in a covering space* can be established as presented in the following theorem [11].

Theorem 3. *If Y is a covering space of topological base space X under surjection $u : Y \rightarrow X$ and $H_{\text{cov}} : [0, 1]^2 \rightarrow (B \subset X)$ is a homotopy with $H_{\text{cov}}(0, 0) = q(0)$ then there is a covering homotopy in the covering space given by $\overline{H_{\text{cov}}} : [0, 1]^2 \rightarrow Y$ such that $\overline{H_{\text{cov}}}(0, 0) = y_0$.*

The fibration in a path-connected topological space can be defined in the respective covering spaces and in terms of homotopy lifting. Moreover, there exists a special variety of coverings called compact-covering in a metrizable space. Let us consider two continuous functions between two topological spaces X, Y which are denoted as $i \in \{0, 1\}, f_i : Y \rightarrow X$. The definitions of fibration in covering space, compact-covering and point-countable cover are presented as follows.

2.2. Definitions: Fibration and Covering Varieties

Let $(f_0 \sim)$ be a lifting of f_0 and the function H is a homotopy from f_0 to f_1 . There is a homotopy H_f such that $p \circ H_f = H$ where $p : (X \sim) \rightarrow X$ is a covering projection [2]. First, we present the definition of covering fibration and its discrete variety in a covering space. The definitions of two different notions related to coverings are presented next.

Fibration [8,13]: A fibering of a topological covering space is a structure given by $(Y, X, u, \Omega, \eta_U)$ where $u : Y \rightarrow X$ is a covering projection from fibered $Y, \Omega = \{U_i : i \in \Lambda, U_i \subset X\}$ is a collection of open sets (i.e., the neighborhood components in base space), and $\eta_U : U_i \times u^{-1}(U_i) \rightarrow Y$ is continuous. A corresponding G -covering space contains discrete fibers in the covering projection $u : Y \rightarrow X$ (i.e., the covering space is a discretely fibered space).

Compact-covering [15]: A function $s : X \rightarrow Y$ is called compact-covering if $\forall A \subset Y, \exists B \subset X$ such that $A \subset s(B)$. It is important to note that the covering maps or covering projections preserve complete metrizability under certain conditions.

Point-countable cover [16]: Suppose P is a cover of space X and $F \subset P$ is a finite subcover. The space X is defined as having a point-countable cover P if $(x \in U \subset X) \Rightarrow (x \in (\bigcup_{\Lambda} F)^0 \subset \bigcup_{\Lambda} F \subset U)$ where $U = U^0$ in X .

The equivalence between multiple covering spaces and the corresponding multiple covering projections can be established in the presence of covering varieties. If two different covering projections are given as $p : Y_1 \rightarrow X$ and $q : Y_2 \rightarrow X$ then they are called equivalent if, and only if, there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $q \circ h = p$. This leads to the conjugacy theorem of fundamental groups in the covering spaces, as presented in the following theorem [11,13].

Theorem 4. *If $p(e_0) = q(l_0) = x_0 \in X$ then $H_1 = p_*(\pi_1(Y_1, e_0))$ and $H_2 = q_*(\pi_1(Y_2, l_0))$ are conjugate to the fundamental group $\pi_1(X, x_0)$.*

Finally, if a covering projection $u : Y \rightarrow X$ is given then any two universal covering spaces of a base space are isomorphic to each other. Furthermore the path connectedness of a topological space (X, τ_X) preserves the isomorphisms of two fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ in the space.

3. Topological Structures and Definitions

Let a second countable Hausdorff as well as compact normal topological (C, R) space be represented as (X, τ_X) such that $X \subset C \times R$ and an open set $X_p \subset X$ contains the point $x_p = (z_p, r_p)$ where $X_p \subset C \times \{r_p\}$. Suppose $X_{\text{cov}} \subset C \times I$ where $I \subset R$, is also a (C, R) space such that $X_{\text{cov}} \cap X = \emptyset$ and X_{cov} is a second countably compact space (i.e., a variety of compact Lindelof space). If $B \subset X_p$ is a topological subspace such that $B = \overline{B}$ and $N_p \subset B$ is an open neighborhood of the point $x_p \in N_p \subset X_p$ then the surjective function

given by $f_c : X_{\text{cov}} \rightarrow B$ is a covering map of $N_p \subset B$ and $f_c^{-1}(N_p) = \{A_i \subset X_{\text{cov}} : i \in \Lambda\}$ represents the respective covering (C, R) spaces. It is considered in this paper that the covering spaces are finite variety, and as a result X_{cov} is compact. Consequently, the covering (C, R) spaces maintain the property that $\forall A_i, A_k \in f_c^{-1}(N_p), A_i \cap A_k = \emptyset$ if $i \neq k$ where $i, k < +\infty$ and each compact cover is a path-connected component. In this paper the i -th path-connected component of the covering spaces is denoted as $A_i \in f_c^{-1}(N_p)$ such that $A_i \subset C \times \{r_i \in I\}$. Note that the compactness of spaces X and X_{cov} enables the formulation of locally homeomorphic embeddings into a non-compact (C, R) space Y . In this paper, the topological embeddings are employed to construct the embedded subspaces, preserving local homeomorphism. It is important to note that the topological injective embeddings forming a Noetherian structure in covering spaces do *not* consider the group algebraic standpoints and such embeddings are completely topological in nature. The reason is that the set of finite groups in covering components in a topological (C, R) space are distinct. The definitions of groups in covering (C, R) spaces, the corresponding homeomorphic embeddings of subspaces and the concept of identity fiber are presented in Sections 3.1–3.4. First we present the concept of compact Lindelof and Noetherian (LN) variety of covering (C, R) spaces through the topological embeddings and the construction of a set of homeomorphic finite groups within such spaces. Note that in this paper all topological spaces are considered to be second-countable Hausdorff spaces in nature and the topological groups are compactible as well as connected.

