

Article

The Approximation of the Nonlinear Singular System with Impulses and Sliding Mode Control via a Singular Polynomial Fuzzy Model Approach

Jiawen Li ¹, Yi Zhang ^{1,*}  and Zhenghong Jin ²¹ School of Science, Shenyang University of Technology, Shenyang 110870, China; songyue@smail.sut.edu.cn² State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang 110819, China; 1810287@stu.neu.edu.cn

* Correspondence: zhangyi@sut.edu.cn

Abstract: In this paper, the Singular-Polynomial-Fuzzy-Model (SPFM) approach problem and impulse elimination are investigated based on sliding mode control for a class of nonlinear singular system (NSS) with impulses. Considering two numerical examples, the SPFM of the nonlinear singular system is calculated based on the compound function type and simple function type. According to the solvability and the steps of two numerical examples, the method of solving the SPFM form of the nonlinear singular system with (and without) impulse are extended to the more general case. By using the Heine–Borel finite covering theorem, it is proven that a class of nonlinear singular systems with bounded impulse-free item (BIFI) properties and separable impulse item (SII) properties can be approximated by SPFM with arbitrary accuracy. The linear switching function and sliding mode control law are designed to be applied to the impulse elimination of SPFM. Compared with some published works, a human posture inverted pendulum model example and Example 3.2 demonstrate that the approximation error is small enough and that both algorithms are effective. Example 3.3 is to illustrate that sliding mode control can effectively eliminate impulses of SPFM.

Keywords: nonlinear singular systems; singular polynomial fuzzy model; sliding mode control; approximation; impulse



Citation: Li, J.; Zhang, Y.; Jin, Z. The Approximation of the Nonlinear Singular System with Impulses and Sliding Mode Control via a Singular Polynomial Fuzzy Model Approach. *Symmetry* **2021**, *13*, 1409. <https://doi.org/10.3390/sym13081409>

Academic Editors: José Carlos R. Alcántud and Basil Papadopoulos

Received: 25 June 2021
Accepted: 29 July 2021
Published: 2 August 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

The Takagi–Sugeno (T-S) fuzzy model was put forward by T. Takagi and M. Sugeno [1] in 1985, and they proposed a new type of fuzzy model representation. Due to the excellent approximation performance of the T-S fuzzy model in nonlinear systems, more and more scholars are interested in the T-S fuzzy model using multiple local linear systems to represent the nonlinear system and then analyzing the characteristics of the nonlinear system by using the analysis method of the linear system. In 1992, an important conclusion was proved by Wang [2], that is, fuzzy systems are a universal approximator, and Wang approximated any continuous function on the compact set with any precision by using the fuzzy system constructed with gaussian membership functions, a product inference machine, and a center-average weighted defuzzifier. Since then, many scholars [3–7] have proven that the above conclusions can be adapted to various fuzzy systems. In [8], the finite-time stabilization of a class of stochastic nonlinear systems was studied by applying the fuzzy-logic systems to approximate the unknown nonlinearities and a novel adaptive finite-time control strategy was proposed. The vector integral sliding mode surface and sliding mode control (SMC) law were proposed for the T-S fuzzy singular system with matched external disturbances in [9]. In recent years, a new fuzzy model has been proposed by K. Tanaka [10], that is, a polynomial fuzzy model. In essence, it is a more extensive form of the T-S fuzzy model. The main difference between the two models is the conclusion: the result of T-S fuzzy model is a linear model, and the result of polynomial fuzzy model is a

polynomial model. Therefore, the number of fuzzy rules in the polynomial fuzzy model is much less than that in the T-S fuzzy model when describing the same nonlinear systems. Hence, it will become more and more popular, and more and more people [11–13] have begun to study the polynomial fuzzy system. In [14,15], the polynomial fuzzy singular system was proposed first and the interval observers for polynomial fuzzy singular systems were designed, in which the external disturbances and unknown parameters were included in the systems. However, Pang [14,15] assumed that the polynomial fuzzy singular systems is regular and impulse-free. In this paper, the SPFM can be obtained for the NSSs with the impulse by the SPFM approximation theorem and related algorithms given.

As the fuzzy modeling develops, polynomial fuzzy models employ polynomial models as local subsystems instead of linear models. It not only makes the modeling process simpler but also handles more nonlinear plant in comparison with the singular T-S fuzzy model. The LMI-based stability analysis method commonly used in T-S fuzzy model has some difficulties in the application of polynomial fuzzy models. Thus far, it has not been directly used in the stability analysis of polynomial model systems. Based on the difficulty of analyzing polynomial fuzzy models, the sum-of-squares based (SOSB) approaches emerges and occupies an important position in the polynomial fuzzy model, which can effectively deal with stability analysis problems [10]. In [10], the SOSB stability analysis of a polynomial fuzzy model is proposed under a parallel, distributed compensation strategy; hence, the fuzzy controller designed has the same membership function and fuzzy rules as the polynomial fuzzy model. In [16], the stability of the polynomial-fuzzy-model-based (PFMB) control system was investigated by using the SOSB stability analysis method and by considering the number of fuzzy rules and the shape of premise fuzzy membership functions. In [17,18], the condition of system conservative stability was obtained based on the piecewise linear membership function and the polynomial approximation membership function.

The singular system is a natural representation of objective system. It can be used to describe further characteristics of the system and has been widely applied in large system theory, singular perturbed theory, circuit theory, and economic theory [19–23]. In 1999, Taniguchi et al. combined the T-S fuzzy system with the singular system and promoted it to propose the T-S fuzzy singular system [24]. In recent years, various novel control techniques have been applied to the fuzzy singular system [25–30]. In many practical models, most of the systems are nonlinear and the nonlinearity of many nonlinear systems can be transformed into polynomials. If the T-S fuzzy system is used to approximate the nonlinear system, there are two challenges, namely, the number of rules is too many and the approximation accuracy is insufficient. However, the birth of the SPFM approximation method can be able to solve such problems and to effectively reduce the number of fuzzy rules for the impulse-free nonlinear system. There are two main characteristics of SPFM: 1. The SPFM is a class of more complex system including dynamic and non-dynamic constraints. 2. The number of fuzzy rules of the SPFM for nonlinear system is less than the normal fuzzy model, and the results are more simple.

The T-S fuzzy singular system is used to approximate the NSS, but there may be two problems for complex NSS. One is too many rules for the T-S fuzzy singular system, and the other is that the approximation error is large. However, the SPFM is able to solve such problems, and the number of fuzzy rules can be reduced for the impulse-free NSS [14,15]. If the NSS has an impulse, there is no method to show that SPFM can effectively approach such systems. Reference [31] deals with the existence of solutions to singular second-order differential equations with impulse effects and with the Dirichlet boundary conditions. However, the requirements of the thesis are very strict, and there are many assumptions. In [32], Zhang and Yuan dealt with the existence and multiplicity of solutions for the nonlinear Dirichlet value problem with impulses by using the variational methods and critical points theory. In [33], Nieto introduced the concept of a weak solution for a damped linear equation with Dirichlet boundary conditions and impulses. The above three papers dealt with impulse problems for second-order differential equations and did not give

the case of higher-order differential equations. In [34,35], this technical note discusses to what extent the high-order sliding mode control may serve as an alternative to the conventional sliding mode control. The definitions of sliding mode order, relative degree, chattering attenuation, filtering, and implementation complexity constitute the scope of that discussion. In [36], Utkin systematically introduced the development of the theory of sliding modes, the design of sliding mode controller, and the application of sliding mode control. In this paper, a theorem using the SPFM to approximate the nonlinear singular system with impulses was proven for any order and the assumptions were easier to satisfy. Our major research interest is to study the SPFM approach problem for a class of NSS with impulses. The main contributions of this paper are summarized below.

- (1) The method using SPFM to approximate the impulse-free nonlinear singular system is basically the same as the T-S fuzzy model, and there are many related studies. One of the important differences between singular systems and normal systems is that the former may contain impulse terms in the solution. Hence, the impulses are a distinctive feature of singular systems, and they are the basis on which singular systems can describe a wider range of physical systems. Current research on singular systems often ignores the impulses or assumes that the system is impulse-free and that there is no method to approximate the singular system with an impulse by using SPFM. The theorem that the SPFM can approximate an NSS with impulses with an arbitrary precision is proven.
- (2) According to the complexity of the NSS, it is divided into a nonlinear model with a compound function type and a simple function type. For two different types of NSS, two different theorems are given to prove the effectiveness of SPFM. It is fully proven that the NSS can be approximated using the SPFM model, even if the NSS has impulses. Then, the algorithms solving SPFM for two kinds of nonlinear functions are given.
- (3) The principle of SMC to eliminate the impulse is given. The designed SMC law not only can effectively eliminate impulses of SPFM but also can make the systems asymptotically stable.

The organizational structure of the paper is as follows. The descriptions of the approximation of SPFM are considered in Section 2. Two numerical examples of different types about NSSs and the corresponding algorithms for solving the SPEM for the NSS with (and without) impulses are given in Section 3. Then, the linear switching function and SMC law are designed, and two important theorems in SMC are proven. Finally, the theorem that the impulse of the SPFM can be eliminated by the SMC law is proven and the principle of SMC to eliminate the impulse is given. In Section 4, a human posture inverted pendulum model and two numerical example are presented to demonstrate the effectiveness of our theoretical results. Section 5 concludes the paper.

Notation: The symbol A^T denotes the transposition of vector or matrix A ; the symbol \mathbb{R}^n represents the n -dimensional vector space; $\|\cdot\|_2$ represents Euclidean norm; $\sup(\cdot)$ represents supremum function; $O(n, m)$ represents $n \times m$ dimensional zero matrix; and the symbol $*$ represents entries.

2. Preliminaries and Problem Formulation

In this paper, we use the SPFM to approximate NSSs with (or without) impulses with any precision. Based on the sector nonlinearity concept, the singular polynomial fuzzy system can be represented.

Model Rules i :

$$\begin{aligned} & \text{IF } p_1(t) \text{ is } M_{i1} \text{ and } \dots \text{ and } p_k(t) \text{ is } M_{ik} \\ & \text{THEN } E\dot{x}(t) = A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, and M_{ik} is the fuzzy set of the rule i corresponding to the premise variable $p_k(t)$. $E \in \mathbb{R}^{n \times n}$ is a matrix, $\text{rank}(E) = r(\leq n)$, $A_i(x(t)) \in \mathbb{R}^{n \times n}$, and $B_i(x(t)) \in \mathbb{R}^{n \times m}$ are the polynomial matrices about $x(t)$.

The term $\hat{x}(x(t)) \in \mathbb{R}^n$ is a monomial about $x(t)$, and each monomial is a function of form $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, where $\alpha_i (> 0)$ are integers, $i = 1, 2, \dots, n$. Thus, $A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)$ is a polynomial vector.

