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Some Properties Concerning the $J_L(X)$ and $Y_J(X)$ Which Related to Some Special Inscribed Triangles of Unit Ball

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Abstract: In this paper, we will make some further discussions on the $J_L(X)$ and $Y_J(X)$ which are symmetric and related to the side lengths of some special inscribed triangles of the unit ball, and also introduce two new geometric constants $L_1(X, \triangle)$, $L_2(X, \triangle)$ which related to the perimeters of some special inscribed triangles of the unit ball. Firstly, we discuss the relations among $J_L(X)$, $Y_J(X)$ and some geometric properties of Banach spaces, including uniformly non-square and uniformly convex. It is worth noting that we point out that uniform non-square spaces can be characterized by the side lengths of some special inscribed triangles of unit ball. Secondly, we establish some inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants, including the James constant $J(X)$ and the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$. Finally, we introduce the two new geometric constants $L_1(X, \triangle)$, $L_2(X, \triangle)$, and calculate the bounds of $L_1(X, \triangle)$ and $L_2(X, \triangle)$ as well as the values of $L_1(X, \triangle)$ and $L_2(X, \triangle)$ for two Banach spaces.

Keywords: Banach space; geometric constant; uniformly non-square; uniformly convex; inequality

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1. Introduction

As we all know, geometric constant is an important tool to study the geometric properties of Banach spaces. Since Clarkson [1] put forward the concept of the modulus of convexity in 1936, more and more scholars began to study the geometric constants. Obviously, the advantage of geometric constant is that when we need to determine whether a space has a certain geometric property, we only need to calculate the value of the geometric constants which can characterize this geometric property. This avoids us using abstract definitions to prove.

After more than half a century of research, a variety of geometric constants have been proposed. Many geometric constants are highly concerned. For example, the modulus of convexity introduced by Clarkson [1] can be used to characterize uniformly convex spaces ([2], Lemma 2), the modulus of smoothness proposed by Day [3] can be used to characterize uniformly smooth spaces ([4], Theorem 2.5), the von Neumann-Jordan constant proposed by Clarkson [5] and the James constant proposed by Gao and Lau [6] can be used to characterize uniformly non-square spaces ([7], Theorem 2 and [8], Proposition 1). Many properties of these constants have been studied, such as the relations between these constants and normal structure or uniform normal structure ([9–12]), the equalities or inequalities for these constants ([12–15]) and so on. Although the study of geometric constants has gone through more than half a century, many new geometric constants still constantly appear in our field of vision. For readers interested in this research direction, we recommend reading references [16–20] as well as the references mentioned in this paper.

In [21], we introduce the following two geometric constants $J_L(X)$ and $Y_J(X)$ which was inspired by the works in [22,23].

$$J_L(X) = \inf\{\|x-y\|, \|y-z\|, \|x-z\| : \|x\| = \|y\| = \|z\| = 1,\ x+y+z = 0\};$$

$$Y_J(X) = \sup\{\|x-y\|, \|y-z\|, \|x-z\| : \|x\| = \|y\| = \|z\| = 1,\ x+y+z = 0\}.$$

The constants $J_L(X)$ and $Y_J(X)$ can be regarded as discussing the infimum and supremum of the side lengths of some special inscribed triangles of unit ball, where the vertices x, y, z of the inscribed triangles need to satisfy $x + y + z = 0$ (i.e., the vectors x, y and z can form a triangle by translation).

In this paper, we follow the work in [21] and continue to explore the properties of $J_L(X)$ and $Y_J(X)$. The arrangement of this paper is as follows:

In Section 2, we introduce some notations and recall some conclusions that we need to use in this paper.

In Section 3, we make some further discussions on some results of $J_L(X)$ and $Y_J(X)$ which given in [21]. Firstly, we prove that the best upper bound of $J_L(X)$ is $\sqrt{3}$. Secondly, we prove that X is uniformly non-square if and only if $J_L(X) > 1$. Thirdly, we point out that the converse propositions of the following two propositions do not hold.

(i) if X is uniformly convex, then $Y_J(X) < 2$;

(ii) if $Y_J(X) < 2$, then X is uniformly non-square.

In Section 4, we discuss the inequalities for $J_L(X)$, $Y_J(X)$, $J(X)$ and $C_{\mathrm{NJ}}(X)$, and obtain some inequalities with simple forms.

In Section 5, we introduce two geometric constants $L_1(X, \triangle)$, $L_2(X, \triangle)$ which related to the perimeters of some special inscribed triangles of the unit ball, and discuss their bounds and compute the values of $L_1(X, \triangle)$, $L_2(X, \triangle)$ for two specific spaces.

2. Notations and Preliminaries

Throughout the paper, let X be a real Banach space with dim $X \geq 2$. The unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. The dual space of X is denoted by X^*. Now, let's recall some concepts and conclusions that we need in this paper.

