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Generalized Lie Triple Derivations of Lie Color Algebras and Their Subalgebras

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Abstract: Consider a Lie color algebra, denoted by $\mathcal{L}$. Our aim in this paper is to study the Lie triple derivations $TDer(\mathcal{L})$ and generalized Lie triple derivations $GTDer(\mathcal{L})$ of Lie color algebras. We discuss the centroids, quasi centroids and central triple derivations of Lie color algebras, where we show the relationship of triple centroids, triple quasi centroids and central triple derivation with Lie triple derivations and generalized Lie triple derivations of Lie color algebras $\mathcal{L}$. A classification of Lie triple derivations algebra of all perfect Lie color algebras is given, where we prove that for a perfect and centerless Lie color algebra, $TDer(\mathcal{L}) = Der(\mathcal{L})$ and $TDer(Der(\mathcal{L})) = Inn(Der(\mathcal{L}))$.

Keywords: Lie triple derivations; centroids; quasi centroids; Lie color algebras; Generalizd Lie triple derivations

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1. Introduction

The generalization of Lie algebra is introduced by Ree [1], which is now known as Lie color algebra. Lie color algebra plays an important role in theoretical physics, as expalained in [2,3]. Montgomery [4] proved that Simple Lie color algebra can be obtained from associative graded algebra, while the Ado theorem and the PBW theorem of Lie color algebra were proven by Scheunert [5]. In the last two decades, Lie color algebra has developed as an interesting topic in mathematics and physics [6–10].

The concept of derivations contributes significantly in the different mathematical fields such as in associative (non-associative) rings and operator algebras. In algebra, derivation is usually a linear map that satisfies the Leibniz rule. Researchers have worked on the concept of derivations, generalized derivations, centroids and quasi centroids with different perspectives in [11–14]. In fact there are various forms of derivations in algebra (Lie algebra) such as double derivations, triple derivations, and n-derivations. In the present article, we focus on the Lie triple derivations, which are first introduced by Müller [15]. Later on, various authors investigated the triple derivations in different algebraic settings. Wang and Xiao [16] studied the Lie triple derivations of incidence algebras, which is a type of operator algebra. Triple derivations of another operator algebra called nest algebra was discussed by Zhang [17]. Xiao and Wei [18] have researched the Lie triple derivations of triangular algebra. Furthermore, Lie triple derivations of some von Neumann algebra are studied by Qi [19], where it is proven that a Lie triple derivation of von Neumann algebra is the sum of a derivation algebra and a special additive map that sends the commutator to zero. Zhou [20] studied the triple derivations of perfect Lie algebra, where it is proven that Lie triple derivations of the perfect Lie algebra are in fact a derivation algebra. Moreover, every Lie triple derivation of the derivation algebra is an inner derivation. Later, this work was extended to Lie superalgebras in [21].

Our purpose in this paper is to discuss the Lie triple derivations $TDer(\mathcal{L})$ (generalized Lie triple derivations $GTDer(\mathcal{L})$) of a Lie color algebra $\mathcal{L}$. We discuss centroids and quasi centroids of Lie color algebras and evaluate some important results. In addition, our main

result consists in the complete classification of Lie triple derivations of a Lie color algebra. We prove that, for perfect Lie color algebra,

- Lie triple derivations algebra coincide with Lie derivations algebra.
- Lie triple derivations of derivations algebra coincide with inner derivations algebra.

This paper is organized as follows; in Section 2, we recall some important definitions and notions related to Lie color algebras $\mathcal{L}$. Along with presenting some interesting propositions, we show that Lie triple derivations $TDer(\mathcal{L})$ (generalized triple derivations $GTDer(\mathcal{L})$) of a Lie color algebra $\mathcal{L}$ form a subalgebra of the general linear Lie color algebra $Kl(\mathcal{L})$. In Section 3, we define triple centroid $TC(\mathcal{L})$ and triple quasi centroid $TQC(\mathcal{L})$ of Lie color algebra. We show that, for centerless Lie color algebra, centroid and quasi centroid belong to commutative Lie color algebra, as explained in the literature [20,22,23]. Furthermore, we obtain the relation of centroid and quasi centroid with Lie triple derivations (generalized Lie triple derivations) of Lie color algebra $\mathcal{L}$. In Section 4, we prove our main results in Theorems 2 and 3, where we prove that for the perfect and centerless Lie color algebras $\mathcal{L}$, we have $TDer(\mathcal{L}) = Der(\mathcal{L})$ and $TDer(Der(\mathcal{L})) = Inn(Der(\mathcal{L}))$. We prove our results by giving some interesting lemmas.

2. On the Lie Triple Derivations of Lie Color Algebras

Consider a Lie color algebra $\mathcal{L}$ over a field $\mathcal{F}$ with a characteristic denoted by $Char(\mathcal{F})$, satisfying $Char(\mathcal{F}) \neq 2$. The operation of $\mathcal{L}$ is denoted by $[.,.]$. Let $\mathcal{F}^* = \mathcal{F}\backslash\{0\}$ be the group of units of $\mathcal{F}$ and $hg(\mathcal{L})$ be the set of all homogeneous elements in $\mathcal{L}$. Suppose that x is a homogeneous element and its degree is represented by $\sigma(x)$. We use $\mathcal{G}$ to denote a fixed abelian group, and θ, μ, λ are some notions for the elements of $\mathcal{G}$. A Lie color algebra $\mathcal{L}$ is called perfect if its derived subalgebra $[\mathcal{L}, \mathcal{L}]$ is equal to itself $\mathcal{L}$. The center of $\mathcal{L}$ is denoted by $Z(\mathcal{L})$. To introduce the concept of a Lie color algebra, we recall the bicharacter of an abelian group.

Definition 1. *Let $\mathcal{F}$ be a field and $\mathcal{G}$ be an abelian group. A map $\epsilon : \mathcal{G} \times \mathcal{G} \to \mathcal{F}^*$ is called a skew-symmetric bicharacter on $\mathcal{G}$ if the following identities hold, for all $f, g, h \in \mathcal{G}$:*

1. $\epsilon(f, g + h) = \epsilon(f, g)\epsilon(f, h)$,
2. $\epsilon(g + h, f) = \epsilon(g, f)\epsilon(h, f)$,
3. $\epsilon(g, h)\epsilon(h, g) = 1$.

With the notation of bicharacter, we can use it to define Lie color algebras as follows. For simplicity, we use $\epsilon(s, t)$ a shorthand notation for $\epsilon(\sigma(s), \sigma(t))$ in the next definition.