3.1. Definition: LN Covering of (C, R) Space

Let (X, τ_X) be a compact topological (C, R) space and the corresponding embedding within covering spaces generated by $f_c^{-1}(N_p)$ be given by the injective function $\forall A_i, A_k \in f_c^{-1}(N_p), g_{ik} : A_i \rightarrow A_k$ such that $\text{hom}(A_i, g_{ik}(A_i))$ condition is preserved. The covering (C, R) space $f_c^{-1}(N_p) = \{A_i \subset X_{\text{cov}} : i \in \mathbb{Z}^+\}$ is defined to be an LN variety if $g_{ik}(A_i) \subset A_k$ whenever $i < k$.

Note that the LN covering path components are finite and countable maintaining the property that $\bigcup_{m \in \Lambda} A_m \subseteq X_{\text{cov}}$. If we consider a topological (C, R) space (Y, τ_Y) such that $X \cap Y = \emptyset$ and the space Y is not compact then a set of suitable injective embeddings can be formulated maintaining local homeomorphisms.

3.2. Definition: Covering (C, R) Space Embeddings

Let $Y \subseteq C \times R$ be a non-compact topological (C, R) space where $i_{\text{cov}} : X_{\text{cov}} \rightarrow Y$ and $i_X : X \rightarrow Y$ are homeomorphic topological embeddings. The corresponding embeddings are called covering (C, R) space embeddings if it preserve $i_{\text{cov}}(X_{\text{cov}}) \cap i_X(X) = \emptyset$ while maintaining $\text{hom}(X_{\text{cov}}, i_{\text{cov}}(X_{\text{cov}}))$ as well as $\text{hom}(X, i_X(X))$ properties.

Remark 1. It is important to note that the locally homeomorphic embeddings retain the covering map as $f_{\text{cov}} : i_{\text{cov}}(X_{\text{cov}}) \rightarrow i_X(B \subset X)$ such that $\forall A_i \in f_c^{-1}(N_p \subset X)$ the injective and homeomorphic topological embedding maintains the property given by $\text{hom}(A_i, i_{\text{cov}}(A_i))$. Moreover it is relatively straightforward to observe that $(f_{\text{cov}} \circ i_{\text{cov}})(A_i) = (i_X \circ f_c)(A_i)$ within the respective topological space.

Suppose the surjection $w : f_c^{-1}(N_p) \rightarrow f_c^{-1}(N_p)$ is given as $w(A_k) = A_i$ where $k > i$. The finite LN covering of (C, R) spaces can suitably admit a sequence of embeddings forming a Noetherian topological structure in Y given by $(i_{\text{cov}} \circ w) : X_{\text{cov}} \rightarrow Y_{\text{cov}}$ where $i_{\text{cov}}(X_{\text{cov}}) \subseteq Y_{\text{cov}} \subset Y$. This property assists to establish a set of finite group algebraic structures in X_{cov} which are embeddable, retaining the respective group homomorphism in Y . First, we define the topological group structures and group homomorphism in the covering spaces under topological embeddings.

3.3. Definition: Finite Covering (C, R) Space Groups

Let $G_{\text{cov}} = \{G_i : i \in \mathbb{Z}^+\}$ be a countable set of finite (locally compactible as well as locally connected) topological groups such that $\forall G_i \in G_{\text{cov}}, G_i = (X_i \subset A_i, *_i)$ where $A_i \in f_c^{-1}(N_p)$ is a covering (C, R) path component and $*_i : X_i^2 \rightarrow X_i$ is closed in $X_i \subset A_i$ where $X_i = (X_i)^0$ and $A_i = \overline{A_i}$. If the LN covering of (C, R) spaces admits the property that $\exists A_i, A_k \in f_c^{-1}(N_p)$ such that $k = i + 1$ and $g_{ik}(A_i) \subset A_k$ then $G_i, G_k \in G_{\text{cov}}$ are homeomorphic groups if, and only if, $h_{ki} : G_k \rightarrow G_i$ is a group homomorphism.

Remark 2. Note that the structures of Noetherian covering spaces under topological embeddings should be maintained by $h_{ki} : G_k \rightarrow G_i$. Hence, we are not restricting to the strictly bijective variety of $h_{ki} : G_k \rightarrow G_i$ and as result it is considered that $\ker(h_{ki}) \subset X_k$ such that $\ker(h_{ki}) \setminus \{a_k\} \neq \emptyset$ maintaining generality, where a_k is the identity element. Note that $\forall G_i \in G_{\text{cov}}$ the homeomorphic embedding $G_{Y,i} = (i_{\text{cov}}(X_i), *_i)$ is also a group in Y . In addition, the locally homeomorphic embeddings maintain that $\exists e_i \in G_{Y,i}$ such that $a_i = i_{\text{cov}}^{-1}(e_i) \in X_i$ preserving identity of $G_i \in G_{\text{cov}}$.