Definition 1. *[1] A Banach space X is said to be uniformly convex, whenever given $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that if $x, y \in S_X$ and $\|x - y\| \geq \varepsilon$, then*

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

The following function $\delta_X(\varepsilon) : [0, 2] \to [0, 1]$, which we call it the modulus of convexity, was introduced by Clarkson [1]. It can be used to characterize the uniformly convex spaces (see [2], Lemma 2) ,

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in S_X, \|x-y\| = \varepsilon\right\} \ (0 \leq \varepsilon \leq 2),$$

where $\delta_X(\varepsilon)$ can be rewritten as following (see [24], Theorem 1),

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} \ : \ x, y \in B_X, \|x-y\| \geq \varepsilon\right\} \ (0 \leq \varepsilon \leq 2).$$

Definition 2. *[25] A Banach space X is said to be uniformly smooth, if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for any $x, y \in S_X$ and $\|x - y\| \leq \eta$, then*

$$1 - \frac{\|x+y\|}{2} \leq \varepsilon \|x-y\|.$$

The following function $\rho_X(\varepsilon) : [0,2] \to [0,1]$, which we call it the modulus of smoothness, was introduced by Banaś [25]. It can be used to characterize the uniformly convex spaces (see [25], Page 5),

$$\rho_X(\varepsilon) = \sup\left\{1 - \frac{\|x+y\|}{2} \ : \ x,y \in S_X, \|x-y\| = \varepsilon\right\} \ (0 \le \varepsilon \le 2).$$

Some important conclusions about $\rho_X(\varepsilon)$ are listed below:

(1) For all Banach spaces X, then $\rho_X(\varepsilon) \ge 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ (see [26], (3.6));

(2) X is an uniformly non-square if and only if $\rho_X(1) = \frac{1}{2}$ (see [26], Theorem 3);

(3) For all Banach spaces X, then

$$\rho_X(\varepsilon) = (1 - \rho_X(\varepsilon))\overline{\rho}_X\left(\frac{\varepsilon}{2(1-\rho_X(\varepsilon))}\right). \tag{1}$$

where $\overline{\rho}_X(\varepsilon) = \sup\{\min(\|x+\varepsilon y\|, \|x-\varepsilon y\|) - 1 \ , x, y \in S_X\}$ $(\varepsilon \ge 0)$ (see [25], Theorem 2).

Definition 3. *[27] A Banach space X is said to be uniformly non-square, if there exists $\delta \in (0,1)$ such that if $x, y \in S_X$ then*

$$\left\|\frac{x+y}{2}\right\| \le 1-\delta \ \ or \ \ \left\|\frac{x-y}{2}\right\| \le 1-\delta.$$

The following two constants James constant $J(X)$ and von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ can be both used to characterize the uniformly non-square spaces, where the James constant $J(X)$ and the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$ were introduced by Gao and Lau [6] and Clarkson [5], respectively.

$$J(X) = \sup\{\min(\|x+y\|, \|x-y\|) \ : \ x,y \in S_X\};$$

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \ : \ x,y \in X, (x,y) \ne (0,0)\right\}.$$

Some important conclusions about $J(X)$ and $C_{\mathrm{NJ}}(X)$ are listed below:

(1) $\sqrt{2} \le J(X) \le 2$, for all Banach spaces X (see [6], Theorem 2.5);

(2) X is an uniformly non-square if and only if $J(X) < 2$ (see [6], Theorem 3.4);

(3) $J(X) = \sup\{\|x+y\| \ : \ \|x+y\| = \|x-y\|, x, y \in S_X\}$, for all Banach spaces X (see [28], Theorem 5);

(4) $\frac{J(X)^2}{2} \le C_{\mathrm{NJ}}(X) \le J(X)$, for all Banach spaces X (see [8], Theorem 3 and [14], Theorem 3);

(5) $1 \le C_{\mathrm{NJ}}(X) \le 2$, for all Banach spaces X (see [29]);

(6) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$ (see [29]);

(7) X is an uniformly non-square if and only if $C_{\mathrm{NJ}}(X) < 2$ (see [7], Theorem 2);

(8) $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$, for all Banach spaces X (see [7], Lemma 2).

3. The Relations among $J_L(X)$, $Y_J(X)$ and Some Geometric Properties of Banach Spaces

In [21], we have introduced the following two constants:

$$J_L(X) = \inf\{\|x-y\|, \|y-z\|, \|x-z\| \ : \ \|x\| = \|y\| = \|z\| = 1, \ x+y+z = 0\};$$

$$Y_J(X) = \sup\{\|x-y\|, \|y-z\|, \|x-z\| \ : \ \|x\| = \|y\| = \|z\| = 1, \ x+y+z = 0\}.$$

According to the symmetry of the two constants $J_L(X)$ and $Y_J(X)$ (i.e., the positions of x, y and z are interchangeable), the constants $J_L(X)$ and $Y_J(X)$ can be rewritten as follows

$$J_L(X) = \inf\{\|x-y\| \ : \ \|x\| = \|y\| = \|x+y\| = 1\};$$

$$Y_J(X) = \sup\{\|x-y\| \ : \ \|x\| = \|y\| = \|x+y\| = 1\}.$$

Some conclusions about $J_L(X)$ and $Y_J(X)$ are listed below (see [21]):
(1) $1 \le J_L(X) \le 2$ and $\sqrt{3} \le Y_J(X) \le 2$;
(2) if X is uniformly non-square, then $J_L(X) > 1$;
(3) if X is uniformly convex, then $Y_J(X) < 2$;
(4) if $Y_J(X) < 2$, then X is uniformly non-square.
In this section, we will discuss the above conclusions (1)–(4) further. Before further research, we need to give the relation between $J_L(X)$ and $\rho_X(\varepsilon)$, and the relation between $Y_J(X)$ and $\delta_X(\varepsilon)$.