Definition 2. *A $\mathcal{G}$-graded vector space $\mathcal{L} = \oplus_{g\in\mathcal{G}}\mathcal{L}_g$ with a graded bilinear map $[.,.] : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ is called a Lie color algebra if the bracket $[.,.]$ satisfies:*

1. $[L_{\theta_1}, L_{\theta_2}] \subseteq L_{\theta_1+\theta_2}, \forall \theta_1, \theta_2 \in G$,
2. $[s, t] = -\epsilon(s, t)[t, s]$, *and*
3. $\epsilon(u, s)[s, [t, u]] + \epsilon(s, t)[t, [u, s]] + \epsilon(t, u)[u, [s, t]] = 0$,

for $s \in L_{\sigma(s)}, t \in L_{\sigma(t)}, u \in L_{\sigma(u)}, \sigma(s), \sigma(t), \sigma(u) \in \mathcal{G}$ [7].

Example 1.

1. *If $\mathcal{G} = Z_2$ (the additive group of integers modulo 2) and if one defines ϵ as $\epsilon(i, j) := (-1)^{ij}$ for all $i, j \in Z_2$, then Lie color algebras are just Lie superalgebras.*
2. *If $\epsilon(i, j) := 1$ for all $i, j \in \mathcal{G}$, then a Lie color algebra is a $\mathcal{G}$-graded Lie algebra [24].*
3. *Suppose that $A = \oplus_{g\in\mathcal{G}}A_g$ is an associative $\mathcal{G}$-graded algebra and ϵ is a skew-symmetric bicharacter on $\mathcal{G}$. Let $[s, t] = st - \epsilon(s, t)ts$ (the ϵ-commutator of s, t) for all $s \in A_\sigma, t \in A_\theta$. Then $(A, [.,.])$ turns out to be a Lie color algebra.*

For any two vector spaces V and W, we use $Hom(V, W)$ to denote the space of all linear mappings from V to W in the sequel.

Definition 3. *Let $\mathcal{L} = \oplus_{\theta\in\mathcal{G}}\mathcal{L}_\theta$ be a $\mathcal{G}$-graded space. Define $Kl(\mathcal{L}) = \oplus_{\theta\in\mathcal{G}}Kl_\theta(\mathcal{L})$, where $Kl_\theta(\mathcal{L}) := \{D \in Hom(\mathcal{L},\mathcal{L}) : D(\mathcal{L}_\mu) \subseteq L_{\theta+\mu} \quad \forall \mu \in \mathcal{G}\}$.*

It is easy to check that $Kl(\mathcal{L}) = \oplus_{\theta\in\mathcal{G}}Kl_\theta(\mathcal{L})$ is a Lie color algebra over $\mathcal{F}$ with the bracket

$$[D_\theta, D_\mu] = D_\theta D_\mu - \epsilon(\theta,\mu)D_\mu D_\theta, \tag{1}$$

for all $D_\theta, D_\mu \in hg(Kl(\mathcal{L}))$.

Definition 4. *A homogeneous derivation of degree θ of a Lie color algebra $\mathcal{L} = \oplus_{\theta\in\mathcal{G}}\mathcal{L}_\theta$ is an element $D \in Kl_\theta(\mathcal{L})$ such that*

$$[D(s), t] + \epsilon(\theta, s)[s, D(t)] = D([s,t]), \tag{2}$$

for all $s \in hg(\mathcal{L}), t \in \mathcal{L}$.

Let $Der_\theta(\mathcal{L})$ be the set of homogeneous derivations in $Kl_\theta(\mathcal{L})$. Then $Der(\mathcal{L}) := \oplus_{\theta\in\mathcal{G}}Der_\theta(\mathcal{L})$ is a Lie color subalgebra of $Kl(\mathcal{L})$ and is called the derivation algebra of $\mathcal{L}$.

As a generalization of derivations of a Lie color algebra, we introduce the concept of Lie triple derivations of a Lie color algebra as follows.

Definition 5. *Let $\mathcal{L} = \oplus_{\theta\in\mathcal{G}}\mathcal{L}_\theta$ be a Lie color algebra and $D \in Kl_\theta(\mathcal{G})$. Then D is called Lie triple derivation of degree θ of $\mathcal{L}$, if*

$$D([[s,t],u]) = [[D(s),t],u] + \epsilon(\theta,s)[[s,D(t)],u] + \epsilon(\theta,s+t)[[s,t],D(u)] \tag{3}$$

for all $s, t \in hg(\mathcal{L}), u \in \mathcal{L}$.

It is obvious that every derivation in Lie color algebra is indeed a Lie triple derivation, but the converse is not always true in general. The set of all Lie triple derivations of degree θ of $\mathcal{L}$ is denoted by $TDer_\theta(\mathcal{L})$. Let $TDer(\mathcal{L}) = \oplus_{\theta\in\mathcal{G}}TDer_\theta(\mathcal{L})$. Next, we will prove that $TDer(\mathcal{L})$ is a Lie color subalgebra of $Kl(\mathcal{L})$.

Proposition 1. *Suppose that $\mathcal{L} := \oplus_{\theta\in\mathcal{G}}\mathcal{L}_\theta$ is a Lie color algebra. Then $TDer(\mathcal{L})$ is a Lie color subalgebra of $Kl(\mathcal{L})$.*

Proof. For any $D_1 \in TDer_{\theta_1}(\mathcal{L})$ and $D_2 \in TDer_{\theta_2}(\mathcal{L})$, consider,

$$
\begin{aligned}
&D_1D_2([[s,t],u])\\
=&D_1([[D_2(s),(t)],(u)]+\epsilon(\theta_2,s)[[(s),D_2(t)],(u)]+\epsilon(\theta_2,s+t)[[(s),(t)],D_2(u)])\\
=&([[D_1D_2(s),t],(u)]+\epsilon(\theta_1,\theta_2+s)[[D_2(s),D_1(t)](u)]\\
&+\epsilon(\theta_1,\theta_2+s+t)[[D_2(s),t],D_1(u)])+\epsilon(\theta_2,s)([[D_1(s),D_2(t)],u]\\
&+\epsilon(\theta_1,s)[[(s),D_1D_2(t)],u]+\epsilon(\theta_1,\theta_2+s+t)[[s,D_2(t)],D_1(u)])\\
&+\epsilon(\theta_2,s+t)([[D_1(s),t],D_2(u)]+\epsilon(\theta_1,s)[[(s),D_1(t)],D_2(u)]\\
&+\epsilon(\theta_1,s+t)[[s,t],D_1D_2(u)])\\
=&[[D_1D_2(s),t],u]+\epsilon(\theta_1,\theta_2+s)[[D_2(s),D_1(t)],u]\\
&+\epsilon(\theta_1,\theta_2+s+t)[[(D_2(s)),(t)],D_1(u)]+\epsilon(\theta_2,s)[[D_1(s),D_2(t)],(u)]\\
&+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_2,s)[[s,D_1D_2(t)],(u)]+\epsilon(\theta_1,s+t+\theta_2)\epsilon(\theta_2,s)[[s,D_2(t)],D_1(u)]\\
&+\epsilon(\theta_2,s+t)[[D_1(s),(t)],D_2(u)]+\epsilon(\theta_1,s)\epsilon(\theta_2,s+t)[[s,D_1(t)],D_2(u)]\\
&+\epsilon(\theta_1,s+t)\epsilon(\theta_2,s+t)[[s,(t)],D_1D_2(u)]\\
=&[[D_1D_2(s),t],u]+\epsilon(\theta_1+\theta_2,s)[[s,D_1D_2(t)],u]\\
&+\epsilon(\theta_1+\theta_2,s+t)[[s,t],D_1D_2(u)]+\epsilon(\theta_1,\theta_2+s)[[D_2(s),D_1(t)],u]\\
&+\epsilon(\theta_1,s+t+\theta_2)[[D_2(s),t],D_1(u)]+\epsilon(\theta_2,s)[[D_1(s),D_2(t)],(u)]\\
&+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_1,\theta_2+t)[[s,D_2(t)],D_1(u)]+\epsilon(\theta_2,s+t)[[D_1(s),t],D_2(u)]\\
&+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_2,t)[[s,D_1(t)],D_2(u)].
\end{aligned}
\tag{4}
$$