3.4. Definition: Identity Fiber and Rigidity

Let the two path components in covering (C, R) spaces generated by $f_c^{-1}(N_p)$ be given as A_i, A_k respectively. A compact fiber $\mu_{p \times I} \subset Y$ at $i_X(x_p) \in Y$ is defined to be an identity fiber if, and only if, $\mu_{p \times I} \cap i_{\text{cov}}(X_i) = \{e_i\}$ such that $i_{\text{cov}}^{-1}(e_i) \in X_i$. An identity fiber is called as an *identity-rigid* variety if $\mu_{p \times I} \cap i_{\text{cov}}(X_i) = \{e_i\}$ and $\mu_{p \times I} \cap i_{\text{cov}}(X_k) = \{e_k\}$ conditions are maintained in $G_{Y,i}$ and $G_{Y,k}$ respectively.

Note that in general we consider that if $\mu_{p \times I}$ and $\mu_{s \times I}$ are two identity fibers then $\mu_{p \times I} \cap \mu_{s \times I} = \emptyset$ in (Y, τ_Y) . However, if the fiber $\mu_{p \times I}$ in (Y, τ_Y) is identity-rigid then there exists a real projection $\pi_R : \mu_{p \times I} \rightarrow R$ such that $\left| T_{\mu_R}^m(\pi_R(e_k)) \right| - \left| \pi_R(e_i) \right| = 0$ and $\left| \pi_R(e_k) \right| - \left| T_{\mu_R}^m(\pi_R(e_i)) \right| = 0$ where $m \in \mathbb{Z}^+, T_{\mu_R} : R \rightarrow R$ is a finite linear projective translation with respect to the corresponding fiber with symmetry. Recall that the condition $\text{hom}(X_{\text{cov}}, i_{\text{cov}}(X_{\text{cov}}))$ preserves the respective group structures due to local homeomorphisms in $Y_{\text{cov}} \subset Y$. Moreover if $\mu_{p \times I}$ is compact identity-rigid then a corresponding compact identity-rigid fiber $i_{\text{cov}}^{-1}(\mu_{p \times I})$ can be found in fibered X_{cov} maintaining the respective group homomorphism if, and only if, X_{cov} is path connected.

4. Main Results

A homeomorphic embedding of LN covering of (C, R) space into a fibered space enables the formulation of the topological properties related to homotopy structures, where $A_i \subset X_{\text{cov}}$ is a locally path-connected variety. It is important to observe that although the covering (C, R) space X_{cov} is a locally path-connected variety, the corresponding embedded subspace in $Y_{\text{cov}} \subset Y$ is a locally dense and completely path-connected variety maintaining local homeomorphisms of (C, R) LN covering components.

Let a non-compact fiber in the topological (C, R) space Y at $i_X(x_p) \in Y$ be given as $\mu_{p \times I} \subset Y$ such that $\forall A_i \in f_c^{-1}(N_p), \mu_{p \times I} \cap i_{\text{cov}}(A_i) \neq \emptyset$. According to definitions, the subspaces generated by $(f_{\text{cov}}^{-1} \circ i_X)(N_p)$ in Y are a finite variety. Suppose a set of continuous functions are formulated as $\forall A_i \in f_c^{-1}(N_p), \exists v_i : [0, 1] \rightarrow i_{\text{cov}}(A_i)$ such that $v_i(0) = v_i(1)$ and $\mu_{p \times I} \cap i_{\text{cov}}(A_i) = \{v_i(t) : t \in [0, 1]\}$. This results in the formation of a discrete variety of fundamental group in Y under homotopic path products, as presented in the following theorem.

Theorem 5. If $(i_{\text{cov}} \circ f_c^{-1})(N_p) \subset Y$ is an LN covering space in fibered topological (C, R) space Y then $\pi_1(E, y_0)$ is a discrete fundamental group where $y_0 \cong i_X(x_p)$ and $(i_X(X) \cup Y_{\text{cov}}) \subset E \subset Y$.

Proof. Let (X, τ_X) be a compact topological (C, R) space and X_{cov} be a finite LN covering space. According to the locally homeomorphic embeddings into a fibered (C, R) space Y , it follows that $\exists x_p \in X_p \in \tau_X, i_X(x_p) \in Y$ and $(i_{\text{cov}} \circ f_c^{-1})(N_p) \subset Y_{\text{cov}}$ is a covering space of $(i_X \circ f_c)(X_{\text{cov}}) \subset Y$ such that $Y_{\text{cov}} \subset Y$. Suppose $\mu_{p \times I} \subset Y$ is a compact fiber such

that $\mu_{p \times I} \cap i_X(X_p) = \{i_X(x_p)\}$ and $\mu_{p \times I} \cap Y_{\text{cov}} = \{y_k : y_k \in Y\}$ where $y_k \in i_{\text{cov}}(A_k \in f_c^{-1}(N_p))$ and $k \in Z^+, k \in [1, +\infty)$. Let a continuous function $q_{pk} : [0, 1] \rightarrow (H \subset \mu_{p \times I})$ be considered such that $q_{pk}(0) = i_X(x_p)$ and $q_{pk}(1) = y_k$ in the subspace $(i_X(X) \cup Y_{\text{cov}}) \subset E \subset Y$. If we consider a set of continuous functions given by $S_v = \{v_k : [0, 1] \rightarrow i_{\text{cov}}(A_k)\}$ such that three conditions given by: (1) $\forall k \forall v_k \in S_v, v_k(0) = v_k(1)$, (2) $\{v_k(t \in \{0, 1\})\} = \{y_k\}$ and (3) $\text{hom}(v_k([0, 1]), S^1)$ are maintained, then the following path-homotopy equivalence relation is attained.

