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A Certain Structure of Bipolar Fuzzy Subrings

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Abstract: The role of symmetry in ring theory is universally recognized. The most directly definable universal relation in a symmetric set theory is isomorphism. This article develops a certain structure of bipolar fuzzy subrings, including bipolar fuzzy quotient ring, bipolar fuzzy ring homomorphism, and bipolar fuzzy ring isomorphism. We define (α, β)-cut of bipolar fuzzy set and investigate the algebraic attributions of this phenomenon. We also define the support set of bipolar fuzzy set and prove various important properties relating to this concept. Additionally, we define bipolar fuzzy homomorphism by using the notion of natural ring homomorphism. We also establish a bipolar fuzzy homomorphism between bipolar fuzzy subring of the quotient ring and bipolar fuzzy subring of this ring. We constituted a significant relationship between two bipolar fuzzy subrings of quotient rings under a given bipolar fuzzy surjective homomorphism. We present the construction of an induced bipolar fuzzy isomorphism between two related bipolar fuzzy subrings. Moreover, to discuss the symmetry between two bipolar fuzzy subrings, we present three fundamental theorems of bipolar fuzzy isomorphism.

Keywords: bipolar fuzzy set; bipolar fuzzy subring; bipolar fuzzy ideal; bipolar fuzzy homomorphism; bipolar fuzzy isomorphism

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1. Introduction

The theory of fuzzy sets and their initiatory results were proposed by Zadeh [1] in 1965. This theory has become a blooming area of research in almost all fields of science. The fuzzy logic provides appropriate solutions in several bio-informatics and computational biological based problems such as medical image processing, cellular reconstruction, protein structure analysis, gene expression analysis, and medical data classification. The idea of the fuzzy subgroup was commenced by Rosenfeld [2] in 1971. In 1983, Liu [3] presented the opinion of fuzzy subrings and fuzzy ideals. A fuzzy subset A of ring R is a fuzzy subring if $A(x-y) \geq min\{A(x), A(y)\}$ and $A(xy) \geq min\{A(x), A(y)\}$. A fuzzy subset A of ring R is fuzzy ideal if $A(x-y) \geq min\{A(x), A(y)\}$ and $A(xy) \geq max\{A(x), A(y)\}$. Atanassov [4] initiated the generalized form of the fuzzy set by including a new component called the intuitionistic fuzzy set. A new abstraction of bipolar fuzzy sets was initiated by Zhang [5]. The enlargement of the fuzzy set to the bipolar fuzzy set is commensurate to the generalization of positive real numbers to negative real numbers. The bipolar fuzzy set was treated as a new appliance to deal with ambiguity in decision science. More developments relative to bipolar fuzzy sets may be viewed in [6–8]. The theory of bipolar fuzzy sets is that it is an effective tool to study the case of vagueness as compared to Zadeh's fuzzy sets because it deals with positive membership grade and negative membership grade. Although intuitionistic fuzzy sets and bipolar fuzzy sets have similar appearances, according to Lee [9], they are fundamentally distinct concepts. In the intuitionistic fuzzy set, both membership and non-membership belongs to $[0,\ 1]$, and their sum is not more than one, but in the bipolar fuzzy set one, the membership value belongs to $[0,\ 1]$ and one membership value belongs to $[-1,\ 0]$. The bipolar fuzzy sets have extensive implementations in real life problems [10]. Many researchers [11–15] made remarkable achievement to generalize the concept of bipolar fuzzy sets to decision making and modern mathematics. In 2009, Fotea and Davvaz [16] proposed the new idea of fuzzy hyperrings. The convictions of self centered bipolar fuzzy graph and distance, diameter, eccentricity, and length of bipolar fuzzy graph were studied in [17]. The idea of fuzzy hyperideal with fuzzy hypercongruences was examined in [18]. Baik [19] developed a link between bipolar fuzzy sets and the ideals of near ring theory in 2012. This link is obviously a core point of classical fuzzy subring as it provides fresh ideas for various challenges in near ring theory. Sardar et al. [20] illustrated the abstraction of bipolar fuzzy translation of sub-semigroup and bipolar fuzzy equivalence relation. Mahmood and Munir [21] proposed the conception of bipolar fuzzy subgroups and investigated their algebraic features. Ameri et al. [22] introduced the concept of Engel fuzzy subgroups and investigated the fundamental results of the left and right fuzzy Engel elements.. Motameni et al. [23] studied a special kinds of fuzzy hyperideals and extended this concept to fuzzy hperring homomorphism for maximal fuzzy hyperideal and prim fuzzy hyperideal. The idea of bipolar fuzzy subring was presented in [24]. The bipolar fuzzy subring is more a generalized form of fuzzy subring. A bipolar fuzzy subset $\theta = \{x, (\theta^P(x), \theta^N(x)), \forall\ x \in R\}$ of ring R is a bipolar fuzzy subring if it satisfied the two axioms for positive membership and two axioms for negative membership $\theta^P(x-y) \geq min\{\theta^P(x), \theta^P(y)\}$, $\theta^P(xy) \geq min\{\theta^P(x), \theta^P(y)\}$, and $\theta^N(x-y) \leq max\{\theta^N(x), \theta^N(y)\}$, $\theta^N(xy) \leq max\{\theta^N(x), \theta^N(y)\}$. The 4-Engel fuzzy subgroup are discussed in [25]. Moreover, Mohamadzahed et al. [26] invented the definition of nilpotent fuzzy subgroup and discussed many algebraic properties of nilpotent fuzzy subgroups. A fuzzy subgroup A of group G is called a good nilpotent fuzzy subgroup of G if there exists a non negative integer n such that $Z_n(A) = G$; the smallest such integer is called the class of A, where $Z_n(A)$ is normal subgroup of G. Subbian and Kamaraj [27] commenced the notion of bipolar fuzzy ideals and explored the extension of bipolar fuzzy ideals. A bipolar fuzzy subset $\theta = \{x, (\theta^P(x), \theta^N(x)), \forall\ x \in R\}$ of ring R is bipolar fuzzy ideal if it satisfied the following axioms $\theta^P(x-y) \geq min\{\theta^P(x), \theta^P(y)\}$, $\theta^P(xy) \geq max\{\theta^P(x), \theta^P(y)\}$, and $\theta^N(x-y) \leq max\{\theta^N(x), \theta^N(y)\}$, $\theta^N(xy) \leq min\{\theta^N(x), \theta^N(y)\}$. Yamin and Sharma [28] studied the intuitionistic fuzzy ring, intuitionistic fuzzy ideal, and intuitionistic fuzzy quotient ring with operators in 2018. The algebraic structure between fuzzy set and normed rings was presented in [29]. Jun et al. [30] depicted the opinion of bipolar fuzzy subalgebra and k-fold bipolar fuzzy ideals.Trevijano et al. [31] invented a annihilator for the fuzzy subgroup of the Abelian group. They also discussed the behavior of annihilator with respect to intersection and union. The different approximation about fuzzy ring homomorphism was studied in [32].

Liu and Shi [33] presented a novel framework with respect to the fuzzification of lattice, which is know as M-hazy lattice. In Demirci's approach [34,35], the characteristic of the degree between the fuzzy binary operation is not used and the inverse element and the identity element may number more than one. In order to remove this drawback, Liu and Shi [36] introduced M-hazy groups by using a M-hazy binary operation. Mehmood et al. [37] initiated a new algebraic structure of M-hazy ring and studied the various algebraic characteristics of this newly defined ring. Alhaleem and Ahmad [38] proposed the idea of intuitionistic fuzzy normed ring in 2020. Mehmood et al. [39] developed a new algebraic structure of M-hazy ring homomorphism. The mapping ϕ from M-hazy ring $(R_1, +, \circ)$ to M-hazy ring $(R_2, \bigoplus, \bullet)$ is called an M-hazy ring homomorphism if the following conditions hold $\phi_M(k+l) = \phi(k) \bigoplus \phi(l)$ and $\phi_M(k \circ l) = \phi(k) \bullet \phi(l), \forall, k, l \in R_1$. Nakkasen [40] studied the properties of Artnians and Noetherian ternary near-rings under intuitionistic fuzzy ideals. A novel class of t-intuitionistic fuzzy subgroups was investigated in [41]. Gulzar et al. [42] introduced the notion of complex intuitionistic fuzzy group theory. The fuzzy homomorphism theorems of fuzzy rings were depicted in [43]. The notion of complex fuzzy subfields was analyzed in [44]. The new development about Q-complex fuzzy subrings were explored in [45]. The recent development of bipolar fuzzy sets in BCK/BCI-algebras and semigroup theory may be viewed in [46,47]. The competency of the bipolar fuzzy sets plays a key role in solving many physical difficulties. Additionally, the study of bipolar fuzzy subrings is significant in terms of its algebraic structure. This motivates us to describe the concept of bipolar fuzzy sets where one can have multiple options to discuss a particular problem of ring theory in a much more efficient manner. Firstly, we shall prove that the (α,β)-cut of bipolar fuzzy subring forms a subring of a given ring and discuss various algebraic properties of this phenomenon. Secondly, we shall define bipolar fuzzy left cosets and determine the bipolar fuzzy subring of quotient ring. We shall also define the support set of bipolar fuzzy set. Thirdly, we shall describe bipolar fuzzy homomorphism and weak bipolar fuzzy homomorphism and show that bipolar fuzzy homomorphism preserves the ring operation. Additionally, we prove the bipolar fuzzy homomorphism and isomorphism theorems of bipolar fuzzy subrings parallel to natural theorems of ring homomorphism and ring isomorphism.

A sketch of this study is as follows. The bipolar fuzzy sets, bipolar fuzzy subring, and related results are defined in Section 2. In Section 3, we studied the concepts of (α,β)-cut of bipolar fuzzy sets and discuss many important algebraic characteristics of bipolar fuzzy subrings (BFSRs). We prove that the direct product of two bipolar fuzzy subrings (BFSRs) are bipolar fuzzy subrings (BFSR) by using the notion of (α,β) of bipolar fuzzy set (BFS). Furthermore, we define support of bipolar fuzzy subset (BFS) and show that support of bipolar fuzzy ideal (BFI) of ring form a natural ideal of ring. In Section 4, we describe bipolar fuzzy homomorphism (BFH) of bipolar fuzzy subring (BFSR) under a natural ring homomorphism and prove that the bipolar fuzzy homomorphism (BFH) preserves the sum and product operation defined on bipolar fuzzy subring (BFSR). We also develop a significant relationship between two bipolar fuzzy subrings (BFSRs) of the quotient rings under given surjective homomorphism and prove more fundamental theorems of bipolar fuzzy homomorphism (BFH) for these specific fuzzy subrings. Finally, we discuss three fundamental theorems of bipolar fuzzy isomorphism of bipolar fuzzy subrings (BFSRs).