As the fuzzy model develops, polynomial fuzzy models employ polynomial models as local subsystems, and polynomial fuzzy models is a more general case of T-S models. According to model rules and process of T-S model, the SPFM can be represented as

$$\begin{aligned}
 E\dot{x}(t) &= \frac{\sum_{i=1}^k w_i(p(t))[A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)]}{\sum_{i=1}^k w_i(p(t))} \\
 &= \sum_{i=1}^k h_i(p(t))[A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)]
 \end{aligned}
 \tag{1}$$

where

$$p(t) = [p_1(t) \quad p_2(t) \quad \dots \quad p_k(t)]^T$$

and

$$w_i(p(t)) = \prod_{j=1}^k M_{ij}(p_j(t)), w_i(p(t)) \geq 0, i = 1, 2, \dots, k$$

So

$$h_i(p(t)) = \frac{w_i(p(t))}{\sum_{i=1}^k w_i(t)} \geq 0, \sum_{i=1}^k h_i(p(t)) = 1$$

Remark 1. The SPFM is the more extensive form of the T-S fuzzy singular system. If $A_i(x(t))$ and $B_i(x(t))$ are constant matrices (A_i and B_i) and $\hat{x}(x(t)) = x(t)$, then $A_i(x(t))\hat{x}(x(t)) + B_i(x(t))u(t)$ becomes $A_i x(t) + B_i u(t)$ and it is the part of the T-S fuzzy singular system.

The SPFM approximation method has an important advantage that the number of fuzzy rules in a singular polynomial fuzzy system is generally fewer than the T-S fuzzy singular system.

Assumption 1. In this paper, we assume that $\hat{x}(x(t)) = 0$ if and only if $x(t) = 0$.

Remark 2. Assumption 1 is mainly to ensure that the system $\hat{x}(t)$ is a polynomial about $x(t)$ and that the constant term is 0. This is an important premise of polynomial systems, and many results have related assumptions [37].

The NSS is given as follows:

$$E\dot{x}(t) = f(x(t), u(t), t) \tag{2}$$

where $f(x(t), u(t), t)$ is a n-dimensional vector function consisting of continuous function $x(t)$ and $u(t)$.

Let P and Q be invertible matrices and $PEQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Let $Q^{-1}x(t) = \tilde{x}(t)$, where $\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$, the Equation (2) is equivalent to the following equation:

$$\begin{aligned}
 \dot{\tilde{x}}_1(t) &= f_1(\tilde{x}_1(t), \tilde{x}_2(t), u(t), t) \\
 0 &= f_2(\tilde{x}_1(t), \tilde{x}_2(t), u(t), t)
 \end{aligned}
 \tag{3}$$

The research interest of this paper is to find the SPFM to approximate the NSS such as Equation (2) or (3).

Definition 1 (singularity induced bifurcations(SIB) [38]). Consider the system

$$\begin{cases} \dot{x}(t) = f(x, y, p) \\ 0 = g(x, y, p) \end{cases}$$

Suppose that the system has an equilibrium (x_0, y_0, p_0) for $p_0 \in \mathbb{R}$ and that the linearization about this equilibrium has an eigenvalue locus $l(p)$. If for some sequences $\{n_i\}, \{m_i\} \in \mathbb{R}$, such that for $n_i < p_0$ and $m_i > p_0$ we have $\lim_{i \rightarrow \infty} n_i = p_0$ and $\lim_{i \rightarrow \infty} m_i = p_0$, and

(I) $l(n_i) \rightarrow \infty, l(m_i) \rightarrow \infty$ for $i \rightarrow \infty$ and

(II) $\text{Re}(l(n_i)) \text{Re}(l(m_i)) < 0$,

then p_0 is said to be a SIB point.

Remark 3. In theory, the fuzzy system can be used to approximate the nonlinear system with arbitrary accuracy. However, the approximation accuracy of most specific nonlinear singular systems needs to be explained from a numerical perspective by using SPFM, especially for nonlinear singular systems with impulses. In order to verify the effect of model approximation, the error between the numerical solution of SPFM and the state response of the nonlinear singular system is mainly used.

In this research, the NSS can be divided into two types: one is a nonlinear model with a compound function type such as $\sin(x(t)), e^{x(t)}$, and $\ln(x(t))$ and the other is a nonlinear model with a simple function type such as $\sin(t), e^t$, and $\ln(t)$. The methods of finding the SPFM for two kinds of nonlinear singular systems are different. For the nonlinear model with a compound function type, the methods of finding SPFM are the variable transformation and we introduce the variable transformation of the six basic compound functions. In view of these two nonlinear systems, two examples are given later.

2.1. Variable Transformation for Basic Compound Function

At this section, the variable transformation methods of the six basic compound functions are given. The six basic compound functions are the fractional function, the power function, the exponential function, the logarithmic function, the trigonometric function, and the hyperbolic function. Any complex functions can be composed of these six basic functions using the basic operations, such as addition, subtraction, multiplication, division, and compound operations. Therefore, we only give the variable transformation methods of six basic functions.

In the Table 1, for convenience, we employ x instead of $x(t)$. The $f(m, x) = (a_1 + 2a_2x^1 + \dots + ma_mx^{m-1})$. a_0, a_1, \dots, a_m are constant, β is any real constant, and $\alpha (\neq 1)$ is a positive. The trigonometric function contains $\sin(x), \cos(x), \tan(x), \text{ctg}(x), \sec(x)$, and $\text{csc}(x)$ and takes $\sin(x)$ as an example; the other computing methods are similar. The hyperbolic function contains $sh(x), ch(x), th(x)$, and $cth(x)$ and takes $sh(x)$ as an example; the other computing methods are similar. $D_1 = (-\infty, 0) \cup (0, +\infty), D_2 = (-\infty, +\infty), D_3 = (0, +\infty), D_4 = [-1, 1]$.

Table 1. Variable transformation for the basic compound function.

Types	Form	Range	New Variables	Domain
The fractional function	$\frac{1}{a_mx^m + \dots + a_0}$	D_1	$x_1 = \frac{1}{a_mx^m + \dots + a_0}$	D_1
The power function	x^β	D_2	$x_1 = x^\beta, x_2 = \frac{1}{x}$	D_2, D_1
The exponential function	α^x	D_3	$x_1 = \alpha^x$	D_3
The logarithmic function	$\log_\alpha x$	D_2	$x_1 = \log_\alpha x, x_2 = \frac{1}{x}$	D_2, D_1
The trigonometric function	$\sin x$	D_4	$x_1 = \sin x, x_2 = \cos x$	D_4, D_4
The hyperbolic function	$shx = \frac{e^x - e^{-x}}{2}$	D_2	$x_1 = e^x, x_2 = e^{-x}$	D_3, D_3

2.2. Nonlinear Model Example with Compound Function Type

Consider the NSS as follows

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} - \frac{3}{2} \sin(x_1(t)) & -4 \\ \frac{19}{2} - \frac{21}{2} \sin(x_1(t)) & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (4)$$

It is well known that a non-polynomial system can be transformed into a polynomial system by variable transformation. For system (4), we introduce two new variables: $x_3(t) = \sin(x_1(t))$ and $x_4(t) = \cos(x_1(t))$. By the variable transformation, we have

$$\begin{aligned} \dot{x}_3(t) &= \dot{x}_1(t) \cos(x_1(t)) = -\left(\frac{7}{2}x_1(t) + \frac{3}{2}x_3(t)x_1(t) + 4x_2(t)\right)x_4(t) \\ &= -\frac{7}{2}x_1(t)x_4(t) - \frac{3}{2}x_1(t)x_3(t)x_4(t) - 4x_2(t)x_4(t) \end{aligned}$$

$$\begin{aligned} \dot{x}_4(t) &= -\dot{x}_1(t) \sin(x_1(t)) = \left(\frac{7}{2}x_1(t) + \frac{3}{2}x_3(t)x_1(t) + 4x_2(t)\right)x_3(t) \\ &= \frac{7}{2}x_1(t)x_3(t) + \frac{3}{2}x_1(t)x_3^2(t) + 4x_2(t)x_3(t) \end{aligned}$$

Therefore, system (4) can be converted into a polynomial NSS as follows:

$$\begin{aligned} \dot{x}_1(t) &= -\left(\frac{7}{2} + \frac{3}{2}x_3(t)\right)x_1(t) - 4x_2(t) \\ 0 \cdot \dot{x}_2(t) &= \left(\frac{19}{2} - \frac{21}{2}x_4(t)\right)x_1(t) - 2x_2(t) \\ \dot{x}_3(t) &= -\frac{7}{2}x_1(t)x_4(t) - \frac{3}{2}x_1(t)x_3(t)x_4(t) - 4x_2(t)x_4(t) \\ \dot{x}_4(t) &= \frac{7}{2}x_1(t)x_3(t) + \frac{3}{2}x_1(t)x_3^2(t) + 4x_2(t)x_3(t) \end{aligned} \quad (5)$$

Then, we can obtain the SPFM:

$$E\dot{x}(t) = A(x(t))x(t) \quad (6)$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$A(x(t)) = \begin{bmatrix} -\frac{7}{2} & -4 & -\frac{3}{2}x_1(t) & 0 \\ \frac{19}{2} & -2 & 0 & -\frac{21}{2}x_1(t) \\ -\frac{7}{2}x_4(t) & -4x_2(t) & -\frac{3}{2}x_1(t)x_4(t) & 0 \\ \frac{7}{2}x_3(t) & 4x_3(t) & \frac{3}{2}x_1(t)x_3(t) & 0 \end{bmatrix}$$

As the number of nonpolynomial terms of system (4) is less, the SPFM has only one subsystem. The Figure 1 shows a comparison of the state trajectory about the NSS (4) and SPFM (6) when $x(0) = [0.1 \ 0 \ 0 \ 0]^T$.

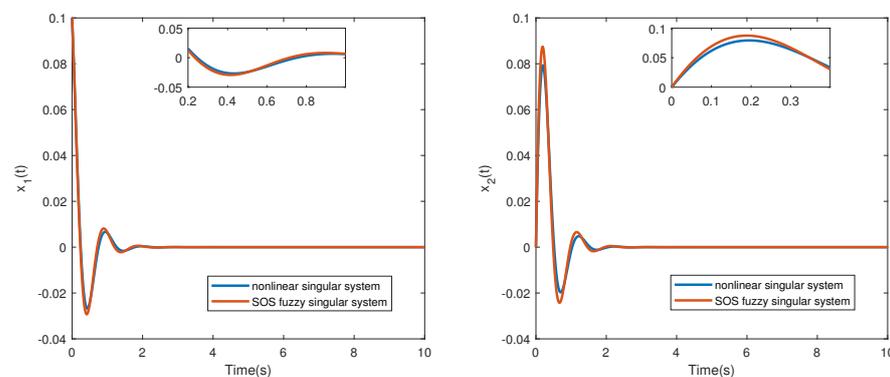


Figure 1. The state trajectory of the NSS (4) and the SPFM (6).

The error figure between two states $x_1(t)$ and $x_2(t)$ is given in Figures 1 and 2. The comparison of state trajectory between NSS and SPFM is given by Figure 1, and the error of $x_1(t)$ and $x_2(t)$ is given by Figure 2. As shown in Figures 1 and 2, the SPFM (6) can reasonably approximate the NSS (4).

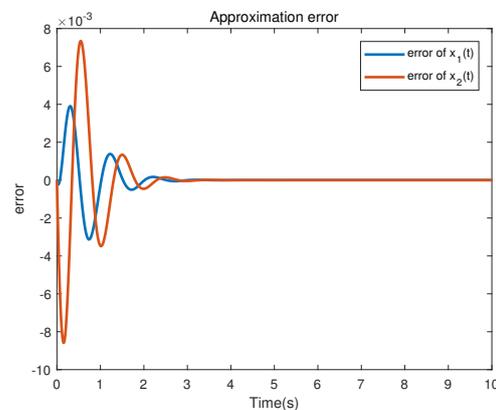


Figure 2. The error figure of the NSS (4) and the SPFM (6).