$$\begin{aligned} \forall t \in [0, 1], \overline{q_{pk}} &= q_{pk}(1 - t), \\ \alpha_k &\cong v_k([0, 1]), y_0 \cong i_X(x_p), \\ [q_{pk}] * [\alpha_k] * [\overline{q_{pk}}] &\cong_H [y_0]. \end{aligned} \quad (2)$$

Thus the set S_v in $Y_{\text{cov}} \subset E$ preserves the path-homotopy equivalence relation in $E \subset Y$ as follows.

$$\begin{aligned} 1 \leq k \leq m, \{a, b\} &\subset Z^+, a < b, \\ u_{ab} : [0, 1] &\rightarrow (H_{ab} \subset \mu_{p \times I}), \\ u_{ab}(0) &= y_a, u_{ab}(1) = y_b, \\ [q_{p1}] * [\alpha_1] * [u_{12}] * [\alpha_2] * \dots * [\overline{q_{pm}}] &\cong_H [y_0]. \end{aligned} \quad (3)$$

Hence, the topological structure $\pi_1(E, y_0)$ admits a discrete variety of fundamental group at the base point $y_0 \cong i_X(x_p)$ in the subspace $E \subset Y$. $\square$

Remark 3. It is important to note that the projections on the corresponding real subspaces denoted as $\pi_R : A_i \rightarrow R$ of the LN covering path components $i_{\text{cov}}(A_a), i_{\text{cov}}(A_b)$ in Y should maintain the property that $\pi_R(A_a) < \pi_R(A_b)$ and $(\pi_R \circ i_{\text{cov}})(A_a) < (\pi_R \circ i_{\text{cov}})(A_b)$ retaining homeomorphism. Moreover, the following algebraic identities of path-homotopy are maintained by $\pi_1(E, y_0)$ for $m < n$.

$$\begin{aligned} [\alpha_m] * [u_{mn}] &\cong_H [y_n], [\alpha_n] * [\overline{u_{mn}}] \cong_H [y_m], \\ [\alpha_m] * [u_{mn}] * [\alpha_n] &\cong_H [\alpha_n], \\ [\alpha_n] * [\overline{u_{mn}}] * [\alpha_m] &\cong_H [\alpha_m], \\ [\alpha_n] * [\overline{u_{mn}}] * [\alpha_m] * [u_{mn}] &\cong_H [y_n]. \end{aligned} \quad (4)$$

If we consider the existence of topological groups in the path-connected LN covering spaces (i.e., the covering (C, R) spaces are in a dense topological subspace under embeddings), then the fiber-connected distributed groups exhibit interesting homotopy properties if two such topological groups maintain group homomorphism under the identity fiber. The following theorem presents this interesting observation.

Theorem 6. If $G_n = (X_n \subset A_n, *_n)$, $G_i = (X_i \subset A_i, *_i)$ are two topological groups in the respective LN covering of (C, R) spaces and $h_{ni} : G_n \rightarrow G_i$ is a homomorphism, then there exists a path-homotopy equivalence with respect to an identity fiber in the topological space (Y, τ_Y) .

Proof. Let $A_n, A_i \in f_c^{-1}(N_p \subset X)$ be two path-components of LN covering of (C, R) spaces in X_{cov} such that $n > i$. Suppose we consider an identity fiber $\mu_{p \times I} \subset Y$ in the respective topological (C, R) space such that $\mu_{p \times I} \cap i_{\text{cov}}(A_i) = \{a_i\}$ and $\mu_{p \times I} \cap i_{\text{cov}}(A_n) = \{a_n\}$. If $G_n = (X_n \subset A_n, *_n)$, $G_i = (X_i \subset A_i, *_i)$ are two topological groups in X_{cov} maintaining the corresponding group homomorphism $h_{ni} : G_n \rightarrow G_i$ then $\exists K_n \subset G_n$ such that $\ker(h_{ni}) = K_n$. As the fiber $\mu_{p \times I} \subset Y$ is an identity fiber so it can be concluded that $(i_{\text{cov}} \circ h_{ni})(K_n) = (a_i \equiv e_i)$ and $e_n \in i_{\text{cov}}(A_n) \setminus \{a_n\}$ maintaining generality. Suppose we consider a continuous function $f_n : [0, 1] \rightarrow (K_n \triangleleft G_n)$ such that $(i_{\text{cov}} \circ f_n)(0) = (i_{\text{cov}} \circ f_n)(1) = a_n$. Note that the covering space $Y_{\text{cov}} \subset Y$ is a path-connected variety which allows us to formulate a continuous function $q : [0, 1] \rightarrow (B \subset Y_{\text{cov}})$ such that $q(0) = a_n, q(1) = a_i$ where $q([0, 1]) \subset \mu_{p \times I}$. Hence, by considering $\forall t \in [0, 1], q(1 - t) \equiv$

