symmetry

Article

On a Functional Integral Equation

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Abstract: In this paper, we establish some results for a Volterra–Hammerstein integral equation with modified arguments: existence and uniqueness, integral inequalities, monotony and Ulam-Hyers-Rassias stability. We emphasize that many problems from the domain of symmetry are modeled by differential and integral equations and those are approached in the stability point of view. In the literature, Fredholm, Volterra and Hammerstein integrals equations with symmetric kernels are studied. Our results can be applied as particular cases to these equations.

Keywords: abstract Gronwall lemma; existence and uniqueness; integral inequalities; monotony; Ulam-Hyers-Rassias stability

MSC: 47H10; 45G10; 47N20

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1. Introduction

Many problems from the domain of symmetry are modeled by integral equations. In this paper, we study a functional integral equation, a generalization of the equations considered in the papers [1–4].

Other functional integral equations were studied in [5–9]. Equations with modified argument were also studied by D. Otrocol and V. A. Ilea in [10,11].

In the following we consider a Banach space $(E, |\cdot|)$, the real number $\tau > 0$ and the set

$$X_\tau := \left\{ u \in C\left(\mathbb{R}_+^3, E\right) \mid \exists M(u) > 0 : |u(x,y,z)|e^{-\tau(x+y+z)} \le M(u), \forall (x,y,z) \in \mathbb{R}_+ \right\}.$$

On X_τ, we consider Bielecki's norm

$$\|u\|_\tau := \sup_{x,y,z \in R_+} \left(|u(x,y,z)|e^{-\tau(x+y+z)} \right).$$

It is clear that $(X_\tau, \|\cdot\|_\tau)$ is a Banch space.

Further, we study a functional integral equation of the form

$$\begin{aligned} u(x,y,z) &= g(x,y,z,h(u)(x,y,z)) \\ &+ \int_0^x \int_0^y \int_0^z K(x,y,z,r,s,t,u(f_1(r,s,t), f_2(r,s,t), f_3(r,s,t)))drdsdt \\ &+ \int_0^\infty \int_0^\infty \int_0^\infty V(x,y,z,r,s,t,u(f_1(r,s,t), f_2(r,s,t), f_3(r,s,t)))drdsdt, \end{aligned} \tag{1}$$

$(x,y,z,r,s,t) \in \Delta$, where
$\Delta = \left\{ (x,y,z,r,s,t) \in \mathbb{R}_+^6 : 0 \le r \le x < \infty, 0 \le s \le y < \infty, 0 \le t \le z < \infty \right\}$, $(E, |\cdot|)$ is a Banach space, $g \in C\left(\mathbb{R}_+^3 \times E, E\right)$, $K, V \in C\left(\mathbb{R}_+^6 \times E, E\right)$, $h \in C(X_\tau, X_\tau)$, $f_1, f_2, f_3 \in C\left(\mathbb{R}_+^3, \mathbb{R}_+\right)$.

The equation is studied using Picard operators technique and Gronwall-type inequalities technique. Picard operators technique was extensively presented in [12] by I.A. Rus. For this reason, we will use the notions and terminology from this paper. We first recall some notions from Picard operators theory.

Definition 1. *([13]). Let X be a nonempty set. Let $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X, n \in \mathbb{N}\}$. Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. The triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:*

(i) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$;
(ii) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

An element of $c(X)$ is called a convergent sequence and $x = Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence. We write $(x_n)_{n\in\mathbb{N}} \to x$ or $x_n \to x$ as $n \to \infty$.

Definition 2. *([14]). Let X be a nonempty set. $(X, \to, \leq)$ is called an ordered L-space if:*

(i) $(X, \to)$ is an L-space;
(ii) $(X, \leq)$ is a partially ordered set;
(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$ for each $n \in \mathbb{N} \Rightarrow x \leq y$.

Let $A : X \longrightarrow X$ be an operator and $F_A = \{x \in X \mid A(x) = x\}$ be the fixed points set of A. Let $A_0 := 1_X$, $A^1 := A, ..., A^{n+1} := A \circ A^n$, $n \in \mathbb{N}$.