2.3. Nonlinear Model Example with Simple Function Type

Consider the NSS as follows

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) - x_2^2(t) \\ 0 \cdot \dot{x}_2(t) &= -2x_1(t) - x_2(t) - \sin(t)x_1(t) \end{aligned} \quad (7)$$

System (7) can be expressed as

$$E\dot{x}(t) = \begin{bmatrix} 1 & -x_2(t) \\ -2 - \sin(t) & -1 \end{bmatrix} x(t) \quad (8)$$

where $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

The steps that construct a SPFM to represent the system are given. To begin with, we let $p(t) = \sin(t)$ and it is known that $p(t) \in [-1, 1]$ for all t .

The rules of SPFM can be given.

Model Rule 1:

$$\text{IF } p(t) \text{ is } M_1, \text{ THEN } E\dot{x}(t) = A_1(x(t))x(t)$$

Model Rule 2:

$$\text{IF } p(t) \text{ is } M_2, \text{ THEN } E\dot{x}(t) = A_2(x(t))x(t)$$

where $-M_1(p(t)) + M_2(p(t)) = p(t)$ and $M_1(p(t)) + M_2(p(t)) = 1$, so

$$M_1(p(t)) = \frac{1-p(t)}{2}, M_2(p(t)) = \frac{1+p(t)}{2}$$

The SPFM is

$$E\dot{x}(t) = \sum_{i=1}^2 h_i(p(t))A_i(x(t))x(t) \quad (9)$$

where $A_1(x(t)) = \begin{bmatrix} 1 & -x_2(t) \\ -1 & -1 \end{bmatrix}$, $A_2(x(t)) = \begin{bmatrix} 1 & -x_2(t) \\ -3 & -1 \end{bmatrix}$, $h_1(p(t)) = M_1(p(t)) = \frac{1-\sin(t)}{2}$, $h_2(p(t)) = M_2(p(t)) = \frac{1+\sin(t)}{2}$.

A contrast figure (see Figure 3) of the state responses of the NSS (7) and the SPFM (9) is given below when $x(0) = [0.1 \quad -0.2]^T$.

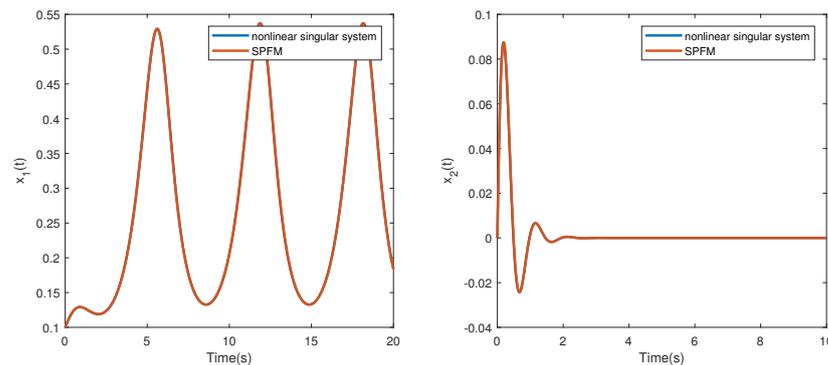


Figure 3. The state trajectory of the NSS (7) and the SPFM (9).

As the state of the two systems is completely overlapped, the SPFM (9) can effectively approximate the NSS (7).

Remark 4. In this paper, we use two methods to solve the SPFM of the NSS, and the two methods have their own advantages and disadvantages. For the method in Section 3.2, the advantage is that the number of fuzzy rules is relatively small and the disadvantage is that the dimension of SPFM is increased; the method in Section 3.3 is just the opposite.

3. Main Results

In this section, the theorem of the SPFM that can effectively approach the model is proven for the nonlinear singular system with impulses. It is proven that a class of nonlinear singular system with a bounded impulse-free item property and a separable impulse item property can be approximated by SPFM with arbitrary accuracy. The linear switching function and sliding mode control law are designed to be applied to the impulse elimination of the SPFM.

3.1. The SPFM Approximation Theorem without Impulse

Before giving the approximation theorem, some necessary symbols are given: $U \subseteq \mathbb{R}^n$ is compact, set and $C(U)$ is the set of continuous function on the U , that is

$$C(U) = \{ f : X \in U, \text{ and } f \text{ is continuous function of } X. \}$$

Γ is the real continuous function set on set U :

$$\Gamma = \{ f_i(X), X \in U, i = 1, 2, \dots, q \}$$

Ω is the solution set:

$$\Omega = \{ X : f_i(X) \in \Gamma, i = 1, 2, \dots, q, X \in U \}$$

Lemma 1 ([39]). Ω is a bounded closed set on \mathbb{R}^n .

Theorem 1. The vector space $V \in \mathbb{R}^n$ is a compact set, where the length of vector is expressed in Euclidean distance and V is the set consisting of $x_i(t)$ and $u_i(t)$ of system (2).

Proof. There are two main aspects to be proven: 1. V is a distance space. 2. Any sequence of V has a subsequence converging to V . First, prove the first part.

For any n -dimensional vector $\alpha, \beta, \gamma \in V$, $\alpha = (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n)$, $\beta = (\beta_1 \ \beta_2 \ \dots \ \beta_n)$, $\gamma = (\gamma_1 \ \gamma_2 \ \dots \ \gamma_n)$, where α_i, β_i and γ_i are constants.

1. Nonnegativity.

$$d(\alpha, \beta) = \|(\alpha_1 - \beta_1 \ \alpha_2 - \beta_2 \ \dots \ \alpha_n - \beta_n)\|_2$$

$$= \sqrt{\sum_{i=1}^n (\alpha_i - \beta_i)^2} \geq 0$$

$d(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.

2. Symmetry.

$$d(\alpha, \beta) = \|(\alpha_1 - \beta_1 \ \alpha_2 - \beta_2 \ \dots \ \alpha_n - \beta_n)\|_2$$

$$= \|(\beta_1 - \alpha_1 \ \beta_2 - \alpha_2 \ \dots \ \beta_n - \alpha_n)\|_2$$

$$= d(\beta, \alpha)$$

3. Trigonometric inequality.

$$d(\alpha, \beta) = \|(\alpha_1 - \beta_1 \ \dots \ \alpha_n - \beta_n)\|_2$$

$$= \sqrt{\sum_{i=1}^n (\alpha_i - \beta_i)^2} = \sqrt{\sum_{i=1}^n ((\alpha_i - \gamma_i) - (\beta_i - \gamma_i))^2}$$

$$\leq \sqrt{\sum_{i=1}^n (\alpha_i - \gamma_i)^2} + \sqrt{\sum_{i=1}^n (\beta_i - \gamma_i)^2}$$

$$= d(\alpha, \gamma) + d(\beta, \gamma) = d(\alpha, \gamma) + d(\gamma, \beta)$$

Through the above analysis, V is a distance space. Next, we prove that any sequence of V has a subsequence converging to V . For the above $\alpha = (\alpha_1 \ \alpha_2 \ \dots \ \alpha_n) \in \mathbb{R}^n$. It is apparent that there exists a subsequence $\{\alpha^1, \alpha^2, \dots, \alpha^n, \dots\}$ that satisfies $\lim_{n \rightarrow \infty} \alpha^n = \alpha$, where $\alpha^n \in \mathbb{R}^n, n = 1, 2, \dots, n, \dots$. That is, any sequence of V has a subsequence converging to V . Therefore, the vector space $V \in \mathbb{R}^n$ is a compact set. \square

Definition 2. The distance between two functions on \mathbb{R}^n is

$$d_\infty(y_1(s) - y_2(s)) = \sup_{s \in \mathbb{R}^n} (|y_1(s) - y_2(s)|)$$

It is easy to prove that d_∞ satisfies nonnegativity, symmetry and triangle inequality, so the (Ψ_U, d_∞) is a metric space. The whole space of the polynomial fuzzy systems is denoted as Ψ_U on a compact set U .

Lemma 2 ([3]). Consider any real continuous function g on the compact set $U \subseteq \mathbb{R}^n$ and any precision $\epsilon > 0$; there exists $f \in \Psi_U$ that satisfies $d_\infty(y(s) - g(s)) < \epsilon$.

In this paper, we discuss the approximation ability of SPFM without impulses. Let $\theta(s)$ be any real continuous function of NSS on set Ω ,

Lemma 3. For the above $\theta(s)$ and any precision $\epsilon > 0$, there exists $f(s) \in \Psi_U$ that satisfies $d_\infty(\theta(s) - f(s)) < \epsilon$.

Proof. Specific proof see literature [39]. \square

Remark 5. In the literature [39], the theorem using the T-S fuzzy singular system to approximate the NSS without impulses has been proven. There are some similarities between our proof process and that in the literature, but the biggest difference is that our results are part of a polynomial model and that our theorem is more general.

3.2. The SPFM Approximation Theorem with Impulse

In this section, we mainly prove the approximation problems for a class of singularity-induced bifurcations system using SPFM. First, a general description of the problem is given:

$$E\dot{x}(t) = f(x(t), u(t), m) \quad (10)$$

The value of bifurcation is m^* , and system (10) is SIB system at $m = m^*$. The following useful definitions are given below.

Definition 3 (bounded impulse-free item). Consider system (10) at $m = m^*$; the NSS appears impulse and the number of impulse-free items is l ($0 < l < n$). If this impulse are bounded, the NSS is called BIFI.

Definition 4 (separable impulse item). Consider system (10) at $m = m^*$; the singular system appears impulse and can be written:

$$E\dot{x}(t) = A(x(t), m^*)x(t) + Bu(t)$$

Consider each impulse item $x_i(t) \in x(t)$; if there is not an impulse item in the matrix $A(x(t), m^*)$, the NSS is called SII.

Theorem 2. For the SIB system (10) at $m = m^*$, if the system is BIFI and SII; the system can be approximated by SPFM.

Proof. System (10) is BIFI and SII; without loss of generality, assume that the number of impulse items is one ($l = 1$) and that the impulse item is called $x_n(t)$. Therefore, we assumed that the number of nonlinear terms without impulse item except for polynomials is k ; that the nonlinear terms are called $g_i(t)$; and that the maximum and minimum are g_i^{\max} and g_i^{\min} ($i = 1, 2, \dots, k$), respectively.