$\bar{q}(t)$ we can formulate a set of path-homotopy equivalences in (Y, τ_Y) which are given as: (1) $[(i_{\text{cov}} \circ f_n)] \cong_H [q] * [\bar{q}]$ and (2) $[(i_{\text{cov}} \circ (h_{ni} \circ f_n))] \cong_H [q]$. $\square$

Note that the function $f_n : [0, 1] \rightarrow (K_n \triangleleft G_n)$ maintains a path-homotopy equivalence at the base point $i_{\text{cov}}^{-1}(a_n)$ which can be represented as $[f_n] \cong_H [i_{\text{cov}}^{-1}(a_n)]$. If we consider that $B \subset Y_{\text{cov}}$ is a dense and simply connected subspace then an arbitrary continuous function in the dense subspace also preserves the path-homotopy equivalence property. This observation is presented in the following lemma.

Lemma 1. *Let the covering space $B \subset Y_{\text{cov}}$ be a simply connected locally dense subspace and the function $q : [0, 1] \rightarrow (B \subset Y_{\text{cov}})$ is continuous. If the function $q([0, 1])$ is arbitrary such that $\forall t_1, t_2 \in (0, 1), (t_1 \neq t_2) \Rightarrow (q(t_1) \neq q(t_2)) \wedge (q(t) \cap \mu_{p \times I} = \emptyset)$ then the path-homotopy $[(i_{\text{cov}} \circ f_n)] \cong_H [q] * [\bar{q}]$ is preserved with respect to the identity fiber $\mu_{p \times I}$.*

Proof. The proof is relatively straightforward. If we consider that $B \subset Y_{\text{cov}}$ is a simply connected locally dense subspace then it is path-connected. Hence a continuous function $q : [0, 1] \rightarrow (B \subset Y_{\text{cov}})$ can be constructed such that it maintains two conditions given as: $q([0, 1]) \cap \mu_{p \times I} = \{q(0), q(1)\}$ and $\forall t_1, t_2 \in (0, 1), (t_1 \neq t_2) \Rightarrow (q(t_1) \neq q(t_2))$. Hence, a path-homotopy is formulated in Y_{cov} which is given by $[(i_{\text{cov}} \circ f_n)] \cong_H [q] * [\bar{q}]$ where $f_n : [0, 1] \rightarrow (K_n \triangleleft G_n)$ is a continuous function such that $\text{hom}(f_n([0, 1]), S^1)$ is preserved. $\square$

Interestingly, there is interplay between the algebraic properties of group homomorphism in LN covering of (C, R) spaces and the homotopy property if the space is a simply connected variety and the fiber is identity-rigid. Suppose the finite linear translation $T : Y \rightarrow Y$ in the respective topological (C, R) space is restricted on the fiber $\mu_{p \times I} \subset Y$ which is denoted as $T_\mu : \mu_{p \times I} \rightarrow \mu_{p \times I}$. We show in the following theorem that such translation establishes a path-homotopy equivalence on the identity-rigid fiber $\mu_{p \times I}$ if $\mu_{p \times I} \cap i_{\text{cov}}(K_n \triangleleft G_n) \neq \emptyset$.

Theorem 7. *If $e_i \in i_{\text{cov}}(X_i)$ and $e_n \in i_{\text{cov}}(X_n)$ are respective identities of $G_{Y,i}$ and $G_{Y,n}$ such that $\mu_{p \times I}$ is identity-rigid then the continuous function $f_n : [0, 1] \rightarrow (K_n \triangleleft G_n)$ preserves the path-homotopy equivalence relation under finite translation $T_\mu : \mu_{p \times I} \rightarrow \mu_{p \times I}$.*

Proof. Let $A_i, A_n \subset X_{\text{cov}}$ be two LN covering components of (C, R) space of a neighborhood $N_p \subset X$ where $A_i \supset X_i$ and $A_n \supset X_n$. Suppose Y is a non-compact topological (C, R) space such that $X \cap Y = \emptyset$. If we consider a compact identity-rigid fiber $\mu_{p \times I} \subset Y$ then we can conclude that $\{e_i, e_n\} \subset \mu_{p \times I}$ where $e_i \in i_{\text{cov}}(X_i)$ and $e_n \in i_{\text{cov}}(X_n)$ are the identities of respective groups $G_{Y,i}$ and $G_{Y,n}$. If the function $h_{ni} : G_n \rightarrow G_i$ is a group homomorphism then $h_{ni}(K_n \triangleleft G_n) = i_{\text{cov}}^{-1}(e_i)$. If the continuous function $f_n : [0, 1] \rightarrow (K_n \triangleleft G_n)$ is constructed such that $f_n(0) = f_n(1) = i_{\text{cov}}^{-1}(e_n)$ then we can conclude that $[f_n] \cong_H [i_{\text{cov}}^{-1}(e_n)]$. Hence, if the finite linear translation on fiber $T_\mu : \mu_{p \times I} \rightarrow \mu_{p \times I}$ maintains the property that $\exists m \in \mathbb{Z}^+, 1 \leq m < +\infty, T_\mu^m(e_n) = e_i$ then we can further conclude that $[T_\mu^m((i_{\text{cov}} \circ f_n))] \cong_H [(i_{\text{cov}} \circ h_{ni})(K_n)]$ in Y . $\square$

Remark 4. *It is important to note that in this case $A_i, A_n \subset X_{\text{cov}}$ are considered to be simply connected topological subspaces such that $\forall g \in [f_n]$ in A_n the continuous function $g : [0, 1] \rightarrow \{i_{\text{cov}}^{-1}(e_n)\}$ is nullhomotopic in nature where $\text{hom}(g([0, 1]), S^1)$ is preserved. If we relax this condition then the aforesaid property may not always be satisfied if we enforce the condition that $\pi_1(A_n, i_{\text{cov}}^{-1}(e_n))$ is also a fundamental group in X_{cov} .*

Interestingly, if a group in the LN covering of (C, R) space is a trivial group then an equivalence relation involving the finite linear translation on an identity-rigid fiber can be established. This observation is presented in the following lemma.