2. Preliminaries

This section recalls some basic ideas of BFS, BFSR, and BFI that are interconnected to the analysis of this article.

Definition 1 ([5])**.** *A BFS θ of universe of discourse P is described as an object of the form $\theta = \{(k,\theta^P(k),\theta^N(k)) : k \in P\}$, where $\theta^P : P \to [0,1]$ and $\theta^N : P \to [-1,0]$. The grades of positive membership $\theta^P(k)$ represents the degree of belief of an element k to the property corresponding to a degree of BFS θ and the grades of negative membership $\theta^N(k)$ represents the degree of disbelief of an element k to some implicit counter-property corresponding to a BFS θ. If $\theta^P(k) \neq 0$ and $\theta^N(k) = 0$, this means that k has only a degree of belief for θ and if $\theta^P(k) = 0$ and $\theta^N(k) \neq 0$, this means that k does not possess belief on the property of θ, but somewhat satisfies the counter property of θ. It is possible for an element k to be such that $\theta^P(k) \neq 0$ and $\theta^N(k) \neq 0$ when the membership function of the attribution overlaps that of its counter attribution over some part of the nonempty set P.*

Definition 2 ([24])**.** *A BFS θ of a ring R is called a BFSR of a R, if the following conditions hold:*

1. $\theta^P(k-t) \geq min\{\theta^P(k),\theta^P(t)\}$,
2. $\theta^P(kt) \geq min\{\theta^P(k),\theta^P(t)\}$,
3. $\theta^N(k-t) \leq max\{\theta^N(k),\theta^N(t)\}$,
4. $\theta^N(kt) \leq max\{\theta^N(k),\theta^N(t)\}$.

Definition 3 ([27])**.** *A BFS θ of a ring R is said to be bipolar fuzzy left ideal(BFLI) of R, if the following conditions hold:*

1. $\theta^P(k-t) \geq min\{\theta^P(k),\theta^P(t)\}$,

2. $\theta^P(kt) \geq \theta^P(t)$,
3. $\theta^N(k-t) \leq max\{\theta^N(k), \theta^N(t)\}$,
4. $\theta^N(kt) \leq \theta^N(t)$, *for all* $k, t \in R$.

Definition 4. *Ref. [27] A BFS θ of a ring R is said to be bipolar fuzzy right ideal(BFRI) of R, if the following conditions hold:*

1. $\theta^P(k-t) \geq min\{\theta^P(k), \theta^P(t)\}$,
2. $\theta^P(kt) \geq \theta^P(k)$,
3. $\theta^N(k-t) \leq max\{\theta^N(k), \theta^N(t)\}$,
4. $\theta^N(kt) \leq \theta^N(k)$, *for all* $k, t \in R$.

Definition 5. *Ref. [27] A BFS θ of a ring R is said to be BFI of R, if the following axioms hold:*

1. $\theta^P(k-t) \geq min\{\theta^P(k), \theta^P(t)\}$,
2. $\theta^P(kt) \geq max\{\theta^P(k), \theta^P(t)\}$,
3. $\theta^N(k-t) \leq max\{\theta^N(k), \theta^N(t)\}$,
4. $\theta^N(kt) \leq min\{\theta^N(k), \theta^N(t)\}$, $\forall\, k, t \in R$.

Definition 6 ([24])**.** *Let $\theta = \{(k, \theta^P(k), \theta^N(k)) : k \in P\}$ and $\eta = \{(t, \eta^P(t), \eta^N(t)) : t \in Q\}$ be any two BFSs of nonempty sets P and Q, respectively. Then the Cartesian product of θ and η is represented by $\theta \times \eta$ and is defined as the following:*

$$\theta \times \eta = \{< (k,t), (\theta^P \times \eta^P)(k,t), (\theta^N \times \eta^N)(k,t) >: k \in P, t \in Q\},$$

where $(\theta^P \times \eta^P)(k,t) = min\{\theta^P(k), \eta^P(t)\}$ and $(\theta^N \times \eta^N)(k,t) = max\{\theta^N(k), \eta^N(t)\}$.

3. Fundamental Algebraic Properties of Bipolar Fuzzy Subrings

In this section, we study the (α, β)-cut of BFS and investigate some important characteristics of this phenomenon. We define support of BFS and justify their corresponding desirable set-theoretic properties under BFSR. We also found the BFSR of quotient ring.

Definition 7. *Let $\theta = \{(k, \theta^P(k), \theta^N(k)) : k \in P\}$ be a BFS of a set P, then a (α, β)-cut of θ is a crisp subset of P and is defined as $C_{\alpha,\beta}(\theta) = \{k \in P \mid \theta^P(k) \geq \alpha,\ \theta^N(k) \leq \beta\}$, where $\alpha \in [0,1]$ and $\beta \in [-1, 0]$.*

Theorem 8. *If θ is a BFS of a ring R, then $C_{\alpha,\beta}(\theta)$ is a subring of R if and only if θ is a BFSR of R.*

Proof. Clearly $C_{\alpha,\beta}(\theta) \neq \emptyset$, because $\theta^P(0) \geq \alpha$, $\theta^N(0) \leq \beta$. Let $k, t \in C_{\alpha,\beta}(\theta)$. Then $\theta^P(k), \theta^P(t) \geq \alpha$ and $\theta^N(k), \theta^N(t) \leq \beta$.

Consider $\theta^P(k-t) \geq \min\{\theta^P(k), \theta^P(t)\} \geq \alpha$ and $\theta^N(k-t) \leq \max\{\theta^N(k), \theta^N(t)\} \leq \beta$.

Thus $k - t \in C_{\alpha,\beta}(\theta)$.
Further, we have the following.

$$\theta^P(kt) \geq \min\{\theta^P(k), \theta^P(t)\} \geq \alpha \text{ and } \theta^N(kt) \leq \max\{\theta^N(k), \theta^N(t)\} \leq \beta.$$

Therefore, $kt \in C_{\alpha,\beta}(\theta)$. As result, $C_{\alpha,\beta}(\theta)$ is subring of R.

Conversely, suppose that $\alpha = \min\{\theta^P(k), \theta^P(t)\}$, $\beta = \max\{\theta^N(k), \theta^N(t)\}$. Then $\theta^P(k) \geq \alpha, \theta^N(k) \leq \beta$ and $\theta^P(t) \geq \alpha, \theta^N(t) \leq \beta$. This gives $k, t \in C_{\alpha,\beta}(\theta)$. Since $C_{\alpha,\beta}(\theta)$ is a subring of R. Then we have $k - t \in C_{\alpha,\beta}(\theta)$. This implies the following.

$$\theta^P(k-t) \geq \alpha = \min\{\theta^P(k), \theta^P(t)\} \text{ and } \theta^N(k-t) \leq \beta = \max\{\theta^N(k), \theta^N(t)\}.$$

Moreover, as $C_{\alpha,\beta}(\theta)$ is a subring of R. This implies that $kt \in C_{\alpha,\beta}(\theta)$. Thus, the following is the case.

$$\theta^P(kt) \geq \alpha = \min\{\theta^P(k), \theta^P(t)\}, \text{ and } \theta^N(kt) \leq \beta = \max\{\theta^N(k), \theta^N(t)\}.$$

Hence, θ is a BFSR of R. □

Theorem 9. *If θ is a BFS of a ring R, then $C_{\alpha,\beta}(\theta)$ is a ideal of R if and only if θ is a BFI of R.*

Proof. From Theorem 8, we have $k - t \in C_{\alpha,\beta}(\theta)$.
Furthermore, if $r \in R$ and $t \in C_{\alpha,\beta}(\theta)$, we have the following.

$$\begin{aligned}\theta^P(rt) &\geq \max\{\theta^P(r), \theta^P(t)\} \geq \theta^P(t) \geq \alpha \text{ and } \theta^N(rt) \leq \min\{\theta^N(r), \theta^N(t)\} \leq \theta^N(t) \leq \beta.\\ \theta^P(tr) &\geq \max\{\theta^P(r), \theta^P(t)\} \geq \theta^P(t) \geq \alpha \text{ and } \theta^N(tr) \leq \min\{\theta^N(r), \theta^N(t)\} \leq \theta^N(t) \leq \beta.\end{aligned}$$

Therefore, $rt, tr \in C_{\alpha,\beta}(\theta)$. As result, $C_{\alpha,\beta}(\theta)$ is an ideal of R.
Conversely, from Theorem 8, we have the following.

$$\theta^P(k-t) \geq \min\{\theta^P(k), \theta^P(t)\} \text{ and } \theta^N(k-t) \leq \max\{\theta^N(k), \theta^N(t)\}.$$

Suppose that $\alpha = \theta^P(t)$, $\beta = \theta^N(t)$. This gives, $t \in C_{\alpha,\beta}(\theta)$, for any $r \in R$. Since $C_{\alpha,\beta}(\theta)$ is an ideal of R. In this case, $rt, tr \in C_{\alpha,\beta}(\theta)$. Thus, the following is the case.

$$\theta^P(rt) \geq \alpha = \theta^P(t), \text{ and } \theta^N(rt) \leq \beta = \theta^N(t), \theta^P(tr) \geq \alpha = \theta^P(t), \text{ and } \theta^N(tr) \leq \beta = \theta^N(t).$$

Hence, θ is bipolar fuzzy ideal. This establishes the proof. □

Theorem 10. *Let θ be an BFSR of R, then $C_{\alpha,\beta}(\theta) \subseteq C_{u,\delta}(\theta)$ if $\alpha \geq u$ and $\beta \leq \delta$, where α, $u \in [0,1]$ and β, $\delta \in [-1,0]$.*

Proof. Let $k \in C_{\alpha,\beta}(\theta)$, then $\theta^P(k) \geq \alpha$ and $\theta^N(k) \leq \beta$. Since $\alpha \geq u$ and $\beta \leq \delta$, we can write, $\theta^P(k) \geq \alpha \geq u$ and $\theta^N(k) \leq \beta \leq \delta$. Therefore $k \in C_{u,\delta}(\theta)$. □