Proposition 1. *Let X be a Banach space. Then $J_L(X) = 2 - 2\rho_X(1)$, $Y_J(X) = 2 - 2\delta_X(1)$.*

Proof. Obviously, we have

$$\begin{aligned} J_L(X) =& \inf\{\|x-y\| \ : \ \|x\| = \|y\| = \|x+y\| = 1\} \\ =& \inf\{\|x+y\| \ : \ \|x\| = \|y\| = \|x-y\| = 1\} \\ =& 2 - 2\rho_X(1). \end{aligned}$$

The proof of $Y_J(X) = 2 - 2\delta_X(1)$ is similar. □

Remark 1. *The Proposition 1 is an improvement of Theorem 2 in [21]. In addition, the Proposition 1 tells us that the values of $\rho_X(1)$ and $\delta_X(1)$ are actually determined by the infimum and supremum of the side lengths of some special inscribed triangles of unit ball, where the vertices x, y, z of the inscribed triangles need to satisfy $x + y + z = 0$.*

Now, we start to make some further research on the above conclusions (1)–(4). For conclusion (1), we have the following improved results.

Proposition 2. *Let X be a Banach space. Then $1 \le J_L(X) \le \sqrt{3}$ and $\sqrt{3} \le Y_J(X) \le 2$.*

Proof. From $\rho_X(\varepsilon) \ge 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ and Proposition 1, we have

$$J_L(X) = 2 - 2\rho(1) \le 2 - 2\left(1 - \sqrt{1 - \frac{1^2}{4}}\right) = \sqrt{3}.$$

This completes the proof. □

Remark 2. *The Proposition 2 is an improvement of Theorem 1 in [21]. Furthermore, $\sqrt{3}$ is the best upper bound of $J_L(X)$, Since $J_L(X) = \sqrt{3}$ holds for any Hilbert spaces X (see [21] Example 1).*

For the conclusion (2), it's easy for us to know the converse of the conclusion (2) holds, since $\rho_X(1) = \frac{1}{2}$ if and only if X is uniformly non-square and $J_L(X) = 2 - 2\rho_X(1)$. Therefore, we have the following Proposition 3 holds, which means that X is uniformly non-square can be characterized by the side lengths of some special inscribed triangles of unit ball, where the vertices x, y, z of the inscribed triangles need to satisfy $x + y + z = 0$.

Proposition 3. *Let X be a Banach space. Then X is uniformly non-square if and only if $J_L(X) > 1$.*

For the conclusion (3), we assert that the converse does not hold. In fact, Kato and Takahashi [30] have proved that for any $\varepsilon > 0$ there exists a Banach space E which is not strictly convex, with $C_{NJ}(E) < 1 + \varepsilon$ (see [30], Theorem 3). Hence, by $Y_J(X) \leq \sqrt{4C_{NJ}(X) - 1}$ (see [21], Proposition 1), it's easy for us to know the converse of the conclusion (5) is not true.

For the conclusion (4), we will give the following counter example to show that its converse does not hold.

Example 1. *Let $\mathbb{R}^2$ with the $l_1 - l_\infty$ norm defined by*

$$\|x\| = \begin{cases} \|x\|_1 = |x_1| + |x_2| & x_1 x_2 \geq 0, \\ \|x\|_\infty = \max\{|x_1|, |x_2|\} & x_1 x_2 < 0. \end{cases}$$

Then $\mathbb{R}^2$ is uniformly non-square and $Y_J(\mathbb{R}^2) = 2$.

Proof. In [31], Mizuguchi has proved that $C_{NJ}(X)(\mathbb{R}^2) = \frac{3+\sqrt{5}}{4} < 2$, hence $\mathbb{R}^2$ is uniformly non-square.

On the other hand, we put $x = (1, 0)$, $y = (0, -1)$, $z = (-1, 1)$, then $\|x\| = \|y\| = \|z\| = 1$, $x + y + z = 0$. Now, from Proposition 2, we obtain

$$2 \geq Y_J(\mathbb{R}^2) \geq \|y - z\| = \|(1, -2)\| = 2,$$

which means that $Y_J(\mathbb{R}^2) = 2$. □

4. Some Inequalities for $J_L(X)$, $Y_J(X)$, $J(X)$ and $C_{NJ}(X)$

In [21], we have pointed out that $J_L(X)$, $Y_J(X)$, $J(X)$ and $C_{NJ}(X)$ have the following relations:

$$\frac{2}{J(X)} \leq J_L(X) \leq Y_J(X) \leq \sqrt{4C_{NJ}(X) - 1} \leq \sqrt{4J(X) - 1}. \tag{2}$$

Now, we will use the (2) and some conclusions that we have recalled in the Section 2 to give some new inequalities for $J_L(X)$, $Y_J(X)$, $J(X)$ and $C_{NJ}(X)$.

According to the $\frac{J(X)^2}{2} \leq C_{NJ}(X)$ and (2), we will get the following result easily.

Corollary 1. *Let X be a Banach space. Then*

$$\sqrt{\frac{2}{C_{NJ}(X)}} \leq \frac{2}{J(X)} \leq J_L(X) \leq Y_J(X) \leq \sqrt{4C_{NJ}(X) - 1} \leq \sqrt{4J(X) - 1}, \tag{3}$$

Remark 3. *Concerning inequalities in (3), we have the following assertion:*

(1) If X is not an uniformly non-square space, then $\sqrt{\frac{2}{C_{NJ}(X)}} = J_L(X)$ holds. That is because $C_{NJ}(X) = 2$ and $J_L(X) = 1$ hold for not uniformly non-square spaces.