On the other hand, we have

$$
\begin{aligned}
&D_2D_1([[s,t],u])\\
=&D_2([[D_1(s),t],u]+\epsilon(\theta_1,s)[[s,D_1(t)],u]+\epsilon(\theta_1,s+t)[[s,t],D_1(u)])\\
=&([[D_2D_1(s),(t)],u]+\epsilon(\theta_1+\theta_2,s)[[s,D_2D_1(t)],u]+\epsilon(\theta_1+\theta_2,s+t)[[s,t],D_2D_1(u)])\\
&+\epsilon(\theta_2,s+\theta_1)[[D_1(s),D_2(t)],(u)]+\epsilon(\theta_2,s+t+\theta_1)[[D_1(s),(t)],D_2(u)]\\
&+\epsilon(\theta_1,s)[[D_2(s),D_1(t)],u]+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_2,\theta1+t)[[s,D_1(t)],D_2(u)]\\
&+\epsilon(\theta_1,s+t)[[D_2(s),t],D_1(u)]+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_1,t)[[s,D_2(t)],D_1(u)]
\end{aligned}
\tag{5}
$$

From Equations (4) and (5), we have

$$
\begin{aligned}
[D_1,D_2]([[s,t],u]) \quad =&[[[D_1,D_2](s),(t)],u]+\epsilon(\theta_1+\theta_2,s)[[s,[D_1,D_2](t)],u]\\
&+\epsilon(\theta_1+\theta_2,s+t)[[(s),t],[D_1,D_2](u)]
\end{aligned}
\tag{6}
$$

which imply that $[D_1,D_2]\in TDer_{\theta_1+\theta_2}(\mathcal{L})$ for any $D_1\in TDer_{\theta_1}(\mathcal{L})$ and $D_2\in TDer_{\theta_2}(\mathcal{L})$. This completes the proof. □

A further generalization of Lie triple derivation is a generalized Lie triple derivation defined as follows.

Definition 6. *Let $\mathcal{L}=\oplus_{\theta\in\mathcal{G}}\mathcal{L}_\theta$ be a Lie color algebra and $D\in Kl_\theta(\mathcal{L})$. Then D is called a generalized Lie triple derivation of $\mathcal{L}$ if there is $E\in TDer_\theta(\mathcal{L})$ related to D such that,*

$$
D([[s,t],u])=[[D(s),t],u]+\epsilon(\theta,s)[[s,E(t)],u]+\epsilon(\theta,s+t)[[s,t],E(u)] \tag{7}
$$

for all $s,t\in hg(\mathcal{L}),u\in\mathcal{L}$.

It is obvious that a Lie triple derivation is a generalized Lie triple derivation with $D=E$, so $TDer(\mathcal{L})\subseteq GTDer(\mathcal{L})$, but its converse is not always true.

We denote all generalized Lie triple derivations on a Lie color algebra by $GTDer(\mathcal{L})=\oplus_{\theta\in\mathcal{G}}GTDer_\theta(\mathcal{L})$. Just like the set of all Lie triple derivations, the set of all generalized Lie triple derivations also forms a Lie color subalgebra of the Lie color algebra of linear maps $Kl(\mathcal{L})$.

Proposition 2. *$GTDer(\mathcal{L})$ is a Lie color subalgebra of $Kl(\mathcal{L})$.*

Proof. To prove $GTDer(\mathcal{L}) \subseteq Kl(\mathcal{L})$, we must satisfy $[D_1, D_2] \in GTDer_{\theta_1+\theta_2}(\mathcal{L})$, for any $D_1 \in GTDer_{\theta_1}(\mathcal{L})$ and $D_2 \in GTDer_{\theta_2}(\mathcal{L})$. For any $s, t \in hg(\mathcal{L})$ and $u \in \mathcal{L}$, we have

$$\begin{aligned}
&D_1D_2([[s,t],u])\\
=&D_1([[D_2(s),t],u]+\epsilon(\theta_2,s)[[s,E_2(t)],u]+\epsilon(\theta_2,s+t)[[s,t],E_2(u)])\\
=&D_1([[D_2(s),t],u])+\epsilon(\theta_2,s)D_1([[s,E_2(t)],u])+\epsilon(\theta_2,s+t)D_1([[s,t],E_2(u)])\\
=&([[D_1D_2(s),t],u]+\epsilon(\theta_1,\theta_2+s)[[D_2(s),E_1(t)],u]\\
&+\epsilon(\theta_1,\theta_2+s+t)[[D_2(s),t],E_1(u)])+\epsilon(\theta_2,s)([[D_1(s),E_2(t)],u]\\
&+\epsilon(\theta_1,s)[[s,E_1E_2(t)],u]+\epsilon(\theta_1,\theta_2+s+t)[[s,E_2(t)],E_1(u)])\\
&+\epsilon(\theta_2,s+t)([[D_1(s),t],E_2(u)]+\epsilon(\theta_1,s)[[s,E_1(t)],E_2(u)]\\
&+\epsilon(\theta_1,s+t)[[s,t],E_1E_2(u)])\\
=&([[D_1D_2(s),t],u]+\epsilon(\theta_1+\theta_2,s)[[s,E_1E_2(t)],u]\\
&+\epsilon(\theta_1+\theta_2,s+t)[[s,t],E_1E_2(u)])+\epsilon(\theta_1,\theta_2+s)[[D_2(s),E_1(t)],u]\\
&+\epsilon(\theta_1,\theta_2+s+t)[[D_2(s),t],E_1(u)]+\epsilon(\theta_2,s)[[D_1(s),E_2(t)],u]\\
&+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_1,\theta_2+t)[[s,E_2(t)],E_1(u)]+\epsilon(\theta_2,s+t)[[D_1(s),t],E_2(u)]\\
&+\epsilon(\theta_1+\theta_2,s)\epsilon(\theta_2,t)[[s,E_1(t)],E_2(u)].
\end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}
&D_2D_1([[s,t],u])\\
=&D_2([[D_1(s),t],u]+\epsilon(\theta_1,s)[[s,E_1(t)],u]+\epsilon(\theta_1,s+t)[[s,t],E_1(u)])\\
=&([[D_2D_1(s),t],u]+\epsilon(\theta_2+\theta_1,s)[[s,E_2E_1(t)],u]\\
&+\epsilon(\theta_2+\theta_1,s+t)[[s,t],E_2E_1(u)])+\epsilon(\theta_2,\theta_1+s)[[D_1(s),E_2(t)],u]\\
&+\epsilon(\theta2,\theta_1+s+t)[[D_1(s),t],E_2(u)]+\epsilon(\theta_1,s)[[D_2(s),E_1(t)],u]\\
&+\epsilon(\theta_2+\theta_1,s)\epsilon(\theta_2,\theta_1+t)[[s,E_1(t)],E_2(u)]+\epsilon(\theta_1,s+t)[[D_2(s),t],E_1(u)]\\
&+\epsilon(\theta_2+\theta_1,s)\epsilon(\theta_1,t)[[s,E_2(t)],E_1(u)]
\end{aligned} \tag{9}$$