Theorem 11. *If θ and η are BFSRs of a ring R, then $C_{\alpha,\beta}(\theta \cap \eta) = C_{\alpha,\beta}(\theta) \cap C_{\alpha,\beta}(\eta)$.*

Proof. We have $C_{\alpha,\beta}(\theta \cap \eta) = \{k \in R | (\theta^P \cap \eta^P)(k) \geq \alpha, (\theta^N \cap \eta^N)(k) \leq \beta\}$. Now, $k \in C_{\alpha,\beta}(\theta \cap \eta)$.

$$\begin{aligned}&\Leftrightarrow (\theta^P \cap \eta^P)(k) \geq \alpha, (\theta^N \cap \eta^N)(k) \leq \beta\\ &\Leftrightarrow \min\{\theta^P(k), \eta^P(k)\} \geq \alpha \text{ and } \max\{\theta^N(k), \eta^N(k)\} \leq \beta\\ &\Leftrightarrow \theta^P(k), \eta^P(k) \geq \alpha \text{ and } \theta^N(k), \eta^N(k) \leq \beta\\ &\Leftrightarrow k \in C_{\alpha,\beta}(\theta) \text{ and } k \in C_{\alpha,\beta}(\eta)\\ &\Leftrightarrow k \in C_{\alpha,\beta}(\theta) \cap C_{\alpha,\beta}(\eta).\end{aligned}$$

Therefore, $C_{\alpha,\beta}(\theta \cap \eta) = C_{\alpha,\beta}(\theta) \cap C_{\alpha,\beta}(\eta)$. □

Theorem 12. *If $\theta \subseteq \eta$ then $C_{\alpha,\beta}(\theta) \subseteq C_{\alpha,\beta}(\eta)$, where θ and η are BFSRs of a ring R.*

Proof. Let $\theta \subseteq \eta$ and $k \in C_{\alpha,\beta}(\theta)$. Then $\theta^P(k) \geq \alpha, \theta^N(k) \leq \beta$. Since $\theta \subseteq \eta$, so we have $\eta^P(k) \geq \theta^P(k) \geq \alpha$ and $\eta^N(k) \leq \theta^N(k) \leq \beta$ and consequently $k \in C_{\alpha,\beta}(\eta)$. Thus, $C_{\alpha,\beta}(\theta) \subseteq C_{\alpha,\beta}(\eta)$. □

Theorem 13. *If θ and η are BFSRs of a ring R, then $C_{\alpha,\beta}(\theta \cup \eta) = C_{\alpha,\beta}(\theta) \cup C_{\alpha,\beta}(\eta)$.*

Proof. Since $\theta \subseteq (\theta \cup \eta)$ and $\eta \subseteq (\theta \cup \eta)$. By the above theorem $C_{\alpha,\beta}(\theta) \subseteq C_{\alpha,\beta}(\theta \cup \eta)$ and $C_{\alpha,\beta}(\eta) \subseteq C_{\alpha,\beta}(\theta \cup \eta)$. Therefore, $C_{\alpha,\beta}(\theta \cup \eta) \supseteq C_{\alpha,\beta}(\theta) \cup C_{\alpha,\beta}(\eta)$.
Now suppose that $k \in C_{\alpha,\beta}(\theta \cup \eta)$. This implies that $(\theta^P \cup \eta^P(k)) \geq \alpha$ and $(\theta^N \cup \eta^N(k)) \leq \beta \Rightarrow \max\{\theta^P(k), \eta^P(k)\} \geq \alpha$ and $\min\{\theta^N(k), \eta^N(k)\} \leq \beta \Rightarrow \theta^P(k) \geq \alpha$ or $\eta^P(k) \geq \alpha$ and $\theta^N(k) \leq \beta$ or $\eta^N(k) \geq \beta$. This implies that $k \in C_{\alpha,\beta}(\theta)$ or $k \in C_{\alpha,\beta}(\eta)$. This implies that $C_{\alpha,\beta}(\theta \cup \eta) \subseteq C_{\alpha,\beta}(\theta) \cup C_{\alpha,\beta}(\eta)$. Consequently, we have $C_{\alpha,\beta}(\theta) \cup C_{\alpha,\beta}(\eta) = C_{\alpha,\beta}(\theta \cup \eta)$. □

Proposition 14. *If θ and η be two BFSs of R_1 and R_2, respectively. Then $C_{\alpha,\beta}(\theta \times \eta) = C_{\alpha,\beta}(\theta) \times C_{\alpha,\beta}(\eta)$, for all $\alpha, \in [0,1]$, and $\beta \in [-1,0]$.*

Proof. Let $(k,t) \in C_{\alpha,\beta}(\theta \times \eta)$ be any element.

$$\begin{aligned}
&\Leftrightarrow \quad (\theta^P \times \eta^P)(k,t) \geq \alpha \text{ and } (\theta^N \times \eta^N)(k,t) \leq \beta \\
&\Leftrightarrow \quad \min\{\theta^P(k), \eta^P(t)\} \geq \alpha \text{ and } \max\{\theta^N(k), \eta^N(t)\} \leq \beta \\
&\Leftrightarrow \quad \theta^P(k) \geq \alpha, \eta^P(t) \geq \alpha \text{ and } \theta^N(k) \leq \beta, \eta^N(t) \leq \beta \\
&\Leftrightarrow \quad \theta^P(k) \geq \alpha, \theta^N(k) \leq \beta \text{ and } \eta^P(t) \geq \alpha, \eta^N(t) \leq \beta \\
&\Leftrightarrow \quad k \in C_{\alpha,\beta}(\theta) \text{ and } n \in C_{\alpha,\beta}(\eta) \\
&\Leftrightarrow \quad (k,t) \in C_{\alpha,\beta}(\theta) \times C_{\alpha,\beta}(\eta).
\end{aligned}$$

Hence, $C_{\alpha,\beta}(\theta \times \eta) = C_{\alpha,\beta}(\theta) \times C_{\alpha,\beta}(\eta)$. □

Theorem 15. *Let θ and η be BFSRs of ring R_1 and R_2, respectively. Then $\theta \times \eta$ is also the BFSR of ring $R_1 \times R_2$.*

Proof. Since θ and η are the BFSRs of ring R_1 and R_2, respectively. Therefore, $C_{\alpha,\beta}(\theta)$ and $C_{\alpha,\beta}(\eta)$ are BFSRs of ring R_1 and R_2, respectively. $\forall\ \alpha, \in [0,1]$ and $\beta \in [-1,0]$. By Theorem 8, we have the following.

$$\begin{aligned}
&\Leftrightarrow \quad C_{\alpha,\beta}(\theta) \times C_{\alpha,\beta}(\eta) \text{ is subring of } R_1 \times R_2. \\
&\Leftrightarrow \quad C_{\alpha,\beta}(\theta \times \eta) \text{ is subring of } R_1 \times R_2. \\
&\Leftrightarrow \quad \theta \times \eta \text{ is BFSR of ring } R_1 \times R_2. \text{ (By Theorem 8.)}
\end{aligned}$$

□

Definition 16. *Let R be a ring and θ be the BFSR of a ring R. Let $k \in R$ be a fixed element. Then, the set $(k+\theta)(t) = \{< t, k+\theta^P(t), k+\theta^N(t) >: t \in R\}$, where $k+\theta^P(t) = \theta^P(t-k)$ and $k+\theta^N(t) = \theta^N(t-k)$, for all $t \in R$ is said to be bipolar fuzzy left coset of R purposed by θ and k.*

Theorem 17. *Let θ be a BFI of R and k be an arbitrary fixed element of R. Then, $k + C_{\alpha,\beta}(\theta) = C_{\alpha,\beta}(k+\theta)$.*

Proof. Consider $k + C_{\alpha,\beta}(\theta)$.

$$\begin{aligned}
&= \quad k + \{< t \in R : \theta^P(t) \geq \alpha \text{ and } \theta^N(t) \leq \beta >\}. \\
&= \quad \{< k+t \in R : \theta^P(t) \geq \alpha \text{ and } \theta^N(t) \leq \beta >\}.
\end{aligned}$$

Place $k+t = a$ so that $t = a-k$. Then, $k + C_{\alpha,\beta}(\theta)$:

$$\begin{aligned}
&= \quad \{< a \in R : \theta^P(a-k) \geq \alpha \text{ and } \theta^N(a-k) \leq \beta >\} \\
&= \quad \{< a \in R : k+\theta^P(a) \geq \alpha \text{ and } k+\theta^N(a) \leq \beta >\}.
\end{aligned}$$

thus, $k + C_{\alpha,\beta}(\theta) = C_{\alpha,\beta}(k+\theta)$, $\forall\ \alpha \in [0,1]$ and $\beta \in [-1,0]$. □

Definition 18. *Let θ and η be BFSs of universal set P. Then, the bipolar fuzzy sum of θ and η is denoted by $\theta+\eta=\{t,(\theta^P+\eta^P)(t),(\theta^N+\eta^N)(t) : t\in P\}$, where the following is the case.*

$$(\theta^P+\eta^P)(t)=\begin{cases} max\{min\{\theta^P(a),\eta^P(b)\}, & if\ t=a+b \\ 0, & otherwise \end{cases}$$

$$(\theta^N+\eta^N)(t)=\begin{cases} min\{max\{\theta^N(a),\eta^N(b)\}, & if\ t=a+b \\ 1, & otherwise \end{cases}$$

Definition 19. *Let θ be a BFS of P. The support set θ_* of θ is defined as the following.*

$$\theta_*=\{k\in P:\theta^P(k)>0,\theta^N(k)<0\}.$$

Remark 20. *Let θ be a BFSR of R. Then θ_* is a BFSR of R.*

The next theorem shows that how a support set of bipolar fuzzy ideal is a ideal of R.

Theorem 21. *Let θ be BFI of R, then θ_* is a BFI of R.*

Proof. Note that θ is BFI. Assume that $k,t\in\theta_*$. Consider $\theta^P(k-t)\geq\min\{\theta^P(k),\theta^P(t)\ \}>0$, and $\theta^N(k-t)\leq\max\{\theta^N(k),\theta^N(t)\ \}<0$. This implies that $k-t\in\theta_*$.
Furthermore, suppose that $k\in\theta_*$ and $t\in R$. Then we have $\theta^P(kt)\geq\max\{\theta^P(k),\theta^P(t)\ \}>0$, and $\theta^N(kt)\leq\min\{\theta^N(k),\theta^N(t)\ \}<0$. Similarly, $\theta^P(tk)>0$ and $\theta^N(tk)<0$ implies that $kt,tk\in\theta_*$. This implies that θ_* is an ideal of R.

