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Numerical Solution of Two-Dimensional Fredholm–Volterra Integral Equations of the Second Kind

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Abstract: The paper presents an iterative numerical method for approximating solutions of two-dimensional Fredholm–Volterra integral equations of the second kind. As these equations arise in many applications, there is a constant need for accurate, but fast and simple to use numerical approximations to their solutions. The method proposed here uses successive approximations of the Mann type and a suitable cubature formula. Mann's procedure is known to converge faster than the classical Picard iteration given by the contraction principle, thus yielding a better numerical method. The existence and uniqueness of the solution is derived under certain conditions. The convergence of the method is proved, and error estimates for the approximations obtained are given. At the end, several numerical examples are analyzed, showing the applicability of the proposed method and good approximation results. In the last section, concluding remarks and future research ideas are discussed.

Keywords: Fredholm–Volterra integral equations; fixed-point theorems; numerical approximations

MSC: 45H05; 47H10; 47H09; 65D32

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1. Preliminaries

Fredholm–Volterra equations are integral equations of the following type:

$$u(t,x) = \int_0^t \int_\Omega K(t,x,\tau,y,u(\tau,y))\,dy\,d\tau + f(t,x),$$

for $(t,x) \in [0,T] \times \Omega$, Ω a closed subset of $\mathbb{R}^n$, $n = 1,\ 2,\ 3$.

One encounters these equations in many applications in areas of physics, engineering or biology. In addition, many reformulations of boundary value problems can be written as Volterra–Fredholm integral equations. They are also used to model the progress of an epidemic and various other biological and physical problems. Integral equations with symmetric kernels are of frequent occurrence in the formulation of electronic and optic problems, as well as in optimization and spectral analysis.

In this paper, we consider mixed Fredholm–Volterra integral equations of the following form:

$$u(t,x) = \int_0^t \int_a^b K(t,x,\tau,y,u(\tau,y))\,dy\,d\tau + f(t,x), \tag{1}$$

$(t,x) \in D = [0,T] \times [a,b]$, where $K \in C(D \times D \times \mathbb{R})$ and $f \in C(D)$.

Given the wide variety of applications, there have been substantial works on the solvability of these equations and on studying their properties. Numerical approximations of their solutions have been studied via collocation methods [1–3], block-pulse functions [4,5], Adomian decomposition methods [6], wavelet-based methods [7–9], iterative methods [10–15], differential quadratures [16], meshless procedures [17], etc. A simplified, one-dimensional case was studied in [18]. More details and considerations can be found, for example, in [19–22].

The aim of the present work is to develop a simple but quite accurate numerical method for approximating the solution of such equations. We derive a method based on fixed point theory for the existence and uniqueness of the solution, and on the use of an appropriate cubature formula for the numerical approximation. As such, the advantage of this new method consists mainly in the fact that it is easy to use and implement but gives good approximations of the solution at a given set of nodes. Compared to other classical methods used for integral equations, such as projection, Nyström or decomposition methods, this procedure does not require solving in the end an algebraic system for the values of the unknown function at the grid points. Such systems can be ill-conditioned, and may require additional procedures, which increase the computational and implementation cost of the resulting method, while decreasing its area of applicability. Instead, the proposed scheme finds the approximations at the nodes iteratively, using previously found values.

The rest of the paper is organized as follows: in Section 2, we discuss the solvability of Equation (1), via fixed point theory. Altman's algorithm [23] is employed instead of the classical Banach's theorem. This uses a Mann-type iteration (see [24]), which, by means of some parameters (the sequences ε_n and y_n, respectively, from Theorem 1 below), allows better control over the speed of convergence. With an appropriate choice of those parameters, we obtain faster successive approximations than the ones provided by the Picard-type iteration. In Section 3, we present a numerical method for approximating the solution of Equation (1), using a suitable cubature formula. Then, we analyze the convergence and give error estimates for the case when the two-dimensional trapezium rule is used for the numerical approximation of the iterates. In Section 4 we apply the proposed method to several numerical examples that are discussed in detail, showing good agreement between the theoretical results and the practical ones. Section 5 contains the concluding remarks on the procedure presented, and a discussion of ideas for future research in this area.