(2) $J_L(X) = \sqrt{4C_{NJ}(X) - 1}$ holds if and only if X is a Hilbert space. If X is a Hilbert space, then $C_{NJ}(X) = 1$ and $J_L(X) = \sqrt{3}$, hence $J_L(X) = \sqrt{4C_{NJ}(X) - 1}$. Conversely, if $J_L(X) = \sqrt{4C_{NJ}(X) - 1}$ holds for X which is not a Hilbert space, then $C_{NJ}(X) > 1$, hence $J_L(X) = \sqrt{4C_{NJ}(X) - 1} > \sqrt{3}$. This contradicts Proposition 2.

Although, the above inequalities seem to be relatively simple, but we still want to ask whether some of them can be improved? The answer is yes. Now, we give the following results.

Proposition 4. *Let X be a Banach space. Then*

$$J(X) \leq Y_J(X) \leq \sqrt{4J(X) - 1}. \tag{4}$$

Proof. First of all, we need to know that for any $x, y \in S_X$ such that $\|x + y\| = \|x - y\|$, we must have $\|x + y\| = \|x - y\| \geq 1$. That is because

$$2 = 2\|x\| \leq \|x - y\| + \|x + y\| = 2\|x - y\|.$$

Now, from the monotonicity of function $\delta_X(\varepsilon)$ and Proposition 1, we have the following inequalities hold for any $x, y \in S_X$ such that $\|x + y\| = \|x - y\|$

$$\|x + y\| \leq 2 - 2\delta_X(\|x - y\|) \leq 2 - 2\delta_X(1) = Y_J(X).$$

According to $J(X) = \sup\{\|x + y\| : \|x - y\| = \|x + y\|, x, y \in S_X\}$, we obtain the conclusion $J(X) \leq Y_J(X)$. For $Y_J(X) \leq \sqrt{4J(X) - 1}$, it dues to (3) (cf. [21], Proposition 2). □

Remark 4. *Concerning inequalities in (4), we have the following assertion:*

(1) If X is not an uniformly non-square space, then $J(X) = Y_J(X)$. That is because $Y_J(X) = 2$ and $J(X) = 2$ hold for not uniformly non-square spaces.

(2) The inequality $J(X) \leq Y_J(X)$ in (4) improves the inequality $\frac{2}{J(X)} \leq Y_J(X)$ in (3), because of $\frac{2}{J(X)} \leq J(X)$ with $J(X) \geq \sqrt{2}$.

(3) In fact, we can also use the same method to prove

$$J(t, x) \leq Y_J(X),$$

where $J(t, x) = \sup\{\min(\|x + ty\|, \|x - ty\|) : x, y \in S_x\}$ $(t \geq 0)$. That is because $J(t, x)$ can also be rewritten as $J(t, x) = \sup\{\|x + ty\| : \|x + ty\| = \|x - ty\|, x, y \in S_x\}$ $(t \geq 0)$ (cf. [28], Theorem 5).

From the (3), (4) and $C_{NJ}(X) \leq J(X)$, we can get the following result easily:

Corollary 2. *Let X be a Banach space. Then*

$$\max\left\{C_{NJ}(X), \sqrt{\frac{2}{C_{NJ}(X)}}\right\} \leq Y_J(X) \leq \sqrt{4C_{NJ}(X) - 1}. \tag{5}$$

Remark 5. *$C_{NJ}(X)$ is not necessarily bigger than $\sqrt{\frac{2}{C_{NJ}(X)}}$. If X is a Hilbert space, then $C_{NJ}(X) < \sqrt{\frac{2}{C_{NJ}(X)}}$. If X is not an uniformly non-square space, then $C_{NJ}(X) > \sqrt{\frac{2}{C_{NJ}(X)}}$.*

Corollary 3. *Let X be a Banach space, X^* be the dual space of X. Then*

$$Y_J(X) \leq \sqrt{4Y_J(X^*) - 1};$$

$$Y_J(X^*) \leq \sqrt{4Y_J(X) - 1};$$

$$|Y_J(X) - Y_J(X^*)| \leq \sqrt{7} - \sqrt{3}.$$

Proof. From (5) and $C_{NJ}(X) = C_{NJ}(X^*)$, we obtain

$$Y_J(X) \leq \sqrt{4C_{NJ}(X) - 1} = \sqrt{4C_{NJ}(X^*) - 1} \leq \sqrt{4Y_J(X^*) - 1},$$

$$Y_J(X^*) \leq \sqrt{4C_{NJ}(X^*) - 1} = \sqrt{4C_{NJ}(X) - 1} \leq \sqrt{4Y_J(X) - 1}.$$

Hence, by $\sqrt{3} \leq Y_J(X) \leq 2$, we compute

$$Y_J(X) - Y_J(X^*) \leq \sqrt{4Y_J(X^*) - 1} - Y_J(X^*) \leq \sqrt{7} - \sqrt{3},$$

$$Y_J(X^*) - Y_J(X) \leq \sqrt{4Y_J(X) - 1} - Y_J(X) \leq \sqrt{7} - \sqrt{3},$$

which means

$$|Y_J(X) - Y_J(X^*)| \leq \sqrt{7} - \sqrt{3}.$$

□

Throughout the above results, it is not difficult to know that $J_L(X)$ and $Y_J(X)$ can be controlled by $C_{\mathrm{NJ}}(X)$ and $J(X)$ respectively. Therefore, we naturally ask whether $C_{\mathrm{NJ}}(X)$ and $J(X)$ can also be controlled by $J_L(X)$ and $Y_J(X)$ respectively? For $Y_J(X)$, we will prove that $C_{\mathrm{NJ}}(X)$ and $J(X)$ can be controlled by some simple inequalities. For $J_L(X)$, from what we've got so far, we need to use $J_L(X)$ and $J(X)$ together to control $C_{\mathrm{NJ}}(X)$. Now, we give the following results.