Definition 3. *([12]). Let $(X, \to, \leq)$ be an ordered L-space. An operator $A : X \longrightarrow X$ is called a Picard operator (PO) if there exists $x_A^* \in X$ such that $F_A = \{x_A^*\}$ and, $A^n(x) \longrightarrow x_A^*$ as $n \longrightarrow \infty$ for all $x \in X$.*

Definition 4. *([12]). Let $(X, \to, \leq)$ be an ordered L-space. An operator $A : X \longrightarrow X$ is called a c-Picard operator (c-PO) if A is PO, $c > 0$ and, $d(x, x_A^*) \leq cd(x, A(x))$ for all $x \in X$.*

Because the working technique used is also that of Gronwall-type inequalities introduced in paper [15], we recall the following two lemmas.

Lemma 1. *(Abstract Gronwall Lemma [12]). Let$(X, \to, \leq)$ be an ordered $L-$space and $A : X \to X$ an operator. We suppose that:*

(i) A is a Picard operator;
(ii) A is an increasing operator.

If we denote by x_A^* the unique fixed point of A, then we have:

(a)
$$x \in X,\, x \leq A(x) \Rightarrow x \leq x_A^*, \tag{2}$$
and

(b)
$$x \in X,\, x \geq A(x) \Rightarrow x \geq x_A^*. \tag{3}$$

Lemma 2. *([12]). Let$(X, \to, \leq)$ be an ordered $L-$space and $A, B, C : X \to X$ three operators such that:*

(i) $A \leq B \leq C$;
(ii) A, B, C are Picard operators;
and
(iii) B is an increasing operator.

If we denote by x_A^* the unique fixed point of A, by x_B^* the unique fixed point of B, and by x_C^* the unique fixed point of C, then
$$x_A^* \leq x_B^* \leq x_C^*.$$

The main objectives of the paper are the study of some properties of the solutions of the Equation (1), among which we mention the existence and uniqueness, integral inequalities, monotony and Ulam stability.

Equations of this type have multiple applications in mathematics, physics, technology, economics, etc. Thus, in papers [16–18] are studied integro-differential models with applications in economics, and in papers [1,5,12] are studied mathematical problems formulated on these equations.

2. Existence and Uniqueness

In the following, we will state a theorem of existence and uniqueness. In this sense we will show that the operator $A : X_\tau \to X_\tau$,

$$
\begin{aligned}
&A(u)(x,y,z) := g(x,y,z,h(u)(x,y,z)) \\
&+ \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt,
\end{aligned} \tag{4}
$$

is a contraction, under certain conditions.

Theorem 1. *If*

(i) $g \in C(\mathbb{R}_+^3 \times E, E), K, V \in C(\mathbb{R}_+^6 \times E, E), h \in C(X_\tau, X_\tau), f_1, f_2, f_3 \in C(\mathbb{R}_+^3, \mathbb{R}_+)$;

(ii) *there exists* $l_h > 0$ *such that*

$$|h(u)(x,y,z) - h(v)(x,y,z)| \le l_h \|u - v\|_\tau e^{\tau(x+y+z)}, \forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau;$$

(iii) *there exists* $l_g > 0$ *such that*

$$
\begin{aligned}
&|g(x,y,z,h(u)(x,y,z)) - g(x,y,z,h(v)(x,y,z))| \\
&\le l_g |h(u)(x,y,z) - h(v)(x,y,z)|,
\end{aligned}
$$

$\forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau$;

(iv) *there exists* $l_K, l_V \in C(\mathbb{R}_+^6, \mathbb{R}_+)$ *such that*

$$
\begin{aligned}
&|K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t))) \\
&-K(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))| \\
&\le l_K(x,y,z,r,s,t)\|u - v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau,
\end{aligned}
$$

$$
\begin{aligned}
&|V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t))) \\
&-V(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))| \\
&\le l_V(x,y,z,r,s,t)\|u - v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau;
\end{aligned}
$$

(v) *there exists* $l_1 > 0$ *and* $l_2 > 0$ *such that*

$$\int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt \le l_1 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty l_V(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt \le l_2 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta;$$

(vi) $l_g l_h + l_1 + l_2 < 1$;

then (1) *has in* X_τ *a unique solution.*

Proof. From conditions (i)–(vi), it follows that the operator A is a contraction in X_τ. Indeed, for every $u, v \in X_\tau$ we have:

$$\begin{aligned}
&|A(u)(x,y,z) - A(v)(x,y,z)| \\
&\leq l_g l_h \|u-v\|_\tau e^{\tau(x+y+z)} \\
&+ \int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)\|u-v\|_\tau e^{\tau(r+s+t)} drdsdt \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)\|u-v\|_\tau e^{\tau(r+s+t)} drdsdt \\
&\leq l_g l_h \|u-v\|_\tau e^{\tau(x+y+z)} + l_1\|u-v\|_\tau e^{\tau(x+y+z)} + l_2\|u-v\|_\tau e^{\tau(x+y+z)}.
\end{aligned}$$