Lemma 2. If $G_{Y,i}$ is a trivial group then $\exists m \in [1, +\infty)$ such that $(i_{\text{cov}} \circ (g_{in} \circ h_{ni}))(K_n) \cong T_\mu^m(i_{\text{cov}}(X_i))$ in (Y, τ_Y) where the fiber is identity-rigid.

Proof. Let in the topological space (Y, τ_Y) two topological groups be presented as $G_{Y,i}$ and $G_{Y,n}$ such that $G_{Y,i}$ is a trivial group and $n > i$ in the corresponding LN covering of (C, R) space X_{cov} . If there is a group homomorphism $h_{ni} : G_n \rightarrow G_i$ where $K_n \triangleleft G_n$ then the function $g_{in} : A_i \rightarrow A_n$ maintains the property given by $g_{in}(X_i) \subset K_n$. Moreover in this case $(h_{ni} \circ g_{in})(X_i) = h_{ni}(K_n)$ condition is preserved by the trivial group $G_{Y,i}$. Suppose we consider a linear finite translation on the identity-rigid fiber $T_\mu : \mu_{p \times I} \rightarrow \mu_{p \times I}$ in (Y, τ_Y) . Thus $\exists m \in [1, +\infty)$ such that $T_\mu^m(i_{\text{cov}}(X_i)) = (g_{in} \circ i_{\text{cov}})(X_i)$ in (Y, τ_Y) where $\mu_{p \times I} \cap i_{\text{cov}}(A_i) = \{e_i \in X_i\}$ and $\mu_{p \times I} \cap i_{\text{cov}}(A_n) = g_{in}(X_i)$. Hence we can conclude that $(i_{\text{cov}} \circ (g_{in} \circ h_{ni}))(K_n) = \mu_{p \times I} \cap i_{\text{cov}}(A_n)$ and this results in the property given by $(i_{\text{cov}} \circ (g_{in} \circ h_{ni}))(K_n) \cong T_\mu^m(i_{\text{cov}}(X_i))$. $\square$

The formulation of path-homotopy equivalences on a single fiber requires the specific condition about the position of identity elements in a fibered topological (C, R) space. However a more relaxed version of path-homotopy equivalences can be formulated involving multiple fibers in the topological (C, R) space where the identity elements are distributed on multiple fibers for respective group structures within the embedded LN covering of (C, R) space. This observation is presented in the following theorem.

Theorem 8. If a fibered topological (C, R) space (Y, τ_Y) is dense then $\exists B \subset Y_{\text{cov}}$ such that the continuous function $q : [0, 1] \rightarrow B$ establishes path-homotopy equivalence where $q(0) = e_i \in \mu_{p \times I}$ and $q(1) = e_n \in \mu_{r \times I}$ on the respective identity fibers for finite topological groups $G_{Y,i}, G_{Y,n}$ under locally homeomorphic embeddings in (Y, τ_Y) .

Proof. Let us consider that $h_{ni} : G_n \rightarrow G_i$ be a group homomorphism in the LN covering (C, R) space X_{cov} . Suppose the space $B \subset Y_{\text{cov}}$ is locally dense and the finite variety of two topological groups $G_n = (X_n, *_n)$ and $G_i = (X_i, *_i)$ are established such that $X_i \subset A_i \subset X_{\text{cov}}, X_n \subset A_n \subset X_{\text{cov}}$ where $X_n \cap X_i = \emptyset$. Suppose the group identities $a_i \in X_i, a_n \in X_n$ of two locally homeomorphic group embeddings are given as $i_{\text{cov}}(a_i) = e_i$ and $i_{\text{cov}}(a_n) = e_n$ for the respective G_i, G_n . Let the two fibers $\mu_{p \times I}, \mu_{r \times I}$ in a dense topological (C, R) space (Y, τ_Y) maintain the following conditions.

$$\begin{aligned} \mu_{p \times I} \cap \mu_{r \times I} &= \emptyset, \\ \{e_i\}, \{e_n\} &\in \tau_Y, \\ \mu_{p \times I} \cap i_{\text{cov}}(X_i) &= \{e_i\}, \\ \mu_{r \times I} \cap i_{\text{cov}}(X_n) &= \{e_n\}. \end{aligned} \tag{5}$$