2. Solvability of the Integral Equation

We analyze the solvability of Equation (1) via fixed point results. To this end, we define the integral operator $F : C(D) \to C(D)$ associated with Equation (1) by the following:

$$Fu(t,x) \quad := \quad \int_0^t \int_a^b K(t,x,\tau,y,u(\tau,y))\, dy\, d\tau + f(t,x). \tag{2}$$

Then, we find a solution of the Equation (1) by finding a fixed point of the operator F:

$$u = Fu. \tag{3}$$

Let $X = C(D)$, endowed with the Chebyshev norm:

$$||u|| \quad := \quad \max_{(t,x)\in D} |u(t,x)|, \ \ u \in X.$$

Then, it is known that $(X, ||\cdot||)$ is a Banach space and for some $\rho > 0$, the ball $B_\rho := \{u \in C(D) \mid ||u - f|| \le \rho\} \subseteq X$ is a closed subset. The well-known contraction principle holds for $F : X \to X$. The speed of convergence can be improved by using the following result due to Mann [24], also known as Altman's algorithm [23]:

Theorem 1. *Consider $(X, ||\cdot||)$ a Banach space and $T : X \to X$ a $q-$contraction. Let $0 < \varepsilon_n \leq 1$ be a sequence of numbers satisfying the following:*

$$\sum_{n=0}^{\infty} \varepsilon_n = \infty. \tag{4}$$

Then, we have the following:

(a) *Equation $u = Tu$ has exactly one solution $u^* \in X$.*
(b) *The sequence of successive approximations*

$$u_{n+1} = (1-\varepsilon_n)u_n + \varepsilon_n T u_n, \;\; n = 0,\, 1, \ldots \tag{5}$$

converges to the solution u^*, for any $u_0 \in X$.
(c) *For every $n \in \mathbb{N}$, the following error estimate holds:*

$$||u_n - u^*|| \;\; \leq \;\; \frac{e^{1-q}}{1-q}\, e^{-(1-q)y_n}\, ||u_0 - Tu_0|| \tag{6}$$

where $y_0 = 0$, $y_n = \sum_{i=0}^{n-1} \varepsilon_i$, for $n \geq 1$.

The error estimate in Equation (6) is better than the classical error $\dfrac{q^n}{1-q}$ given by the contraction principle. We will use this result for our integral operator F with $\varepsilon_n = \dfrac{1}{n+1}$, which satisfies the requirements of Theorem 1. Then, we have the following:

Theorem 2. *Let $K \in C(D \times D \times \mathbb{R})$, $f \in C(D)$ and $\rho_1 := \min_{(t,x)\in D} f(t,x)$, $\rho_2 := \max_{(t,x)\in D} f(t,x)$. Assume the following:*

(i) *there exists a constant $L > 0$ such that*

$$\big|K(t,x,\tau,y,u) - K(t,x,\tau,y,v)\big| \;\; \leq \;\; L||u-v||, \tag{7}$$

for all $(t,x), (\tau,y) \in D$ and all $u, v \in [\rho_1 - \rho, \rho_2 + \rho]$;

(ii)

$$q := \;\; LT(b-a) < \;\; 1; \tag{8}$$

(iii)

$$M_K T(b-a) \;\; \leq \;\; \rho, \tag{9}$$

where $M_K := \max|K(t,x,\tau,y,u)|$ over all $(t,x), (\tau,y) \in D$ and all $u, v \in [\rho_1 - \rho, \rho_2 + \rho]$.

Then, the operator F in Equation (2) has exactly one fixed point, i.e., Equation (3) has exactly one solution $u^* \in B_R$, which can be obtained as the limit of the sequence of successive approximations as follows:

$$u_{n+1} \;\; = \;\; \left(1 - \frac{1}{n+1}\right)u_n + \frac{1}{n+1} F u_n, \;\; n = 0,\, 1, \ldots, \tag{10}$$

starting with any arbitrary initial point $u_0 \in B_R$. Moreover, for every $n \in \mathbb{N}$, the following error estimate holds:

$$||u_n - u^*|| \;\; \leq \;\; \frac{e^{1-q}}{1-q}\, e^{-(1-q)y_n}\, ||u_0 - Fu_0|| \tag{11}$$

where the sequence $\{y_n\}$ is defined by the following:

$$y_0 = 0, \ y_n = \sum_{i=0}^{n-1} \frac{1}{i+1}, \ n \geq 1. \tag{12}$$

Proof. Let u be any arbitrary point in B_ρ. For a fixed $(t,x) \in D$, we have the following:

$$|Fu(t,x) - f(t,x)| \leq \int_0^t \int_a^b \Big| K\big(t,x,\tau,y,u(\tau,y)\big) \Big| \, dy \, d\tau \leq M_K T(b-a).$$