Then we have:

$$\|A(u) - A(v)\|_\tau \leq (l_g l_h + l_1 + l_2)\|u-v\|_\tau,$$

for all $u, v \in X_\tau$. Using (vi) it follows that A is a contraction. Hence (1) has a unique solution in X_τ. □

3. Integral Inequalities

Theorem 2. *Let $(E, |\cdot|, \leq)$ be an ordered Banach space. We suppose that:*

(i) the conditions (i)–(vi) from Theorem 1 are satisfied;
(ii) the operators

$$\begin{aligned}
&g(x,y,z,\cdot) : E \to E, \\
&h : X_\tau \to X_\tau, \\
&K(x,y,z,r,s,t,\cdot) : E \to E, \\
&V(x,y,z,r,s,t,\cdot) : E \to E,
\end{aligned}$$

are increasing.

If $u^* \in X_\tau$ is the unique solution of Equation (1) and $u \in X_\tau$ is a solution of the inequality

$$u(x,y,z) \leq A(u)(x,y,z),$$

then

$$u(x,y,z) \leq u^*(x,y,z). \tag{5}$$

Proof. The operator A is a Picard operator. This operator it is also increasing. We apply Lemma 1 and we get that (5) is satisfied. □

4. Monotony

Let $(E, |\cdot|, \leq)$ be an ordered Banach space.
Consider the following integral equations

$$\begin{aligned}
u_i(x,y,z) &= g_i(x,y,z,h(u_i)(x,y,z)) \\
&+ \int_0^x\int_0^y\int_0^z K_i(x,y,z,r,s,t,u_i(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty V_i(x,y,z,r,s,t,u_i(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt,
\end{aligned} \tag{6}$$

$i = 1,2,3.$

Theorem 3. *We suppose that:*

(i) $g_i, h, K_i, V_i, i = 1,2,3$ satisfy the conditions (i)–(vi) from Theorem 1;

(ii) *the operators*

$$\begin{aligned} & g_2(x,y,z,\cdot) : E \to E, \\ & h : X_\tau \to X_\tau, \\ & K_i(x,y,z,r,s,t,\cdot) : E \to E, i = 1,2,3, \\ & V_i(x,y,z,r,s,t,\cdot) : E \to E, i = 1,2,3, \end{aligned}$$

are increasing.

(iii) $g_1 \le g_2 \le g_3, K_1 \le K_2 \le K_3, V_1 \le V_2 \le V_3.$

Then (6) *has a unique solution* u_i^*, *for each* $i = 1,2,3$ *and*

$$u_1^* \le u_2^* \le u_3^*. \tag{7}$$

Proof. We consider the operators $A_i : X_\tau \to X_\tau, i = 1,2,3,$

$$A_i(u)(x,y,z) = \text{second part of (6)}.$$

These operators are as presented in Lemma 2. Indeed, $A_1 \le A_2 \le A_3$, by assumption (iii).

The operators $A_i, i = 1,2,3$, are Picard operators since the conditions (i)–(vi) from Theorem 1 are satisfied. From Theorem 1 we have that (6) has a unique solution $u_i^* \in X_\tau$, for each $i = 1,2,3$.

The operator A_2 is increasing.

We apply Lemma 2 and we get that (7) is satisfied. □

5. Hyers-Ulam-Rassias Stability

In what follows we consider the Equation (1) and the inequality

$$\begin{aligned} & \Big| u(x,y,z) - g(x,y,z,h(u)(x,y,z)) \\ & - \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt \\ & - \int_0^\infty\int_0^\infty\int_0^\infty V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt \Big| \\ & \le \varphi(x,y,z), \end{aligned} \tag{8}$$

where $(E,|\cdot|)$ is a Banach space and $\varphi \in C\left(\mathbb{R}_+^3, \mathbb{R}_+\right)$.