Corollary 4. *Let X be a Banach space. Then*

$$1 \leq \max\left(\frac{Y_J(X)^2+1}{4}, \frac{2}{Y_J(X)}\right) \leq J(X) \leq Y_J(X) \leq 2. \tag{6}$$

$$1 \leq \frac{Y_J(X)^2+1}{4} \leq C_{\mathrm{NJ}}(X) \leq Y_J(X) \leq 2; \tag{7}$$

Proof. (6) can be obtained directly from (3) and (4). For (7), by (5), it is clear that

$$\frac{Y_J(X)^2+1}{4} = \max\left(\frac{Y_J(X)^2+1}{4}, \frac{2}{Y_J(X)^2}\right) \leq C_{\mathrm{NJ}}(X) \leq Y_J(X),$$

where $\dfrac{Y_J(X)^2+1}{4} = \max\left(\dfrac{Y_J(X)^2+1}{4}, \dfrac{2}{Y_J(X)^2}\right)$ is due to $\sqrt{3} \leq Y_J(X) \leq 2$. □

Remark 6. *$\frac{Y_J(X)^2+1}{4}$ is not necessarily bigger than $\frac{2}{Y_J(X)}$. If X is a Hilbert space, then $\frac{Y_J(X)^2+1}{4} < \frac{2}{Y_J(X)}$. If $Y_J(X) = 2$ (c.f. Example 1), then $\frac{Y_J(X)^2+1}{4} > \frac{2}{Y_J(X)}$.*

In the sequel we need the following Lemmas 1 and 2.

Lemma 1. *Let X be a Banach space, $a > 0$, $t_1 = \dfrac{-a+\sqrt{a^2+4}}{2}$, $f(t) = \dfrac{a+(1+t)^2}{2(1+t^2)}$. Then the following statements hold.*

(1) $f(t)$ is increasing on $[0, t_1]$ and decreasing on $[t_1, 1]$;
(2) if $a = 4J_L(X)^{-2} > 0$, then $f_{max}(t) = f(t_1)$, where $t \in [0, J_L(X)^{-1}]$;
(3) if $a = J(X)^2 > 0$, then $f_{max}(t) = f(J_L(X)^{-1})$, where $t \in [J_L(X)^{-1}, 1]$.

Proof. (1) The monotonicity of the function $f(t)$ in the interval $[0, 1]$ can be known by derivation. $t_1 \in (0, 1)$ can be known directly from $a > 0$. They turn out to be very direct, we omit it.

(2) According to the monotonicity of the function $f(t)$, we just need to prove $t_1 \leq J_L(X)^{-1} \leq 1$. Since $a = 4J_L(X)^{-2} > 0$, we obtain $t_1 \leq J_L(X)^{-1}$ is equivalent to $\frac{-a+\sqrt{a^2+4}}{2} \leq \frac{\sqrt{a}}{2}$ and $\frac{-a+\sqrt{a^2+4}}{2} \leq \frac{\sqrt{a}}{2}$ is equivalent to $2a\sqrt{a} + a - 4 \geq 0$.

Let $g(a) = 2a\sqrt{a} + a - 4$, where $a = 4J_L(X)^{-2} \in \left[\frac{4}{3}, 4\right]$. Then, it is easy for us to know $g_{min}(a) = g(\frac{4}{3}) > 0$. So, by Proposition 2, $t_1 \leq J_L(X)^{-1} \leq 1$ holds.

(3) Also by the monotonicity of the function $f(t)$, we just need to prove $t_1 \leq J_L(X)^{-1} \leq 1$. Since $a = J(X)^2 > 0$, we have $t_1 \leq J_L(X)^{-1}$ is equivalent to $\frac{-J(X)^2+\sqrt{J(X)^4+4}}{2} - \frac{1}{J_L(X)} \leq 0$ and $\frac{-J(X)^2+\sqrt{J(X)^4+4}}{2} - \frac{1}{J_L(X)} \leq 0$ is equivalent to $J_L(X)^2 \leq 1 + J(X)^2 J_L(X)$.

It's easy to prove that $J_L(X)^2 \leq 1 + J(X)^2 J_L(X)$ is true, because

$$J_L(X)^2 \leq 3 = 1 + (\sqrt{2})^2 \times 1 \leq 1 + J(X)^2 J_L(X).$$

So, by Proposition 2, $t_1 \leq J_L(X)^{-1} \leq 1$ holds. □

According to (1) and Proposition 1, we have the following Lemma holds.