Then, by Equation (9), $Fu \in B_\rho$ and, thus, $F(B_\rho) \subseteq B_\rho$. Now, for every fixed $(t,x) \in D$, we use Equation (7) to obtain the following:

$$\begin{aligned} |Fu(t,x) - Fv(t,x)| &\leq \int_0^t \int_a^b \Big| K\big(t,x,\tau,y,u(\tau,y)\big) - K\big(t,x,\tau,y,v(\tau,y)\big) \Big| \, dy \, d\tau \\ &\leq L||u-v|| \int_0^t \int_a^b dy \, d\tau \\ &\leq q||u-v||. \end{aligned}$$

Thus,

$$||Fu - Fv|| \leq q||u-v||$$

and since $q < 1$, all the conclusions follow from Theorem 1. □

Remark 1. *Let us note that the Lipschitz and contraction conditions* (7) *and* (8) *can be quite restrictive if required on the entire space. This is why we use only a local existence and uniqueness result so that these conditions need only be satisfied for $u \in B_\rho$, for some $\rho > 0$, which is much more reasonable. This observation will also be important in the next section, when we discuss the numerical approximation of the solution at the nodes (see Remark 2).*

For more considerations and details on fixed points, see [21,24].

3. A Numerical Method for Solving the Integral Equation

In order to use the iterative procedure Equation (10), we have to approximate the integrals numerically. Consider the following numerical integration scheme:

$$\int_a^b \int_c^d \varphi(s,w) \, dw \, ds = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} a_{ij} \varphi(s_i, w_j) + R_\varphi, \tag{13}$$

with nodes $a = s_0 < s_1 < \cdots < s_{m_1} = b$, $c = w_0 < w_1 < \cdots < w_{m_2} = d$, coefficients $a_{ij} \in \mathbb{R}, i = 0,1,\ldots,m_1, j = 0,1,\ldots,m_2$, such that there exists $M > 0$ with the following:

$$|R_\varphi| \leq M, \tag{14}$$

where $M \to 0$ as $m_1, m_2 \to \infty$.

For our purposes, let $0 = t_0 < t_1 < \cdots < t_{m_1} = T$ and $a = x_0 < x_1 < \cdots < x_{m_2} = b$ be partitions of $[0,T]$ and $[a,b]$, respectively, and let $u_0 = \tilde{u}_0 \equiv f$ be the initial approximation. We will use the successive iterations (10) and the numerical integration formula (13) to approximate $u_n(t_l, x_k)$ by $\tilde{u}_n(t_l, x_k)$, for $l = \overline{0, m_1}$, $k = \overline{0, m_2}$ and $n = 0, 1, \ldots$ Let $l \in \{0, 1, \ldots, m_1\}$ and $k \in \{0, 1, \ldots, m_2\}$ be fixed. The following approximations hold:

$$\begin{aligned} u_1(t_l, x_k) &= Fu_0(t_l, x_k) \\ &= \int_0^{t_l} \int_a^b K(t_l, x_k, \tau, y, f(\tau, y))\, dy\, d\tau + f(t_l, x_k) \\ &= \sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K(t_l, x_k, t_i, x_j, f(t_i, x_j)) + R_K + f(t_l, x_k) \\ &= \tilde{u}_1(t_l, x_k) + \tilde{R}_1, \end{aligned}$$

where

$$\tilde{u}_1(t_l, x_k) = \sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K(t_l, x_k, t_i, x_j, f(t_i, x_j)) + f(t_l, x_k).$$

We make the following notation for the maximum error at the nodes:

$$err(u_n, \tilde{u}_n) := \max_{(t_l, x_k) \in D} |u_n(t_l, x_k) - \tilde{u}_n(t_l, x_k)|.$$