Our next theorem provides the significant importance of support of intersection of any two BFSR of a ring R. □

Theorem 22. *If θ and η are BFSR of R. Then $(\theta\cap\eta)_*=\theta_*\cap\eta_*$.*

Proof. For any arbitrary element, $k\in(\theta\cap\eta)_*$, implies that $(\theta^P\cap\eta^P)(k)>0$ and $(\theta^N\cap\eta^N)(k)<0$. We have $\theta^P(k),\ \eta^P(k)\geq\min\{\theta^P(k),\eta^P(k)\}=(\theta^P\cap\eta^P)(k)>0$. This implies that $\theta^P(k),\ \eta^P(k)>0$, and $\theta^N(k),\ \eta^N(k)\leq\max\{\theta^N(k),\eta^N(k)\}=(\theta^N\cap\eta^N)(k)<0$. This implies that $\theta^N(k),\ \eta^N(k)<0$. This implies that $k\in\theta_*\cap\eta_*$. Consequently, $(\theta\cap\eta)_*\subseteq\theta_*\cap\eta_*$. Moreover, $k\in\theta_*\cap\eta_*$. This implies that $\theta^P(k),\ \eta^P(k)>0$ and $\theta^N(k),\ \eta^N(k)<0$. This implies that $\min\{\theta^P(k),\eta^P(k)\}>0$, and $\max\{\theta^N(k),\eta^N(k)\}<0$. this implies that $(\theta^P\cap\eta^P)(k)>0$ and $(\theta^N\cap\eta^N)(k)<0$. This implies that $k\in(\theta\cap\eta)_*$. Therefore, $(\theta\cap\eta)_*\supseteq\theta_*\cap\eta_*$. Consequently, $(\theta\cap\eta)_*=\theta_*\cap\eta_*$. This concludes the proof. □

Remark 23. *If θ and η are BFSRs of R, then $(\theta+\eta)_*=\theta_*+\eta_*$.*

Definition 24. *Let θ and η be the BFS and BFSR of R, respectively, with $\theta\subseteq\eta$. Then θ is called a BFI of η if the following axiom holds:*

1. $\theta^P(k-t)\geq min\{\theta^P(t),\theta^P(k)\},\ \forall k,\ t\in R,$
2. $\theta^P(kt)\geq max\{min\{\theta^P(t),\eta^P(k)\},min\{\eta^P(t),\theta^P(k)\}\}\ \forall\ k,\ t\in R,$
3. $\theta^N(k-t)\leq max\{\theta^N(t),\theta^N(k)\},\ \forall\ k,\ t\in R,$
4. $\theta^N(kt)\leq min\{max\{\theta^N(t),\eta^N(k)\},max\{\eta^N(t),\theta^N(k)\}\}\ \forall\ k,\ t\in R.$

Theorem 25. *Let θ and η be the BFSR of a ring R and θ is a BFI of η. Then θ_* is ideal of ring η_*.*

Theorem 26. *Let θ be a BFI and η be a BFSR of R, then $\theta\cap\eta$ is BFI of η.*

Proof. Consider an elements $k, t \in R$. We have the following.

$$\begin{aligned}(\theta^P \cap \eta^P)(k-t) =& \min\{\theta^P(k-t), \eta^P(k-t)\},\\ \geq& \min\{\min\{\theta^P(t), \theta^P(k)\}, \min\{\eta^P(t), \eta^P(k)\}\}\\ =& \min\{\min\{\theta^P(t), \eta^P(t)\}, \min\{\theta^P(k), \eta^P(k)\}\}\\ =& \min\{\left(\theta^P \cap \eta^P\right)(t), \left(\theta^P \cap \eta^P\right)(k)\}, \ \forall\, k, t \in R.\end{aligned}$$

Moreover, we also have the following.

$$\begin{aligned}(\theta^P \cap \eta^P)(kt) =& \min\{\theta^P(kt), \eta^P(kt)\},\\ \geq& \min\{\max\{\theta^P(k), \theta^P(t)\}, \min\{\eta^P(k), \eta^P(t)\}\}\\ =& \max\{\min\{\theta^P(k), \min\{\eta^P(k), \eta^P(t)\}\}, \min\{\theta^P(t), \min\{\eta^P(t), \eta^P(k)\}\}\\ =& \max\{\min\{(\theta^P \cap \eta^P)(k), \eta^P(t)\}, \min\{(\theta^P \cap \eta^P)(t), \eta^P(k)\}\}.\end{aligned}$$

Furthermore, we have the following.

$$\begin{aligned}(\theta^N \cap \eta^N)(k-t) =& \max\{\theta^N(k-t), \eta^N(k-t)\},\\ \leq& \max\{\max\{\theta^N(t), \theta^N(k)\}, \max\{\eta^N(t), \eta^N(k)\}\}\\ =& \max\{\max\{\theta^N(t), \eta^N(t)\}, \max\{\theta^N(k), \eta^N(k)\}\}\\ =& \max\{\left(\theta^N \cap \eta^N\right)(t), \left(\theta^N \cap \eta^N\right)(k)\}, \ \forall\, k, t \in R.\end{aligned}$$

In addition, we have the following.

$$\begin{aligned}(\theta^N \cap \eta^N)(kt) =& \max\{\theta^N(kt), \eta^N(kt)\},\\ \leq& \max\{\min\{\theta^N(k), \theta^N(t)\}, \max\{\eta^N(k), \eta^N(t)\}\}\\ =& \min\{\max\{\theta^N(k), \max\{\eta^N(k), \eta^N(t)\}\}, \max\{\theta^N(t), \max\{\eta^N(t), \eta^N(k)\}\}\\ =& \min\{\max\{(\theta^N \cap \eta^N)(k), \eta^N(t)\}, \max\{(\theta^N \cap \eta^N)(t), \eta^N(k)\}\}.\end{aligned}$$

This concludes the proof. □

Remark 27. *Let θ, η and ψ be BFSR of R such that θ and η are BFI of ψ, then $\theta \cap \eta$ is BFI of ψ.*

Theorem 28. *Let L be an ideal of a ring R. If $\theta = \{(k, \theta^P(k), \theta^N(k)) : k \in R\}$ is a BFSR of R, then the BFS $\bar{\theta} = \{(K + L, \bar{\theta^P}(k + L), \bar{\theta^N}(k + L)) : k \in R\}$ of R/L is also BFSR of R/L, where $\bar{\theta^P}(k + L) = max\{\theta^P(k + a)|a \in L\}$ and $\bar{\theta^N}(k + L) = min\{\theta^N(k + a)|a \in L\}$.*

Proof. First we shall show that $\bar{\theta^P} : R/L \to [0, 1]$ and $\bar{\theta^N} : R/L \to [0, 1]$ are well defined. Let $k + L = t + L$, then $t = k + a$ for some $a \in L$.

Consider the following:

$$\begin{aligned}\bar{\theta^P}(t + L) &= \max\{\theta^P(t + b)|b \in L\}\\ &= \max\{\theta^P(k + a + b)|b \in L\}\\ &= \max\{\theta^P(k + c)|c = a + b \in L\}\\ &= \bar{\theta^P}(k + L)\end{aligned}$$

and the following.

$$\begin{aligned}\bar{\theta^N}(t+L) &= \min\{\theta^N(t+b)|b\in\ L\}\\ &= \min\{\theta^N(k+a+b)|b\in\ L\}\\ &= \min\{\theta^N(k+c)|c=a+b\in\ L\}\\ &= \bar{\theta^N}(k+L).\end{aligned}$$

Therefore, $\bar{\theta^P}$ and $\bar{\theta^N}$ are well defined. Now we shall prove that $\bar{\theta}$ is BFSR of R/L. We have $\bar{\theta^P}\{(k+L)-(t+L)\}$.

$$\begin{aligned}&=\bar{\theta^P}\{(k-t)+L\}\\ &=\max\{\theta^P(k-t+u)|u\in\ L\}\\ &=\max\{\theta^P(k-t+v-w)|u=v-w\in\ L\}\\ &=\max\{\theta^P(k+v)-(t+w)|v,w\in\ L\}\\ &\geq\max\{\min\{\theta^P(k+v),\theta^P(t+w)|v,w\in\ L\}\}\\ &=\min\{\max\{\theta^P(k+v)|v\in\ L\},\max\{\theta^P(t+w)|w\in\ L\}\}\\ &\text{since } v \text{ and } w \text{ vary independently.}\\ &=\min\{\bar{\theta^P}(k+L),\bar{\theta^N}(k+L)\}\end{aligned}$$

Moreover, we have $\bar{\theta^P}\{(k+L)(t+L)\}$.

$$\begin{aligned}&= \bar{\theta^P}\{(kt)+L\}\\ &= \max\{\theta^P(kt+u)|u\in\ L\}\\ &\geq \max\{\min\{\theta^P(k+v),\theta^P(t+w)|v,w\in\ L\}\}\\ &= \min\{\max\{\theta^P(k+v),v\in L\},\max\{\theta^P(t+w)|w\in\ L\}\}\\ &= \min\{\bar{\theta^P}(k+L),\bar{\theta^P}(t+L)\}.\end{aligned}$$

In addition, we have $\bar{\theta^N}\{(k+L)-(t+L)\}.=\bar{\theta^N}\{(k-t)+L\}$

$$\begin{aligned}&=\min\{\theta^N(k-t+u)|u\in\ L\}\\ &=\min\{\theta^N(k-t+v-w)|u=v-w\in\ L\}\\ &=\min\{\theta^N(k+v)-(t+w)|v,w\in\ L\}\\ &\leq\min\{\max\{\theta^N(k+v),\theta^N(t+w)|v,w\in\ L\}\}\\ &=\max\{\min\{\theta^N(k+v)|v\in\ L\},\min\{\theta^N(t+w)|w\in\ L\}\}\\ &\text{since } v \text{ and } w \text{ vary independently.}\\ &=\max\{\bar{\theta^N}(k+L),\bar{\theta^N}(k+L)\}\end{aligned}$$

Furthermore, $\bar{\theta^N}\{(k+L)(t+L)\}$.

$$\begin{aligned}&= \bar{\theta^N}\{(kt)+L\}\\ &= \min\{\theta^N(kt+u)|u\in\ L\}\\ &\leq \min\{\max\{\theta^N(k+v),\theta^N(t+w)|v,w\in\ L\}\}\\ &= \max\{\min\{\theta^N(k+v),v\in L\},\min\{\theta^N(n+w)|w\in\ L\}\}\\ &= \max\{\bar{\theta^N}(k+L),\bar{\theta^N}(t+L)\}.\end{aligned}$$

Hence, $\bar{\theta}=\{(k+L,\bar{\theta^P}(k+L),\bar{\theta^N}(k+L)):k\in R\}$ is a BFSR of R/L. $\square$

4. Fundamental Theorems of Bipolar Fuzzy Homomorphism and Bipolar Fuzzy Isomorphism of Bipolar Fuzzy Subrings

In this section, we investigate the concept of BFH of BFSR and prove that this homomorphism preserves the operation of fuzzy sum and fuzzy product of BFSR of ring R. We clarify bipolar fuzzy homomorphism for these fuzzy subrings and investigate the idea of BFH relation between any two BFSRs. We also presented the bipolar fuzzy isomorphism theorem of BFSRs.