Thus,

$$\begin{aligned}
[D_1,D_2]([[s,t],u]) = \quad &[[[D_1,D_2](s),t],u]+\epsilon(\theta2+\theta1,s)[[s,[E_1,E_2](t)],u]\\
&+\epsilon(\theta2+\theta1,s+t)[[s,t],[E_1,E_2](u)].
\end{aligned} \tag{10}$$

Hence, $[D_1, D_2] \in GTDer_{\theta_1+\theta_2}(\mathcal{L})$, i.e., $GTDer(\mathcal{L})$ is a subalgebra of the Lie color algebra $Kl(\mathcal{L})$. This completes the proof. □

Lemma 1 ([25]). *For any $s, t, u \in hg(\mathcal{L})$, $D \in GTDer(\mathcal{L})$ and $E \in TDer(\mathcal{L})$ related to D, we can obtain that*

1. $(D-E)([[s,t],u]) = [[(D-E)(s),t],u] = \epsilon(\theta,s)[[s,(D-E)(t)],u]$
   $= \epsilon(\theta,s+t)[[s,t],(D-E)(u)]$.
2. $D([s,[t,u]]) = [D(s),[t,u]]+\epsilon(\theta,s)[s,[E(t),u]]+\epsilon(\theta,s+t)[s,[t,E(u)]]$.
3. $(D-E)([s,[t,u]]) = [(D-E)(s),[t,u]] = \epsilon(\theta,s)[s,[(D-E)(t),u]]$
   $= \epsilon(\theta,s+t)[s,[t,(D-E)(u)]]$.

Proof. Consider that $D \in GTDer_\theta(\mathcal{L})$ and E is the Lie triple derivation related to D. For $s, t, u \in hg(\mathcal{L})$, we have

$$\begin{aligned}
D([[s,t],u]) =&[[D(s),t],u]+\epsilon(\theta,s)[[s,E(t)],u]+\epsilon(\theta,s+t)[[s,t],E(u)]\\
=&[[D(s),t],u]+E([[s,t],u])-[[E(s),t],u];
\end{aligned} \tag{11}$$

this implies that

$$(D-E)([[s,t],u]) = [[(D-E)(s),t],u]. \tag{12}$$

Now consider,

$$\begin{aligned}(D-E)([[s,t],u]) =& -\epsilon(s,t)(D-E)([[t,s],u]) = -\epsilon(s,t)[[(D-E)(t),s],u]\\ =&\epsilon(\theta,s)[[s,(D-E)(t)],u].\end{aligned} \tag{13}$$

By the Jacobi identity $[x,[y,z]] = [[x,y],z] + \epsilon(x+y,z)[[z,x],y]$ and (12), we observe that:

$$\begin{aligned}&(D-E)([[s,t],u])\\ =&-\epsilon(s+t,u)(D-E)([u,[s,t]])\\ =&-\epsilon(s+t,u)(D-E)([[u,s],t]+\epsilon(u+s,t)[[t,u],s])\\ =&-\epsilon(s+t,u)(D-E)([[u,s],t]) - \epsilon(s,u+t)(D-E)([[t,u],s])\\ =&-\epsilon(s+t,u)[[(D-E)(u),s],t] - \epsilon(s,u)\epsilon(\theta+s,t)[[t,(D-E)(u)],s]\\ =&-\epsilon(s+t,u)[[(D-E)(u),s],t] + \epsilon(s,u)\epsilon(\theta+s,t)\epsilon(t+\theta+u,s)[s,[t,(D-E)(u)]]\\ =&-\epsilon(s+t,u)[[(D-E)(u),s],t] + \epsilon(\theta,s+t)[s,[t,(D-E)(u)]]\\ =&-\epsilon(s+t,u)[[(D-E)(u),s],t] + \epsilon(\theta,s+t)([[s,t],(D-E)(u)]\\ &+\epsilon(s+t,\theta+u)[[(D-E)(u),s],t])\\ =&-\epsilon(s+t,u)[[(D-E)(u),s],t] + \epsilon(\theta,s+t)[[s,t],(D-E)(u)]\\ &+\epsilon(s+t,u)[[(D-E)(u),s],t]\\ =&\epsilon(\theta,s+t)[[s,t],(D-E)(u)].\end{aligned} \tag{14}$$

Equations (12)–(14) give the proof of (1). Similarly, we can prove (3).

Applying $D \in GTDer(\mathcal{L})$ on Jacobi identity:

$$[s,[t,u]] = [[s,t],u] + \epsilon(s+t,u)[[u,s],t], \tag{15}$$

we get that

$$\begin{aligned}&D[s,[t,u]]\\ =&D([[s,t],u]) + \epsilon(s+t,u)D([[u,s],t])\\ =&[[D(s),t],u] + \epsilon(\theta,s)[[s,E(t)],u] + \epsilon(\theta,s+t)[[s,t],E(u)]\\ &+\epsilon(s+t,u)([[D(u),s],t] + \epsilon(\theta,u)[[u,E(s)],t] + \epsilon(\theta,u+s)[[u,s],E(t)])\\ =&[D(s),[t,u]] + \epsilon(\theta,s)[s,[E(t),u]] + \epsilon(\theta,s+t)[s,[t,E(u)]].\end{aligned} \tag{16}$$

Hence, (2) follows. □

Definition 7. *A linear map $D \in Kl_\theta(\mathcal{L})$ is said to be a* central triple derivation *of $\mathcal{L}$ if, for $s,t \in hg(\mathcal{L})$ and $u \in \mathcal{L}$, we have*

$$D([[s,t],u]) = [[D(s),t],u] = 0, \tag{17}$$

The set of all central Lie triple derivations of Lie color algbera $\mathcal{L}$ is denoted by $ZTDer(\mathcal{L}) = \oplus_{\theta\in\mathcal{G}} ZTDer_\theta(\mathcal{L})$.