The general scheme of finding the SPFM is given:

1. When $m = m^*$, system (10) can be written:

$$E\dot{x}(t) = A(x(t), m^*)x(t) + B(x(t))u(t)$$

2. Defining that $p_i(t), i = 1, 2, \dots, k$ represents the nonlinear terms, therefore,

$$E\dot{x}(t) = A(p(t), x(t), m^*)x(t) + B(p(t), x(t))u(t) \quad (11)$$

3. Calculating the minimum and maximum values of $p_i(t), i = 1, 2, \dots, k$ under $g_i(t) \in [g_i^{\min}, g_i^{\max}]$, the minimum and maximum of $p_i(t)$ are p_i^{\min} and p_i^{\max} ($i = 1, 2, \dots, k$).
4. Calculating the membership functions, based on step 3, we have

$$p_i(t) = M_{i1}(p_i(t)) \cdot p_i^{\max} + M_{i2}(p_i(t)) \cdot p_i^{\min}$$

where

$$M_{i1}(z_i(t)) + M_{i2}(z_i(t)) = 1$$

and ($i = 1, 2, \dots, k$). Hence, the membership functions can be written as

$$M_{i1}(p_i(t)) = f_i(p_i(t)), M_{i2}(p_i(t)) = 1 - f_i(p_i(t))$$

5. Obtain the fuzzy rules of model. For $i = 1, 2, \dots, k$, we have Model Rule i1:

IF $p_1(t)$ is $M_{11}(p_1(t))$ and $p_2(t)$ is $M_{21}(p_2(t))$ and ... and $p_i(t)$ is $M_{i,1}(p_i(t))$... and $p_{n-1}(t)$ is $M_{n-1,1}(p_{n-1}(t))$

$$\text{THEN } E\dot{x}(t) = A_{i1}(x(t))x(t) + B_{i1}(x(t))u(t)$$

Model Rule i2:

IF $p_1(t)$ is $M_{12}(p_1(t))$ and $p_2(t)$ is $M_{22}(p_2(t))$ and ... and $p_i(t)$ is $M_{i2}(p_i(t))$... and $p_{n-1}(t)$ is $M_{n-1,2}(p_{n-1}(t))$

$$\text{THEN } E\dot{x}(t) = A_{i2}(x(t))x(t) + B_{i2}(x(t))u(t)$$

where $A_{i1}(x(t))$, $A_{i2}(x(t))$, $B_{i1}(x(t))$, and $B_{i2}(x(t))$ can be given using p_i^{max} and p_i^{min} to replace $p_i(t)$ in $A(p_i(t), (x(t)))$ and $B(p_i(t), (x(t)))$, respectively.

6. Calculate the T-S fuzzy singular system.

$$E\dot{x}(t) = \sum_{i=1}^k \begin{pmatrix} h_{i1}(p(t))[A_{i1}(x(t))x(t) + B_{i1}(x(t))u(t)] \\ +h_{i2}(p(t))[A_{i2}(x(t))x(t) + B_{i2}(x(t))u(t)] \end{pmatrix} \quad (12)$$

where

$$h_{i1}(p(t)) = \prod_{i=1}^{n-1} M_{i1}(p_i(t)), h_{i2}(p(t)) = \prod_{i=1}^{n-1} M_{i2}(p_i(t))$$

□

The proof of Theorem 2 provides a scheme to solve the SPFM for a class of NSSs with an impulse which is BIFI and SII.

3.3. The Algorithms of SPFM Approximation

Consider the two types of NSSs; the algorithms of SPFM approximation are given.

1. The Algorithm 1 for a nonlinear model with a compound function type

For NSSs (2), the nonlinear function $f(x(t), u(t), t)$ contains some compound function so the compound function must be considered. The algorithm is as follows.

Algorithm 1 The SPFM approximation with a compound function type

Step 1: The variable transformation. Without loss of generality, assume that the number of compound function of system (2) is one and that the compound function is called $g(x(t))$. Introducing a new variable $z_1(t) = g(x(t))$, and if $\dot{g}(x(t)) \neq ax^\alpha(t)g(x(t))$, $\forall a$, then $z_2(t) = \dot{g}(x(t))$, where a and α are constants.

Step 2: Derivative. Calculating the derivative of new variables $z_1(t)$ and $z_2(t)$ (if any), we have

$$\begin{aligned} \dot{z}_1(t) &= h_1(x(t), z_1(t), z_2(t), u(t), t) \\ \dot{z}_2(t) &= h_2(x(t), z_1(t), z_2(t), u(t), t) \end{aligned} \quad (13)$$

Step 3: Transformation. Replacing the compound function of $\hat{f}(x(t), z_1(t), z_2(t), u(t), t)$, $\dot{z}_1(t)$, and $\dot{z}_2(t)$ (if any) with the new variable, we have

$$E\dot{x}(t) = \hat{f}(x(t), z_1(t), z_2(t), u(t), t) \quad (14)$$

Step 4: Obtain the SPFM. Combining Equations (13) and (14) and writing it in matrix form, we have

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}(\bar{x}(t))\bar{x}(t) + B(\bar{x}(t))u(t) \quad (15)$$

$$\text{where } \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I_{2 \times 2} \end{bmatrix}, \bar{x}(t) = \begin{bmatrix} x(t) \\ z_1(t) \\ z_2(t) \end{bmatrix}.$$

According to the Algorithm 1, the SPFM of NSS can be obtained and the SPFM has only one subsystem.

2. The Algorithm 2 for a nonlinear model with a simple function type

For NSSs (2), the nonlinear function $f(x(t), u(t), t)$ does not contain the compound function, so the SPFM approximation of this system is very similar to the T-S fuzzy approximation. However, there is only one difference, which is that the SPFM does not consider the nonlinear polynomials term. This is also an advantage of the SPFM. The algorithm of the SPFM for nonlinear model with simple function type is given.

Algorithm 2 The SPFM approximation with a simple function type

Step 1: System (3) can be written as

$$E\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) \quad (16)$$

Step 2: Consider the nonlinear terms of matrix $A(x(t))$ and $B(x(t))$ by defining that $p_i(t), i = 1, 2, \dots, k$ denotes the nonlinear terms. Therefore,

$$E\dot{x}(t) = A(p(t), x(t))x(t) + B(p(t), x(t))u(t) \quad (17)$$

Step 3: Calculating the minimum and maximum values of $p_i(t), i = 1, 2, \dots, k$ under $g_i(t) \in [g_i^{\min}, g_i^{\max}]$, the maximum and minimum of $p_i(t)$ are p_i^{\min} and p_i^{\max} ($i = 1, 2, \dots, k$).

Step 4: Calculating the membership functions, based on the **Step 3**, we have

$$p_i(t) = M_{i1}(p_i(t)) \cdot p_i^{\max} + M_{i2}(p_i(t)) \cdot p_i^{\min}$$

where

$$M_{i1}(p_i(t)) + M_{i2}(p_i(t)) = 1$$

and ($i = 1, 2, \dots, k$).

Therefore, the membership functions can be represented as

$$\begin{aligned} M_{i1}(p_i(t)) &= f_i(p_i(t)) \\ M_{i2}(p_i(t)) &= 1 - f_i(p_i(t)) \end{aligned}$$

Step 5: Obtain the fuzzy rules of model. For $i = 1, 2, \dots, k$, we have Model Rule i1:

IF $p_1(t)$ is $M_{11}(p_1(t))$ and $p_2(t)$ is $M_{21}(p_2(t))$ and ... and $p_i(t)$ is $M_{i1}(p_i(t))$... and $p_k(t)$ is $M_{k1}(p_k(t))$
THEN $E\dot{x}(t) = A_{i1}(x(t))x(t) + B_{i1}(x(t))u(t)$

Model Rule i2:

IF $p_1(t)$ is $M_{11}(p_1(t))$ and $p_2(t)$ is $M_{21}(p_2(t))$ and ... and $p_i(t)$ is $M_{i2}(p_i(t))$... and $p_k(t)$ is $M_{k1}(p_k(t))$
THEN $E\dot{x}(t) = A_{i2}(x(t))x(t) + B_{i2}(x(t))u(t)$

where $A_{i1}(x(t)), A_{i2}(x(t)), B_{i1}(x(t))$, and $B_{i2}(x(t))$ can be given by using p_i^{\max} and p_i^{\min} to replace $p_i(t)$ in $A(p_i(t), x(t))$ and $B(p_i(t), x(t))$, respectively.

Step 6: Calculate the SPFM.

$$E\dot{x}(t) = \sum_{i=1}^k \left(\begin{array}{l} h_{i1}(p(t))[A_{i1}(x(t))x(t) + B_{i1}(x(t))u(t)] \\ + h_{i2}(p(t))[A_{i2}(x(t))x(t) + B_{i2}(x(t))u(t)] \end{array} \right) \quad (18)$$

where

$$\begin{aligned} h_{i1}(p(t)) &= \prod_{i=1}^k M_{i1}(p_i(t)) \\ h_{i2}(p(t)) &= \prod_{i=1}^k M_{i2}(p_i(t)) \end{aligned}$$

The system does not have the compound function, and the nonlinear terms are bounded except for the polynomials. Without loss of generality, we assume that the number of nonlinear terms except for polynomials is k ; the nonlinear terms are called $p_i(t)$, and the minimum and maximum are p_i^{\min} and p_i^{\max} ($i = 1, 2, \dots, k$), respectively.

Remark 6. In general, the singular system can be transformed into a normal system by solving the algebraic constraints, and then, it approaches the normal system via SPFM. However, there is one problem with this method. If the algebraic constraints are too complicated, the new variables and the number of fuzzy rules increase and the SPFM becomes more complex. Therefore, we deal with the singular system directly.

3.4. Design of the Sliding Surface and the SMC Law

In this subsection, we use a state feedback sliding mode control law to eliminate the impulse of SPFM. For system (1) with an impulse, we design the linear switching function as

$$s(t) = MEx(t) \quad (19)$$

where $M \in \mathbb{R}^{m \times n}$ is constant matrix and needs to be designed such that $\det(M\bar{B}(x(t))) \neq 0$ for any $t \geq 0$.

Therefore,

$$\dot{s}(t) = ME\dot{x}(t) = M(\bar{A}(x(t))x(t) + \bar{B}(x(t))u(t)) \quad (20)$$

where $\bar{A}(x(t)) = \sum_{i=1}^k h_i(p(t))A_i(x(t))$, $\bar{B}(x(t)) = \sum_{i=1}^k h_i(p(t))B_i(x(t))$.

Let $\dot{s}(t) = 0$; then, the equivalent control law can be obtained:

$$u_{eq}(t) = -(M\bar{B}(x(t)))^{-1}M\bar{A}(x(t))x(t) \quad (21)$$

Substitute Equation (21) with Equation (1); the sliding mode motion is then

$$\begin{cases} E\dot{x}(t) = (\bar{A}(x(t)) - \bar{B}(x(t))(M\bar{B}(x(t)))^{-1}M\bar{A}(x(t)))x(t) \\ s(t) = 0 \end{cases} \quad (22)$$

Consider the following SMC law

$$u(t) = -(M\bar{B}(x(t)))^{-1}[M\bar{A}(x(t))x(t) + ks(t) + \varepsilon \text{sgn}(s(t))] \quad (23)$$

where k and ε are positive real numbers.

Therefore, the closed-loop control system can be expressed as

$$E\dot{x}(t) = \begin{pmatrix} \bar{A}(x(t)) - \bar{B}(x(t))(M\bar{B}(x(t)))^{-1} \\ M\bar{A}(x(t)) - k\bar{B}(x(t))(M\bar{B}(x(t)))^{-1}ME \\ -\varepsilon\bar{B}(x(t))(M\bar{B}(x(t)))^{-1}\text{sgn}(s(t)) \end{pmatrix} x(t) \quad (24)$$

Remark 7. The initial condition incompatibility and input discontinuity are two reasons for the existence of impulses in the singular system. However, the initial conditions are all compatible, and the input consists of continuous and discontinuous in this paper. Next, we analyze how to eliminate the impulse by SMC and give relevant proof in the following subsection. The control input is divided into two parts:

$$u(t) = u_c(t) + u_d(t) \quad (25)$$

where $u_c(t)$ and $u_d(t)$ represent a continuous part and a discontinuous part of the input, respectively.