Definition 29. *Let $f : R \to R'$ be the ring homomorphism from R to R'. Let θ and η be BFSRs of rings R and R', respectively. The image and inverse image of θ and η, respectively, are described as $\omega(\theta)(t) = \{(t, \omega(\theta^P)(t), \omega(\theta^N)(t)), t \in R'\}$ and $\omega^{-1}(\theta)(k) = \{(k, \omega^{-1}(\theta^P)(k), \omega^{-1}(\theta^N)(k)),$ $k \in R\}$ where we have the following:*

$$\omega(\theta^P)(t) = \begin{cases} max\{\theta^P(k) & k \in \omega^{-1}(t) \neq \emptyset\}, for\ all\ t \in R' \\ 0 & otherwise \end{cases}$$

$$\omega(\theta^N)(t) = \begin{cases} min\{\theta^N(k) & k \in \omega^{-1}(t) \neq \emptyset\}, for\ all\ t \in R' \\ 1 & otherwise. \end{cases}$$

and the following is the case.

$$\omega^{-1}(\eta^P)(k) = \eta^P(\omega(k)), \omega^{-1}(\eta^N)(k) = \eta^N(\omega(k)) \quad \forall\, k \in R.$$

The homomorphism ω is called a BFH from θ onto η if $\omega(\theta) = \eta$ and is denoted by $\theta \approx \eta$. A homomorphism ω from bipolar fuzzy subring θ to η is said to be a bipolar fuzzy isomorphism from θ to η if $\omega(\theta) = \eta$. In this situation, θ is bipolar fuzzy isomorphic to η and is represented by $\theta \cong \eta$. The homomorphism ω is called weak BFH from θ to η if $\omega(\theta) \subseteq \eta$.

In the next theorem,we illustrate the fuzzy homomorphism relation between bipolar fuzzy subring of ring and any of its factor ring.

Theorem 30. *Let ω be a homomorphism from ring R to ring S. Let θ and η be two BFSs of ring R. Then, the following is the case.*

$$\begin{aligned} \omega(\theta + \eta) &= \omega(\theta) + \omega(\eta) \\ \omega(\theta \circ \eta) &= \omega(\theta) \circ \omega(\eta). \end{aligned}$$

Proof. For $t \in S$, we have the following:

$$\begin{aligned} \omega(\theta + \eta)(t) &= (\omega(\theta^P + \eta^P)(t), \omega(\eta^N + \eta^N)(t)) \\ \text{and } (\omega(\theta) + \omega(\eta))(t) &= (\omega(\theta^P) + \omega(\eta^P)(t), \omega(\eta^N) + \omega(\eta^N)(t)). \end{aligned}$$

Consider the following.
$\omega(\theta^P + \eta^P)(t) = \max\{(\theta^P + \eta^P)(k) : k \in R, t = \omega(k)\}$

$= \max\{\max\{\min\{\theta^P(k_1), \eta^P(k_2) : k_1, k_2 \in R, k = k_1 + k_2\}k \in R, t = \omega(k)\}\}$
$= \max\{\max\{\min\{\theta^P(k_1), \eta^P(k_2) : k_1, k_2 \in R, t_1 = \omega(k_1), t_2 = \omega(k_2)\}, t_1, t_2 \in S,$
$t = t_1 + t_2\}\}$
$= \max\{\min\{\max\{\theta^P(m_1) : k_1 \in R, t_1 = \omega(k_1)\}, \max\{\eta^P(k_2) : k_2 \in R : t_2 = \omega(k_2)\}\}\}$
$= \max\{\min\{\omega(\theta^P)(t_1), \omega(\eta^P)(t_2)\} : t_1, t_2 \in S, t = t_1 + t_2\}$
$= (\omega(\theta^P) + \omega(\eta^P))(t).$

Moreover, we have the following.

$$\omega(\theta^N+\eta^N)(t) = \min\{(\theta^N+\eta^N)(k) : k \in R, t=\omega(k)\}$$
$$= \min\{\min\{\max\{\theta^N(k_1), \eta^N(k_2) : k_1, k_2 \in R, k = k_1+k_2\} k \in R, t = \omega(k)\}\}$$
$$= \min\{\min\{\max\{\theta^N(k_1), \eta^N(k_2) : k_1, k_2 \in R, t_1 = \omega(k_1), t_2 = \omega(k_2)\}, t_1, t_2 \in S,$$
$$t = t_1+t_2\}\}\}$$
$$= \min\{\max\{\min\{\theta^N(m_1) : k_1 \in R, t_1 = \omega(k_1)\}, \max\{\eta^N(m_2) : k_2 \in R : t_2 = \omega(k_2)\}\}\}.$$
$$= \min\{\max\{\omega(\theta^N)(t_1), \omega(\eta^N)(t_2)\} : t_1, t_2 \in S, n = t_1+t_2\}$$
$$= (\omega(\theta^N)+\omega(\eta^N))(t).$$

Therefore, $\omega(\theta+\eta) = \omega(\theta)+\omega(\eta)$.

(ii) For $t \in S$, we have the following.

$$\begin{aligned} \omega(\theta \circ \eta)(t) &= (\omega(\theta^P \circ \eta^P)(t), \omega(\theta^N \circ \eta^N)(t)) \\ \text{and } \omega(\theta)\circ\omega(\eta)(t) &= (\omega(\theta^P)\circ\omega(\eta^P)(t), \omega(\theta^N)\circ\omega(\eta^N)(t)). \end{aligned}$$

Consider $\omega(\theta^P \circ \eta^P)(t)$.

$$\begin{aligned}
&= \max\{(\theta^P \circ \eta^P)(k) : k \in R, t = \omega(k)\} \\
&= \max\{\max\{\min\{\theta^P(k_1), \eta^P(k_2) : k_1, k_2 \in R, k = k_1k_2\} k \in R, t = \omega(k)\}\} \\
&= \max\{\max\{\min\{\theta^P(k_1), \eta^P(k_2) : k_1, k_2 \in R, t_1 = \omega(k_1), t_2 = \omega(k_2)\}, t_1, t_2 \in S, \\
&\quad t = t_1t_2\}\} \\
&= \max\{\min\{\max\{\theta^P(k_1) : k_1 \in R, t_1 = \omega(k_1)\}, \max\{\eta^P(k_2) : k_2 \in R : k_1, k_2 \in R, \\
&\quad t_2 = \omega(k_2)\}\}\}. \\
&= \max\{\min\{\omega(\theta^P)(t_1), \omega(\eta^P)(t_2)\} : t_1, t_2 \in S, t = t_1t_2\} = \omega(\theta^P)\circ\omega(\eta^P)(t).
\end{aligned}$$

Furthermore, we have $\omega(\theta^N \circ \eta^N)(t)$.

$$\begin{aligned}
&= \min\{(\theta^N \circ \eta^N)(k) : k \in R, t = \omega(k)\} \\
&= \min\{\min\{\max\{\theta^N(k_1), \eta^N(k_2) : k_1, k_2 \in R, k = k_1k_2\} k \in R, t = \omega(k)\}\} \\
&= \min\{\min\{\max\{\theta^N(k_1), \eta^N(k_2) : k_1, k_2 \in R, t_1 = \omega(k_1), t_2 = \omega(k_2)\}, t_1, t_2 \in S, \\
&\quad t = t_1t_2\}\} \\
&= \min\{\max\{\min\{\theta^N(k_1) : k_1 \in R, t_1 = \omega(k_1)\}, \min\{\eta^N(k_2) : k_2 \in R : k_1, k_2 \in R, \\
&\quad t_2 = \omega(k_2)\}\}\}. \\
&= \min\{\max\{\omega(\theta^N)(t_1), \omega(\eta^N)(t_2)\} : t_1, t_2 \in S, t = t_1t_2\} \\
&= (\omega(\theta^N)\circ\omega(\eta^N))(t).
\end{aligned}$$

Consequently, $\omega(\theta \circ \eta) = \omega(\theta)\circ\omega(\eta)$. □

Theorem 31. *Let $\pi : R \to R/L$ be ring homomorphism from R onto R/L, where L is an ideal of ring R. Let θ and θ_ρ be a BFSRs of R and R/L, respectively. Then, π is a BFH from θ onto θ_ρ.*

Proof. Since π is a homomorphism from R onto R/L described by the rule $\pi(a) = a+L$, for any $a \in R$. We have $\pi(\theta)(\mathrm{a}+\mathrm{L}) = (\pi(\theta^P)(\mathrm{a}+\mathrm{L}), \pi(\theta^N)(\mathrm{a}+\mathrm{L}))$. Where

$\pi(\theta^P(\mathrm{a}+\mathrm{L}) = \max\{\theta^P(u) : u \in \pi^{-1}(\mathrm{a}+\mathrm{L})\}$ and $\pi(\theta^N(\mathrm{a}+\mathrm{L}) = \min\{\theta^N(u) : u \in \pi^{-1}(\mathrm{a}+\mathrm{L})\}$. Consider the following case:

$$\begin{aligned}\pi(\theta^P(\mathrm{a}+\mathrm{L}) =& \max\{\theta^P(u) : u \in \pi^{-1}(\mathrm{a}+\mathrm{L})\}\\ =& \max\{\theta^P(u) : \pi(u) = \mathrm{a}+\mathrm{L}\}\\ =& \max\{\theta^P(u) : u + L = \mathrm{a}+\mathrm{L}\}\\ =& \max\{\theta^P(u) : u = a + n, t \in L\}\\ =& \max\{\theta^P(a+n) : t \in L\}\\ =& \theta^P_\rho(a+L).\end{aligned}$$

which implies that $\pi(\theta^P) = \theta^P_\rho$.