Then, by Equation (14), we have the following:

$$err(u_1, \tilde{u}_1) \leq |\tilde{R}_1| \leq M. \tag{15}$$

We continue with the next iteration:

$$\begin{aligned} u_2(t_l, x_k) &= \left(1 - \frac{1}{2}\right) u_1(t_l, x_k) + \frac{1}{2} \left(\int_0^{t_l} \int_a^b K(t_l, x_k, \tau, y, u_1(\tau, y))\, dy\, d\tau + f(t_l, x_k) \right) \\ &= \left(1 - \frac{1}{2}\right) \tilde{u}_1(t_l, x_k) + \left(1 - \frac{1}{2}\right) (u_1(t_l, x_k) - \tilde{u}_1(t_l, x_k)) \\ &+ \frac{1}{2} \left(\sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K(t_l, x_k, t_i, x_j, u_1(t_i, x_j)) + R_K + f(t_l, x_k) \right) \\ &= \left(1 - \frac{1}{2}\right) \tilde{u}_1(t_l, x_k) + \left(1 - \frac{1}{2}\right) \Big(u_1(t_l, x_k) - \tilde{u}_1(t_l, x_k)\Big) \\ &+ \frac{1}{2} \left(\sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K(t_l, x_k, t_i, x_j, \tilde{u}_1(t_i, x_j)) + R_K + f(t_l, x_k) \right. \\ &+ \left. \sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K\Big(t_l, x_k, t_i, x_j, u_1(t_i, x_j) - \tilde{u}_1(t_i, x_j)\Big) \right) \\ &= \left(1 - \frac{1}{2}\right) \tilde{u}_1(t_l, x_k) + \frac{1}{2} \left(\sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K(t_l, x_k, t_i, x_j, \tilde{u}_1(t_i, x_j)) + f(t_l, x_k) \right) \\ &+ \left(1 - \frac{1}{2}\right) \Big(u_1(t_l, x_k) - \tilde{u}_1(t_l, x_k)\Big) \\ &+ \frac{1}{2} \left(\sum_{i=0}^{l} \sum_{j=0}^{m_2} a_{ij} K\Big(t_l, x_k, t_i, x_j, u_1(t_i, x_j) - \tilde{u}_1(t_i, x_j)\Big) + R_K \right) \\ &= \tilde{u}_2(t_l, x_k) + \tilde{R}_2, \end{aligned} \tag{16}$$

with

$$\begin{aligned}
\tilde{u}_2(t_l, x_k) &= \left(1-\frac{1}{2}\right)\tilde{u}_1(t_l, x_k) + \frac{1}{2}\left(\sum_{i=0}^{l}\sum_{j=0}^{m_2} a_{ij}K\big(t_l, x_k, t_i, x_j, \tilde{u}_1(t_i, x_j)\big) + f(t_l, x_k)\right), \\
\tilde{R}_2 &= \left(1-\frac{1}{2}\right)\Big(u_1(t_l, x_k) - \tilde{u}_1(t_l, x_k)\Big) \\
&+ \frac{1}{2}\left(\sum_{i=0}^{l}\sum_{j=0}^{m_2} a_{ij}K\Big(t_l, x_k, t_i, x_j, u_1(t_i, x_j) - \tilde{u}_1(t_i, x_j)\Big) + R_K\right).
\end{aligned}$$

The values $\tilde{u}_2(t_l, x_k)$ can be then computed from the values obtained in the previous step. For the error estimate, let $\theta := L\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}|a_{ij}|$. We have, by Equation (15), the following:

$$\begin{aligned}
err(u_2, \tilde{u}_2) &\leq |\tilde{R}_2| \\
&\leq \left(1-\frac{1}{2}\right)|\tilde{R}_1| + \frac{1}{2}\left(\sum_{i=0}^{k}\sum_{j=0}^{m}|a_{ij}|\cdot L\cdot|\tilde{R}_1| + |R_K|\right) \\
&\leq \left(1-\frac{1}{2}\right)M + \frac{1}{2}\left(LM\sum_{i=0}^{m}\sum_{j=0}^{m}|a_{ij}| + M\right) \\
&= M + \frac{1}{2}M\theta \\
&\leq M(1+\theta).
\end{aligned} \tag{17}$$