The normal motion part

$$E\dot{x}(t) = \bar{A}(x(t))x(t) + \bar{B}(x(t))u(t), s(t) \neq 0 \quad (26)$$

The sliding mode part

$$E\dot{x}(t) = \bar{A}(x(t))x(t) + \bar{B}(x(t))u_c(t), s(t) = 0 \quad (27)$$

When $s(t) \neq 0$, the control input $u(t)$ is discontinuous, but the control input to system (1) is able to reach the sliding surface before the impulse occurs. If the system reaches the sliding surface, then $s(t) = 0$ and the trajectory of the SPFM transitions from the normal motion phase to the sliding mode phase. Additionally, the controller $u_c(t)$ is continuous; therefore the impulse of the SPFM (1) can be eliminated by the SMC law (23).

Theorem 3. For the closed-loop control system (24), the sliding surface (21) converges to zero in finite time.

Proof. The Lyapunov function candidate is chosen as

$$V(t) = s^T(t)s(t) \quad (28)$$

In fact

$$\begin{aligned} \dot{V}(t) &= 2s^T(t)\dot{s}(t) = 2s^T(t)[ME\dot{x}(t)] = 2s^T(t)[M\bar{A}(x(t))x(t) + M\bar{B}(x(t))u(t)] \\ &= 2s^T(t)[M\bar{A}(x(t))x(t) - M\bar{A}(x(t))x(t) - ks(t) - \varepsilon \operatorname{sgn}(s(t))] \\ &= 2s^T(t)[-ks(t) - \varepsilon \operatorname{sgn}(s(t))] \leq -2k\|s(t)\|^2 - 2\varepsilon\|s(t)\| < 0 \end{aligned}$$

Therefore, the switching function $s(t)$ converges to zero in finite time. The proof is completed. \square

3.5. Stability Analysis of the Sliding Motion

In the Theorem 3, the sliding surface has been proven, which converges to zero in finite time. In this subsection, the stability analysis of the sliding motion is given.

Theorem 4. The closed-loop control system (24) is asymptotically stable if and only if there exist a nonsingular matrix $X \in \mathbb{R}^{n \times n}$, two full column rank matrices $V, U \in \mathbb{R}^{n \times (n-r)}$, and a matrix $S \in \mathbb{R}^{(n-r) \times (n-r)}$ such that, for any $t \geq 0$,

$$\mathcal{A}(E^T + VSU^T) + (XE^T + VSU^T)\mathcal{A}^T < 0 \quad (29)$$

where $\mathcal{A} = \bar{A}(x(t)) - \bar{B}(x(t))(M\bar{B}(x(t)))^{-1}M\bar{A}(x(t)) - k\bar{B}(x(t))(M\bar{B}(x(t)))^{-1}ME$.

Proof. According to the [40], the Equation (29) is equivalent to the following equation: there exists a nonsingular matrix P such that

$$E^T P = P^T E \geq 0 \quad (30)$$

The Lyapunov function candidate is chosen as

$$V(t) = x^T(t)E^T P x(t) \quad (31)$$

because

$$\dot{V}(t) = 2x^T(t)P^T E \dot{x}(t) = 2x^T(t)P^T \begin{pmatrix} (A - B(MB)^{-1}MA - kB(MB)^{-1}) \\ ME)x(t) - \varepsilon B(MB)^{-1} \operatorname{sgn}(s(t)) \end{pmatrix}$$

where $A = \bar{A}(x(t))$ and $B = \bar{B}(x(t))$. Bases on Theorem 3, it is known that the switching function $s(t)$ converges to zero in finite time. Then,

$$\dot{V}(t) \leq 2\|x(t)\|\|P\| \left[-k\|B(MB)^{-1}ME\| \right] \|x(t)\| = -k\|P\| \|B(MB)^{-1}ME\| \|x(t)\|^2 < 0$$

Therefore, the closed-loop control system (24) is asymptotically stable. The proof is completed. \square

3.6. Impulse Elimination via SMC

In Theorem 4, it has been proven that the closed-loop control system (24) is asymptotically stable when (30) is satisfied. In this subsection, the closed-loop control system (14) is impulse-free. In other words, the impulse of SPFM (1) can be eliminated by the SMC law (23).

Theorem 5. *The closed-loop control system (24) is impulse-free if and only if there exists a nonsingular matrix $P \in \mathbb{R}^{n \times n}$ such that, for any $t \geq 0$,*

$$E^T P = P^T E \geq 0 \tag{32}$$

$$P^T A + A^T P < 0 \tag{33}$$

Proof. As $\text{rank}(E) = n_0 \leq n$, there exist two nonsingular matrices $M_1 \in \mathbb{R}^{n \times n}$ and $M_2 \in \mathbb{R}^{n \times n}$ such that

$$\tilde{E} = M_1 E M_2 = \begin{bmatrix} I_{n_0} & 0 \\ 0 & 0_{n-n_0} \end{bmatrix} \tag{34}$$

Let

$$\tilde{P} = M_1^{-T} P M_2 = \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{bmatrix} \tag{35}$$

$$M_1 A M_2 = \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{bmatrix} \tag{36}$$

It is obvious that matrix \tilde{P} is nonsingular; therefore, the conditions $\tilde{P}_1 > 0$, $\tilde{P}_2 = 0$, and $\det(\tilde{P}_4) \neq 0$ are satisfied according to the (32), (34), and (35) for any $t \geq 0$. We can know $\tilde{P}_3 = 0$ because P is symmetric and $\tilde{P}_2 = 0$. The left of the inequality (33) is multiplied by M_2^T , and the right is multiplied by M_2 , so

$$\begin{aligned} M_2^T P^T A M_2 + M_2^T A^T P M_2 &= M_2^T P^T M_1^{-1} M_1 A M_2 + M_2^T A^T M_1^T M_1^{-T} P M_2 \\ &= \begin{bmatrix} \tilde{P}_1^T & \tilde{P}_2^T \\ \tilde{P}_3^T & \tilde{P}_4^T \end{bmatrix} \begin{bmatrix} \tilde{A}_{i1} & \tilde{A}_{i2} \\ \tilde{A}_{i3} & \tilde{A}_{i4} \end{bmatrix} + \begin{bmatrix} \tilde{A}_{i1}^T & \tilde{A}_{i2}^T \\ \tilde{A}_{i3}^T & \tilde{A}_{i4}^T \end{bmatrix} \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{bmatrix} \\ &= \begin{bmatrix} * & * \\ * & \tilde{P}_4^T \tilde{A}_{i4} + \tilde{A}_{i4}^T \tilde{P}_4 \end{bmatrix} < 0 \end{aligned} \tag{37}$$

According to the (37), the matrix \tilde{A}_{i4}^T is nonsingular for any $t \geq 0$, so the closed-loop control system (24) is impulse-free. The proof is completed. \square

Theorem 6. *The SMC law $u(t)$ is able to the SPFM (1) to reach the sliding surface before the impulse occurs, when the initial value of the switching function $s_0 = \frac{\varepsilon}{k} (1 - e^{kt_m})$, where t_m indicates the moment when the system (2) first appears as an impulse.*

Proof. Without loss of generality, t_m denotes the moment when the SPFM (1) first appears as an impulse, that is

$$E \dot{x}(t_m) = Ax(t_m) + Bu(t_m) \tag{38}$$

where $A = \bar{A}(x(t_m))$ and $B = \bar{B}(x(t_m))$. When

$$u(t_m) = -(MB)^{-1}[MAx(t) + ks(t_m) + \varepsilon \text{sgn}(s(t_m))]$$

Therefore,

$$E \dot{x}(t_m) = (A - B(MB)^{-1}MA - kB(MB)^{-1}ME)x(t_m) - \varepsilon B(MB)^{-1} \text{sgn}(s(t_m)) \tag{39}$$

and

$$\dot{s}(t_m) = ME\dot{x}(t_m) = -kMEs(t_m) - \varepsilon \operatorname{sgn}(s(t_m)) = -ks(t_m) - \varepsilon \operatorname{sgn}(s(t_m)) \quad (40)$$

The solution of Equation (40) is divided into the following two cases.

Case 1: When $s(t_m) = 0$, $\dot{s}(t_m) = 0$.

Case 2: When $s(t_m) \neq 0$,

$$s(t_m) = \frac{\varepsilon}{k} + \left(s_0 - \frac{\varepsilon}{k}\right)e^{-kt_m} = \frac{\varepsilon}{k} \left(1 - e^{-kt_m}\right) + s_0 e^{-kt_m}$$

where s_0 represents the initial value of the switching function.

Then,

$$\begin{cases} s(t_m) < 0 \\ s_0 < \frac{\varepsilon}{k} \left(1 - e^{kt_m}\right) \end{cases}$$

and

$$\begin{cases} s(t_m) > 0 \\ s_0 > \frac{\varepsilon}{k} \left(1 - e^{kt_m}\right) \end{cases}$$

As a result of the initial value of the switching function $s_0 = \frac{\varepsilon}{k} \left(1 - e^{kt_m}\right)$, Case 2 does not hold. In summary, $s(t_m)$ and $\dot{s}(t_m)$ satisfy the following conditions

$$s(t_m) = 0, \dot{s}(t_m) = 0$$

Based on the theory of SMC, system (1) reaches the sliding surface at time t_m , that is, the SMC law $u(t)$ enables system (1) to reach the sliding surface before an impulse occurs. The proof is completed. \square

Remark 8. Compared with the existing design methods for eliminating the impulse, the controller designed in this paper has the following advantage: it can not only eliminate the impulse of the SPFM but also make the system asymptotically stable.

4. Simulation Examples

4.1. A Double Inverted Pendulum Model of Human Standing

As in Figure 4, the inputs of the double inverted pendulum model [41] are the torques $u_1(t)$ and $u_2(t)$, and then its output are the associated angular positions. Segment 1 represents the lower limbs (not including the feet), and segment 2 represents the upper limbs (including the trunk and head) [41]. The NSS of the double inverted pendulum model of human standing can be written as

$$E\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) \quad (41)$$

$$\begin{aligned} \text{where } E(x(t)) &= \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, A(x(t)) = \begin{bmatrix} 0 & I \\ G & -S \end{bmatrix}, B(x(t)) = \begin{bmatrix} 0 \\ R \end{bmatrix}, u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(t) &= [\theta_1(t) \quad \theta_2(t) \quad \theta_3(t) \quad \theta_4(t)]^T, R = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, G = \begin{bmatrix} d \frac{\sin(\theta_1(t))}{\theta_1(t)} & 0 \\ 0 & e \frac{\sin(\theta_2(t))}{\theta_2(t)} \end{bmatrix}, \\ S &= c \sin(\theta_1(t) - \theta_2(t)) \begin{bmatrix} 0 & \theta_3(t) \\ -\theta_4(t) & 0 \end{bmatrix}, \theta_3(t) = \dot{\theta}_1(t), \theta_4(t) = \dot{\theta}_2(t), a = I_1 + m_1 K^2 L_1^2 + \\ & m_2 L_1^2, b = I_2 + m_2 L_2^2, c = m_2 L_1 L_2, d = (m_2 + m_1 K) g L_1, e = m_2 g L_2, \\ M &= \begin{bmatrix} a & c \cos(\theta_1(t) - \theta_2(t)) \\ c \cos(\theta_1(t) - \theta_2(t)) & b \end{bmatrix}. \end{aligned}$$

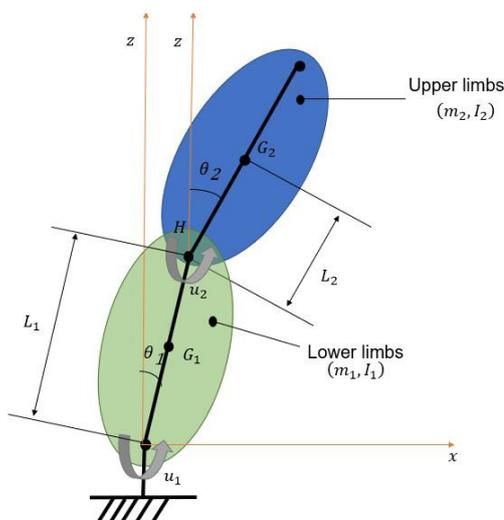


Figure 4. A human posture inverted pendulum model.