Note that the respective fibers are identity fibers in this case. Thus there exists a continuous function $q : [0, 1] \rightarrow B$ such that $q(0) = e_i$ and $q(1) = e_n$ in $B \subset Y_{\text{cov}}$. Now consider another continuous function in $B \subset Y_{\text{cov}}$ given by $f_n : [0, 1] \rightarrow i_{\text{cov}}(K_n \triangleleft G_n)$ such that $f_n(0) = f_n(1) = e_n$. Note that the locally homeomorphic embeddings preserve group homomorphism as $(i_{\text{cov}} \circ h_{ni})(K_n) = \{e_i\}$. Hence the continuous function $f_n([0, 1])$ in dense $B \subset Y_{\text{cov}}$ maintains a path-homotopy equivalence relation given by $[f_n] \cong_H [\bar{q}] * [q]$ involving the corresponding two identity fibers $\{\mu_{p \times I}, \mu_{r \times I}\}$. $\square$

The locally homeomorphic embeddings of LN covering of (C, R) spaces and the corresponding projections of fibers on a path component of covering (C, R) spaces generate a cyclic group structure under certain conditions. First, the projections of fibers on real subspace need to be finite, and second the resulting planar subspace forms a 2-simplex. This interesting property is presented in the following theorem.

Theorem 9. If (Y, τ_Y) is a dense topological (C, R) space containing $i_{\text{cov}}(X_{\text{cov}})$ then there exists a cyclic group structure under finite real projections of fibers into embedded LN covering (C, R) spaces.

Proof. Let (Y, τ_Y) be a dense topological (C, R) space such that $(i_{\text{cov}} \circ f_c^{-1})(N_p)$ is the embedded LN covering (C, R) spaces of corresponding $i_X(B \subset X)$ such that $N_p \subset B$. If (Y, τ_Y) is a fibered space then one can select three fibers $\mu_{p \times I}, \mu_{r \times I}, \mu_{s \times I} \subset Y$ such that $\mu_{p \times I} \cap (i_{\text{cov}}(A_p)) = \{y_p\}$, $\mu_{r \times I} \cap (i_{\text{cov}}(A_r)) = \{y_r\}$ and $\mu_{s \times I} \cap (i_{\text{cov}}(A_s)) = \{y_s\}$ where $\bigcup_{k \in \{p, r, s\}} A_k \subset X_{\text{cov}}$. Moreover if the respective projections on real subspace $\pi_R : \mu_{p \times I} \rightarrow R$ maintains the property that $\forall y_p \in Y, \pi_R(\mu_{p \times I}) \in R \setminus \{-\infty, +\infty\}$ then we can infer that $\exists \{r_p, r_r, r_s\} \subset R \setminus \{-\infty, +\infty\}$ such that $y_p \in C \times \{r_p\}$, $y_r \in C \times \{r_r\}$ and $y_s \in C \times \{r_s\}$. Thus a planar 2-simplex is formed in Y_{cov} denoted as $\sigma_Y^2 = \langle y_p, y_r, y_s \rangle$ and a corresponding planar as well as closed triangulated topological subspace is generated which is given by $|\sigma_Y^2| \subset i_{\text{cov}}(X_{\text{cov}})$. Hence a prime ordered cyclic group $G_\sigma = (Y_\sigma, \cdot)$ is formed where $Y_\sigma = \pi_R(\sigma_Y^2)$. $\square$

Remark 5. Note that the linear translations restricted to fibers given by $\forall y_p \in Y_{\text{cov}}, T_\mu|_p : \mu_{p \times I} \rightarrow \mu_{p \times I}$ admits $k \in \{p, r, s\}$, $G_{T_\sigma} = (T_\mu^m|_k(Y_\sigma), \cdot)$ as long as $\|T_\mu^m|_k\| < +\infty$ where $T_\mu^m|_p(y_p) \neq T_\mu^m|_r(y_r) \neq T_\mu^m|_s(y_s)$. However the restriction to be maintained is that the finite fiber-restricted translations within the space must maintain symmetry condition given by $(\pi_R(T_\mu^m|_p(y_p))) \cdot (\pi_R(T_\mu^m|_r(y_r))) = \pi_R(T_\mu^m|_s(y_s))$ so that the cyclic group structure is preserved considering that $\pi_R(y_s)$ represents the identity element.

Corollary 1. Even if the structure $\sigma_Y^2 = \langle y_p, y_r, y_s \rangle$ is generated by $\mu_{p \times I}, \mu_{r \times I}, \mu_{s \times I} \subset Y$ in Y_{cov} such that $r_p = r_r = r_s$ and $y_p \neq y_r \neq y_s$ then $\exists A_m \subset X_{\text{cov}}$ forming $|\sigma_Y^2| \subset i_{\text{cov}}(A_m)$ which preserves cyclic group structure successfully as $G_{A_\sigma} = (Y_\sigma \subset i_{\text{cov}}(A_m), \cdot)$ where the group operation is an abstract algebraic operation.

Proof. The proof is straightforward if we consider that the finite real projection $\pi_R : Y \rightarrow R$ generates $\{y_p, y_r, y_s\} \subset i_{\text{cov}}(A_m)$ such that $|\sigma_Y^2| \subset i_{\text{cov}}(A_m)$ and the group operation is an abstract algebraic operation retaining the prime ordered cyclic group structure. $\square$

Remark 6. It is interesting to note that $G_{A_\sigma} = (Y_\sigma \subset i_{\text{cov}}(A_m), \cdot)$ is largely a relaxed variety because it does not require the topological condition that every continuous function $v : S^1 \rightarrow \langle \sigma_Y^2 \rangle$ is nullhomotopic in nature. However, in this case there is a restriction with respect to the translation of $|\sigma_Y^2| \subset i_{\text{cov}}(A_m)$ by employing any finite linear function $T : i_{\text{cov}}(A_m) \rightarrow i_{\text{cov}}(A_m)$ in order to retain the $G_{TA_\sigma} = (T^n(Y_\sigma \subset i_{\text{cov}}(A_m)), \cdot)$ structure for some $1 \leq n < +\infty$. Note that some forms of finite linear translations support $G_{TA_\sigma} = (T^n(Y_\sigma \subset i_{\text{cov}}(A_m)), \cdot)$ structure; however some other varieties of translations will not. This property is presented in the following theorem.