Lemma 2. *Let X be a Banach space. Then*

$$\overline{\rho}_X\left(J_L(X)^{-1}\right) = 2J_L(X)^{-1} - 1.$$

Proposition 5. *Let X be a Banach space. Then*

$$\begin{aligned}
&\min\left(\frac{2}{J_L(X)^2}, \frac{J_L(X)^2+1}{4}\right) \leq C_{\mathrm{NJ}}(X) \\
&\leq \max\left(\frac{8J_L(X)^2 + 2J_L(X)^2(\sqrt{16+4J_L(X)^2}-4)}{4J_L(X)^4 + (\sqrt{16+4J_L(x)^4}-4)^2}, \frac{J(X)^2J_L(X)^2 + (J_L(X)+1)^2}{2(J_L(X)^2+1)}\right) \\
&\leq \max\left(\frac{2}{J_L(X)^2} + 1, \frac{J(X)^2J_L(X)^2 + (J_L(X)+1)^2}{2(J_L(X)^2+1)}\right).
\end{aligned} \tag{8}$$

Proof. We only prove the second and third inequalities, the first inequality can be easily known by (3). For any $x, y \in S_X$, $t \in [0,1]$, we have

$$\begin{aligned}
&\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} \\
=&\frac{\min(\|x+ty\|^2, \|x-ty\|^2)}{2(1+t^2)} + \frac{\max(\|x+ty\|^2, \|x-ty\|^2)}{2(1+t^2)} \\
\leq&\frac{(\overline{\rho}_X(t)+1)^2 + (1+t)^2}{2(1+t^2)}.
\end{aligned} \tag{9}$$

To obtain the second and third inequalities, we consider the following two cases.

Case 1: $t \in [0, J_L(X)^{-1}] \subseteq [0,1]$.

From (9), Lemma 1 (2) and Lemma 2, we obtain

$$\begin{aligned}
&\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} \\
\leq&\frac{(\overline{\rho}_X(t)+1)^2 + (1+t)^2}{2(1+t^2)} \leq \frac{\left(\overline{\rho}_X\left(J_L(X)^{-1}\right)+1\right)^2 + (1+t)^2}{2(1+t^2)} \\
\leq&\frac{\left(2J_L(X)^{-1}\right)^2 + (1+t)^2}{2(1+t^2)} \leq \max_{0\leq t\leq J_L(X)^{-1}} \frac{\left(2J_L(X)^{-1}\right)^2 + (1+t)^2}{2(1+t^2)} \\
=&\frac{8J_L(X)^2 + 2J_L(X)^2(\sqrt{16+4J_L(X)^2}-4)}{4J_L(X)^4 + (\sqrt{16+4J_L(x)^4}-4)^2},
\end{aligned} \tag{10}$$

where $(\overline{\rho}_X(t)+1)^2$ is less than $\left(\overline{\rho}_X\left(J_L(X)^{-1}\right)+1\right)^2$ because of the monotonicity of $\overline{\rho}_X(\varepsilon)$ (c.f. [28], Theorem 8).

In addition, we also have

$$\max_{0\le t\le J_L(X)^{-1}} \frac{\left(2J_L(X)^{-1}\right)^2+(1+t)^2}{2(1+t^2)} \le \frac{2}{J_L(X)^2}+1, \tag{11}$$

that is because

$$\frac{\left(2J_L(X)^{-1}\right)^2+(1+t)^2}{2(1+t^2)} \le \frac{2}{J_L(X)^2}+1,\ t\in[0,J_L(X)^{-1}].$$

Case 2: $t\in[J_L(X)^{-1},1]\subseteq[0,1]$.
From (9), Lemma 1 (3), we obtain

$$\begin{aligned}
&\frac{\|x+ty\|^2+\|x-ty\|^2}{2(1+t^2)}\\
\le&\frac{(\overline{\rho}_X(t)+1)^2+(1+t)^2}{2(1+t^2)} \le \frac{(\overline{\rho}_X(1)+1)^2+(1+t)^2}{2(1+t^2)}\\
=&\frac{(J(X))^2+(1+t)^2}{2(1+t^2)} \le \max_{J_L(X)^{-1}\le t\le 1}\frac{(J(X))^2+(1+t)^2}{2(1+t^2)}\\
=&\frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}.
\end{aligned} \tag{12}$$

where $(\overline{\rho}_X(t)+1)^2$ is less than $(\overline{\rho}_X(1)+1)^2$, also dues to the monotonicity of $\overline{\rho}_X(\varepsilon)$ (c.f. [28], Theorem 8).

Finally, notice that $C_{\mathrm{NJ}}(X)=\sup\left\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2(1+t^2)}\ :\ x,y,\in S_X, t\in[0,1]\right\}$, so combining the (10)–(12), we have the second and third inequalities hold. □

Remark 7. *(1)* $\frac{J_L(X)^2+1}{4}$ *is not necessarily bigger than* $\frac{2}{J_L(X)^2}$. *If X is not an uniformly non-square space,* $\frac{J_L(X)^2+1}{4}<\frac{2}{J_L(X)^2}$. *If X is a Hilbert space,* $\frac{J_L(X)^2+1}{4}>\frac{2}{J_L(X)^2}$.

(2) $\frac{8J_L(X)^2+2J_L(X)^2(\sqrt{16+4J_L(X)^2}-4)}{4J_L(X)^4+(\sqrt{16+4J_L(x)^4}-4)^2}$ *is not necessarily bigger than* $\frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}$. *If X is a Hilbert space,*

$$\frac{8J_L(X)^2+2J_L(X)^2(\sqrt{16+4J_L(X)^2}-4)}{4J_L(X)^4+(\sqrt{16+4J_L(x)^4}-4)^2} > \frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}.$$

If X is not an uniformly non-square space, then

$$\frac{8J_L(X)^2+2J_L(X)^2(\sqrt{16+4J_L(X)^2}-4)}{4J_L(X)^4+(\sqrt{16+4J_L(x)^4}-4)^2} < \frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}.$$