According to [41], the table of human standing model parameters is given as Table 2. Using the data of Table 2, the relevant parameters and matrices can be obtained

$$a = 40.53, b = 5.42, c = 10.36, d = 488.98, e = 116.88$$

$$A(x(t)) = \begin{bmatrix} \mathbf{0}_{2 \times 2} & I_2 \\ A_{21} & A_{22} \end{bmatrix}$$

$$B(x(t)) = \begin{bmatrix} \mathbf{0}_{2 \times 2} \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} d \frac{\sin(\theta_1(t))}{\theta_1(t)} & 0 \\ 0 & e \frac{\sin(\theta_2(t))}{\theta_2(t)} \end{bmatrix}$$

$$A_{22} = c \sin(\theta_1(t) - \theta_2(t)) \begin{bmatrix} 0 & -\theta_4(t) \\ \theta_3(t) & 0 \end{bmatrix}$$

Table 2. Human standing model parameters [41].

Notation	Value	Designation
L_1 (m)	0.87	Lower limbs length
L_2 (m)	0.26	Distance between the hip(H) and the upper pendulum centre of gravity G_2
I_1 (kg/m ²)	1.25	Lower limbs inertia
I_2 (kg/m ²)	2.32	Upper pendulum inertia
m_1 (kg)	21.87	Lower limbs mass
m_2 (kg)	45.87	Upper pendulum mass
K	0.525	Lower limbs centre of mass G_1
g (m/s ²)	0.98	Gravitational acceleration

It is known that the $A(x(t))$ of NSS (41) contains the compound function $\frac{\sin(\theta_1(t))}{\theta_1(t)}$ and $\sin(\theta_1(t) - \theta_2(t))$. Therefore, we can obtain SPFM by using the Algorithm 1. The specific steps are as follows:

1. The variable transformation. We introduce six new variables $x_1(t) = \sin(\theta_1(t))$, $x_2(t) = \cos(\theta_1(t))$, $x_3(t) = \sin(\theta_2(t))$, $x_4(t) = \cos(\theta_2(t))$, $x_5(t) = \sin(\theta_1(t) - \theta_2(t))$,

and $x_6(t) = \cos(\theta_1(t) - \theta_2(t))$.

2. Derivative.

$$\begin{cases} \dot{x}_1(t) = \dot{\theta}_1(t) \cos(\theta_1(t)) = \theta_3(t)x_2(t) \\ \dot{x}_2(t) = -\dot{\theta}_1(t) \sin(\theta_1(t)) = -\theta_3(t)x_1(t) \\ \dot{x}_3(t) = \dot{\theta}_2(t) \cos(\theta_2(t)) = \theta_4(t)x_4(t) \\ \dot{x}_4(t) = -\dot{\theta}_2(t) \sin(\theta_2(t)) = -\theta_4(t)x_3(t) \\ \dot{x}_5(t) = (\dot{\theta}_1(t) - \dot{\theta}_2(t)) \cos(\theta_1(t) - \theta_2(t)) \\ \quad = (x_3(t) - x_4(t))x_6(t) \\ \dot{x}_6(t) = (\dot{\theta}_2(t) - \dot{\theta}_1(t)) \sin(\theta_1(t) - \theta_2(t)) \\ \quad = (x_4(t) - x_3(t))x_5(t) \end{cases} \quad (42)$$

3. Obtain the SPFM. Combining Equations (41) and (42) written as a matrix form, we have

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}(\bar{x}(t))\bar{x}(t) + \bar{B}(\bar{x}(t))u(t) \quad (43)$$

$$\text{where } \bar{E} = \begin{bmatrix} E & 0 \\ 0 & I_6 \end{bmatrix}, \bar{x}(t) = \begin{bmatrix} x(t) \\ x_1(t) \\ \vdots \\ x_6(t) \end{bmatrix}, \bar{A}(\bar{x}(t)) = \begin{bmatrix} A_1(\bar{x}(t)) & A_3(\bar{x}(t)) \\ \mathbf{0}_{6 \times 4} & A_2(\bar{x}(t)) \end{bmatrix},$$

$$\bar{B}(\bar{x}(t)) = \begin{bmatrix} \mathbf{0}_{2 \times 2} \\ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \\ \mathbf{0}_{6 \times 2} \end{bmatrix}, A_1(\bar{x}(t)) = \begin{bmatrix} \mathbf{0}_{2 \times 2} & I_2 \\ \mathbf{0}_{2 \times 2} & \begin{bmatrix} 0 & -c\theta_4(t)x_5(t) \\ c\theta_3(t)x_6(t) & 0 \end{bmatrix} \end{bmatrix},$$

$$A_3(\bar{x}(t)) = \begin{bmatrix} \mathbf{0}_{2 \times 2} \\ \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix} & \mathbf{0}_{4 \times 4} \end{bmatrix}$$

$$A_2(\bar{x}(t)) = \begin{bmatrix} 0 & \theta_3(t) & 0 & 0 & 0 & 0 \\ -\theta_3(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_4(t) & 0 & 0 \\ 0 & 0 & -\theta_4(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_3(t) - x_4(t) \\ 0 & 0 & 0 & 0 & x_4(t) - x_3(t) & 0 \end{bmatrix}$$

To illustrate the accuracy of the SPFM approximation, the state trajectory and error are given when $x(0) = O(1, 10)$.

The contrast of state trajectory between human standing system and the SPFM is given by Figure 5, and the error of $x_1(t)$ and $x_2(t)$ is given by Figure 6. As shown in Figures 5 and 6, the SPFM (43) can effectively approximate the NSS (41).

Remark 9. As the number of nonlinear items of system (41) is 6, if we use the T-S fuzzy model to approach system (41), the rules of the T-S fuzzy system are at least 2^6 . If we want to reach about the same accuracy of approximation with the SPFM, the number of fuzzy rules needed is more when using T-S fuzzy model. Therefore, using the SPFM to approach system (41) can effectively reduce the number of fuzzy rules with a high accuracy.

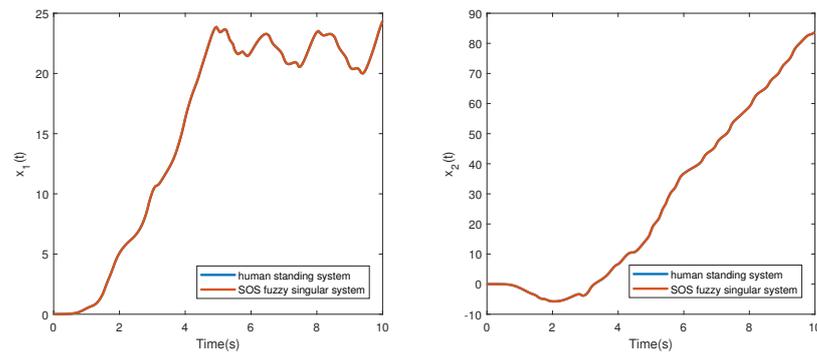


Figure 5. The state trajectory of systems (41) and (43).

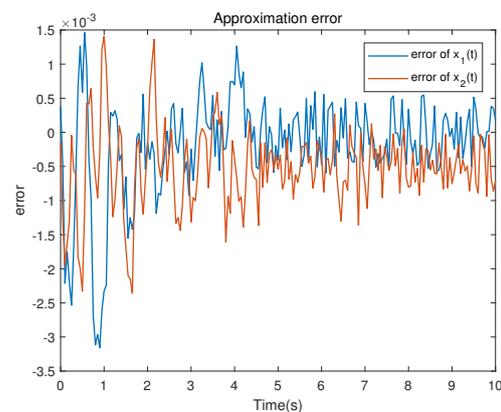


Figure 6. Approximation error of $x_1(t)$ and $x_2(t)$.

4.2. A Comparison Example with T-S Fuzzy Model

In this subsection, in order to compare the performance of the T-S fuzzy model method, an example of the second-order NSS is considered.

$$\begin{aligned}\dot{x}_1(t) &= (x_1(t) + x_2(t))x_1(t) - \sin(t)x_2(t) - \cos(t)u(t) \\ 0 &= x_1(t) + x_2(t)\end{aligned}\quad (44)$$

The algorithm processes of SPFM and T-S fuzzy model of system (44) are given below.

(1) SPFM

It is known that system (44) does not contain the compound function and that the nonlinear terms $\sin(t)$ and $\cos(t)$ are bounded except for polynomials. Let $p_1(t) = \sin(t)$ and $p_2(t) = \cos(t)$, so $p_1(t) \in [-1, 1]$, $p_2(t) \in [-1, 1]$. The membership functions can be calculated

$$\begin{aligned}M_{11}(p_1(t)) &= \frac{1-p_1(t)}{2}, M_{12}(p_1(t)) = \frac{1+p_1(t)}{2} \\ M_{21}(p_2(t)) &= \frac{1-p_2(t)}{2}, M_{22}(p_2(t)) = \frac{1+p_2(t)}{2}\end{aligned}$$

The model rules can be given:

Model Rule 1:

$$\begin{aligned}\text{IF } p_1(t) \text{ is } M_{11} \text{ and } p_2(t) \text{ is } M_{21} \\ \text{THEN } E\dot{x}(t) &= A_1(x(t))x(t) + B_1(x(t))u(t)\end{aligned}$$

Model Rule 2:

$$\begin{aligned}\text{IF } p_1(t) \text{ is } M_{11} \text{ and } p_2(t) \text{ is } M_{22} \\ \text{THEN } E\dot{x}(t) &= A_2(x(t))x(t) + B_2(x(t))u(t)\end{aligned}$$

Model Rule 3:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{21}
THEN $E\dot{x}(t) = A_3(x(t))x(t) + B_3(x(t))u(t)$

Model Rule 4:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{22}
THEN $E\dot{x}(t) = A_4(x(t))x(t) + B_4(x(t))u(t)$

where $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A_1(x(t)) = A_2(x(t)) = \begin{bmatrix} x_1(t) + x_2(t) & 1 \\ 1 & 1 \end{bmatrix}$,
 $A_3(x(t)) = A_4(x(t)) = \begin{bmatrix} x_1(t) + x_2(t) & -1 \\ 1 & 1 \end{bmatrix}$, $B_1(x(t)) = B_3(x(t)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_2(x(t)) =$
 $B_4(x(t)) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

Therefore, the SPFM of system (44) is

$$E\dot{x}(t) = \sum_{i=1}^4 h_i(p(t))[A_i(x(t))x(t) + B_i(x(t))u(t)] \quad (45)$$

where $h_1(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1-p_2(t)}{2}$, $h_2(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1+p_2(t)}{2}$, $h_3(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1-p_2(t)}{2}$, $h_4(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1+p_2(t)}{2}$.