Again, in a similar way, denoting by

$$\begin{aligned}
\tilde{u}_n(t_l, x_k) &= \left(1-\frac{1}{n}\right)\tilde{u}_{n-1}(t_l, x_k) \\
&+ \frac{1}{n}\left(\sum_{i=0}^{l}\sum_{j=0}^{m_2} a_{ij}K\big(t_l, x_k, t_i, x_j, \tilde{u}_{n-1}(t_i, x_j)\big) + f(t_l, x_k)\right),
\end{aligned} \tag{18}$$

for $l = 0, 1, \dots, m_1$, $k = 0, 1, \dots, m_2$, by induction, we find the following:

$$\begin{aligned}
err(u_n, \tilde{u}_n) &\leq |\tilde{R}_n| \\
&\leq \left(1-\frac{1}{n}\right)|\tilde{R}_{n-1}| + \frac{1}{n}\Big(\theta|\tilde{R}_{n-1}| + M\Big) \\
&\leq M\big(1+\theta+\cdots+\theta^{n-2}\big)\left(1-\frac{1}{n}+\frac{1}{n}\right) + \frac{1}{n}M\theta^{n-1} \\
&\leq M\big(1+\theta+\cdots+\theta^{n-2}\big) + M\theta^{n-1} \\
&= M\big(1+\theta+\cdots+\theta^{n-1}\big).
\end{aligned} \tag{19}$$

Then, we have the following approximation result:

Theorem 3. *Assume the conditions of Theorem 2 hold. In addition, assume that the coefficients in the numerical integration formula* (13) *satisfy the following:*

$$\theta = L\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}|a_{ij}| < 1. \tag{20}$$

Then, the following error estimate holds for every $n \in \mathbb{N}$:

$$err(u^*, \tilde{u}_n) \leq \frac{e^{1-q}}{1-q} e^{-(1-q)y_n} ||u_0 - Fu_0|| + \frac{M}{1-\theta} \tag{21}$$

where u^* is the true solution of Equation (3), $\tilde{u}_n$ is the approximation given by Equation (18) and the sequence $\{y_n\}$ is defined in Equation (12).

Proof. By Equations (19) and (20), for all $l = 0, 1, \dots, m_1$ and $k = 0, 1, \dots, m_2$

$$|u_n(t_l, x_k) - \tilde{u}_n(t_l, x_k)| \leq \frac{M}{1-\theta}. \tag{22}$$

Since

$$\left|u^*(t_l, x_k) - \tilde{u}_n(t_l, x_k)\right| \leq \left|u^*(t_l, x_k) - u_n(t_l, x_k)\right| + \left|u_n(t_l, x_k) - \tilde{u}_n(t_l, x_k)\right|,$$

the estimate in Equation (21) now follows from Equation (22) and Theorem 2. □

Remark 2. *Let us discuss condition (20), which can seem to be quite restrictive, especially since it also involves the constant L. As we will see below, when the quadrature scheme used is the trapezoidal rule, this condition reduces to the contraction condition (8) (whose applicability was discussed earlier in Remark 1), and, thus, does not introduce any new restrictions. In fact, the same thing is true for other fairly easy quadrature formulas, such as the midpoint or Simpson's rule (see [25]).*

A Numerical Method Based on the Trapezoidal Rule

As discussed previously, we can use any numerical integration formula to approximate the iterates $u_n(x_k)$, as long as it satisfies condition (20). In what follows, we propose one of the simplest formulas, the two-dimensional trapezoidal rule:

$$\begin{aligned}\int_a^b \int_c^d \varphi(\tau, y)\, dy\, d\tau &= \frac{(b-a)(d-c)}{4m_1 m_2}\Big[\varphi(a,c) + \varphi(b,c) + \varphi(a,d) + \varphi(b,d) \\ &+ 2\sum_{i=1}^{m_1-1} \left(\varphi(\tau_i, c) + \varphi(\tau_i, d)\right) \\ &+ 2\sum_{j=1}^{m_2-1} \left(\varphi(a, y_j) + \varphi(b, y_j)\right) + 4\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1} \varphi(\tau_i, y_j)\Big] + R_\varphi,\end{aligned} \tag{23}$$

using the nodes $s_i = a + \dfrac{b-a}{m_1} i, w_j = c + \dfrac{d-c}{m_2} j,\ i = \overline{0, m_1}, j = \overline{0, m_2}$. The remainder is the following:

$$\begin{aligned}R_\varphi &= -\Big[\frac{(b-a)^3(d-c)}{12m_1^2 m_2}\varphi^{(2,0)}(\xi, \eta_1) + \frac{(b-a)(d-c)^3}{12m_1 m_2^2}\varphi^{(0,2)}(\xi_1, \eta) \\ &+ \frac{(b-a)^3(d-c)^3}{144m_1^2 m_2^2}\varphi^{(2,2)}(\xi, \eta)\Big],\ \ \xi, \xi_1 \in (a,b), \eta, \eta_1 \in (c,d),\end{aligned} \tag{24}$$

where we use the notation $\varphi^{(\alpha,\beta)}(t,x) = \dfrac{\partial^{\alpha+\beta}\varphi}{\partial t^\alpha \partial x^\beta}(t,x)$.