Remark 1. *It could be remarked that a central Lie triple derivation is a generalized Lie triple derivation with $E = 0$.*

As stated in [26], central Lie derivations algebra is an ideal of Lie derivation algebra, and we want to check whether central Lie triple derivations $ZTDer(\mathcal{L})$ are ideal of $TDer(\mathcal{L})$ algebra or not.

Proposition 3. *$ZTDer(\mathcal{L})$ is an ideal of $TDer(\mathcal{L})$ and $GTDer(\mathcal{L})$.*

Proof. By Equation (2), it is not hard to show that $ZTDer(\mathcal{L})$ is subalgebra of $TDer(\mathcal{L})$ and $GTDer(\mathcal{L})$. Next, we show that $ZTDer(\mathcal{L})$ is an ideal. For any $D_1 \in ZTDer_{\theta 1}(\mathcal{L})$ and $D_2 \in TDer_{\theta 2}(\mathcal{L})$, we have

$$\begin{aligned}
&[[[D_1, D_2](s), t], u] \\
=&[[D_1 D_2(s), t], u] - \epsilon(\theta 1, \theta 2)[[D_2 D_1(s), t], u] \\
=&-\epsilon(\theta 1, \theta 2)(D_2([[D_1(s), t], u]) - \epsilon(\theta 2, \theta 1 + s)([[D_1(s), D_2(t)], u]) \\
&-\epsilon(\theta 2, \theta 1 + s + t)[[D_1(s), t], D_2(u)]) \\
=&0
\end{aligned} \tag{18}$$

for any $s, t \in hg(\mathcal{L})$, $u \in \mathcal{L}$. Similarly, we have

$$[D_1, D_2]([[s, t], u]) = 0. \tag{19}$$

Thus, we have $[D_1, D_2] \in ZTDer_{\theta 1+\theta 2}(\mathcal{L})$. This completes the proof. □

3. Centroids and Quasi Centroids

In this section, we define the Lie triple centroid and Lie triple quasi centroid of a Lie color algebra $\mathcal{L}$. We determine their relationship with $TDer(\mathcal{L})$ and $GTDer(\mathcal{L})$. We also discuss the relation between the center $Z(\mathcal{L})$ and these maps. In order to achieve our goal, we first give some definitions, and then we move forward to our main results [23,27,28].

Definition 8. *A linear map $D \in Kl_\theta(\mathcal{L})$ is called a* Lie triple s-map *of $\mathcal{L}$ if it satisfies the following axiom:*

$$D([[s, t], u]) = [[D(s), t], u] = \epsilon(\theta, s + t)[[s, t], D(u)], \tag{20}$$

for any $s, t \in hg(\mathcal{L})$, $u \in \mathcal{L}$.

It should be noted that the second condition in Equation (20) follows from the use of Jacobi identity.

Definition 9. *A linear map $D \in Kl_\theta(\mathcal{L})$ is said to be a* Lie triple qs-map *of $\mathcal{L}$ if it satisfies the following axiom:*

$$[[D(s), t], u] = \epsilon(\theta, s + t)[[s, t], D(u)] \tag{21}$$

for any $s, t \in hg(\mathcal{L})$, $u \in \mathcal{L}$.

The set of all Lie triple s-maps and Lie triple qs-maps of the Lie color algebra $\mathcal{L}$ are called Lie triple centroid and Lie triple quasi centroid, denoted by $TC(\mathcal{L})$ and $TQC(\mathcal{L})$, respectively. Moreover, $TC(\mathcal{L}) = \oplus_{\theta \in \mathcal{G}} TC_\theta(\mathcal{L})$ and $TQC(\mathcal{L}) = \oplus_{\theta \in \mathcal{G}} TQC_\theta(\mathcal{L})$.

Proposition 4. *$TC(\mathcal{L})$ and $TQC(\mathcal{L})$ are Lie color subalgebras of $Kl(\mathcal{L})$.*

Proof. Let $D_1 \in TC_{\theta_1}(\mathcal{L})$, $D_2 \in TC_{\theta_2}(\mathcal{L})$. Then, to prove that $TC(\mathcal{L})$ is a Lie color subalgebra of $Kl(\mathcal{L})$, we must satisfy $[D_1, D_2] \in TC_{\theta_1+\theta_2}(\mathcal{L})$. For any $s, t, u \in hg(\mathcal{L})$, we have

$$\begin{aligned}
&[[[D_1, D_2](s), t], u] \\
=&[[D_1 D_2(s), t], u] - \epsilon(\theta_1, \theta_2)[[D_2 D_1(s), t], u] \\
=&D_1[[D_2(s), t], u] - \epsilon(\theta_1, \theta_2) D_2[[D_1(s), t], u] \\
=&D_1 D_2[[s, t], u] - \epsilon(\theta_1, \theta_2) D_2 D_1[[s, t], u] \\
=&[D_1, D_2]([[s, t], u]).
\end{aligned} \tag{22}$$

Which imply that, $[D_1, D_2] \in TC_{\theta_1+\theta_2}(\mathcal{L})$. This completes the proof and other cases can be proved similarly. □

Proposition 5. *Let $\mathcal{L}$ be a Lie color algebra, for any generalized Lie triple derivation D and Lie triple derivation E related to D; we have $D - E \in TC(\mathcal{L})$.*

Proof. Proof is obvious from Equation (12). □

Proposition 6. *Let $\mathcal{L}$ be a Lie color algebra, for any generalized Lie triple derivation $D \in GTDer(\mathcal{L})$ and Lie triple derivation $E \in TDer(\mathcal{L})$ related to D; we have $(D - E) \in TQC(\mathcal{L})$.*

Proof. From Lemma 1, it is clear that $[[(D-E)(s), t], u] = \epsilon(\theta, s+t)[[s, t], (D-E)(u)]$. Therefore, by Definition 9, we can say $(D - E) \in TQC(\mathcal{L})$. This completes the proof. □

Proposition 7. *If $Z(\mathcal{L}) = 0$, then $TC(\mathcal{L})$ and $TQC(\mathcal{L})$ are commutative Lie color algebras.*