The error of $x_1(t)$ and $x_2(t)$ is given by Figure 7 when $u(t) = \sin(t)$. As shown in Figure 7, system (44) can be approximated system (45) using SPFM effectively.

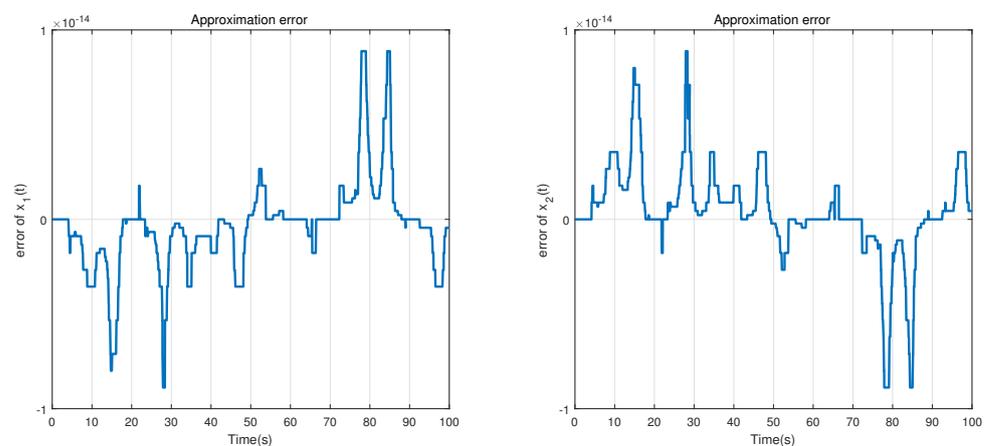


Figure 7. Approximation error of $x_1(t)$ and $x_2(t)$ using SPFM.

(2) T-S fuzzy model

Let $p_1(t) = \sin(t)$, $p_2(t) = \cos(t)$ and $p_3(t) = x_1(t) + x_2(t)$, so $p_1(t) \in [-1, 1]$, $p_2(t) \in [-1, 1]$ and $p_3(t) \in [-1, 1]$. The membership functions can be calculated

$$\begin{aligned} M_{11}(p_1(t)) &= \frac{1-p_1(t)}{2}, M_{12}(p_1(t)) = \frac{1+p_1(t)}{2} \\ M_{21}(p_2(t)) &= \frac{1-p_2(t)}{2}, M_{22}(p_2(t)) = \frac{1+p_2(t)}{2} \\ M_{31}(p_3(t)) &= \frac{1-p_3(t)}{2}, M_{32}(p_3(t)) = \frac{1+p_3(t)}{2} \end{aligned}$$

The model rules can be given

Model Rule 1:

IF $p_1(t)$ is M_{11} and $p_2(t)$ is M_{21} and $p_3(t)$ is M_{31}
THEN $E\dot{x}(t) = A_1(x(t))x(t) + B_1(x(t))u(t)$

Model Rule 2:

IF $p_1(t)$ is M_{11} and $p_2(t)$ is M_{21} and $p_3(t)$ is M_{32}
THEN $E\dot{x}(t) = A_2(x(t))x(t) + B_2(x(t))u(t)$

Model Rule 3:

IF $p_1(t)$ is M_{11} and $p_2(t)$ is M_{22} and $p_3(t)$ is M_{31}
THEN $E\dot{x}(t) = A_3(x(t))x(t) + B_3(x(t))u(t)$

Model Rule 4:

IF $p_1(t)$ is M_{11} and $p_2(t)$ is M_{22} and $p_3(t)$ is M_{32}
THEN $E\dot{x}(t) = A_4(x(t))x(t) + B_4(x(t))u(t)$

Model Rule 5:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{21} and $p_3(t)$ is M_{31}
THEN $E\dot{x}(t) = A_5(x(t))x(t) + B_5(x(t))u(t)$

Model Rule 6:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{21} and $p_3(t)$ is M_{32}
THEN $E\dot{x}(t) = A_6(x(t))x(t) + B_6(x(t))u(t)$

Model Rule 7:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{22} and $p_3(t)$ is M_{31}
THEN $E\dot{x}(t) = A_7(x(t))x(t) + B_7(x(t))u(t)$

Model Rule 8:

IF $p_1(t)$ is M_{12} and $p_2(t)$ is M_{22} and $p_3(t)$ is M_{32}
THEN $E\dot{x}(t) = A_8(x(t))x(t) + B_8(x(t))u(t)$

where $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $A_1(x(t)) = A_3(x(t)) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_2(x(t)) = A_4(x(t)) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_5(x(t)) = A_7(x(t)) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$, $A_6(x(t)) = A_8(x(t)) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B_1(x(t)) = B_2(x(t)) = B_5(x(t)) = B_6(x(t)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B_3(x(t)) = B_4(x(t)) = B_7(x(t)) = B_8(x(t)) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Therefore, the T-S fuzzy model of system (44) is

$$E\dot{x}(t) = \sum_{i=1}^8 h_i(p(t)) [A_i(x(t))x(t) + B_i(x(t))u(t)] \quad (46)$$

where $h_1(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1-p_2(t)}{2} \times \frac{1-p_3(t)}{2}$, $h_2(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1-p_2(t)}{2} \times \frac{1+p_3(t)}{2}$, $h_3(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1+p_2(t)}{2} \times \frac{1-p_3(t)}{2}$, $h_4(p(t)) = \frac{1-p_1(t)}{2} \times \frac{1+p_2(t)}{2} \times \frac{1+p_3(t)}{2}$, $h_5(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1-p_2(t)}{2} \times \frac{1-p_3(t)}{2}$, $h_6(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1-p_2(t)}{2} \times \frac{1+p_3(t)}{2}$, $h_7(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1+p_2(t)}{2} \times \frac{1-p_3(t)}{2}$, $h_8(p(t)) = \frac{1+p_1(t)}{2} \times \frac{1+p_2(t)}{2} \times \frac{1+p_3(t)}{2}$.

The errors of $x_1(t)$ and $x_2(t)$ are given in Figure 8 when $u(t) = \sin(t)$ and using T-S model. The following compares the differences between SPFM and T-S fuzzy model from three aspects (accuracy, number of fuzzy rules, and running time), as shown in the Table 3. By comparison, it can be found that the accuracy of SPFM is much higher than the T-S fuzzy model, and the fuzzy rules of SPFM are half that of T-S fuzzy model. The run time of two models is relatively low, and the time of SPFM is lower than the T-S fuzzy model. Therefore, the approximation effect of SPFM in nonlinear systems is better than the T-S fuzzy model method.

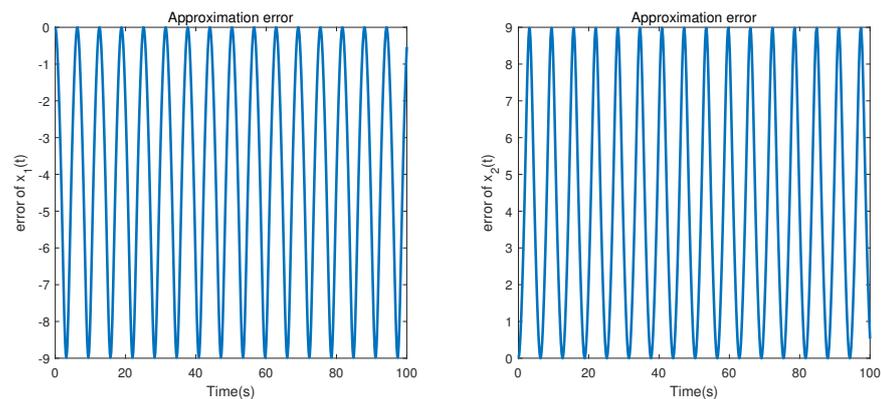


Figure 8. Approximation error of $x_1(t)$ and $x_2(t)$ using the T-S fuzzy model.

Table 3. The performance comparison of SPFM and T-S fuzzy model.

Model	Accuracy	Number of Fuzzy Rules	Running Time (s)
SPFM	10^{-14}	4	0.0245
T-S fuzzy model	1	8	0.0429

4.3. Eliminate Impulse by Using SMC

Consider the SPFM:

$$E\dot{x}(t) = \sum_{i=1}^2 \mu_i(z(t)) [A_i(x(t))x(t) + B_i(x(t))u(t)] \quad (47)$$

where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_1(x(t)) = \begin{bmatrix} -x_1 & -1 \\ 2 & 0 \end{bmatrix}, A_2(x(t)) = \begin{bmatrix} -x_2 & 1 \\ 2 & 0 \end{bmatrix}$$

$$B_1(x(t)) = B_2(x(t)) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mu_1 = \frac{1-\sin(t)}{2}, \mu_2 = \frac{1+\sin(t)}{2}$$

According to the discrimination method of the singular system impulse, the calculation of matrix rank is given below:

$$\text{rank} \left(\begin{bmatrix} E & 0 \\ \bar{A}(x(t)) & E \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{x_2-x_1}{2} + \frac{\sin(t)(x_1+x_2)}{2} & \sin(t) & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix} \right) = 2 < 3$$

Therefore, the SPFM (47) is impulsive. The objective is to design a linear switching function as in (19) and SMC law (23) such that the closed-loop control system (24) is asymptotically stable. First, the parameter of switching function is given

$$M = [2 \quad 3]$$

and $\det(MB) = 5$; therefore, the matrix M satisfies the design requirement. Then, the parameters of the SMC law (23) are designed with $k = 0.05$ and $\varepsilon = 0.01$. Figure 9 shows the state trajectories of SPFM (47) with the initial value $x(0) = [27 \quad 26]^T$. The SMC law $u(t)$ and the sliding surface $s(t)$ are given as Figures 10 and 11. Figures 9–11 show that the states of SPFM (47) converge to zero within 4s under the sliding mode controller (23) and linear sliding mode surface (19), and the SMC laws can eliminate the impulses of SPFM (47).

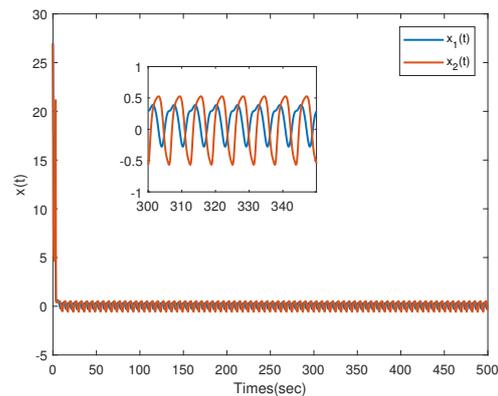


Figure 9. The state trajectories of system (47).

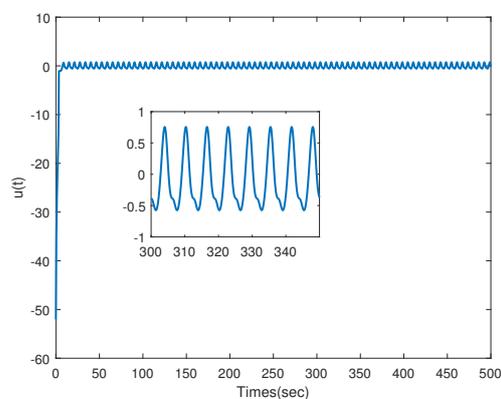


Figure 10. Sliding mode control law (23).