For fixed m_1, m_2, we consider the nodes $t_l = \dfrac{T}{m_1} l, x_k = a + \dfrac{b-a}{m_2} k,\ l = \overline{0, m_1}, k = \overline{0, m_2}$. For simplicity, we will use the notation $K_{l,k,i,j} = K\big(t_l, x_k, t_i, x_j, u_n(t_i, x_j)\big)$. Then we have the following:

$$
\begin{aligned}
\int_0^{t_l}\int_a^b K\big(t_l, x_k, \tau, y, u_n(\tau, y)\big)\, dy\, d\tau \quad &= \quad \frac{t_l(b-a)}{4lm_2}\Big[K_{l,k,0,0} + K_{l,k,l,0} + K_{l,k,0,m_2} \\
&\quad + K_{l,k,l,m_2} + 2\sum_{i=0}^{l-1}\Big(K_{l,k,i,0} + K_{l,k,i,m_2}\Big) \\
&\quad + 2\sum_{j=0}^{m_2-1}\Big(K_{l,k,0,j} + K_{l,k,l,j}\Big) \\
&\quad + 4\sum_{i=0}^{l-1}\sum_{j=0}^{m_2-1} K_{l,k,i,j}\Big] + R_K,
\end{aligned} \tag{25}
$$

for each $l = 0,\ 1, \ldots, m_1, k = 0,\ 1, \ldots, m_2$. Since $\dfrac{t_l}{l} = \dfrac{T}{m_1}$, in this case, $\theta \le LT(b-a) = q$, which, by Equation (8) is strictly less than 1.

Next, let us discuss the bound M from Equation (14). By Equation (24), if $K^{(2,0)}\big(\tau, y, u_n(\tau, y)\big)$, $K^{(0,2)}\big(\tau, y, u_n(\tau, y)\big)$ and $K^{(2,2)}\big(\tau, y, u_n(\tau, y)\big)$ are bounded, then the remainder R_K is of the form $\mathcal{O}\left(\frac{1}{m_1^2}\right) + \mathcal{O}\left(\frac{1}{m_2^2}\right)$. For simplicity, we write the function K emphasizing only the variables that it is to be differentiated with respect to, i.e., $K(\tau, y, u(\tau, y))$. We have the following:

$$
\begin{aligned}
K^{(2,0)}\big(\tau, y, u_n(\tau, y)\big) \quad &= \quad \frac{\partial^2 K}{\partial \tau^2}\big(\tau, y, u_n(\tau, y)\big) + 2\frac{\partial^2 K}{\partial \tau \partial u}\big(\tau, y, u_n(\tau, y)\big)\frac{\partial u}{\partial \tau}(\tau, y) \\
&+ \quad \frac{\partial^2 K}{\partial u^2}\big(\tau, y, u_n(\tau, y)\big)\left(\frac{\partial u}{\partial \tau}(\tau, y)\right)^2 \\
&+ \quad \frac{\partial K}{\partial u}\big(\tau, y, u_n(\tau, y)\big)\frac{\partial^2 u}{\partial \tau^2}(\tau, y), \\
K^{(0,2)}\big(\tau, y, u_n(\tau, y)\big) \quad &= \quad \frac{\partial^2 K}{\partial y^2}\big(\tau, y, u_n(\tau, y)\big) + 2\frac{\partial^2 K}{\partial y \partial u}\big(\tau, y, u_n(\tau, y)\big)\frac{\partial u}{\partial y}(\tau, y) \\
&+ \quad \frac{\partial^2 K}{\partial u^2}\big(\tau, y, u_n(\tau, y)\big)\left(\frac{\partial u}{\partial y}(\tau, y)\right)^2 \\
&+ \quad \frac{\partial K}{\partial u}\big(\tau, y, u_n(\tau, y)\big)\frac{\partial^2 u}{\partial y^2}(\tau, y),
\end{aligned}
$$

and a similar (albeit much longer) formula can be found for $K^{(2,2)}\big(\tau, y, u_n(\tau, y)\big)$, involving partial derivatives of K and u_n of up to order 4. For the partial derivatives of u_n, we have the following:

$$
\begin{aligned}
u_n(t, x) \quad &= \quad \int_0^t\int_a^b K\big(t, x, \tau, y, u_{n-1}(\tau, y)\big)\, dy\, d\tau + f(t, x), \\
\frac{\partial u_n}{\partial x}(t, x) \quad &= \quad \int_0^t\int_a^b \frac{\partial K}{\partial x}\big(t, x, \tau, y, u_{n-1}(\tau, y)\big)\, dy\, d\tau + \frac{\partial f}{\partial x}(t, x), \\
\frac{\partial u_n}{\partial t}(t, x) \quad &= \quad \int_a^b K\big(t, x, t, y, u_{n-1}(t, y)\big)\, dy \\
&+ \quad \int_0^t\int_a^b \frac{\partial K}{\partial t}\big(t, x, \tau, y, u_{n-1}(\tau, y)\big)\, dy\, d\tau + \frac{\partial f}{\partial t}(t, x),
\end{aligned}
$$

and so on, up to the partial derivatives of order 4.

It is now obvious that if K and f are C^4 functions with bounded fourth order partial derivatives, then there exists $M > 0$, independent of n, such that

$$|R_K| \quad \leq \quad M, \tag{26}$$

with $M \to 0$ as $m_1, m_2 \to \infty$. Thus, under these assumptions and those in Theorem 2, we have the following error estimate:

$$err(u^*, \tilde{u}_n) \quad \leq \quad \frac{e^{1-q}}{1-q}\, e^{-(1-q)y_n}\, ||u_0 - Fu_0|| + \frac{M}{1-\theta}, \tag{27}$$

for all $n = 1, 2, \ldots,$ and $\{y_n\}$ given in Equation (12).

4. Numerical Examples

We now illustrate the applicability of the proposed method on several numerical examples. All computations are completed in Matlab, in double precision arithmetic. In general, the number of nodes is chosen such that the mesh size is around 0.05, which is small enough to achieve good accuracy but not so small as to increase the number of operations.

Example 1. *First, let us consider the linear mixed Fredholm–Volterra equation:*

$$u(t,x) \quad = \quad \int_0^t \int_0^2 xe^{-y} u(\tau, y)\, dy\, d\tau + t(e^x - tx), \;\; t \in [0,1], \tag{28}$$

with exact solution $u^*(t,x) = te^x$.

We take $\rho = 15.5$. We have $K(t,x,\tau,y,u) = xe^{-y}u$, $\dfrac{\partial K}{\partial u} = xe^{-y}$ and the following:

$$\begin{aligned} LT(b-a) &\approx 0.74 < 1, \\ M_K T(b-a) &\approx 15.37 \leq \rho, \end{aligned}$$

so all the hypotheses of Theorem 3 are satisfied. Additionally, for $\rho = 15.5$, we have that $u^* \in B_\rho$.

We consider the two-dimensional trapezoidal rule with $m_1 = 18$ and $m_2 = 36$, with the corresponding nodes $t_i = \dfrac{1}{m_1} i, i = \overline{0, m_1}$ and $x_j = \dfrac{2}{m_2} j, j = \overline{0, m_2}$. Table 1 contains the errors $err(u^*, \tilde{u}_n)$, with initial approximation $u_0(t,x) = f(t,x) = t(e^x - tx)$. The CPU time per iteration is approximately 1.01.

Table 1. Errors for Example 1, $m_1 = 18$, $m_2 = 36$.

n	$err(u^*, \tilde{u}_n)$
1	1.080492×10^{-1}
5	1.210778×10^{-4}
10	5.837723×10^{-6}

Example 2. *Next, consider the following nonlinear integral equation:*

$$u(t,x) \quad = \quad 2 \int_0^t \int_0^1 x^2 y \tau e^{-\tau} e^{u(\tau,y)}\, dy\, d\tau + x^2(1 - e^{-t}), \;\; t \in [0, 1/4], \tag{29}$$

whose exact solution is $u^*(t,x) = tx^2$.

Here, $K = \frac{\partial K}{\partial u} = 2x^2 y\tau e^{-\tau} e^u$ Thus, for $\rho = 1$, we have the following:

$$\begin{aligned} LT(b-a) &\approx 0.33 < 1, \\ M_K T(b-a) &\approx 0.33 \leq \rho, \end{aligned}$$

thus, Theorem 3 is applicable and $u^* \in B_\rho$.