Proof. We only need to prove the first case; the other case can be proven in a similar way. For any $s, t \in hg(\mathcal{L})$ and $u \in \mathcal{L}$, $D_1 \in TC_{\theta_1}(\mathcal{L})$, $D_2 \in TC_{\theta_2}(\mathcal{L})$, we find that

$$\begin{aligned}
[[D_1D_2(s), t], u] &= D_1([[D_2(s), t], u]) \\
&= -\epsilon(\theta_2 + s, t)D_1([[t, D_2(s)], u]) \\
&= -\epsilon(\theta_2 + s, t)([[D_1(t), D_2(s)], u]) \\
&= \epsilon(\theta_2 + s, t)\epsilon(\theta 1 + t, \theta_2 + s)([[D_2(s), D_1(t)], u]) \\
&= \epsilon(\theta_1, \theta_2 + s)D_2([[s, D_1(t)], u]) \\
&= -\epsilon(\theta_1, \theta_2)\epsilon(s, t)D_2([[D_1(t), s], u]) \\
&= \epsilon(\theta_1, \theta_2)D_2D_1([[s, t], u]) \\
&= \epsilon(\theta_1, \theta_2)[[D_2D_1(s), t], u]
\end{aligned} \tag{23}$$

Thus, we have $[[[D_1, D_2](s), t], u] = 0$. Since $Z(\mathcal{L}) = 0$, we get $[D_1, D_2] = 0$. Hence, the proof is completed. □

Theorem 1. *Let $\mathcal{L}$ be a perfect Lie color algebra, $Z(\mathcal{L})$ the center of $\mathcal{L}$. Then $[TC(\mathcal{L}), TQC(\mathcal{L})] \subseteq Hom(\mathcal{L}, Z(\mathcal{L}))$. Moreover, if $Z(\mathcal{L}) = 0$, then $[TC(\mathcal{L}), TQC(\mathcal{L})] = 0$.*

Proof. Let $D_1 \in TC_{\theta_1}(\mathcal{L})$ and $D_2 \in TQC_{\theta_2}(\mathcal{L})$. Then for $s, t \in hg(\mathcal{L})$ and $u \in \mathcal{L}$, we have

$$\begin{aligned}
[[[D_1, D_2](s), t], u] &= [[D_1D_2(s), t], u] - \epsilon(\theta_1, \theta_2)[[D_2D_1(s), t], u] \\
&= D_1([[D_2(s), t], u]) - \epsilon(\theta_1, \theta_2)\epsilon(\theta_2, \theta_1 + s + t)([[D_1(s), t], D_2(u)]) \\
&= D_1([[D_2(s), t], u]) - \epsilon(\theta_2, s + t)D_1([[s, t], D_2(u)]) \\
&= D_1([[D_2(s), t], u] - \epsilon(\theta_2, s + t)[[s, t], D_2(u)]) \\
&= 0.
\end{aligned} \tag{24}$$

From Equation (21), we get that $[[[D_1, D_2](s), t], u] = \epsilon(\theta_1 + \theta_2, s + t)[[s, t], [D_1, D_2](u)]$. Since $\mathcal{L}$ is perfect, $[D_1, D_2](u) \in Z(\mathcal{L})$. Hence $[D_1, D_2](u) \in Z_{\theta_1+\theta_2}(\mathcal{L})$ and $[D_1, D_2] \in Hom(\mathcal{L}, Z(\mathcal{L}))$. In addition, if $Z(\mathcal{L}) = 0$, then clearly $[TC(\mathcal{L}), TQC(\mathcal{L})] = \{0\}$. □

Proposition 8. *Let $\mathcal{L}$ be a Lie color algebra. Then we have the following results:*

1. $[GTDer(\mathcal{L}), TC(\mathcal{L})] \subseteq TC(\mathcal{L})$.
2. $[TDer(\mathcal{L}), TC(\mathcal{L})] \subseteq TC(\mathcal{L})$.

Proof. Consider the first case. Other cases can be proven in a similar way. Suppose that $D_1 \in GTDer_{\theta_1}(\mathcal{L})$, $D_2 \in TC_{\theta_2}(\mathcal{L})$, $s,t \in hg(\mathcal{L}), u \in \mathcal{L}$. Then

$$\begin{aligned}
&[[D_1D_2(s),t],u]\\
=&D_1([[D_2(s),t],u]) - \epsilon(\theta_1,\theta_2+s)[[D_2(s),E_1(t)],u]\\
&-\epsilon(\theta_1,\theta_2+s+t)[[D_2(s),t],E_1(u)]\\
=&D_1D_2([[s,t],u]) - \epsilon(\theta_1,\theta_2+s)D_2([[s,E_1(t)],u])\\
&-\epsilon(\theta_1,\theta_2+s+t)D_2([[s,t],E_1(u)]).
\end{aligned} \tag{25}$$

In the same way, we have

$$\begin{aligned}
&[[D_2D_1(s),t],u]\\
=&D_2([[D_1(s),t],u])\\
=&D_2D_1([[s,t],u]) - \epsilon(\theta_1,s)D_2([[s,E_1(t)],u])\\
&-\epsilon(\theta_1,s+t)D_2([[s,t],E_1(u)]).
\end{aligned} \tag{26}$$

By now, we have

$$[[[D_1,D_2](s),t],u] = [D_1,D_2]([[s,t],u]). \tag{27}$$

Thus, $[D_1,D_2] \in TC_{\theta 1+\theta 2}(\mathcal{L})$. This completes the proof. □

Proposition 9. *Let $\mathcal{L}$ be a Lie color algebra. Then we have the following results:*

1. $[GTDer(\mathcal{L}), TQC(\mathcal{L})] \subseteq TQC(\mathcal{L})$.
2. $[TDer(\mathcal{L}), TQC(\mathcal{L})] \subseteq TQC(\mathcal{L})$.