(3) $\frac{2}{J_L(X)^2}+1$ *is not necessarily bigger than* $\frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}$. *If X is a Hilbert space,*

$$\frac{2}{J_L(X)^2}+1 > \frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}.$$

If X is not an uniformly non-square space, then

$$\frac{2}{J_L(X)^2}+1 < \frac{J(X)^2J_L(X)^2+(J_L(X)+1)^2}{2(J_L(X)^2+1)}.$$

5. The Constants $L_1(X, \triangle)$ and $L_2(X, \triangle)$

In the last two section, we have discussed $J_L(X)$ and $Y_J(X)$ and got some results. As we said at the beginning of Section 2, the constants $J_L(X)$ and $Y_J(X)$ can be regarded as discussing the infimum and supremum of the side lengths of some special inscribed triangles of unit ball, where the vertices x, y, z of the inscribed triangles need to satisfy $x + y + z = 0$. Further, it's natural for us to consider the following two constants which can be regarded as discussing the infimum and supremum of the perimeters of these special inscribed triangles.

$$L_1(X, \triangle) = \inf\{\|x - y\| + \|x - z\| + \|y - z\| : \|x\| = \|y\| = \|z\| = 1, x + y + z = 0\};$$

$$L_2(X, \triangle) = \sup\{\|x - y\| + \|x - z\| + \|y - z\| : \|x\| = \|y\| = \|z\| = 1, x + y + z = 0\}.$$

In this section, we will make a discussion on the bounds of $L_1(X, \triangle)$ and $L_2(X, \triangle)$ and calculate the values of $L_1(X, \triangle)$ and $L_2(X, \triangle)$ for two Banach spaces.

Lemma 3. *[32] Let X be a normed space. If $x, y, z \in X$ which are not all zero satisfy $x + y + z = 0$, then*

$$\|x - y\| + \|y - z\| + \|z - x\| \geq \frac{3}{2}(\|x\| + \|y\| + \|z\|). \tag{13}$$

According to the above lemma 3 and the triangle inequality of the norm, it is easy to get the following conclusion.

Proposition 6. *Let X be a Banach space. Then $\frac{9}{2} \leq L_1(X, \triangle) \leq L_2(X, \triangle) \leq 6$.*

In the proof of Proposition 6, we use inequality (13). Therefore, it's natural for us to consider the following constants.

$$\triangle_{LYJ}(X) = \inf\left\{\frac{\|x - y\| + \|y - z\| + \|x - z\|}{\|x\| + \|y\| + \|z\|} \ : \ x + y + z = 0, (x, y, z) \neq (0, 0, 0)\right\};$$

$$\triangle'_{LYJ}(X) = \sup\left\{\frac{\|x - y\| + \|y - z\| + \|x - z\|}{\|x\| + \|y\| + \|z\|} \ : \ x + y + z = 0, (x, y, z) \neq (0, 0, 0)\right\};$$

$$\triangle_{Li}(X) = \inf\left\{\frac{\|x - y\|}{\|x\| + \|y\| + \|z\|}, \frac{\|y - z\|}{\|x\| + \|y\| + \|z\|}, \frac{\|x - z\|}{\|x\| + \|y\| + \|z\|} : x + y + z = 0, \right.$$
$$\left. (x, y, z) \neq (0, 0, 0)\right\};$$

$$\triangle'_{Li}(X) = \sup\left\{\frac{\|x - y\|}{\|x\| + \|y\| + \|z\|}, \frac{\|y - z\|}{\|x\| + \|y\| + \|z\|}, \frac{\|x - z\|}{\|x\| + \|y\| + \|z\|} : x + y + z = 0, \right.$$
$$\left. (x, y, z) \neq (0, 0, 0)\right\}.$$

What's interesting is that these four constants are all fixed values.

Proposition 7. *Let X be Banach space. Then $\triangle_{LYJ}(X) = \frac{3}{2}$, $\triangle'_{LYJ}(X) = 2$, $\triangle_{Li}(X) = 0$, $\triangle'_{Li}(X) = 1$.*

Proof. According to Lemma 3 and the nonnegativity of norm, it is easy to see $\triangle_{LYJ}(X) \geq \frac{3}{2}$, $\triangle_{Li}(X) \geq 0$. Now, we put $y = z = -\frac{1}{2}x$, then

$$\frac{\|x - y\| + \|y - z\| + \|x - z\|}{\|x\| + \|y\| + \|z\|} = \frac{3}{2}, \ \frac{\|y - z\|}{\|x\| + \|y\| + \|z\|} = 0,$$

which implies $\triangle_{LYJ}(X) = \frac{3}{2}$, $\triangle_{Li}(X) = 0$,

On the other hand, since the triangle inequality of the norm and $\|x - y\| \leq \|x\| + \|y\| \leq \|x\| + \|y\| + \|z\|$, it is easy to see $\triangle'_{LYJ}(X) \leq 2$, $\triangle'_{Li}(X) \leq 1$. Now, we put $y = -x$, $z = 0$, then

$$\frac{\|x - y\| + \|y - z\| + \|x - z\|}{\|x\| + \|y\| + \|z\|} = 2, \quad \frac{\|x - y\|}{\|x\| + \|y\| + \|z\|} = 1,$$

which means $\triangle'_{LYJ}(X) = 2$, $\triangle'_{Li}(X) = 1$. □

Finally, we end this article with two examples. These two examples show that the bounds given in the Proposition 6 are optimal.