Theorem 4. *We suppose that:*

(i) *the conditions* (i)–(vi) *from Theorem* 1 *are satisfied;* *and*

(ii) *there exists* $m \in \mathbb{R}_+$ *such that*

$$\varphi(x,y,z)e^{-\tau(x+y+z)} \le m, \forall x,y,z \in \mathbb{R}_+.$$

If u *is a solution of* (8) *and* u^* *is the unique solution of* (1), *we have*

$$\|u - u^*\|_\tau \le \frac{m}{1 - l_g l_h - l_1 - l_2},$$

so the Equation (1) *is Hyers-Ulam-Rassias stable.*

Proof. We denote $w = (f_1(r,s,t), f_2(r,s,t), f_3(r,s,t))$. We have

$$\begin{aligned}
&|u(x,y,z) - u^*(x,y,z)| \\
&\le \Big|u(x,y,z) - g(x,y,z,h(u)(x,y,z)) \\
&- \int_0^x\int_0^y\int_0^z K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt \\
&- \int_0^\infty\int_0^\infty\int_0^\infty V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))drdsdt\Big| \\
&+ |g(x,y,z,h(u)(x,y,z)) - g(x,y,z,h(u^*)(x,y,z))| \\
&+ \int_0^x\int_0^y\int_0^z |K(x,y,z,r,s,t,u(w)) - K(x,y,z,r,s,t,u^*(w))|drdsdt \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty |V(x,y,z,r,s,t,u(w)) - V(x,y,z,r,s,t,u^*(w))|drdsdt \\
&\le \varphi(x,y,z) + l_g l_h \|u-u^*\|_\tau e^{\tau(x+y+z)} \\
&+ \int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)\|u-u^*\|_\tau e^{\tau(r+s+t)}drdsdt \\
&+ \int_0^\infty\int_0^\infty\int_0^\infty l_V(x,y,z,r,s,t)\|u-u^*\|_\tau e^{\tau(r+s+t)}drdsdt \\
&\le \varphi(x,y,z) + l_g l_h \|u-u^*\|_\tau e^{\tau(x+y+z)} + l_1\|u-u^*\|_\tau e^{\tau(x+y+z)} \\
&+ l_2\|u-u^*\|_\tau e^{\tau(x+y+z)}.
\end{aligned}$$

Hence we have:

$$\|u-u^*\|_\tau \le \frac{\varphi(x,y,z)}{1-l_gl_h-l_1-l_2}e^{-\tau(x+y+z)} \le \frac{m}{1-l_gl_h-l_1-l_2}$$

so the Equation (1) is Hyers-Ulam-Rassias stable. □

Below, we consider an example for the case where the function φ is symmetric.

Example 1. *Let*

$$\begin{aligned}
&\varphi(x,y,z) = e^{\tau(x+y+z)}, h(u) = \frac{u}{5}, g(x,y,z,h(u)(x,y,z)) = \frac{1}{5}h(u)(x,y,z), \\
&f_1(x,y,z) = \frac{x}{2}, f_2(x,y,z) = \frac{y}{2}, f_3(x,y,z) = \frac{z}{2}, \\
&K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t))) = \frac{e^{\tau(-r-s-t+x+y+z)}}{5xyz}u\left(\frac{r}{2},\frac{s}{2},\frac{t}{2}\right), \\
&V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t))) = \frac{e^{\tau(-r-s-t+x+y+z)}}{5(r+1)^2(s+1)^2(t+1)^2}u\left(\frac{r}{2},\frac{s}{2},\frac{t}{2}\right).
\end{aligned}$$

The conditions (i)–(vi) *from Theorem 1 are satisfied:*

(i) $g \in C(\mathbb{R}_+^3 \times E, E), K, V \in C(\mathbb{R}_+^6 \times E, E), h \in C(X_\tau, X_\tau), f_1, f_2, f_3 \in C(\mathbb{R}_+^3, \mathbb{R}_+)$;

(ii) *there exists* $l_h = \frac{1}{5} > 0$ *such that* $|h(u)(x,y,z) - h(v)(x,y,z)| \le l_h\|u-v\|_\tau e^{\tau(x+y+z)}$, $\forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau$. *Indeed we have*

$$\begin{aligned}
&|h(u)(x,y,z) - h(v)(x,y,z)| = \frac{1}{5}|u(x,y,z) - v(x,y,z)| \\
&\le \frac{1}{5}\|u-v\|_\tau e^{\tau(x+y+z)}, \forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau.
\end{aligned}$$