Theorem 10. If $T : i_{\text{cov}}(A_m) \rightarrow i_{\text{cov}}(A_m)$ is a finite linear translation then $G_{TA_\sigma} = (T^n(Y_\sigma \subset i_{\text{cov}}(A_m)), \cdot)$ is a cyclic group if, and only if, $i_{\text{cov}}(A_m)$ is simply connected.

Proof. Let us consider a separation of topological (C, R) subspace $W \subset i_{\text{cov}}(A_m)$ such that $W \subset Y_{m1} \cup Y_{m2}$ where $Y_{m1} \cap Y_{m2} = \{y_w\}$ and $W \cap Y_{m1} \neq \emptyset, W \cap Y_{m2} \neq \emptyset$. Suppose we consider that $|\sigma_Y^2| \subset Y_{m1}$ and $T^n(Y_\sigma) \subset Y_{m2}$ such that $\sigma_{TY}^2 = \langle T^n(y_p), T^n(y_r), T^n(y_s) \rangle$ and $1 \leq n < +\infty$. Now consider a continuous function in the subspace given by $v : S^1 \rightarrow i_{\text{cov}}(A_m)$. If we impose two contradictory conditions on the restriction of the function $v|_W : S^1 \rightarrow i_{\text{cov}}(A_m)$ within the separation such that $v : S^1 \rightarrow \langle \sigma_Y^2 \rangle$ is nullhomotopic and $v : S^1 \rightarrow \langle \sigma_{TY}^2 \rangle$ is not nullhomotopic then the finite linear translation function $T : i_{\text{cov}}(A_m) \rightarrow i_{\text{cov}}(A_m)$ violates $\text{hom}(|\sigma_Y^2|, |\sigma_{TY}^2|)$ property and as result such a translation does not exist in $i_{\text{cov}}(A_m)$ supporting $G_{TA_\sigma} = (T^n(Y_\sigma \subset i_{\text{cov}}(A_m)), \cdot)$. Hence, the subspace $i_{\text{cov}}(A_m)$ is simply connected by the

nullhomotopic function $v : S^1 \rightarrow i_{\text{cov}}(A_m)$ and as a result $v : S^1 \rightarrow W$ is also nullhomotopic in its separations preserving $G_{TA\sigma} = (T^n(Y_\sigma \subset i_{\text{cov}}(A_m)), \cdot)$ cyclic group structure under finite translations. $\square$

Corollary 2. *If the fiber $\mu_{p \times I} \subset Y$ is arbitrary (i.e., general fiber) such that $\{y_i = i_{\text{cov}}(A_i) \cap \mu_{p \times I}, y_k = i_{\text{cov}}(A_k) \cap \mu_{p \times I}\} \subset \mu_{p \times I}$ where $\{y_i, y_k\} \subset f_{\text{cov}}^{-1}(y_0)$ then the covering projection of fundamental group $H_k = f_{\text{cov}*}(\pi_1(Y_{\text{cov}}, y_k))$ maintains the property that $H_k \subset \pi_1(i_X(X), y_0)$ if, and only if, there is a continuous function $\beta : [0, 1] \rightarrow Y_{\text{cov}}$ such that $\beta(0) = y_i, \beta(1) = y_k$.*

The proof of aforesaid corollary is straightforward and it shows that a fibered topological (C, R) space containing embedded LN covering spaces and base space successfully preserves the classification of covering spaces in terms of fundamental groups. In other words the interplay of homotopy and topological groups in a fibered topological (C, R) covering space of LN variety with embeddings does not interfere with the classical results related to the covering space classifications based on fundamental groups.

5. Conclusions

The topological covering (C, R) spaces enable suitable incorporation of additional structures enhancing the richness of their properties. The compact Lindelof variety of path-connected components of covering (C, R) spaces enables the formulation of finite group algebraic structures within the spaces. The groups can be equipped with homomorphism and the corresponding finite Noetherian covering spaces formed by homeomorphic embeddings allowtable the formulation of various path-homotopy equivalences in the fibered topological (C, R) space. The interplay of finite homomorphic groups in the path-connected components of covering spaces and the topological fibers generates a discrete variety of fundamental group structure within the embedded dense subspaces. The topological fibers get classified into several varieties depending on the position of identity elements of the embedded homomorphic groups. As a result a wide array of path-homotopy equivalences is formulated within the embedded LN covering spaces including the base space. The rigidity of fibers based on identity and the multiplicity of fibers support path-homotopy equivalences considering that the path connected covering components are simply connected in view of nullhomotopy. Interestingly, the resulting 2-simplex structures and finite as well as symmetric translations within the fibered covering space assists in determining the simple connectedness of the path-connected covering components under topological embeddings.

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