Example 2. *Let l_∞ be the linear space of all bounded sequences in $\mathbb{R}$ with the norm defined by*

$$\|x\| = \sup_{1 \leq i < \infty} |x_i|.$$

Then $L_1(l_\infty, \triangle) = \frac{9}{2}$ and $L_2(l_\infty, \triangle) = 6$.

Proof. Let $x = \left(\frac{1}{2}, \frac{1}{2}, -1, 0, \cdots\right)$, $y = \left(\frac{1}{2}, -1, \frac{1}{2}, 0, \cdots\right)$, $z = \left(-1, \frac{1}{2}, \frac{1}{2}, 0, \cdots\right)$, then $\|x\| = \|y\| = \|z\| = 1$, $x + y + z = 0$. So, via the Proposition 6, we have

$$\begin{aligned}
\frac{9}{2} &\leq L_1(l_\infty, \triangle) \\
&\leq \|x - y\| + \|y - z\| + \|x - z\| \\
&= \left\|\left(0, \frac{3}{2}, -\frac{3}{2}, 0, \cdots\right)\right\| + \left\|\left(\frac{3}{2}, -\frac{3}{2}, 0, 0, \cdots\right)\right\| + \left\|\left(\frac{3}{2}, 0, -\frac{3}{2}, 0, \cdots\right)\right\| \\
&= \frac{9}{2},
\end{aligned}$$

which implies that $L_1(l_\infty, \triangle) = \frac{9}{2}$.

On the other hand, we put $x_n = \left(-\frac{1}{n}, 1, 1, 0, \cdots\right)$, $y_n = (1, 0, -1, 0, \cdots)$, $z_n = \left(\frac{1}{n} - 1, -1, 0, 0, \cdots\right)$, then $\|x_n\| = \|y_n\| = \|z_n\| = 1$, $x_n + y_n + z_n = 0$. Now, from the Proposition 6, we obtain

$$\begin{aligned}
6 &\geq L_2(l_\infty, \triangle) \\
&\geq \|x_n - y_n\| + \|y_n - z_n\| + \|x_n - z_n\| \\
&= \left\|\left(-\frac{1}{n} - 1, 1, 2, 0, \cdots\right)\right\| + \left\|\left(2 - \frac{1}{n}, 1, -1, 0, \cdots\right)\right\| + \left\|\left(-\frac{2}{n} + 1, 2, 1, 0, \cdots\right)\right\|.
\end{aligned}$$

Let $n \to \infty$, yield $6 \geq L_2(l_\infty, \triangle) \geq 6$, which means that $L_2(l_\infty, \triangle) = 6$. □

Example 3. *Let $\mathbb{R}^2$ with the $l_\infty - l_1$ norm*

$$\|x\| = \begin{cases} \|x\|_\infty = \max\{|x_1|, |x_2|\} & x_1 x_2 \geq 0, \\ \|x\|_1 = |x_1| + |x_2| & x_1 x_2 < 0. \end{cases}$$

Then $L_1(\mathbb{R}^2, \triangle) = \frac{9}{2}$ and $L_2(\mathbb{R}^2, \triangle) = 6$.

Proof. Let $x = \left(-\frac{1}{2}, \frac{1}{2}\right)$, $y = \left(1, \frac{1}{2}\right)$, $z = \left(-\frac{1}{2}, -1\right)$, then $\|x\| = \|y\| = \|z\| = 1$, $x + y + z = 0$. So, via the Proposition 6, we have

$$\begin{aligned}\frac{9}{2} \leq L_1(\mathbb{R}^2, \triangle) &\leq \|x - y\| + \|y - z\| + \|x - z\| \\ &= \left\|\left(-\frac{3}{2}, 0\right)\right\| + \left\|\left(\frac{3}{2}, \frac{3}{2}\right)\right\| + \left\|\left(0, \frac{3}{2}\right)\right\| = \frac{9}{2},\end{aligned}$$

which implies that $L_1(\mathbb{R}^2, \triangle) = \frac{9}{2}$.

On the other hand, we put $x = (-1, 0)$, $y = (0, -1)$, $z = (1, 1)$, then $\|x\| = \|y\| = \|z\| = 1$, $x + y + z = 0$. Now, from the Proposition 6, we obtain

$$\begin{aligned}6 \geq L_2(\mathbb{R}^2, \triangle) &\geq \|x - y\| + \|y - z\| + \|x - z\| \\ &= \|(-1, 1)\| + \|(-1, -2)\| + \|(-2, -1)\| = 6,\end{aligned}$$

which means that $L_2(\mathbb{R}^2, \triangle) = 6$. □

6. Conclusions

In this paper, we have done some further discussions on the $J_L(X)$ and $Y_J(X)$ and introduced two new geometric constants $L_1(X, \triangle)$, $L_2(X, \triangle)$. These four geometric constants are all related to some special inscribed triangles of the unit ball. Our results in this paper mainly include the following three aspects. First, the relations among $J_L(X)$, $Y_J(X)$ and some geometric properties of Banach spaces. Second, some simple inequalities for $J_L(X)$, $Y_J(X)$ and some significant geometric constants. Third, the bounds of $L_1(X, \triangle)$, $L_2(X, \triangle)$ and the values of $L_1(X, \triangle)$ and $L_2(X, \triangle)$ for two Banach spaces. Among these results, the most notable one is that uniform non-square spaces can be characterized by the sides lengths of some special inscribed triangles of unit ball, where the vertices x, y, z of the inscribed triangles need to satisfy $x + y + z = 0$.

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