(iii) *there exists* $l_g = \frac{1}{5} > 0$ *such that*

$$|g(x,y,z,h(u)(x,y,z)) - g(x,y,z,h(v)(x,y,z))|$$
$$\leq l_g |h(u)(x,y,z) - h(v)(x,y,z)|, \forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau.$$

Indeed we have

$$|g(x,y,z,h(u)(x,y,z)) - g(x,y,z,h(v)(x,y,z))|$$
$$= \frac{1}{5}|h(u)(x,y,z) - h(v)(x,y,z)|, \forall x,y,z \in \mathbb{R}_+, \forall u,v \in X_\tau.$$

(iv) *there exists* $l_K, l_V \in C(\mathbb{R}_+^6, \mathbb{R}_+)$, $l_K(x,y,z,r,s,t) = \dfrac{e^{\tau(-r-s-t+x+y+z)}}{5xyz}$, $l_V(x,y,z,r,s,t) = \dfrac{e^{\tau(-r-s-t+x+y+z)}}{5(r+1)^2(s+1)^2(t+1)^2}$, *such that*

$$|K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))$$
$$-K(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))|$$
$$\leq l_K(x,y,z,r,s,t)\|u-v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau,$$

$$|V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))$$
$$-V(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))|$$
$$\leq l_V(x,y,z,r,s,t)\|u-v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau.$$

Indeed we have

$$|K(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))$$
$$-K(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))|$$
$$\leq \frac{e^{\tau(-r-s-t+x+y+z)}}{5xyz}\|u-v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau,$$

$$|V(x,y,z,r,s,t,u(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))$$
$$-V(x,y,z,r,s,t,v(f_1(r,s,t),f_2(r,s,t),f_3(r,s,t)))|$$
$$\leq \frac{e^{\tau(-r-s-t+x+y+z)}}{5(r+1)^2(s+1)^2(t+1)^2}\|u-v\|_\tau e^{\tau(r+s+t)}, \forall (x,y,z,r,s,t) \in \Delta, \forall u,v \in X_\tau.$$

(v) *there exists* $l_1 = \frac{1}{5} > 0$ *and* $l_2 = \frac{1}{5} > 0$ *such that*

$$\int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt \leq l_1 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta,$$

$$\int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt \leq l_2 e^{\tau(x+y+z)}, \forall (x,y,z,r,s,t) \in \Delta.$$

Indeed we have

$$\int_0^x\int_0^y\int_0^z l_K(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt$$
$$= \int_0^x\int_0^y\int_0^z \frac{e^{\tau(-r-s-t+x+y+z)}}{5xyz}e^{\tau(r+s+t)}drdsdt \leq \frac{1}{5}e^{\tau(x+y+z)},$$

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty\int_0^\infty l_F(x,y,z,r,s,t)e^{\tau(r+s+t)}drdsdt \\
&= \int_0^\infty\int_0^\infty\int_0^\infty \frac{e^{\tau(-r-s-t+x+y+z)}}{5(r+1)^2(s+1)^2(t+1)^2}e^{\tau(r+s+t)}drdsdt \\
&\le \frac{1}{5}e^{\tau(x+y+z)}, \forall(x,y,z,r,s,t) \in \Delta.
\end{aligned}
$$

(vi) $l_g l_h + l_1 + l_2 = \frac{1}{5}\cdot\frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{11}{25} < 1.$
Also there exists $m = 1 \in \mathbb{R}_+$ *such that*

$$\varphi(x,y,z)e^{-\tau(x+y+z)} \le m, \forall x,y,z \in \mathbb{R}_+.$$

Hence the conditions from Theorem 4 are satisfied, so if u is a solution of (8) and u^* is the unique solution of (1), we have

$$\|u-u^*\|_\tau \le \frac{25}{14}.$$

6. Conclusions

In this paper, we have considered a functional Volterra–Hammerstein integral equation with modified arguments. We have proved an existence and uniqueness theorem, we have established integral inequalities and a monotonicity result. We also have studied Hyers-Ulam-Rassias stability of this equation. The result regarding the stability is illustrated by Example 1, if the function φ is symmetric. Our results can be applied as particular cases to integrals equations with symmetric kernels. We will study this in a future paper.

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