Again, we use the trapezoidal rule with $m_1 = m_2 = 18$ and nodes $t_i = \frac{1}{4m_1} i, i = \overline{0, m_1}$, $x_j = \frac{1}{m_2} j, j = \overline{0, m_2}$. The errors $err(u^*, \tilde{u}_n)$ are given in Table 2, with initial approximation $u_0(t,x) = f(t,x) = x^2(1 - e^{-t})$. The CPU time per iteration is approximately 0.89.

Table 2. Errors for Example 2, $m_1 = m_2 = 18$.

n	$err(u^*, \tilde{u}_n)$
1	2.034743×10^{-1}
5	9.354733×10^{-4}
10	3.077314×10^{-5}

Example 3. *Last, consider the nonlinear mixed Fredholm–Volterra equation as follows:*

$$u(t,x) = 2 \int_0^t \int_0^1 x \cos\tau (u(\tau,y))^2 \, dy \, d\tau + \frac{x \sin t}{9} (9 - \sin^2 t), \tag{30}$$

for $t \in [0, 1/2]$. *The exact solution of Equation (30) is* $u^*(t,x) = x \sin t$.

We have $K(t,x,\tau,y,u) = 2xu^2 \cos\tau$ and $\frac{\partial K}{\partial u} = 4xu\cos\tau$. Choosing $\rho = 0.3$, we obtain the following:

$$\begin{aligned} LT(b-a) &\approx 0.53 < 1, \\ M_K T(b-a) &\approx 0.28 \leq \rho, \end{aligned}$$

so Theorem 3 can be used and $u^* \in B_\rho$.

Again, the trapezoidal rule is used with $m_1 = m_2 = 18$ and nodes $t_i = \frac{1}{2m_1} i, i = \overline{0, m_1}$ and $x_j = \frac{1}{m_2} j, j = \overline{0, m_2}$. In Table 3 we give the errors $err(u^*, \tilde{u}_n)$ with initial approximation $u_0(t,x) = f(t,x) = \frac{x \sin t}{9}(9 - \sin^2 t)$. The CPU time per iteration is approximately 0.98.

Table 3. Errors for Example 3, $m_1 = m_2 = 18$.

n	$err(u^*, \tilde{u}_n)$
1	2.733605×10^{-1}
5	7.890241×10^{-4}
10	2.766358×10^{-5}

5. Conclusions

We presented a numerical method for approximating solutions of two-dimensional mixed Fredholm–Volterra integral equations of the second kind, using a combination of successive approximations for fixed points and cubature formulas. In this paper, we used Altman's algorithm and the Mann iteration for finding fixed points of an integral operator and the two-dimensional trapezium rule for the numerical integration of the iterates. This has many advantages: in the first place, the fixed point result we used not only guarantees the existence of a unique solution, but also gives a procedure for finding it by successive iterations. Moreover, Mann iterates converge faster than Picard ones (see [24]), so better accuracy is obtained with fewer iterations. In addition, by using the trapezoidal rule, the contraction condition for the integral operator also guarantees the convergence of the numerical approximations. Secondly, the choice of the trapezoidal scheme makes the method easy to use and implement since most mathematical software have this rule built-in. Last, but not least, many popular approximation methods, such as Nyström, collocation, Galerkin or Adomian decomposition methods, lead to difficult-to-solve systems of algebraic equations that are many times ill-conditioned. Such problems are avoided here since the computation of an approximate value only requires the values obtained at the previous step. This reduces the computational and implementation cost of the method. Still, the method proposed converges with order $\mathcal{O}\left(e^{-(1-q)y_n}\right) + \mathcal{O}\left(\frac{1}{m_1^2}\right) + \mathcal{O}\left(\frac{1}{m_2^2}\right)$ (with $\{y_n\}$ given in Equation (12)), producing good resulting approximations as the numerical examples show. On the downside, there are some limitations to the types of equations that this method can be applied to, due to the constraints in Theorem 2.

These ideas can be continued in studying other types of mixed integral equations, such as equations in higher dimensions ($\Omega \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$), equations with singular kernels (arising, for example, in reformulations of the heat equation), or kernels with modified argument, etc. Other types of successive approximations or other numerical integration schemes can also be explored.

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