Moreover, we have the following:

$$\begin{aligned}\pi(\theta^N)(\mathrm{a}+\mathrm{L}) =& \min\{\theta^N(u) : u \in \pi^{-1}(\mathrm{a}+\mathrm{L})\}\\ =& \min\{\theta^N(u) : \pi(u) = \mathrm{a}+\mathrm{L}\}\\ =& \min\{\theta^N(u) : u + I = \mathrm{a}+\mathrm{L}\}\\ =& \min\{\theta^N(u) : u = a + t, n \in L\}\\ =& \max\{\theta^N(a+t) : t \in L\}\\ =& \theta^N_\rho(a+L).\end{aligned}$$

which implies that $\pi(\theta^N) = \theta^N_\rho$. Therefore, $\pi(\theta) = \theta_\rho$. Hence, the above is proved. □

We explain the above algebraic fact of BFH in the next example.

Example 32. *Consider the factor ring $Z/2Z = \{2Z, 1+2Z\}$, where $R = Z$ is a ring of integer and $S = 2Z = \{2k | k \in Z\}$ is an ideal of ring Z. Then BFS of Z is defined as follows:*

$$\theta^P(k) = \begin{cases} 0.7, & \text{if} \quad k \in 2Z, \\ 0.4, & \text{if} \quad k \in 1+2Z. \end{cases}$$

and the following.

$$\theta^N(k) = \begin{cases} -0.8, & \text{if} \quad k \in 2Z, \\ -0.5, & \text{if} \quad k \in 1+2Z. \end{cases}$$

Define BFSR θ_ρ of $Z/2Z$ as follows:

$$\theta^P_\rho(t) = \begin{cases} 0.7, & \text{if} \quad t = 2Z, \\ 0.4, & \text{if} \quad t = 1+2Z. \end{cases}$$

and the following.

$$\theta^N_\rho(t) = \begin{cases} -0.8, & \text{if} \quad t = 2Z, \\ -0.5, & \text{if} \quad t = 1+2Z. \end{cases}$$

The natural homomorphism π from Z to $Z/2Z$ is described as the following: $\pi(k) = k + 2Z$, for all $k \in Z$. This implies that $\pi(\theta^P)(2Z) = max\{\theta^P(k) : k \in 2Z\}$, implies that $\pi(\theta^P)(2Z) = 0.7$ and $\pi(\theta^N)(2Z) = min\{\theta^N(k) : k \in 2Z\ \}$, and implies that $\pi(\theta^N)(2Z) = -0.8$. Moreover, $\pi(\theta^P)(1+2Z) = max\{\theta^P(k) : k \in 1+2Z\ \}$, which implies that $\pi(\theta^P)(1+2Z) = 0.4$ and $\pi(\theta^N)(1+2Z) = min\{\theta^N(k) : k \in 1+2Z\ \}$, and implies that $\pi(\theta^N)(1+2Z) = -0.5$. Thus, $\pi(\theta) = \theta_\rho$.

Theorem 33. *Let θ and η be BFSR of rings R and R', respectively, and ω be a BFH from θ onto η. Then a mapping $\varphi : R/L \to R'$ is a BFH from θ_ρ onto η, where θ_ρ is a BFSR of R/L.*

Proof. Since $\omega(\theta) = \eta$. In addition, we have φ is homomorphism from R/L onto $R^{'}$ defined by the rule $\varphi(k+L) = \omega(k) = t,\ \forall\ k \in R$. The image of θ_ρ under the function φ may be described as the following.

$$\varphi(\theta_\rho)(t) = (\varphi(\theta_\rho^P)(t), \varphi(\theta_\rho^N)(t)), \text{for all } t \in R^{'}$$

Now, we have the following.

$$\begin{aligned}
\varphi(\theta_\rho^P)(t) =& \max\{\theta_\rho^P(k+L) : k+L \in \varphi^{-1}(t), t \in R'\} \\
=& \max\{\theta_\rho^P(k+L) : \varphi(k+L) = t, t \in R'\} \\
=& \max\{\theta_\rho^P(k+t) : t \in L, \omega(k) = t\} \\
=& \max\{\theta_\rho^P(u) : u \in \omega^{-1}(t)\} \\
=& \omega(\theta^P)(t) \\
=& \eta^P(t).
\end{aligned}$$

This Implies that $\varphi(\theta_\rho^P)(t) = \eta^P(t)\ \forall\ t \in R'$, which implies that $\varphi(\theta_\rho) = \eta$. Moreover, we have the following.

$$\begin{aligned}
\varphi(\theta_\rho^N)(t) =& \max\{\theta_\rho^N(k+L) : k+L \in \varphi^{-1}(t), t \in R'\} \\
=& \max\{\theta_\rho^N(k+L) : \varphi(k+L) = t, t \in R'\} \\
=& \max\{\theta_\rho^N(k+t) : t \in L, \omega(k) = t\} \\
=& \max\{\theta_\rho^N(u) : u \in \omega^{-1}(t)\} \\
=& \omega(\eta^N)(t) \\
=& \eta^N(t).
\end{aligned}$$

Thus, $\varphi(\theta_\rho) = \eta$. This establishes the proof. □

Example 34. *Consider the rings* $Z = \{0, \pm 1, \pm 2, ...\}$ *and also* $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ *is the ring of integers modulo* 4*. Define a homomorphism from Z onto* Z_4 *as follows.* $\omega(k) = k \ (mod\ 4)$ *the BFS of Z is given as the following:*

$$\theta^P(k) = \begin{cases} 0.6, & if \quad k \in 2Z, \\ 0.5, & if \quad k \notin 2Z. \end{cases}$$

and the following.

$$\theta^N(k) = \begin{cases} -0.7, & if \quad k \in 2Z, \\ -0.4, & if \quad k \notin 2Z. \end{cases}$$

The BFS η *of* Z_4 *is given as follows:*

$$\eta^P(k) = \begin{cases} 0.6, & if \quad k \in 2Z_4, \\ 0.5, & if \quad k \notin 2Z_4. \end{cases}$$

and also given as follows.

$$\eta^N(k) = \begin{cases} -0.7, & if \quad k \in 2Z_4, \\ -0.4, & if \quad k \notin 2Z_4. \end{cases}$$

Consider the following case.

$$\begin{aligned}
\omega(\theta^P)(0) &= max\{\theta^P(u) : u \in 4Z\} = 0.6, \\
\omega(\theta^P)(1) &= max\{\theta^P(u) : u \in 1 + 4Z\} = 0.5
\end{aligned}$$

Similarly, $\omega(\theta^P)(2) = 0.6$ *and* $\omega(\theta^P)(3) = 0.5$. *Moreover, we have the following.*

$$\begin{aligned}\omega(\theta^N)(0) &= min\{\theta^N(u) : u \in 4Z\} = -0.7,\\ \omega(\theta^N)(1) &= min\{\theta^N(u) : u \in 1+4Z\} = -0.4,\\ \omega(\theta^N)(2) &= -0.7,\ \omega(\theta^N)(3) = -0.4.\end{aligned}$$

Thus, $\omega(\theta) = \eta$. *The quotient ring of* $Z = \{0, \pm1, \pm2, ...\}$ *is given by* $Z/4Z = \{4Z, 1+4Z, 2+4Z, 3+4Z\}$ *where* $4Z$ *is an ideal of the ring of integers* Z. *Define BFS* θ_ρ *of* $Z/4Z$ *as follows:*

$$\theta_\rho^P(u) = \begin{cases} 0.6: & u \in \{4Z, 2+4Z\} \\ 0.5: & u \in \{1+4Z, 3+4Z\}. \end{cases}$$

and also as follows.

$$\theta_\rho^N(u) = \begin{cases} -0.7: & u \in \{4Z, 2+4Z\} \\ -0.4: & u \in \{1+4Z, 3+4Z\}. \end{cases}$$

Define a mapping ϕ *from* $Z/4Z$ *onto* Z_4 *as follows* $\phi(k+4Z) = \omega(k) = k \ (mod\ 4)$, *for all* $k \in Z$.

From the above information, we have the following.

$$\phi(\theta_\rho^P)(0) = max\{\theta_\rho^P(k+4Z) : k+4Z \in \phi^{-1}(0), \quad k \in Z\} = 0.6,\ and$$

$$\begin{aligned}&\phi(\theta_\rho^N)(0) = min\{\theta_\rho^N(k+4Z) : k+4Z \in \phi^{-1}(0), \qquad k \in Z\} = -0.7,\\ &\phi(\theta_\rho^P)(1) = 0.5 = \phi(\theta_\rho^P)(3),\ \phi(\theta_\rho^P)(2) = 0.6\\ &and\ \phi(\theta_\rho^N)(1) = \phi(\theta_\rho^N)(3) = -0.4, \phi(\theta_\rho^N)(2) = -0.7.\end{aligned}$$

Therefore, $\phi(\theta_\rho) = \eta$.

Remark 35. *Let* θ *and* η *are BFSRs of rings* R *and* R', *respectively, and* f *be a BFH from* θ *onto* η *with* $S = \{k \in R, \omega(k) = 0_{R'}\}$ *as a kernel of* f. *Then the mapping* φ *from* R/S *to* R' *is a BFH from* θ^S *onto* η, *where* θ^S *is a BFSR of* R/S.

In following result, we develop an important link between BFSRs of a ring R' and any of its factor ring.

Theorem 36. *Let* θ *and* η *be BFSRs of rings* R *and* R', *respectively. Let* ω *be a BFH from* θ *onto* η *and the natural homomorphism* π *from* R' *onto* R'/L' *be a BFH from* η *onto* $\theta_{\rho'}$, *where* $\theta_{\rho'}$ *is a BFSR of* R'/L'. *Then* $\phi = \pi \circ \omega$ *is a BFH from* θ *onto* $\theta_{\rho'}$, *where* L *is a ideal of* R *with* $\omega(L) = L'$.