Proof. As with previous results, we only show the proof of first result; the second case can be obtained similarly. Suppose that $D_1 \in GTDer_{\theta_1}(\mathcal{L})$, $D_2 \in TQC_{\theta_2}(\mathcal{L})$, $s,t \in hg(\mathcal{L})$, $u \in \mathcal{L}$. Then

$$\begin{aligned}
&[[D_1D_2(s),t],u]\\
=&D_1([[D_2(s),t],u]) - \epsilon(\theta_1,\theta_2+s)[[D_2(s),E_1(t)],u]\\
&-\epsilon(\theta_1,\theta_2+s+t)[[D_2(s),t],E_1(u)]\\
=&\epsilon(\theta_2,s+t)D_1[[s,t],D_2(u)] - \epsilon(\theta_1+\theta_2,s)[[s,E_1(t)],D_2(u)]\\
&-\epsilon(\theta_1,\theta_2)\epsilon(\theta_1+\theta_2,s+t)[[s,t],D_2E_1(u)].
\end{aligned} \tag{28}$$

On the other hand, we have

$$\begin{aligned}
&[[D_2D_1(s),t],u]\\
=&\epsilon(\theta_2,\theta_1+s+t)[[D_1(s),t],D_2(u)]\\
=&\epsilon(\theta_2,\theta_1+s+t)(D_1([[s,t],D_2(u)]) - \epsilon(\theta_1,s)[[s,E_1(t)],D_2(u)]\\
&-\epsilon(\theta_1,s+t)[[s,t],E_1D_2(u)]).
\end{aligned} \tag{29}$$

By using Definition 9, we find that

$$[[[D_1,D_2](s),t],u] = \epsilon(\theta 1+\theta 2,s+t)[[s,t],[E_1,D_2](u)]. \tag{30}$$

Thus, we have $[D_1,D_2] \in TQC_{\theta 1+\theta 2}(\mathcal{L})$. This completes the proof. □

4. Classification of Triple Derivations of Perfect Lie Color Algebras

In the final section, we classify (generalized) Lie triple derivations of all perfect Lie color algebras. Let us recall some useful definitions. For the similar results related to Lie algebras and Lie superalgebras, readers are referred to [20,21].

Definition 10. *For any $s \in \mathcal{L}$, we can define adjoint derivation $ad : \mathcal{L} \to \mathcal{L}$ such that $ad(s)(t) = [s, t]$, for all $t \in \mathcal{L}$. The set of all such derivations of $\mathcal{L}$ is denoted by $Inn(\mathcal{L})$.*

For convenience, we assume $\mathcal{L}$ to be a finite dimensional perfect Lie color algebra throughout this section. It is easy to see that $Der(\mathcal{L})$ and $Inn(\mathcal{L})$ are subalgebras of the Lie color algebras of $TDer(\mathcal{L})$. Furthermore, we have the following lemmas.

Lemma 2. *$Inn(\mathcal{L})$ is an ideal of the Lie color algebra $TDer(\mathcal{L})$.*

Proof. Let $D \in TDer_\theta(\mathcal{L})$ and $s \in \mathcal{L}$. Since $\mathcal{L}$ is a perfect Lie color algebra, there exists $s = \sum_{i\in I}[s_{i_1}, s_{i_2}]$ for some finite index set I such that $s_{i_1}, s_{i_2} \in hg(\mathcal{L})$. For any arbitrary $t \in \mathcal{L}$, we have

$$\begin{aligned}
&[D, ad(s)](t)\\
=&Dads(t) - \epsilon(\theta, s)ad(s)(D(t))\\
=&D([s, t]) - \epsilon(D, t)[s, D(t)]\\
=&D([\sum_{i\in I}[s_{i_1}, s_{i_2}], t]) - \epsilon(\theta, s)[\sum_{i\in I}[s_{i_1}, s_{i_2}], D(t)]\\
=&\sum_{i\in I}([[D(s_{i_1}), s_{i_2}], t] + \epsilon(\theta, s_{i_1})[[s_{i_1}, D(s_{i_2})], t]\\
&+\epsilon(\theta, s_{i1} + s_{i_2})[[s_{i_1}, s_{i_2}], D(t)]) - \epsilon(\theta, s)[\sum_{i\in I}[s_{i_1}, s_{i_2}], D(t)]\\
=&\sum_{i\in I}[[D(s_{i_1}), s_{i_2}], t] + \epsilon(\theta, s_{i_1})[[s_{i_1}, D(s_{i_2})], t]\\
=&ad(\sum_{i\in I}([D(s_{i_1}), s_{i_2}] + \epsilon(\theta, s_{i_1})[s_{i_1}, D(s_{i_2})]))(t).
\end{aligned} \tag{31}$$

By the arbitrariness of t, $Inn(\mathcal{L})$ is an ideal of Lie color algebra $TDer(\mathcal{L})$. □

In fact, there is another connection between $Der(\mathcal{L})$ and $TDer(\mathcal{L})$. First let us give some lemmas.

Lemma 3. *For a perfect Lie color algebras $\mathcal{L}$ with zero center, there exists a linear map $\delta : TDer(\mathcal{L}) \to Der(\mathcal{L})$, $D \mapsto \delta_D$ such that for all $s \in \mathcal{L}$, $D \in TDer(\mathcal{L})$, we have $[D, ad(s)] = ad\delta_D(s)$.*

Proof. By the proof of Lemma 2, if $\mathcal{L}$ is perfect and has zero center, then we can define a linear endomorphism δ_D on $\mathcal{L}$ as follows. For any $s = \sum_{i\in I}[s_{i_1}, s_{i_2}] \in \mathcal{L}$, we have

$$\delta_D(s) = \sum_{i\in I}([D(s_{i_1}), (s_{i_2})] + \epsilon(\theta, s_{i_1})[s_{i_1}, D(s_{i_2})]). \tag{32}$$

This definition of $\delta_D(s)$ does not depend on the choice of expression of s. To prove it, we take

$$\begin{aligned}
\gamma &= \sum_{i\in I}([D(s_{i_1}), s_{i_2}] + \epsilon(\theta, s_{i_1})[s_{i_1}, D(s_{i_2})]),\\
\omega &= \sum_{j\in J}([D(t_{j_1}), t_{j_2}] + \epsilon(\theta, t_{j_1})[t_{j_1}, D(t_{j_2})]),
\end{aligned} \tag{33}$$

where s can also be expressed in the other form $s = \sum_{j\in J}[t_{j_1}, t_{j_2}]$. Since $D \in TDer(\mathcal{L})$, for any $u \in \mathcal{L}$, we have that

$$[\gamma, u] = D([s, u]) - \epsilon(\theta, s)[s, D(u)] = [\omega, u]. \tag{34}$$

Hence, $[(\gamma - \omega), u] = 0$, for any $u \in \mathcal{L}$, i.e., $\gamma - \omega \in Z(\mathcal{L})$. As the center is zero, we have that $\gamma = \omega$. Hence, δ_D is well defined. Furthermore, it follows from the proof of Lemma (2) immediately that $[D, ad(s)] = ad\delta_D(s)$.