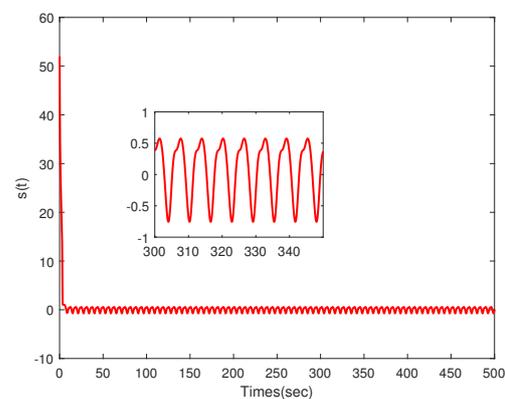


Figure 11. Linear switching function (19).

5. Conclusions

In this paper, the SPFM approach problem was studied for a class of nonlinear singular system with impulses and without impulses. Consider the nonlinear singular system without impulses; some scholars directly use SOSTOOLS to solve the problem based on the T-S fuzzy model. However, this paper theoretically proves the accuracy of the approximation and gives the algorithm. For the nonlinear singular system with impulses, the theorem of the SPFM that can effectively approach the model has been proven for the first time. It is proved that a class of nonlinear singular system with a bounded impulse-free item property and a separable impulse item property can be approximated by SPFM with arbitrary accuracy. This enables the nonlinear singular system with impulses to be locally linearized with relatively few fuzzy rules and high precision. Then, the designed sliding

mode controller can effectively eliminate impulses of the SPFM. Finally, a human posture inverted pendulum model and two numerical example were carried out to demonstrate the effectiveness of the proposed algorithms.

Author Contributions: Conceptualization, J.L. and Y.Z.; methodology, J.L. and Y.Z.; software, Z.J.; validation, J.L. and Y.Z.; formal analysis, J.L. and Y.Z.; investigation, J.L. and Y.Z.; data curation, J.L. and Y.Z.; writing original draft preparation, J.L.; writing review and editing, J.L.; visualization, Z.J.; supervision, Y.Z.; project administration, Y.Z.; funding acquisition, Y.Z. and J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China under grant No.61673099.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no potential conflicts of interest.

References

1. Takagi, T.; Sugeno, M. Fuzzy identification of systems and its application to modeling and control. *IEEE Trans. Syst. Man Cybern.* **1985**, *15*, 116–132. [[CrossRef](#)]
2. Wang, L.X. Fuzzy systems are universal approximators. In Proceedings of the IEEE International Conference on Fuzzy System, San Diego, CA, USA, 8–12 March 1992; pp. 1153–1162.
3. Gao, G.S.; Rees, N.W.; Feng, G. Analysis and design for a class of complex control system. *Automatica* **1997**, *33*, 1017–1028.
4. Castro, J.L. Fuzzy logic controllers are universal approximators. *IEEE Trans. Syst. Man Cybern.* **1995**, *25*, 629–635. [[CrossRef](#)]
5. Kosko, B. Fuzzy systems as universal approximators. *IEEE Trans. Comput.* **1994**, *43*, 1329–1333. [[CrossRef](#)]
6. Ying, H. General Takagi-Sugeno fuzzy systems are universal approximators. In Proceedings of the 1998 IEEE International Conference on Fuzzy Systems Proceedings, IEEE World Congress on Computational Intelligence (Cat. No.98CH36228), Anchorage, AK, USA, 4–9 May 1998; Volume 1, pp. 819–823.
7. Zeng, X.J.; Singh, M.G. Approximation theory of fuzzy systems-MIMO case. *IEEE Trans. Fuzzy Syst.* **1995**, *3*, 219–235. [[CrossRef](#)]
8. Wang, F.; Chen, B.; Sun, Y.M.; Gao, Y.L.; Lin, C. Finite-time fuzzy control of stochastic nonlinear systems. *IEEE Trans. Cybern.* **2020**, *50*, 2617–2626. [[CrossRef](#)]
9. Zhang, Y.; Zhang, Q.L.; Zhang, J.Y.; Wang, Y.Y. Sliding-mode control for fuzzy singular systems with time-delay based on vector integral sliding mode surface. *IEEE Trans. Fuzzy Syst.* **2020**, *28*, 768–782. [[CrossRef](#)]
10. Tanaka, K.; Yoshida, H.; Ohtake, H.; Wang, H.O. A sum of squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems. *IEEE Trans. Fuzzy Syst.* **2009**, *17*, 911–922. [[CrossRef](#)]
11. Feng, G. A survey on analysis and design of model-based fuzzy control systems. *IEEE Trans. Fuzzy Syst.* **2006**, *14*, 676–697. [[CrossRef](#)]
12. Wang, H.O.; Tanaka, K.; Griffen, M.F. An approach to fuzzy control of nonlinear system: Stability and design issues. *IEEE Trans. Fuzzy Syst.* **1996**, *4*, 14–23. [[CrossRef](#)]
13. Qiu, J.B.; Feng, G.; Yang, J. A new design of delay-dependent robust H filtering for discrete-time T-S fuzzy systems with time-varying delay. *IEEE Trans. Fuzzy Syst.* **2009**, *17*, 1044–1058.
14. Pang, B.; Zhang, Q.L. Interval observers design for polynomial fuzzy singular systems by utilizing Sum-of-Squares program. *IEEE Trans. Syst. Man, Cybern. Syst.* **2018**, *50*, 1–8. [[CrossRef](#)]
15. Pang, B.; Zhang, Q.L. Sliding mode control for polynomial fuzzy singular systems with time delay. *IET Control. Theory Appl.* **2018**, *12*, 1483–1490. [[CrossRef](#)]
16. Lam, H.K.; Tsai, S.H. Stability analysis of polynomial-fuzzy-model based control systems with mismatch premise membership functions. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 223–229. [[CrossRef](#)]
17. Lam, H.K. Polynomial fuzzy-model-based control systems: Stability analysis via piecewise-linear membership function. *IEEE Trans. Fuzzy Syst.* **2011**, *19*, 588–593. [[CrossRef](#)]
18. Lam, H.K.; Liu, C.; Wu, L.; Zhao, X.D. Polynomial fuzzy-model based control systems: Stability analysis via approximated membership functions considering sector nonlinearity of control input. *IEEE Trans. Fuzzy Syst.* **2015**, *23*, 2202–2214. [[CrossRef](#)]
19. Zheng, G.; Boutat, D.; Wang, H. A nonlinear Luenberger-like observer for nonlinear singular systems. *Automatica* **2017**, *86*, 11–17. [[CrossRef](#)]
20. Muoi, N.H.; Phat, V.N.; Niamsup, P. Criteria for robust finite-time stabilisation of linear singular systems with interval time-varying delay. *IET Control Theory Appl.* **2017**, *11*, 1968–1975. [[CrossRef](#)]
21. Campbell, S.L. *Singular Systems of Differential Equations*; Pitman: London, UK, 1980.

22. Singh, S.; Liu, R.W. Existence of state equation representation of linear large-scale dynamical systems. *IEEE Trans. Circuit Theory* **1973**, *20*, 239–246. [[CrossRef](#)]
23. Feng, Z.; Shi, Z. Two equivalent sets: Application to singular systems. *Automatica* **2017**, *2017*, 198–205. [[CrossRef](#)]
24. Taniguchi, T.; Tanaka, K.; Wang, H.O. Fuzzy descriptor systems and nonlinear model following control. *IEEE Trans. Fuzzy Syst.* **2000**, *8*, 442–452. [[CrossRef](#)]
25. Lin, C.; Wang, Q.G.; Lee, T.H. Stability and stabilization of a class of fuzzy time-delay descriptor systems. *IEEE Trans. Fuzzy Syst.* **2006**, *14*, 542–551.
26. Zhai, D.; An, L.W.; Li, J.H.; Zhang, Q.L. Fault detection for stochastic parameter-varying Markovian jump systems with application to networked control systems. *Appl. Math. Model.* **2016**, *40*, 2368–2383. [[CrossRef](#)]
27. Wu, J.R. Robust stabilization for uncertain T-S fuzzy singular system. *Int. J. Mach. Learn. Cybern.* **2016**, *7*, 699–706. [[CrossRef](#)]
28. Li, W.X.; Feng, Z.G.; Sun, W.C.; Zhang, J.W. Admissibility analysis for Takagi-Sugeno fuzzy singular systems with time delay. *Neurocomputing* **2016**, *205*, 336–340. [[CrossRef](#)]
29. Wang, Y.Y.; Zhang, J.Y.; Zhang, H.G. Sliding-mode control for fuzzy stochastic systems with different local input matrices. *IEEE Trans. Syst. Man Cybern. Syst.* **2019**, *51*, 4761–4772. [[CrossRef](#)]
30. Wang, Y.Y.; Zhang, J.Y.; Zhang, H.G.; Sun, J.Y. A new stochastic sliding-mode design for descriptor fuzzy systems with time-varying delay. *IEEE Trans. Cybern.* **2019**, *51*, 3858–3870. [[CrossRef](#)]
31. Rachunkova, I. Singular Dirichlet second-order BVPs with impulses. *J. Differ. Equ.* **2003**, *193*, 435–459. [[CrossRef](#)]
32. Zhang, Z.H.; Yuan, R. An application of variational methods to Dirichlet boundary value problem with impulses. *Nonlinear Anal. Real World Appl.* **2010**, *11*, 155–162. [[CrossRef](#)]
33. Juan, J.N. Variational formulation of a damped Dirichlet impulsive problem. *Appl. Math. Lett.* **2010**, *23*, 940–942.
34. Utkin, V. Discussion aspects of high-order sliding mode control. *IEEE Trans. Autom. Control* **2016**, *61*, 829–833. [[CrossRef](#)]
35. Incremona, G.P.; Rubagotti, M.; Ferrara, A. Sliding mode control of constrained nonlinear systems. *IEEE Trans. Autom. Control* **2017**, *62*, 2965–2972. [[CrossRef](#)]
36. Utkin, V. *Sliding Modes in Control and Optimization*; Springer: Berlin/Heidelberg, Germany, 1992; pp. 12–13.
37. Lam, H.K.; Narimani, M.; Li, H.; Liu, H. Stability analysis of Polynomial-Fuzzy-Model-Based control systems using switching polynomial Lyapunov function. *IEEE Trans. Fuzzy Syst.* **2013**, *21*, 800–813. [[CrossRef](#)]
38. Beardmore, R.E. Double singularity-induced bifurcation points and singular Hopf bifurcations. *Dyn. Stab. Syst.* **2000**, *15*, 319–342. [[CrossRef](#)]
39. Ma, J.F.; Zhang, Q.L. Approximation property of T-S fuzzy singular systems. *Control. Theory Appl.* **2008**, *25*, 837–844.
40. Uezato, E.; Ikeda, M. LMI conditions for stability, robust stabilization, and H-infinity control of descriptor systems. *IEEE Conf. Decis. Control* **1999**, *4*, 4092–4097.
41. Guelton, K.; Delprat, S.; Guerra, T.M. An alternative to inverse dynamics joint torques estimation in human stance based on a Takagi-Sugeno unknown-inputs observer in the descriptor form. *Control. Eng. Pract.* **2008**, *16*, 1414–1428. [[CrossRef](#)]