Proof. Since π is natural homomorphism from R onto R'/L'.

For any $a' + L' \in R'/L'$, we have $(\pi \circ \omega)(\theta)(a' + L') = ((\pi \circ \omega)(\theta^P)(a' + L'), (\pi \circ \omega)(\theta^N)(a' + L'))$, where the following is the case:

$$(\pi \circ \omega)(\theta^P)(a' + L') = \max\{\theta^P(u) : u \in (\pi \circ \omega)^{-1}(a' + L')\}$$

and the following is also the case.

$$(\pi \circ \omega)(\theta^N)(a' + L') = \min\{\theta^N(u) : u \in (\pi \circ \omega)^{-1}(a' + L')\}.$$

Consider the following.

$$\begin{aligned}
(\pi \circ \omega)(\theta^P)(\mathrm{a}' + L') &= \max\{\theta^P(u) : u \in (\pi \circ \omega)^{-1}(\mathrm{a}' + L')\} \\
&= \max\{\theta^P(u) : u \in \omega^{-1}(\pi^{-1}(\mathrm{a}' + L'))\} \\
&= \omega(\theta^P)(\pi^{-1}(\mathrm{a}' + L')) \\
&= \eta^P(\pi^{-1}(\mathrm{a}' + L')) \\
&= (\pi^{-1})^{-1}(\eta^P)(\mathrm{a}' + L') \\
&= \pi(\eta^P)(\mathrm{a}' + L') = \theta^P_{\rho'}(\mathrm{a}' + L'). \\
&= \theta^P_{\rho'}(\mathrm{a}' + L'). \\
&\Rightarrow \phi(\theta^P) = \theta^P_{\rho'}.
\end{aligned}$$

Moreover, the following is also the case.

$$\begin{aligned}
(\pi \circ \omega)(\theta^N)(\mathrm{a}' + L') &= \min\{\theta^N(u) : u \in (\pi \circ \omega)^{-1}(\mathrm{a}' + L')\} \\
&= \min\{\theta^N(u) : u \in \omega^{-1}(\pi^{-1}(\mathrm{a}' + I'))\} \\
&= {}_{\omega(\theta^N)}(\pi^{-1}(\mathrm{a}' + L')). \\
&= \eta^N(\pi^{-1}(\mathrm{a}' + L')) \\
&= (\pi^{-1})^{-1}(\eta^N)(\mathrm{a}' + L') \\
&= \pi(\eta^N)(\mathrm{a}' + L') \\
&= \eta^N_{\rho'}(\mathrm{a}' + L').
\end{aligned}$$

This implies that $(\pi \circ \omega)(\theta^N)(\mathrm{a}' + L') = \eta^N_{\rho'}(\mathrm{a}' + L')$, which implies that $\phi(\eta^N) = \eta^N_{\rho'}$. Hence, we have proved our claim. □

Theorem 37. *Let θ and η are BFSR of R and R', respectively, and ω be a BFH from θ onto η. Let $\pi : R' \to R'/\omega'$ be a natural homomorphism and $L = \{k \in R : \omega(k) \in L'\}$. Then, a mapping $\sigma : R/L \to R'/L'$ is a BFH from θ_ρ onto $\theta_{\rho'}$ where θ_ρ and $\theta_{\rho'}$ are BFSR of R/L and R'/L', respectively.*

Proof. From Theorem 36, we define a mapping $\phi : R \to R'/L'$ such that ϕ is a composition of mapping ω and π such that $\phi(\theta) = (\pi \circ \omega)(\theta) = a' + L', \forall\, a' \in R'$. Moreover, $\phi(L) = (\pi \circ \omega)(L) = \pi(\omega(L)) = \pi(L') = L'$. Consider the BFSR θ_ρ of R/L as: $\theta_\rho(k + L) = (\theta^P_\rho(k + L), \theta^N_\rho(k + L)))$, where we have the following.

$$\theta^P_\rho(k + L) = \max\{\theta^P(u) : u \in k + L\}$$

and the following is also the case.

$$\theta^N_\rho(k + L) = \min\{\theta^N(u) : u \in k + L\}.$$

This prove that ϕ is a BFH with $ker(\phi) = L$. Define a mapping σ from R/L to R'/L' as follow $\sigma(a + L) = a' + L', a \in R, a' \in R'$. Where $\sigma(\theta_\rho) \to \theta_{\rho'}$ is described by the rule:

$$\sigma(\theta_\rho)(\mathrm{a}' + L') = (\sigma(\theta^P_\rho)(\mathrm{a}' + L'), \sigma(\theta^N_\rho)(\mathrm{a}' + L')),$$

where

$$\sigma(\theta^P_\rho)(\mathrm{a}' + L') = \max\{\theta^P_\rho(k + L) : k + L \in \sigma^{-1}(\mathrm{a}' + L')\}$$

and

$$\sigma(\theta^N_\rho)(\mathrm{a}' + L') = \min\{\theta^N_\rho(k + L) : k + L \in \sigma^{-1}(\mathrm{a}' + L')\}.$$

Consider the following.

$$\begin{aligned}
\sigma(\theta_\rho^P)(\mathrm{a}'+L') =& \max\{\theta_\rho^P(\mathrm{a}+L) : \mathrm{a}+L \in \sigma^{-1}(\mathrm{a}'+L')\}\\
=& \max\{\max\{\theta^P(\mathrm{a}+p) : p \in L, \mathrm{a} \in R, \omega(\mathrm{a})\} = \mathrm{a}', \sigma(\mathrm{a}+L) = \omega(\mathrm{a}) + L'\}\\
=& \max\{\theta^P(\mathrm{a}+p) : p \in L, \mathrm{a} \in R, \omega(\mathrm{a}) = \mathrm{a}'\}\\
=& \max\{\theta^P(u) : u \in \omega^{-1}(\mathrm{a}')\}\\
=& \omega(\theta^P)(\mathrm{a}')\\
=& \eta^P(\mathrm{a}'), \quad \mathrm{a}' \in R'\\
=& \max\{\eta^P(\mathrm{a}') : \pi(\mathrm{a}') = \mathrm{a}' + L'\}\\
=& \max\{\eta^P(\mathrm{a}') : \mathrm{a}' \in \pi^{-1}(\mathrm{a}'+L')\}\\
=& \pi(\eta^P)(\mathrm{a}'+L')\\
=& \theta_{\rho'}^P(\mathrm{a}'+L')
\end{aligned}$$

This implies that $\sigma(\theta_\rho^P) = \theta_{\rho'}^P$. Moreover, we have the following.

$$\begin{aligned}
\sigma(A_\rho^N)(\mathrm{a}'+L') =& \min\{\theta_\rho^N(\mathrm{a}+L) : \mathrm{a}+L \in \sigma^{-1}(\mathrm{a}'+L')\}\\
=& \min\{\min\{\theta^N(\mathrm{a}+p) : p \in L, \mathrm{a} \in R, \omega(\mathrm{ta})\} = \mathrm{a}', \sigma(\mathrm{a}+L) = \omega(\mathrm{a}) + L'\}\\
=& \min\{\theta^N(\mathrm{a}+p) : p \in L, \mathrm{a} \in R, \omega(\mathrm{a}) = \mathrm{a}'\}\\
=& \min\{\theta^N(u) : u \in \omega^{-1}(\mathrm{a}')\}\\
=& \omega(\theta^N)(\mathrm{a}')\\
=& \eta^N(\mathrm{a}'), \quad \mathrm{a}' \in R'\\
=& \min\{\eta^N(\mathrm{a}') : \pi(\mathrm{a}') = \mathrm{a}' + L'\}\\
=& \min\{\eta^N(\mathrm{a}') : \mathrm{a}' \in \pi^{-1}(\mathrm{a}'+L)\}\\
=& \pi(\eta^N)(\mathrm{a}'+L')\\
=& \eta_{\rho'}^N(\mathrm{a}'+L').
\end{aligned}$$

And this mplies that $\sigma(\theta_\rho^N) = \theta_{\rho'}^N$. Hence, $\sigma(A_\rho^N) = \eta_{\rho'}^N$. □

Remark 38. *There are possible applications of bipolar fuzzy homomorphism. For example, a bipolar fuzzy homomorphism is used in the positioning of the image. A photograph of a person is in fact his homomorphic image that explains his many real qualities such as being tall or short, male or female, and thin or heavy. Sometimes, the homomorphic image become destroyed due to many distortions in the lenses such as scale and pincushion distortion. A distortion in which magnification increases with distances from the axis is called a pincushion distortion. We can apply a bipolar fuzzy homomorphism on a destroyed photograph to remove scale and pincushion distortion in order to obtain its original form.*

Lemma 39. *Let θ and η be any two BFSRs of bipolar rings R and R', respectively, and ω be a epimorphism from R to R' such that $\omega(\theta) = \eta$, where θ and η are BFSRs of R and R', respectively. Then $\omega(\theta_*) = \eta_*$.*

Proof. Given that $\omega(\theta) = \eta$. Let $p \in \omega(\theta) \Rightarrow p = \omega(a)$, for some $a \in \theta_*$. Consider, $\omega(\theta^P)(p) = \max\{\theta^P(a), a \in \omega^{-1}(p)\} \geq \theta^P(a) > 0$ and $\omega(\theta^N)(p) = \min\{\theta^N(a), a \in \omega^{-1}(p)\} \leq \theta^N(a) < 0$. Therefore, $p \in \eta_*$. Thus, $\omega(\theta_*) \subseteq \eta_*$. Moreover, from fact of Definition 29, the epimorphism f develops $\omega(\theta_*) \supseteq \eta_*$. This establishes the proof. □

Theorem 40. *(First Bipolar Fuzzy Isomorphism Theorem): Let θ and η be BFSRs of rings R and R', respectively, and h be a BFH from θ onto η, where $\ker h = T_1$ is kernel of fuzzy homomorphism. Then $\theta/\psi \approx \eta$, where ψ is a BFI of θ.*

Proof. Given that h is a BFH from θ to η. Consider BFSR ψ of R as follows:

$$\psi^P(k) = \begin{cases} \theta^P(k) & if\ k \in T_1 \\ 0 & if\ k \notin T_1 \end{cases}$$

and the following is also the case.