Finally, to prove that δ_D is a derivation of $\mathcal{L}$. for any $D \in TDer(\mathcal{L})$, $s, t \in \mathcal{L}$, we have $[D, ad[s,t]] = ad\delta_D([s,t])$. By using Jacobi identity for Lie color algebras, we have

$$[s,[t,u]] = [[s,t],u] + \epsilon(s+t,u)[[u,s],t]. \tag{35}$$

Then

$$\begin{aligned}] =& [D,[ad(s),ad(t)]] \\ =& [[D,ad(s)],ad(t)] + \epsilon(\theta+s,t)[[adt,D],ad(s)] \\ =& [ad\delta_D(s),ad(t)] - \epsilon(s,t)[[D,ad(t)],ad(s)] \\ =& [ad\delta_D(s),ad(t)] - \epsilon(s,t)[[ad\delta_D(t),ad(s)] \\ =& ad([\delta_D(s),t] + \epsilon(\theta,s)[s,\delta_D(t)]). \end{aligned} \tag{36}$$

Therefore, $ad\delta_D([s,t]) = ad([\delta_D(s),t] + \epsilon(\theta,s)[s,\delta_D(t)])$. By the arbitrariness of s, t, we can say that $\delta_D \in Der(\mathcal{L})$. □

Lemma 4. *Suppose that $\mathcal{L}$ is a perfect Lie color algebra. Then the centralizer of $Inn(\mathcal{L})$ in $TDer(\mathcal{L})$, i.e., $C_{TDer(\mathcal{L})}(Inn(\mathcal{L})) = 0$. More specifically, the center of $TDer(\mathcal{L})$ is zero.*

Proof. Suppose that $D \in C_{TDer_\theta(\mathcal{L})}(Inn(\mathcal{L}))$. Then $[D, ads] = 0$ for any $s \in \mathcal{L}$. Thus, $D([s,t]) = \epsilon(\theta,s)[s,D(t)] = [D(s),t]$ for any $t \in \mathcal{L}$. On the one hand,

$$D([[s,t],u]) = [[D(s),t],u] = \epsilon(\theta,s)[[s,D(t)],u] = \epsilon(\theta,s+t)[[s,t],D(u)], \tag{37}$$

for any $s, t \in hg(\mathcal{L}), u \in \mathcal{L}$. Since $D \in TDer(\mathcal{L})$, we can obtain that

$$D([[s,t],u]) = 3D([[s,t],u]). \tag{38}$$

This means that $D([[s,t],u]) = 0$ for any $s, t, u \in \mathcal{L}$. Since $\mathcal{L}$ is a perfect Lie color algebra, every element of $\mathcal{L}$ can be written as the linear combination of elements of the form $s = \sum_{i\in I}[s_{i_1}, s_{i_2}]$. This implies that $D = 0$. □

Now, we will present the first main result of this section.

Theorem 2. *Let $\mathcal{L}$ be a perfect Lie color algebra with zero center. Then $TDer(\mathcal{L}) = Der(\mathcal{L})$.*

Proof. For $D \in TDer(\mathcal{L})$, $\delta_D \in Der(\mathcal{L})$, and by using Lemma 3, it is clear that $[D, ad(s)] = ad(\delta_D(s))$ and $\delta_D \in Der(\mathcal{L})$. Moreover for any $t \in \mathcal{L}$, we have $ad(\delta_D(s))(t) = [\delta_D(s), t] = \delta_D[s,t] - \epsilon(\theta,s)[s,\delta_D(t)] = [\delta_D, ad(s)](t)$. By the arbitrariness of t, we have that $ad\delta_D(s) = [\delta_D, ad(s)]$. Thus, we have $[D, ad(s)] = [\delta_D, ad(s)]$. So $[(D - \delta_D), ad(s)] = ad(D - \delta_D)(s)$, for any $s \in \mathcal{L}$. By Lemma 4, it is clear that $D - \delta_D = 0$; this implies $D = \delta_D$. □

The second main result of this article is given as follows.

Theorem 3. *If $\mathcal{L}$ is a perfect Lie color algebra and $Z(\mathcal{L}) = 0$, then $TDer(Der(\mathcal{L})) = Inn(Der(\mathcal{L}))$.*

The proof of the theorem follows from the following lemmas.

Lemma 5. *If $\mathcal{L}$ is a perfect Lie color algebra, $D \in TDer(Der(\mathcal{L}))$, then $D(ad(\mathcal{L})) \subseteq Inn(\mathcal{L})$.*

Proof. Since $\mathcal{L}$ is a perfect Lie color algebra, for any element $s = \sum_{i\in I}[s_{i_1}, s_{i_2}] \in \mathcal{L}$, we have

$$\begin{aligned} D(ad(s)) =&([[D(ad(s_{i_1})), ad(s_{i_2})], ad(s_{i_3})]) + \epsilon(\theta, s_{i_1})[[ad(s_{i_1}), D(ad(s_{i_2}))], ad(s_{i_3})] \\ &+\epsilon(\theta, s)[[ad(s_{i_1}), ad(s_{i_2})], D(ad(s_{i_3}))]. \end{aligned} \tag{39}$$

From Lemma 2, it is not hard to see that $D(Inn(\mathcal{L})) \subseteq Inn(\mathcal{L})$. □

Lemma 6. *Let $\mathcal{L}$ be a perfect Lie color algebra with the zero center. If $D \in TDer(Der(\mathcal{L}))$, then for any $s \in \mathcal{L}$, there exists $d \in Der(\mathcal{L})$, such that $D(ad(s)) = ad(d(s))$.*

Proof. For any $D \in TDer(Der(\mathcal{L}))$, $s \in \mathcal{L}$, and by Lemma 5, $D(ad(s)) \in Inn(\mathcal{L})$. Then there exists $t \in \mathcal{L}$, such that $D(ad(s)) = ad(t)$. Since the center is zero, such a t is unique. Let $s_1, s_2 \in hg(\mathcal{L})$ and $s_3 \in \mathcal{L}$. Therefore,

$$\begin{aligned} ad(d([[s_1, s_2], s_3])) =&D(ad([[s_1, s_2], s_3])) \\ =&D([[ads_1, ads_2], ads_3]) \\ =&[[D(ads_1), ads_2], ads_3] + \epsilon(\theta, s_1)[[ads_1, D(ads_2)], ads_3] \\ &+\epsilon(\theta, s_1 + s_2)[[ads_1, ads_2], D(ads_3)]. \\ =&[[ad(d(s_1)), ads_2], ads_3] + \epsilon(\theta, s_1)[[ads_1, ad(d(s_2))], ads_3] \\ &+\epsilon(\theta, s_1 + s_2)[[ads_1, ads_2], ad(d(s_3))] \\ =&ad([[d(s_1), s_2], s_3] + \epsilon(\theta, s_1)[[s_1, d(s_2)], s_3] \\ &+\epsilon(\theta, s_1 + s_2)[[s_1, s_2], d(s_3)]). \end{aligned} \tag{40}$$

Since $Z(\mathcal{L}) = 0$, we have, $d \in TDer(\mathcal{L})$. By Theorem 2, $d \in Der(\mathcal{L})$. □

Now it is not difficult to prove Theorem 3 by using above lemmas.

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