$$\psi^N(k) = \begin{cases} \theta^N(k) & if\ k \in T_1 \\ 1 & if\ k \notin T_1 \end{cases}$$

Obviously, $\psi \subseteq \theta$. Moreover, for any $k \in T_1 and\ t \in R$.

$$\begin{aligned} \text{Consider } \psi^P(k+t) =& \theta^P(k+t) \geq \min\{\theta^P(k), \theta^P(t)\} \\ \geq& \min\{\psi^P(k), \theta^P(t)\}. \end{aligned}$$

$$\text{Similarly } \psi^N(k+t) \leq \max\{\psi^N(k), \theta^N(t)\}.$$

If $m \notin M_1$, then $\psi^P(k) = 0$ and $\psi^N(k) = 1$. This show that ψ is a bipolar fuzzy ideal of θ. Since $\theta \approx \eta \Rightarrow h(\theta) = \eta$. In view of Lemma 39, $h(\theta_*) = \eta_*$. Let $\chi = h'_{\theta_*}$ then $\chi : \theta_* \to \eta_*$ is a homomorphism with kernal $\chi = \psi$. Then there exists an isomorphism χ from θ_*/ψ_* to η_* that can be described as $\chi(k+\psi_*) = z = \chi(k) = h(k))\ \forall\ k \in \theta_*$. We have $\chi(\theta/\psi)(z) = (\chi(\theta^P/\psi^P)(z), \chi(\eta^N/\psi^N)(z))$. Consider the following.

$$\begin{aligned} \chi(\theta^P/\psi^P)(z) =& \max\{(\theta^P/\psi^P)(k+\psi_*) : k \in \theta_*, \chi(k+\psi_*) = z\} \\ =& \max\{(\theta^P/\psi^P)(q), q \in k+\psi_* : k \in \theta_*, \chi(q) = z\} \\ =& \max\{(\theta^P/\psi^P)(q) : q \in \theta_*, \chi(q) = z\} \\ =& \max\{(\theta^P/\psi^P)(q) : q \in R, h(q) = z\} \\ =& h(\theta^P)(z) \\ =& \eta^P(z), \forall\ z \in \eta_*. \end{aligned}$$

This implies that $\chi(\theta^P/\psi^P) = \eta^P$ and the following is the case.

$$\begin{aligned} \chi(\theta^N/\psi^N)(z) =& \min\{(\theta^N/\psi^N)(k+\psi_*) : k \in \theta_*, \chi(k+\psi_*) = z\} \\ =& \min\{(\theta^N/\psi^N)(q), q \in k+\psi_* : k \in \theta_*, \chi(q) = z\} \\ =& \max\{(\theta^N/\psi^N)(q) : q \in \theta_*, \chi(q) = z\} \\ =& \min\{(\theta^N/\psi^N)(q) : q \in R, h(q) = z\} \\ =& h(\eta^N)(z) \\ =& \eta^N(z), \forall\ z \in \eta_*. \end{aligned}$$

This implies that $\chi(\theta^N/\psi^N) = \eta^N$. Thus, $\chi(\theta/\psi) = \eta$. Hence, $(\theta/\psi) \approx \eta$. □

Theorem 41. *(Second Bipolar Fuzzy Isomorphism Theorem) Let θ be a BFI and η be a BFSR of a ring R such that $\theta \subseteq \eta$. Then $\eta/(\theta \cap \eta) \subseteq (\theta + \eta)/\theta$.*

Proof. From Remark 27 and the fact $\theta \subseteq \eta$, one can obtain that quotient rings $\eta_*/(\theta_* \cap \eta_*)$ and $(\theta_* + \eta_*)/\theta_*$. Therefore, by applying the second fundamental theorem of classical ring isomorphism on these specific factor rings, we obtain the following.

$$\eta_*/(\theta_* \cap \eta_*) \cong (\theta_* + \eta_*)/\theta_*$$

The above stated result leads us to obtain the existence of a ring isomorphism h from $\eta_*/((\theta \cap \eta))_*$ to $(\theta_* + \eta_*)/\theta_*$, which can be described as follows.

$$h(k + (\theta \cap \eta)^*) = k + \theta_*, \quad \forall k \in \eta_*.$$

Consider $h(\eta^P/(\theta^P \cap \eta^P))(k + \theta_*)$.

$$\begin{aligned}
&=(\eta^P/(\theta^P \cap \eta^P))(k + (\theta \cap \eta)_*)\\
&=\max\{\eta^P(z) : z \in (k + (\theta \cap \eta)_*)\}\\
&\leq\max\{(\theta^P + \eta^P)(z) : z \in (k + (\theta \cap \eta)_*)\}\\
&\leq\max\{(\theta^P + \eta^P)(z) : z \in k + \theta_*\}\\
&=((\theta^P + \eta^P)/\theta^P)(k + \theta_*), \forall\, k \in \eta_*.
\end{aligned}$$

This implies that $h(\eta^P/(\theta^P \cap \eta^P))(k + \theta_*) \leq ((\theta^P + \eta^P)/\theta^P)(k + \theta_*), \forall\, k \in \eta_*$. Moreover, $h(\eta^N/(\theta^N \cap \eta^N))(k + \theta_*)$.

$$\begin{aligned}
&=(\eta^P/(\theta^P \cap \eta^P))(k + (\theta \cap \eta)_*)\\
&=\min\{\eta^N(z) : z \in (k + (\theta \cap \eta)_*)\}\\
&\geq\min\{(\theta^N + \eta^N)(z) : z \in (k + (\theta \cap \eta)_*)\}\\
&\geq\min\{(\theta^N + \eta^N)(z) : z \in k + \theta_*\}\\
&=((\theta^N + \eta^N/(\theta^N \cap \eta^N))(k + \theta_*), \forall\, k \in \eta_*.
\end{aligned}$$

This implies that $h(\eta^N/(\theta^N \cap \eta^N))(k+\theta_*) \geq ((\theta^N + \eta^N)/\eta^N)(k+\theta_*), \forall\, k \in \eta_*$. Thus, $h(\eta/(\theta \cap \eta)) \subseteq (\theta + \eta)/\theta$. As result, we obtain a weak bipolar fuzzy isomorphism between $(\eta/(\theta \cap \eta))$ and $(\theta + \eta)/\theta$. □

Theorem 42. *(Third Bipolar Fuzzy Isomorphism Theorem): Let θ, η, and ψ be BFSRs of R such that θ and η are BFIs of ψ with $\theta \subseteq \eta$. Then, $(\psi/\theta)/(\eta/\theta) \cong (\psi/\eta)$.*

Proof. From Remark 27, and the fact that θ and η are BFI of ψ with $\theta \subseteq \eta$ and one can obtain the quotient rings $(\psi_*/\theta_*)/(\eta_*/\theta_*)$ and (ψ_*/η_*). Therefore, by applying the third fundamental theorem of classical ring isomorphism on these specific factor rings, we obtain the following.

$$(\psi_*/\theta_*)/(\eta_*/\theta_*) \cong (\psi_*/\eta_*)$$

The above stated results leads us to obtain the existence of a ring isomorphism R from $(\psi_*/\theta_*)/(\eta_*/\theta_*)$ to (ψ_*/η_*), which may be described as follows.

$$h(k + \theta_* + (\eta_*/\theta_*)) = k + \eta_*, \quad \forall\, k \in \psi_*.$$

Consider $h((\psi^P/\theta^P)/(\eta^P/\theta^P))(k+\eta_*)$.

$$
\begin{aligned}
&=((\psi^P/\theta^P)/(\eta^P/\theta^P))(k+\theta_*+(\eta_*/\theta_*))\\
&=\max\{(\psi^P/\theta^P)(t+\theta_*) : t\in\psi_*, t+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\max\{\max\{\psi^P(z) : z\in t+\theta_*\} : t\in\psi_*, t+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\max\{\psi^P(z) : z\in\psi_*, z+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\max\{\psi^P(z) : z\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\max\{\psi^P(z) : z\in\psi_*, h(z)\in k+\eta_*\}\\
&=(\psi^P/\eta^P)(k+\eta_*), \forall\, k\in\psi_*.
\end{aligned}
$$

This implies that $h((\psi^P/\theta^P)/(\eta^P/\theta^P))(k+\eta_*)=(\psi^P/\eta^P)(k+\eta_*), \forall\, k\in\psi_*$.
Moreover, $h((\psi^N/\theta^N)/(\eta^N/\theta^N))(k+\eta_*)$.

$$
\begin{aligned}
&=((\psi^N/\theta^N)/(\eta^N/\theta^N))(k+\theta_*+(\eta_*/\theta_*))\\
&=\min\{(\psi^N/\theta^N)(t+\theta_*) : t\in\psi_*, t+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\min\{\min\{\psi^N(z) : z\in t+\theta_*\} : t\in\psi_*, t+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\min\{\psi^N(z) : z\in\psi_*, z+\theta_*\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\min\{\psi^N(z) : z\in(k+\theta_*+(\eta_*/\theta_*))\}\\
&=\min\{\psi^N(z) : z\in\psi_*, h(z)\in k+\eta_*\}\\
&=(\psi^N/\eta^N)(k+\eta_*), \forall\, k\in\psi_*.
\end{aligned}
$$

This implies that $h((\psi^N/\theta^N)/(\eta^N/\theta^N))(k+\eta_*)=(\psi^N/\eta^N)(k+\eta_*), \forall\, k\in\psi_*$.
Thus, $h(\psi/\theta)/(\eta/\theta)=(\psi/\eta)$.
$(\psi/\theta)/(\eta/\theta)\cong(\psi/\eta)$. □

5. Conclusions

The concept of BFS is a convenient extrapolation of conventional fuzzy sets which evaluates the uncertainty and ambiguity of a fuzzy fact in a more effective manner. In this paper, we have explained (α,β)-cut of a BFS and have demonstrated that (α,β)-cut of a BFSR works as a subring of a given ring. We have developed the BFSR of quotient ring and have proved that product of two BFSRs is a subring. We have developed the BFH between any two bipolar fuzzy subrings, which is actually an important generalization of the natural ring homomorphism. The fundamental theorems of BFI of BFSRs have been developed. Potential future research will be the applications of these algebraic structures to solve certain decision-making problems in order to provide a significant addition to current existing theories for handling uncertainties, especially in the area of bioinformatics, medical imaging, and decision making [48–52].

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